Complete blow-up and thermal avalanche for heat equations with nonlinear boundary conditions

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Abstract

We study the continuation after blow-up of solutions $u(x,t)$ of the heat equation, $u_t = u_{xx}$, with a nonlinear flux condition at the boundary, $-u_x(0,t) = f(u(0,t))$. We prove that such a continuation is trivial (i.e., identically infinite for all $t > T$, where T is the blow-up time) in the following sense: if we replace $f(u)$ by a sequence of functions $f_n(u)$ that do not cause blow-up and f_n converges to f uniformly on compact sets, then the corresponding solutions u_n satisfy $\lim_{n\to\infty} u_n(x,t) = +\infty$ for every $x > 0$ and every $t > T$. This is called complete blow-up and happens in the present problem for all positive and continuous functions f for which solutions blow up.

An interesting phenomenon related to complete blow-up is the thermal avalanche: in the cases in which there is single-point blow-up as $t \nearrow T$ the singularity at the origin propagates instantaneously at time $t = T$ to cover the whole space. We describe the formation of the avalanche in the case $f(u) = u^p$, $p > 1$, as a boundary layer which appears in the limit of the approximate problems by choosing a suitable scaling and passing to self-similar variables. We then show that the layer is described by the solution of a limit problem. We also describe the asymptotic behaviour for the approximate problems as t goes to infinity.

We also consider the nonlinear diffusion equation $u_t = (u^m)_{xx}$, $m > 0$, with nonlinear boundary condition $-(u^m)_x(0,t) = f(u(0,t))$. We prove that blow-up solutions have a nontrivial continuation if and only if $m < 1$, independently of f. The thermal avalanche when $m > 1$ and f is a power is also described in this case, as well as the asymptotic behaviour for the approximations for all m.

AMS Subject Classification: 35B40, 35K60, 35K65.

Keywords and phrases: blow-up, heat equation, nonlinear boundary condition, avalanche, boundary layer.

1 Introduction

This paper is mainly concerned with the description of the form in which blow-up occurs for the reaction-diffusion problem

$$(1.1) \qquad \begin{cases} u_t = u_{xx} & (x,t) \in (0,1) \times (0,T), \\ -u_x(0,t) = f(u(0,t)) & t \in (0,T), \\ u_x(1,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & x \in (0,1), \end{cases}$$

where f is positive and continuous. We assume that u_0 is continuous, nonnegative and nontrivial ($u_0 \not\equiv 0$). Hence, the solution of problem (1.1) exists for a certain time interval $0 < t < T$ and is C^1 up to the boundary for all $t > 0$ as long as it is bounded. The problem can be thought of as a model to describe heat propagation with constant thermal conductivity (also referred to as linear diffusion) in a medium with a nonlinear radiation law at the left boundary, the right one being thermally insulated, [Pa]. Let T be the maximal existence time for the solution. If $T < \infty$, then u becomes unbounded in finite time and we say that it *blows up*, while for $T = \infty$ we say that the solution is *global.* The study of blow-up solutions of the heat equation with a nonlinear boundary condition was initiated in [LP] and since then it has attracted considerable attention, see the survey papers [FF], [CF2].

An important aspect of blow-up problems is the possibility of having a nontrivial extension of the solution for times $t > T$. If such a continuation exists we say that blow-up is *incomplete*; otherwise, it is called *complete*. Our first aim is to study whether blow-up for problem (1.1) is complete or not. A natural way of obtaining a continuation consists of approximating the reaction nonlinearity $f(u)$ in the boundary condition by a sequence of functions $f_n(u)$ such that

$$(1.2) \qquad \begin{cases} f_n(u) \text{ is positive and continuous for every } n; \\ \text{solutions of problem (1.1) with } f \text{ replaced by } f_n \text{ are global;} \\ f_n(u) \to f(u) \text{ uniformly on bounded intervals.} \end{cases}$$

Under these assumptions we solve the approximate problems with f replaced by f_n and obtain a sequence of global solutions $\{u_n\}$. We want to extend $u(x,t)$ as the limit

$$(1.3) \qquad u(x,t) = \lim_{n\to\infty} u_n(x,t)$$

if this limit (finite or infinite) exists for every $(x,t) \in [0,1] \times \mathbb{R}^+$. It is easy to see that for $t < T$ we recover the standard solution of the problem. The question of complete/incomplete blow-up is then reduced to determining whether the limit (1.3) becomes identically infinite for $t > T$ or not. As in [GV2] we will call the limit (1.3) the *proper solution* of (1.1), even if it becomes infinite completely or only somewhere after T.

Complete blow-up was first studied for problems where the nonlinearity occurs in the equation as a reaction term, $u_t = u_{xx} + f(u)$, see [BC], [GV1]-[GV3], [LT], [M]. For the present problem (1.1) in the special case of where the flux function is a power, $f(u) = u^p$, and f_n is a truncation at level n, complete blow-up is proved in [FG].

Our first result states that blow-up for problem (1.1) is complete for any radiation function f as above for which there is blow-up, in the sense that the proper solution is identically infinite for $t > T$.

Our second aim is to describe how the lack of continuation happens by studying the behaviour of the global solutions of the sequence of approximate problems. When f is a power we will be able to precisely describe how the catastrophic event, so-called *avalanche*, takes place near $t = T$ in terms of the most natural approximations. More precisely, we will show that for a sequence of truncations f_n of f which are linear at infinity with increasing slope $f_n'(s) \sim cn^{p-1}$, the global solutions u_n *undergo at times $t \approx T$ the transition from an approximation of the blow-up profile of u towards an approximate traveling wave which propagates at speed $v = v(n)$. As $n \to \infty$ we have $v(n) \to \infty$ and complete blow-up occurs.* This intermediate profile evolves later towards different asymptotics of parabolic type which we also describe. Numerical results illustrate these stages. Let us remark that the study of the approximate problems for large n is in many cases (combustion, chemistry) more realistic than the blow-up problem, which is a mathematical idealization.

Finally, we consider the effect of a nonlinear diffusion in the equation. For simplicity we take the usual power form $u_t = (u^m)_{xx}$ with diffusion exponent $m > 0$. For $m > 1$ this equation is called the porous medium equation, while for $m < 1$ it is known as the fast-diffusion equation. In any case, we arrive to

$$(1.4) \qquad \begin{cases} u_t = (u^m)_{xx} & (x,t) \in (0,1) \times (0,T), \\ -(u^m)_x(0,t) = f(u(0,t)) & t \in (0,T), \\ (u^m)_x(1,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & x \in (0,1). \end{cases}$$

We find the same phenomenon of complete blow-up for all f provoking blow-up if and only if $m \geq 1$, i.e., for porous medium equations. The thermal avalanche is also described in this case. On the contrary, for the so-called fast diffusion equations, $0 < m < 1$, continuation after blow-up is always possible. Moreover, for $m < 1$ the only blow-up point is the origin $x = 0$ for all $t \geq T$ and we are able to describe the form of the isolated singularity after blow-up.

The last results show the fundamental role played by diffusion in the propagation of the ∞-level set. In case $m > 1$ the infinite level propagates with infinite speed and there is no possible continuation, no matter which is the nonlinearity f that we are considering at the boundary. However, when $m < 1$ the ∞-level set does not propagate at all and there is an extension beyond $t = T$ that remains finite everywhere except at the boundary, $x = 0$. Actually, as explained in [ChV], the label fast-diffusion is misleading for blow-up problems since the diffusivity $D(u) = mu^{m-1}$ goes to zero as $u \to \infty$ for $m < 1$.

There is a certain similarity between these problems and the reaction-diffusion equation

$$(1.5) \qquad u_t = \phi(u)_{xx} + f(u).$$

However, there is a major difference concerning the occurrence of complete blow-up. It was proved in [GV1] that the Cauchy problem for (1.5) has complete blow-up under a balance condition between diffusivity and reaction rate, which for powers $\phi(u) = u^m$ and $f(u) = u^p$, $m > 0$, $p > 1$ reads exactly $m + p \geq 2$. On the other hand, our results for problem (1.4) show that complete blow-up happens for $m \geq 1$ independently of the radiation function $f(u)$. We want to remark that in the cases of incomplete blow-up studied in [GV1] the blow-up set expands after the blow-up time T, while in our case for $m < 1$ it is stationary, a single point. To our knowledge there are no mathematical results on the formation of avalanches for the reaction-diffusion equation. Travelling waves and their stability are studied in [BLL]. On the other hand, the formation and properties of travelling waves are a main topic in the mathematical theory of combustion, cf. for instance the study of plane deflagration waves in [BL].

To be specific, the problems are first posed in an interval. The length $L = 1$ is inessential. The same results apply in the half-line $x \in (0, \infty)$ to blowing up solutions with radiation data at $x = 0$ and some reasonable behaviour of u_0 at infinity. Details on the necessary adaptations will be given at the end of each section.

Finally, let us remark that the results can be extended to higher space dimension for radial solutions in the interior or the exterior of a ball. However, the analysis of the non-radial case becomes more delicate. In this respect, complete blow-up with non expanding blow-up set can happen for the reaction-diffusion equation in several space dimensions, [GV2], [LT], but the analysis is difficult and a number of important questions remain open.

The paper is organized as follows: in Section 2 we prove complete blow-up for problem (1.1). In Section 3 we describe the thermal avalanche. In Section 4 we deal with the asymptotic behaviour for a limit function z that describes the avalanche. In Section 5 we study the asymptotics for large times for the approximate problems. In Section 6 we analyze the porous medium equation. Finally in Section 7 we prove incomplete blow-up for the fast diffusion equation.

2 Complete blow-up for linear diffusivity

In this section we prove the complete blow-up result for the heat equation with general flux function f. As additional information, we recall that when f is convex it was shown in [W] that a necessary and sufficient condition for blow-up is $\int^{+\infty} 1/ff' < +\infty$. A sufficient condition for blow-up without convexity assumptions on f can be found in [RR].

Theorem 2.1 *For any positive and continuous f for which there is blow-up, complete blow-up occurs, i.e., for any approximating sequence f_n that satisfies* (1.2) *we have that for $t \neq T$ the limit $\lim_{n\to\infty} u_n(x,t)$ exists and*

$$\lim_{n\to\infty} u_n(x,t) = \begin{cases} u(x,t) & t < T, \\ \\ \infty & t > T. \end{cases}$$

Proof. We proceed in two steps. First, we prove that the proper solution is identically infinite at the origin for $t > T$. The second step consists in propagating this property to the whole interval.

To verify the first step let us prove that given $M \gg 1$, $\delta > 0$, we have $u_n(0,t) \geq M$ for all n large enough and all $t \in [T, T+\delta]$. To see this, we use the explicit stationary solution

$$\psi_n(x) = M - f_n(M)x.$$

This function ψ_n is a subsolution of our problem with f_n. Recall now that for every fixed M we have $f_n(M) \to f(M)$ as $n \to \infty$. Let us observe that, as f is blow-up nonlinearity,

$f(M)/M$ goes to infinity at least along a sequence, $M_j \to \infty$. On the other hand by the regularizing effect of the heat equation, $u_n(x, t_0)$ is Lipschitz for any $t_0 > 0$. Therefore, choosing $M = M_j$ and n large enough, $\psi_n(x)$ and $u_n(x, t_0)$ intersect exactly once (it suffices to choose $M > u_n(0, t_0)$ and $f_n(M) > \|u(\cdot, t_0)\|_{W^{1,\infty}}$). Next we observe that since $u(0, t)$ blows up when $t \nearrow T$, there must exist a time $t_1 < T$ such that $u_n(0, t_1) > u(0, t_1) - \varepsilon > M$ for every n large enough. Now, we use an intersection comparison (lap number) argument: the number of sign changes of the difference $u_n - \psi_n$ does not increase in time, unless it increases at the boundary. At the boundary $x = 1$ the ordering $u_n(1, t) > \psi_n(1)$ holds for $M = M_j$, j large. Let us see that we cannot gain an intersection at the boundary $x = 0$. Indeed, at the first time t_2 such that $u_n(0, t_2) = \psi_n(0)$, if there is another intersection at the interior of the interval, then the difference $\psi_n - u_n$ attains a minimum at $x = 0$, $t = t_2$, while the boundary conditions imply that the normal derivative of the difference is zero at that point. This contradicts Hopf's lemma. Therefore, we conclude that the intersection has moved to the boundary and we must have $u_n(x, t_2) \geq \psi_n(x)$. By a comparison argument, $u_n(x, t) \geq \psi_n(x)$ for every $t \geq t_2$ and n large. Therefore

$$u_n(0, t) \geq \psi_n(0) = M \qquad \text{for all } t \geq t_2, \ n \text{ large.}$$

Observe that this implies that

$$\lim_{n \to \infty} u_n(0, t) = +\infty \qquad \text{for all } t > T.$$

For the second step, the idea is to use the representation formula obtained from the heat kernel. Let $\Gamma(x, t)$ be the fundamental solution of the heat equation, namely

$$\Gamma(x, t) = \frac{1}{(\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right).$$

For $x \in (0, 1)$ we have

$$u_n(x, t) = \int_0^1 u_n(y, 0)\Gamma(x - y, t)\, dy - \int_0^t \frac{\partial u_n}{\partial x}(0, \tau)\Gamma(x, t - \tau)\, d\tau$$
$$- \int_0^t u_n(0, \tau)\frac{\partial \Gamma}{\partial x}(x, t - \tau)\, d\tau + \int_0^t u_n(1, \tau)\frac{\partial \Gamma}{\partial x}(x - 1, t - \tau)\, d\tau.$$

Now we observe that Γ satisfies

$$\Gamma(x, t - \tau) \geq 0, \qquad \frac{\partial \Gamma}{\partial x}(x - 1, t - \tau) \geq 0 \qquad \text{for } x \in [0, 1].$$

Hence, using the boundary conditions and the positivity of f_n we get, for $x \in (0, 1)$, that

$$u_n(x, t) \geq -\int_0^t u_n(0, \tau)\frac{\partial \Gamma}{\partial x}(x, t - \tau)\, d\tau.$$

For the point $x = 1$, taking limits as $x \to 1$, we can conclude as in [Fr] that

$$\frac{1}{2}u_n(1, t) \geq -\int_0^t u_n(0, \tau)\frac{\partial \Gamma}{\partial x}(1, t - \tau)\, d\tau.$$

Now we use the first step to get, for every $x \in (0,1]$, that

$$u_n(x,t) \geq M \int_T^{T+\delta} \left(-\frac{\partial \Gamma}{\partial x}(x, t-\tau) \right) d\tau,$$

for $t \geq T + \delta$. We conclude that

$$\lim_{n \to \infty} u_n(x,t) = +\infty \qquad t > T.$$

This completes the proof. □

Remark 1. The case where $f(u) = u^p$ and $f_n(s) = \min\{s^p, n^p\}$, $n \in \mathbb{N}$, has already been proved in [FG]. We show here that complete blow-up occurs for the whole class of blow-up nonlinearities f, all the approximations satisfying the natural conditions (1.2) and all initial data as above. In particular, the assumption of monotonicity on f which is usually made in blow-up problems has been eliminated.

Remark 2. The assumptions that we have made on f are natural but can be weakened. Continuity allows us to apply standard existence and regularity theory. Concerning the positivity assumption, we remark that in a blow-up problem $f(u)$ must be eventually positive, $f(u) > 0$ for large values of u. Indeed, every zero of the function f, say $f(c) = 0$, produces a constant solution $u \equiv c$ which does not blow-up. It follows that whenever f has arbitrarily large zeros then no solution blows up. The positivity assumption on f can be weakened into the condition that f is eventually positive. In that case not all solutions necessarily blow up, but theorem and its proof hold for blowing up solutions.

Remark 3. With the positivity assumption on f blow-up does not depend on the initial data: a comparison argument shows that if there is blow-up for some initial data, then the same is true for every nontrivial initial data since solutions eventually become large.

Remark 4. When f is increasing we have a stronger result: the previous argument can be done at all levels $M \gg 1$ and not only for a sequence $M_j \to \infty$. It follows that $u_t(0,t) \geq 0$ for all t close to T. Hence, blow-up is achieved in a monotone way in this case.

Remark 5. For the sake of completeness we briefly sketch the argument which shows that the solutions of our problem are Lipschitz continuous in x for $0 < \tau < t < T$. By approximation we may always argue on a sequence of regular solutions u_j corresponding to approximating the initial data. The only problem arises near the lateral boundary $x = 0$. It is clear that the u_j are uniformly bounded for $0 \leq t \leq T - \varepsilon$. Now, since the functions $v_j = u_{j,x}$ are also solutions of the heat equation with uniformly bounded Dirichlet data, they are also uniformly bounded in a neighbourhood of the lateral boundary. This gives a uniform Lipschitz bound in x for u_j away from $t = 0$.

Behaviour at the blow-up time. In order to study what happens exactly at $t = T$, we need additional assumptions on the approximation. We assume that the sequence f_n verifies (1.2), f_n is increasing with n with $f_n \leq f$ and f is such that u has single point blow-up at time $t = T$. By a blow-up point we mean a point $a \in [0,1]$ such that there are $\{x_n\} \subset [0,1]$ and $t_n \nearrow T$ such that $x_n \to a$ and $u(x_n, t_n) \to \infty$ as $n \to \infty$. Under these assumptions we have that the limit $\lim_{n\to\infty} u_n(x,T)$ exists and

$$\lim_{n \to \infty} u_n(x,T) = \begin{cases} \text{finite} & x \neq 0, \\ \infty & x = 0. \end{cases}$$

To see this fact we just observe that if the blow-up set for u is a single point $x = 0$ then, as $f_n(u) \le f(u)$, using a comparison argument, we have $u_n(x,t) \le u(x,t) \le K(x)$ for $x \ne 0$, $t < T$. To finish the argument we recall that in the course of the proof we have shown that $u_n(0,T) \ge M$ for every n large enough, proving that $\lim_{n\to\infty} u_n(0,T) = +\infty$.

The problem in the half-line. In this case there is a difference in the sense that there can be functions f for which some solutions blow up while others do not. When $f(u) = u^p$ this depends on the so-called Fujita exponent, which for linear diffusion is $p = 2$, see [GL], and for general references [L] and [DL]. Therefore, we have to talk about blow-up solutions u, not only about blow-up radiation functions f. In performing the previous proof, when we need to show that there is blow-up at $x = 0$ for all $t > T$ we observe that the function

$$\psi_n(x) = M - f_n(M)x$$

used for the case of the bounded domain, is also a subsolution when considered for the whole space. In the intersection argument, only the behaviour of u near $x = 0$ is used. Then, the propagation of the blow-up to the whole space is done by using a representation formula, exactly as for the case of the bounded domain. This is where some assumption on the behaviour of u_0 for large x is needed.

3 The avalanche at the blow-up time

Next we turn to the question of how complete blow-up takes place at times near $t = T$. We recall that in combustion problems blow-up appears usually when we perform a limit in equations or systems that have global existence. This means that the real situation in the applications corresponds to nonlinearities like f_n for large n. Hence the interest in understanding the behaviour of the solutions u_n at times $t \sim T$ when the limit u blows up at $t = T$. Also the behaviour of u_n at later times is important and will be studied in Section 5.

It is known that the only blow-up point is $x = 0$ provided that $\int^{+\infty} 1/f < +\infty$, see [LMW], [RR]. We have thus *single point blow-up* in this case and our first theorem implies that there is an instantaneous propagation of the singularity to the whole interval, what we call a *thermal avalanche*. For large but finite n this avalanche shows up in the form of a *boundary layer*, the short time interval around $t = T$ in which the enormous growth of the solution in the whole interval takes place. Our main aim is to study the thermal avalanche, in the particular case $f(u) = u^p$ with $p > 1$, in which scaling methods can be used to describe the phenomenon in a smooth way. We will see that the particular way in which the avalanche takes places depends strongly on the approximating sequence $\{f_n\}$. In order to perform the blow-up analysis we need to impose the following additional conditions on the sequence f_n:

$$\text{(3.1)} \qquad \begin{cases} f_n(s) = s^p & \text{for } s < n, \\ \dfrac{f_n(ns)}{n^p} \nearrow g(s) & n \to \infty, \text{ for all } s \ge 1, \end{cases}$$

for some finite function g. A particular choice consists of continuing f beyond $s = n$ in a linear way, that is

$$\text{(3.2)} \qquad f_n(s) = \min\{s^p, n^p + C_n(s-n)\}, \qquad C_n \ge 0, \qquad \lim_{n\to\infty} n^{1-p} C_n = \alpha.$$

In this case $g(s) = 1 + \alpha(s-1)$, $\alpha \geq 0$. As special cases we have two natural extensions of the nonlinearity for big values: (i) $C_n = 0$, which is a truncation, and (ii) $C_n = pn^{p-1}$, which is the extension by the tangent line. The suitable scaling turns out to be

$$w_n(y,\tau) = \frac{1}{n} u_n(n^{(1-p)}y, n^{2(1-p)}\tau + T). \tag{3.3}$$

With this recipe $t = n^{2(1-p)}\tau + T$, so that the blow-up time, T, is shifted to $\tau = 0$. There is also a zoom in the space variable, $x = n^{1-p}y$. The new variable w_n verifies

$$\begin{cases} (w_n)_\tau = (w_n)_{yy} & (y,\tau) \in (0, n^{p-1}) \times (-Tn^{2(p-1)}, \infty), \\ -(w_n)_y(0,\tau) = \begin{cases} (w_n)^p(0,\tau) & \text{if } w_n \leq 1 \\ f_n(nw_n)/n^p & \text{if } w_n \geq 1 \end{cases} & \tau \in (-Tn^{2(p-1)}, \infty), \\ (w_n)_y(n^{p-1}, \tau) = 0 & \tau \in (-Tn^{2(p-1)}, \infty), \\ w_n(y, -Tn^{2(p-1)}) = u_0(n^{1-p}y)/n & y \in (0, n^{p-1}). \end{cases}$$

If we let formally $n \to \infty$, the sequence w_n converges to a function z which is a solution of

$$\begin{cases} z_\tau = z_{yy} & (y,\tau) \in \mathbb{R}^+ \times \mathbb{R}, \\ -z_y(0,\tau) = \begin{cases} z^p(0,\tau) & \text{if } z(0,\tau) \leq 1 \\ g(z(0,\tau)) & \text{if } z(0,\tau) \geq 1 \end{cases} & \tau \in \mathbb{R}, \\ \lim_{\tau \to -\infty} z(y,\tau) = 0 & y \in \mathbb{R}^+. \end{cases} \tag{3.4}$$

This problem does not have uniqueness because it is invariant under time translations. Hence, in order to characterize completely the behaviour in the boundary layer we need some extra information. This information comes from the behaviour of the solutions u as $t \nearrow T$. Actually, we know that the blow-up rate is given by

$$c(T-t)^{\frac{1}{2(1-p)}} \leq \|u(\cdot,t)\|_\infty \leq C(T-t)^{\frac{1}{2(1-p)}}, \tag{3.5}$$

see [CF1], [HY], [FQ]. This implies that the blow-up profile is self-similar,

$$\lim_{t \nearrow T} (T-t)^{\frac{1}{2(p-1)}} u(x(T-t)^{1/2}, t) = \varphi(x), \tag{3.6}$$

uniformly on compact intervals $|x| \leq C$, where $\varphi(x)$ is the unique bounded positive solution with bounded derivative to

$$\begin{cases} -\varphi''(x) + \frac{1}{2}x\varphi'(x) + \frac{1}{2(p-1)}\varphi(x) = 0, \\ -\varphi'(0) = \varphi^p(0). \end{cases} \tag{3.7}$$

The explicit form of φ can be found in [FQ].

We will prove through a suitable matching that the particular solution of (3.4) giving the asymptotic behaviour is the only one that coincides for $\tau < \tau_0$ with the self-similar function

$$z(y,\tau) = (-\tau)^{-\frac{1}{2(p-1)}} \varphi(y(-\tau)^{-1/2}), \tag{3.8}$$

where φ is the unique solution to (3.7). The number $\tau_0 < 0$ is a constant connected with the blow-up rate, see below. Notice that at later times the limit function z depends on the approximating sequence through the function g. The function z gives a description of the thermal avalanche in the following sense. First, we observe that in the cases where f_n is the linear modification of f described above we have g linear at infinity, hence z exists globally in time. The following result holds.

Theorem 3.1 *Let w_n be as in (3.3) and z as in (3.4) such that (3.8) holds for $\tau < \tau_0$ and z exists globally in time. Then, as $n \to \infty$ we have*

$$\lim_{n\to\infty} w_n(y,\tau) = z(y,\tau)$$

uniformly for (y,τ) on compact sets of $\mathbb{R}^+ \times \mathbb{R}$.

However, we can use different linear truncations producing a function $g(z)$ such that z blows up in finite time. For instance we can take a sequence $\tilde{f}_n(s) = f_{n^2}(s)$ where f_n is the linear truncation (3.2). Then, $g(s) = s^p$ and z is just the self-similar solution which blows up at $\tau = 0$ given in (3.8). We also have a convergence result in this case.

Theorem 3.2 *Let w_n be as in (3.3) and z as in (3.4) such that (3.8) holds for $\tau < \tau_0$ and z blows up at $\tau = c$. Then, as $n \to \infty$ we have*

$$\lim_{n\to\infty} w_n(y,\tau) = z(y,\tau)$$

uniformly for (y,τ) on compact sets of $\mathbb{R}^+ \times (-\infty, c)$. For $\tau > c$ the limit is infinite everywhere (complete blow-up).

Proof of Theorems 3.1 and 3.2. First, we claim that there exists τ_0 independent of n such that

$$w_n(y,\tau) = \frac{1}{n} u_n(n^{(1-p)}y, n^{2(1-p)}\tau + T) = \frac{1}{n} u(n^{(1-p)}y, n^{2(1-p)}\tau + T)$$

for every $\tau < \tau_0$. This claim is a consequence of the blow-up rate (3.5). Indeed, if t is such that $C(T-t)^{1/(2(1-p))} \le n$ (this is equivalent to $t \le T - (C/n)^{2(p-1)}$) we have

$$u(x,t) \le C(T-t)^{\frac{1}{2(1-p)}} \le n,$$

so that u_n equals u and the claim holds if we choose $\tau_0 \le -C^{2(p-1)}$. Next, we observe that the w_n are uniformly (in n) bounded in compact subsets, K, of $\mathbb{R}^+ \times (-\infty, c)$ for a c that we determine independently of n. In fact, our assumptions on f_n imply that $w_n \le z$ as long as z is finite, so we can take c as the maximal time of existence of z, maybe $c = +\infty$. Hence by Schauder estimates (see [Li]) we have that

$$\|w_n\|_{C^{2+\alpha,1+\alpha/2}(K)} \le C.$$

Using that $C^{2+\alpha,1+\alpha/2}(K)$ is compactly included in $C^{2+\beta,1+\beta/2}(K)$ we can pass to the limit and obtain a function z that is defined in $\mathbb{R}^+ \times (-\infty, c)$ and is a solution of (3.4). To see that the limit is unique, we recall that the asymptotic profile near blow-up is self-similar, see (3.6). In terms of w_n the self-similar asymptotic behaviour reads

$$\lim_{n\to\infty} w_n(y,\tau) = \lim_{n\to\infty} \frac{1}{n} u(n^{(1-p)}y, n^{2(1-p)}\tau + T) = (-\tau)^{-\frac{1}{2(p-1)}} \varphi(y(-\tau)^{-1/2})$$

for every $\tau \le \tau_0$. We can conclude that

$$z(y,\tau) = \lim_{n\to\infty} w_n(y,\tau) = (-\tau)^{-\frac{1}{2(p-1)}} \varphi(y(-\tau)^{-1/2})$$

for every $\tau < \tau_0$. This gives the uniqueness of the limit z. □

Remark. These results imply that

$$\frac{1}{n}u_n(x,t) \approx z(y,\tau), \qquad x = y/n^{p-1}, \quad t = T + \tau/n^{2(p-1)}, \tag{3.9}$$

for $x \approx 0$ and $t \approx T$.

The problem in the half-line. The argument to study the boundary layer is local, hence is also valid for the problem in the half-line.

4 Long-time asymptotics for z

We discuss next the asymptotics as τ goes to infinity for the limit solution z. There exists a first time τ_1 at which $z(0,\tau)$ reaches the reference level 1, i.e., such that $z(0,\tau) < 1$ for $\tau \in (-\infty, \tau_1)$ and $z(0,\tau_1) = 1$. The monotonicity argument in the proof of Theorem 2.1 shows that $z(0,\tau) \geq 1$ for every $\tau \geq \tau_1$. Hence, to study the asymptotic behaviour as $\tau \to \infty$ we can restrict ourselves to problem

$$\begin{cases} z_\tau = z_{yy} & (y,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ -z_x(0,\tau) = g(z(0,\tau)) & \tau \in \mathbb{R}^+, \end{cases} \tag{4.1}$$

with initial data given at a time τ_1. To simplify we take $\tau_1 = 0$. As in (3.2) we use approximations f_n which produce a limit flux function of the form $g(s) = 1 + \alpha(s-1)$. We have to distinguish the cases $\alpha = 0$ and $\alpha > 0$.

Flat truncation. We start with the case $\alpha = 0$, then $g(z) \equiv 1$. We have a representation formula for $z(y,\tau)$. In fact, multiplying (4.1) by the fundamental solution of the heat equation as in Section 2, we obtain

$$\begin{aligned} z(y,\tau) = \int_0^{+\infty} \Gamma(y-s,\tau) z(s,0)\, ds - \int_0^\tau \frac{\partial z}{\partial y}(0,t)\Gamma(x,\tau-t)\, dt \\ - \int_0^\tau \frac{\partial \Gamma}{\partial y}(y,\tau-t) z(0,t)\, dt. \end{aligned} \tag{4.2}$$

Taking the limit $y \to 0$ we have

$$\frac{1}{2} z(0,\tau) = \int_0^{+\infty} \Gamma(-s,\tau) z(s,0)\, ds + \frac{1}{\sqrt{\pi}} \int_0^\tau \frac{1}{(\tau-t)^{1/2}}\, dt.$$

Therefore,

$$z(0,\tau) \sim \frac{2}{\sqrt{\pi}} \tau^{1/2}.$$

Using that $z(y,\tau)$ verifies (4.2) and the boundary condition we obtain the following result.

Theorem 4.1 *For flat truncations we have the long time behaviour*

$$z(y,\tau) \sim \tau^{1/2} \psi\left(\frac{y}{\tau^{1/2}}\right). \tag{4.3}$$

where ψ solves $\psi''(\xi) + \frac{1}{2}\xi\psi'(\xi) - \frac{1}{2}\psi(\xi) = 0$, $-\psi'(0) = 1$.

Notice that the function

$$Z(y,\tau) = \tau^{1/2}\psi\left(\frac{y}{\tau^{1/2}}\right)$$

is a self-similar solution of the problem in the half-line. Using this asymptotic behaviour we see that the level sets of u behave like those of Z, that is,

$$y = \tau^{1/2}\psi^{-1}\left(\frac{k}{\tau^{1/2}}\right).$$

TANGENT TRUNCATION. Next we deal with the extension of the nonlinearity by the tangent line, $\alpha = p$, or more generally with any $\alpha > 0$. Let us define $h = z_y + \alpha z - (\alpha - 1)$. It satisfies

$$\begin{cases} h_\tau = h_{yy} & (y,\tau) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ h(0,\tau) = 0 & \tau \in \mathbb{R}^+, \\ h(y,0) = h_0(y) & y \in \mathbb{R}^+. \end{cases}$$

This problem has a well-known representation formula in terms of h_0 and is known to converge as $\tau \to \infty$ to the so-called dipole solution, a space derivative of the gaussian kernel. Using this expression for h we get an ordinary differential equation for z which can be easily solved. From the resulting expression for z we get the following result.

Theorem 4.2 *The long time behaviour of the solutions z with α truncation is given in first approximation by a traveling wave*

$$z(y,\tau) \approx K(\alpha)\, e^{\alpha(\alpha\tau - y)}. \tag{4.4}$$

Thus, the level curves behave like

$$y \approx \alpha\tau + C. \tag{4.5}$$

This traveling wave, which originates at the end of the avalanche period, can be translated into the behaviour of the approximate problems near $x = 0$ after the blow-up time, thanks to the convergence result, Theorem 3.1.

Corollary 4.1 *Given large constants $C > 0$ and $\tau_2 < \tau < \tau_3$ we have the approximate behaviour for $n \gg 1$*

$$\frac{1}{n}\, u_n(x,t) \approx K(\alpha)\, e^{\alpha(\alpha\tau - y)}, \tag{4.6}$$

with $y = n^{p-1}x \in (0, C)$ and $\tau = n^{2(p-1)}(t - T) \in (\tau_2, \tau_3)$.

It follows that in such regions the level lines are approximately straight lines with slope

$$dx/dt \approx \alpha n^{p-1}. \tag{4.7}$$

For the problem in a half-line this behaviour will give the long-time asymptotics of the approximate solutions, but this will not be the case in a bounded interval, see Section 5.

5 Long-time asymptotics for the approximate problems

For $t < T$ the behaviour of the approximate functions u_n is similar to that of u, which is given by a self-similar profile for t close to T. On the other hand, if we translate the convergence of w_n to z to the original variables, we see how the thermal avalanche takes place close to the origin for times near T, cf. (4.6). Our next aim is to describe the asymptotic behaviour of the approximate solutions as t goes to infinity. For simplicity we restrict ourselves to two particular choices, f_n given by (3.2) with $C_n = 0$ (flat truncation) and $C_n = pn^{p-1}$ (tangent truncation).

Since u_n coincides with u below level n, there exists a first time t_0 such that $u_n(0,t) < n$ for $t \in [0, t_0)$ and $u_n(0, t_0) = n$. The arguments in the proof of Theorem 2.1 show that $u_n(0,t) \geq n$ for every $t \geq t_0$. Hence, to study the asymptotic behaviour as $t \to \infty$ we can restrict ourselves to problem

$$(5.1) \qquad \begin{cases} (u_n)_t = (u_n)_{xx} & (x,t) \in (0,1) \times \mathbb{R}^+, \\ -(u_n)_x(0,t) = f_n(u_n(0,t)) & t \in \mathbb{R}^+, \\ (u_n)_x(1,t) = 0 & t \in \mathbb{R}^+, \end{cases}$$

with initial data $u_n(x,0) = u(x,t_0)$, $x \in (0,1)$.

Flat truncation. We start by considering the case $C_n = 0$, hence $f_n(u) = n^p$. Let

$$v(x,t) = n^p t + n^p \frac{(1-x)^2}{2}.$$

This function v is a solution of (5.1) and it gives the asymptotic behaviour as $t \to \infty$ of u_n. To see this just check that the difference $u_n - v$ satisfies the heat equation with homogeneous Neumann conditions at both boundaries. Hence, $u_n - v$ converges to the mean value of $u_n(x,0) - v(x,0) \sim n^p/6$.

Theorem 5.1 *For flat truncations we have the long time behaviour in an interval*

$$u_n(x,t) \approx n^p t + n^p \frac{(1-x)^2}{2} - \frac{n^p}{6}, \qquad t \gg 1.$$

The k-level curves are then approximately given by the inverted parabolas

$$t = \frac{k}{n^p} + \frac{1}{6} - \frac{(1-x)^2}{2}, \qquad t \gg 1, \;\; k \gg 1.$$

Summing up, there are *three different regimes* for the approximate solutions u_n: (i) a blow-up self-similar behaviour for $t < T$, $t \approx T$; (ii) the formation of the thermal avalanche, described by z; and finally (iii) a uniform in x growth of the u_n to infinity with rate $n^p t$.

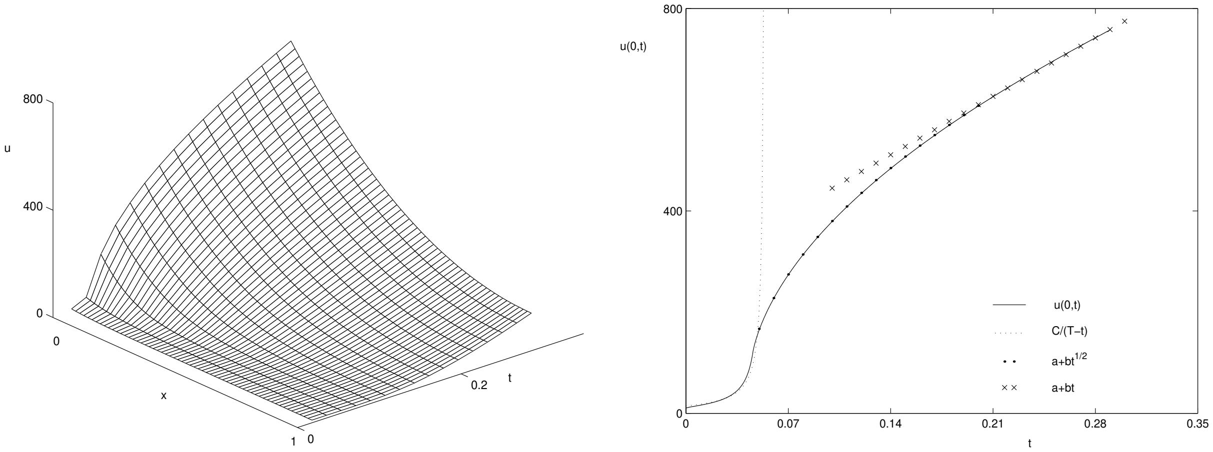

Figure 1. Numerical experiments for the heat equation.

In Figure 1 we show some numerical experiments for u_n with $p = 1.5$, $n = 100$ and initial data $u_0(x) = 10$. In the first picture we display u as a function of x and t. For a better understanding, in the second picture we concentrate on $u(0,t)$. We can see the three different stages, the first one (before the truncation takes place) where the behaviour is of the form $(T-t)^{-1/(2(p-1))}$; the second one (the thermal avalanche) where $u(0,t) \sim t^{1/2}$; and the last one, where $u(0,t) \sim t$. All numerical computations in the paper have been performed on a PC with Matlab using finite differences in space and ODE45 as time integrator.

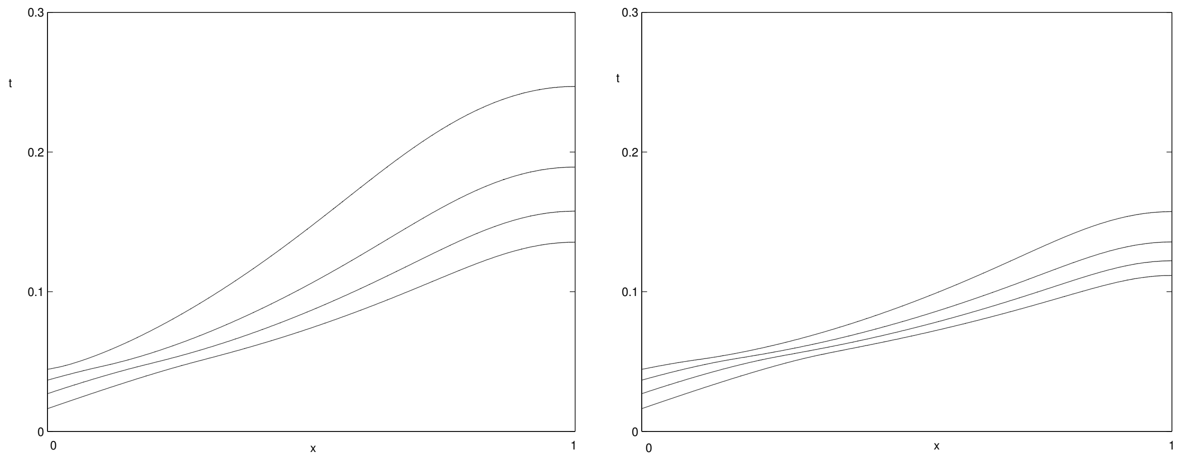

Figure 2. Level curves for different values of n.

In Figure 2 we show the behaviour of the level curves for two different values of n, ($n = 100$ and $n = 400$). We can appreciate that they become more horizontal as n increases.

Tangent truncation. If we extend the nonlinearity by the tangent, then $C_n = pn^{p-1}$ and $f_n(u_n) = pn^{p-1}u_n + (1-p)n^p$. The constant term $(1-p)n^p$ becomes unimportant as $t \to \infty$ and therefore the asymptotic behaviour is given by the only positive eigenvalue of the simplified problem, that is,

$$u_n(x,t) \approx e^{ct}\theta(x), \qquad t \gg 1,$$

where $c \approx p^2 n^{2(p-1)}$ and θ is the associated eigenfunction. Hence the growth is uniform, and u_n behaves like e^{ct}.

The problem in the half-line. In this case we have to deal with the problem

$$
(5.2) \qquad \begin{cases} (u_n)_t = (u_n)_{xx} & (x,t) \in \mathbb{R}^+ \times \mathbb{R}^+, \\ -(u_n)_x(0,t) = f_n(u_n(0,t)) & t \in \mathbb{R}^+, \\ u_n(x,0) = u(x,t_0) & x \in \mathbb{R}^+. \end{cases}
$$

In contrast with the analysis performed in previous sections of the behaviour near T, the asymptotics as $t \to \infty$ in the half-line are different from those on a bounded interval. Rather, it is similar to the behaviour of the limit function z performed in Section 4, since both u_n and z solve similar problems in a half-line. Therefore, we state the results for the approximate problems with only brief details.

Flat truncation. We have $C_n = 0$, $f_n = n^p$. Using the representation formula for $u_n(x,t)$, we obtain:

Theorem 5.2 *For flat truncations we have the long time behaviour in a half-line*

$$
u_n(x,t) \sim n^p t^{1/2} \psi\left(\frac{x}{t^{1/2}}\right),
$$

where ψ solves

$$
\begin{cases} \psi''(\xi) + \frac{1}{2}\xi\psi'(\xi) - \frac{1}{2}\psi(\xi) = 0, \\ -\psi'(0) = 1. \end{cases}
$$

The function

$$
U(x,t) = n^p t^{1/2} \psi\left(\frac{x}{t^{1/2}}\right)
$$

is a self-similar solution of the problem in the half-line. Hence the level sets of u_n behave like those of U, that is,

$$
x = t^{1/2} \psi^{-1}\left(\frac{k}{n^p t^{1/2}}\right).
$$

We observe that *these level curves become horizontal as n increases to infinity.*

Tangent truncation. Next we deal with the extension of the nonlinearity by the tangent line, $C_n = pn^{p-1}$, $f_n(u_n) = pn^{p-1}u_n + (1-p)n^p$. The same ideas performed in Section 4 give us next result.

Theorem 5.3 *For the problem in a half-line the long time behaviour of the solutions u_n with tangent truncation is given in first approximation by a traveling wave*

$$
u_n(x,t) \approx K_n e^{pn^{p-1}(pn^{p-1}t - x)}.
$$

Thus, the level curves behave like

$$
x \approx pn^{p-1}t + C.
$$

This traveling wave originates at the end of the avalanche period, as we have proved in Section 4. The validity of the traveling wave as a representation of the behaviour of u_n extends from a small region near $x = 0$ at $t \approx T$ to wider regions for $t \gg T$, always on the condition that $n \gg 1$.

6 Porous medium equation: $m > 1$

In this section we deal with problem (1.4) in the porous medium case, $m > 1$. For $f(u) = u^p$ it is known that nontrivial solutions blow up if and only if $p > 1$. Moreover, the blow-up set is a single point if and only if $p > m$, [Fi]. The behaviour is analogous to the heat equation: we have complete blow-up and we are able to describe the avalanche and the asymptotics for the approximate problems.

Theorem 6.1 *Let $m > 1$. Assume that f is a blow-up nonlinearity and that f_n satisfies* (1.2)*. Then blow-up for* (1.4) *is complete.*

Proof. As for the heat equation, we proceed in two steps. First we prove that the proper solution is identically infinite at the origin, just as is done for the heat equation, but now taking

$$\psi_n^m(x) = M^m - f_n(M)x.$$

Next we propagate the blow-up to the interior by comparing with a self-similar solution of the PME in the half-line with Dirichlet boundary condition $u(0,t) = M$. This replaces the argument with the representation formula used in the linear case, $m = 1$, which does not apply here. Let us be more precise. Let $U_A(x,t)$ be the solution of

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in \mathbb{R}^+ \times (T, +\infty), \\ u(0,t) = A & t \in (T, +\infty), \\ u(x,T) = 0 & x \in \mathbb{R}^+. \end{cases}$$

The solution of this problem has self-similar form

$$U_A(x,t) = f_A(y), \qquad y = x/(t-T)^{1/2},$$

has compact support, and f_A is monotone decreasing, [AP], [B], [PK]. Moreover, we have the scaling

$$U_A(x,t) = AU_1(x, A^{m-1}t).$$

Let y_* be such that $f_1(y_*) = 1/2$. Given $\bar{t} > T$, let M_0 be such that

$$\frac{1}{M_0^{(m-1)/2}(\bar{t}-T)^{1/2}} < y_*.$$

When U_A is restricted to the interval $(0,1)$ it is a subsolution of (1.4). Now, let $M > M_0$. If n is large enough, then

$$u_n(0,t) \geq M \qquad t \in (T, \bar{t}).$$

Comparison with U_M yields

$$u_n(x,\bar{t}) \geq U_M(x,\bar{t}) = Mf_1\left(\frac{x}{M^{(m-1)/2}(\bar{t}-T)^{1/2}}\right) \geq Mf_1(y_*) \geq M/2.$$

This shows that for any $t > T$, and $x \in [0,1]$ the proper solution becomes identically infinite, that is,

$$\lim u_n(x,t) = \infty.$$

This completes the proof. $\square$

Avalanche analysis. Next, we consider the thermal avalanche for the nonlinearity $f(u) = u^p$ in the case in which there is single-point blow-up, i.e., when $p > m$. The suitable scaling turns out to be

$$w_n(y,\tau) = \frac{1}{n} u_n(n^{m-p}y, n^{-(2p-m-1)}\tau + T). \tag{6.1}$$

The new variable w_n verifies

$$\begin{cases} (w_n)_\tau = (w_n^m)_{yy} & (y,\tau) \in (0, n^{p-m}) \times (-Tn^{2p-m-1}, \infty), \\ -(w_n^m)_y(0,\tau) = \begin{cases} (w_n)^p(0,\tau) & \text{if } w_n \leq 1 \\ f_n(nw_n)/n^p & \text{if } w_n \geq 1 \end{cases} & \tau \in (-Tn^{2p-m-1}, \infty), \\ (w_n)_y(n^{p-m}, \tau) = 0 & \tau \in (-Tn^{2p-m-1}, \infty), \\ w_n(y, -Tn^{2p-m-1}) = u_0(n^{m-p}y)/n & y \in (0, n^{p-m}). \end{cases}$$

If we let formally $n \to \infty$, the w_n converges to a function z which is a solution of

$$\begin{cases} z_\tau = (z^m)_{yy} & (y,\tau) \in \mathbb{R}^+ \times \mathbb{R}, \\ -(z^m)_y(0,\tau) = \begin{cases} z^p(0,\tau) & \text{if } z(0,\tau) \leq 1 \\ g(z(0,\tau)) & \text{if } z(0,\tau) \geq 1 \end{cases} & \tau \in \mathbb{R}, \\ \lim_{\tau \to -\infty} z(y,\tau) = 0 & y \in \mathbb{R}^+. \end{cases} \tag{6.2}$$

This problem does not have uniqueness. Hence, in order to characterize completely the behaviour in the boundary layer we need some extra information. This information comes from the known behaviour of the solutions u as $t \nearrow T$. What we assume is the blow-up rate,

$$c(T-t)^{-\frac{1}{2p-m-1}} \leq \|u(\cdot,t)\|_\infty \leq C(T-t)^{-\frac{1}{2p-m-1}} \tag{6.3}$$

(see [Fi] and [QR] for hypotheses that guarantee this fact) and this implies that the blow-up profile is self-similar,

$$\lim_{t \nearrow T} (T-t)^{\frac{1}{2p-m-1}} u(x(T-t)^{\frac{p-m}{2p-m-1}}, t) = \varphi(x),$$

uniformly on compact intervals $|x| \leq C$, where $\varphi(x)$ is the unique bounded positive solution with bounded derivative (see [PQR]) to

$$\begin{cases} -(\varphi^m)''(x) + \frac{p-m}{2p-m-1} x\varphi'(x) + \frac{1}{2p-m-1}\varphi(x) = 0, \\ -(\varphi^m)'(0) = \varphi^p(0). \end{cases} \tag{6.4}$$

As before, the particular solution of (6.2) giving the asymptotic behaviour is the only one that coincides for $\tau < \tau_0$ ($\tau_0 < 0$ a fixed constant connected with the blow-up rate), with the self-similar function

$$z(y,\tau) = (-\tau)^{\frac{1}{2p-m-1}} \varphi(y(-\tau)^{-\frac{p-m}{2p-m-1}}) \tag{6.5}$$

where φ is the unique solution to (6.4). The existence and uniqueness of the profile φ is given in [GP1], [GP2], [G]. The function z gives a description of the thermal avalanche in the following sense.

Theorem 6.2 *Let u satisfy* (6.3). *Let w_n be as in* (6.1) *and z as in* (6.2) *such that* (6.5) *holds for $\tau < \tau_0$. Then, as $n \to \infty$ we have*

$$\lim_{n\to\infty} w_n(y,\tau) = z(y,\tau)$$

uniformly for (y,τ) on compact sets of $\mathbb{R}^+ \times (-\infty, c)$ where c is the maximal existence time of z.

Proof. The proof is analogous to that of Theorem 3.1. □

If $p < m$ we have global blow-up. So the avalanche has already taken place before T and is described by the solution u itself, [Fi]. In the case $p = m$ the situation is a bit different because we have regional or global blow-up depending on the length of the interval. Taking the interval $[0, L]$ with L small, blow-up is global and the avalanche already took place before T. For larger L the above scaling can be still used. However it does not change the space variable. Since things depend on L only in this case $p = m$, for simplicity we have taken $L = 1$ along the paper.

Asymptotics for the approximate problems. Now we describe the asymptotic behaviour as $t \to \infty$. For the reasons presented when dealing with the heat equation, we just have to look at the asymptotic behaviour of

$$\begin{cases} (u_n)_t = (u_n^m)_{xx} & (x,t) \in (0,1) \times (0,+\infty), \\ -(u_n^m)_x(0,t) = f_n(u_n(0,t)) & t \in (0,+\infty), \\ (u_n^m)_x(1,t) = 0 & t \in (0,+\infty), \\ u_n(x,0) = u(x,t_0) & x \in (0,1). \end{cases}$$

We start with the analysis of a truncation with $C_n = 0$, $f_n = n^p$. This solution obviously goes to infinity. To see how it does so, we first consider $u_n(x,0)$ such that the corresponding solution verifies $(u_n)_t \geq 0$. Therefore $(u_n^m)_{xx} \geq 0$ and $\min_{[0,1]}(u_n^m)_x(\cdot,t) = (u_n^m)_x(0,t) = -n^p$. Hence

$$u_n^m(x,t) - u_n^m(0,t) = (u_n^m)_x(\xi,t)x \geq -n^p x,$$

and the solution goes to infinity uniformly, $u(x,t) \sim u(0,t)$. Integrating the equation in time we get

$$\frac{d}{dt}\int_0^1 u_n(s,t)\,ds = n^p,$$

and then, using the uniform growth,

$$u(x,t) \approx n^p t.$$

For a general initial data we compare with solutions that satisfy $u_t \geq 0$ with initial data below and above $u_n(x,0)$. Again, as happens for the heat equation, there are three different regimes for the approximate solutions u_n. The final stage has the same asymptotic behaviour, $u_n(x,t) \approx n^p t$, both for $m = 1$ and for $m > 1$. However, the first two stages differ because the self-similar exponents and profiles are not the same.

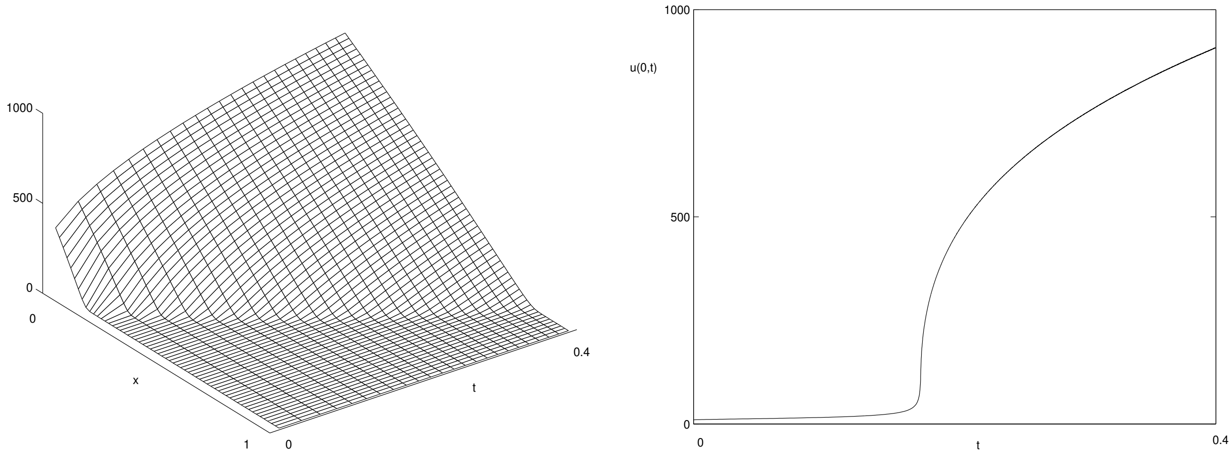

Figure 3. Numerical experiments for the porous medium equation.

The three stages are displayed in Figure 3, where we show some numerical experiments for u_n with $m = 2$, $p = 2.5$, $n = 100$ and initial data $u_0(x) = 10$. In the first picture we show the evolution of $u(x,t)$ and in the second one the evolution of $u(0,t)$. In comparison with Figure 1 we see on the left a sharper front formation typical of porous medium equations.

The problem on the half-line. Blow-up at $x = 0$ for all $t > T$ is proved through comparison with the subsolution

$$\psi_n(x) = M - f_n(M)x.$$

Blow-up is propagated to the whole space through comparison with a self-similar solution.

The thermal avalanche can be described exactly as in the case of a bounded interval by means of the same function z.

The asymptotic behaviour for the approximate solutions is given by a self-similar solution of the form

$$U(x,t) = t^{\alpha}\psi(\xi), \qquad \xi = \frac{x}{t^{\beta}}.$$

In the truncation case, $\alpha = 1/(m+1)$, $\beta = m/(m+1)$ and ψ solves

$$\begin{cases} (\psi^m)''(\xi) = \alpha\psi(\xi) - \beta\xi\psi'(\xi), \\ -(\psi^m)'(0) = n^p, \end{cases}$$

while in the case of extension by the tangent $\alpha = 1/(m-1)$, $\beta = 1$ and ψ solves

$$\begin{cases} (\psi^m)''(\xi) = \alpha\psi(\xi) - \beta\xi\psi'(\xi), \\ -(\psi^m)'(0) = pn^{p-1}\psi(0). \end{cases}$$

The existence of such profiles ψ is proved in [GP1], [GP2], [G].

Notice that the behaviour is different from that of the heat equation, and also from that of the porous medium equation in a bounded interval. The self-similar profile starts at the end of the avalanche as described by the behaviour of z as $\tau \to +\infty$.

7 Incomplete blow-up. The case $m < 1$

In this section we deal with problem (1.4) in the fast diffusion case, $m < 1$. Solutions with $f(u) = u^p$ blow up if and only if $p > (m+1)/2$ and u becomes unbounded only at the origin, see [Fi]. We prove that the solution can be continued for all $t > T$ independently of the radiation function f.

Theorem 7.1 *Let $m < 1$. Blow-up is incomplete in the sense that any proper solution is finite for all $(x,t) \in (0,1] \times [0,\infty)$. Moreover, the blow-up set of any proper solution is a single point at the boundary for all $t \geq T$.*

Proof. First we prove that $u(0,t) = \infty$ for all $t > T$. In order to do so we consider $\phi(x)$ such that

$$\phi_n^m(x) = M - f_n(M)x.$$

Then ϕ_n is a subsolution of the problem. Hence

$$u_n(0,t) \geq M$$

for all $t \geq T$. In order to control the behaviour of the solutions away from the boundary we will use as barrier $U_2(x, t+\tau)$ with $\tau > 0$ large enough, where U_2 is a singular solution of the form

$$U_R(x,t) = t^{\frac{1}{1-m}} g_R(x) \tag{7.1}$$

such that $g_R(0) = g_R(R) = +\infty$ and $(g^m)'' + \frac{1}{1-m} g = 0$ for $0 < x < R$, cf. [ChV]. Observe that $g_2'(1) = 0$ and hence $(U_2)_x(1,t) = 0$. Therefore

$$u_\infty \leq U_R(x, t+\tau).$$

This shows that the burnt zone does not move, it is always a single point, $x = 0$. □

Remark. This is an example of creation of a singularity that stays in time, a *very hot spot* in the terminology of [ChV].

Asymptotics for large t. If we define

$$M(t) = \int_0^L u(x,t)\,dx,$$

integrating the equation on $x \in (0,L)$ we get

$$\frac{d}{dt} M(t) = f(u(0,t)).$$

Integrating in time we obtain

$$M(t) = M(0) + \int_0^t f(u(0,s))\,ds.$$

As $u(0,s) = \infty$ for all $s > T$ we have that $M(t) = \infty$ for all $t > T$. We can apply a mass comparison argument [V] to prove that

$$u(x,t) \geq u_c(x, t-T), \qquad t > T,$$

where u_c is a solution with initial data $c\delta(x)$. Passing to the limit $c \to \infty$ we get

$$u(x,t) \geq (t-T)^{1/(1-m)} g_2(x),$$

see [ChV] for the details. This proves that the singularity created at $x=0$, $t=T$ behaves like a very hot spot.

The estimates from above and below provide immediately the asymptotic behaviour of solutions as $t \to \infty$.

Theorem 7.2 *Let $m<1$ and u be a proper solution to problem* (1.4). *Then*

$$\lim_{t\to\infty} \frac{u(x,t)}{t^{1/(1-m)}} = g_2(x)$$

uniformly in x.

Remark. From the blow-up rate which is, for $f(u)=u^p$,

$$u(0,t) \sim C(T-t)^{-\alpha}, \qquad \alpha = 1/(2p-m-1),$$

see [FPQR], [Fi], we get that

$$M(t) = M(0) + c\int_0^t (T-s)^{-\alpha p}.$$

Therefore, $M(T^-)=\infty$ if and only if $p \leq m+1$. In the other case, $p > m+1$ there is a *finite asymptotic mass* as $t \nearrow T$. There is in this case a mass jump at $t=T$ since, as we have already shown, $M(T+)=\infty$.

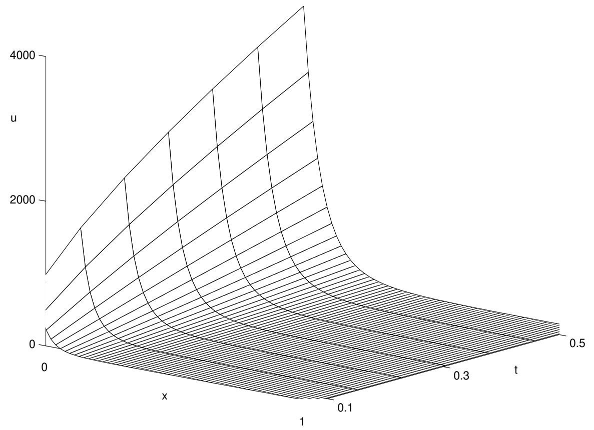

Figure 4. Numerical experiment for the fast diffusion equation.

In Figure 4 we show the picture of the evolution of an approximate problem with $m=0.8$, $p=1.2$, $n=100$ and initial data $u_0(x)=10$. We can appreciate that the singularity at the origin does not propagate to the interior of the domain, contrary to Figures 1 and 3.

The problem in the half-line. In order to control the behaviour of the solutions away from the boundary we will use as a barrier the following explicit solution of the fast diffusion equation

$$U(x,t) = C_m \left(\frac{t}{x^2}\right)^{\frac{1}{1-m}}, \qquad C_m = \left(\frac{2m(m+1)}{1-m}\right)^{\frac{1}{1-m}}.$$

This solution $U(x,t)$ is called Infinite Point Source Solution because $U(x,0) = 0$ for $x \neq 0$ and $\int_0^\infty U(x,t)\,dx = \infty$ for all $t > 0$.

Now we can compare u_n from above with $U(x,t+\tau)$ with $\tau > 0$ large enough so that $U(x,\tau) \geq u_0(x)$. This is only valid for solutions with initial data with the proper decay at infinity. If u_0 does not decay we may use as barriers the singular solutions of the form (7.1). In [ChV] it is proved that $U_R \to U$ as $R \to \infty$.

Let us investigate the behaviour of the solution for $t > T$. The same mass analysis performed in a bounded interval holds here, so again we apply a mass comparison argument to prove that

$$u(x,t) \geq u_c(x, t-T), \qquad t > T,$$

where u_c is a solution with initial data $c\delta(x)$. Passing to the limit $c \to \infty$ we get

$$u(x,t) \geq C_m \left(\frac{t-T}{x^2}\right)^{1/(1-m)}.$$

The estimates from above and below provide the asymptotic behaviour of solutions as $t \to \infty$.

Theorem 7.3 *Let $m < 1$ and u be a proper solution to problem* (1.4). *Then*

$$\lim_{t\to\infty} \frac{u(x,t)}{t^{1/(1-m)}} = C_m x^{-2/(1-m)}$$

uniformly in x.

Acknowledgments

JLV and FQ have been partially supported by DGES grant PB94-0153 (Spain) and TMR Program FMRX-CT 91-0201 (EU). JDR is supported by Universidad de Buenos Aires under grant EX046, CONICET and Fundación Antorchas (Argentina). This research was started while the third author (JDR) was a visitor at Universidad Autónoma de Madrid. He is grateful to this institution for its hospitality. We thank M. Fila for a copy of his unpublished results, E. Chasseigne for useful discussions on the subject of fast diffusion, and R. Ferreira for his help with the numerical computations.

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