Non-simultaneous quenching

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Abstract

We study the possibility of non-simultaneous quenching for positive solutions of a coupled system of two semilinear heat equations, $u_t = u_{xx} - v^{-p}$, $v_t = v_{xx} - u^{-q}$, $p, q > 0$, with homogeneous Neumann boundary conditions and positive initial data. Under some assumptions on the initial data, we prove that if $p, q \geq 1$ then quenching is always simultaneous, if $p < 1$ or $q < 1$ then there exists a wide class of initial data with non-simultaneous quenching, and finally if $p < 1 \leq q$ or $q < 1 \leq p$, then quenching is always non-simultaneous. We also give the quenching rates in all cases.

1 Introduction.

In this paper we study the solutions of the parabolic system

$$\begin{cases} u_t = u_{xx} - v^{-p} \\ v_t = v_{xx} - u^{-q} \end{cases} \quad \text{in } (0,1) \times (0,T), \tag{1.1}$$

with Neumann boundary conditions,

$$\begin{cases} u_x(0,t) = v_x(0,t) = 0 \\ u_x(1,t) = v_x(1,t) = 0 \end{cases} \quad \text{in } (0,T), \tag{1.2}$$

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and initial data

$$\begin{cases} u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases} \quad \text{in } [0,1]. \tag{1.3}$$

Throughout the paper we assume that p, $q > 0$ and that the initial data u_0, v_0 are positive, smooth and compatible with the boundary data.

We are interested in studying the *quenching* phenomenon, which is formulated as follows, see for instance [1]: the solution (u, v) exists in the classical sense and is positive for all $0 \le t < T$, and satisfies

$$\liminf_{t \nearrow T} \min_{0 \le x \le 1} \{u(x,t), v(x,t)\} = 0.$$

At the *quenching time* $t = T$ a singularity appears in the absorption term and the classical solution no longer exists. Our first result shows that this is the case for our system.

Proposition 1 *If $p, q > 0$ quenching happens for system* (1.1)–(1.3) *for every initial data.*

In the sequel we denote by T the quenching time for system (1.1)–(1.3).

Some authors understand quenching as the blow-up of some derivative while the functions remain bounded, see for example [2], [3]. This indeed happens for our problem if u and v are strictly decreasing in time.

Proposition 2 *Let u and v be strictly decreasing in time. If $v(x,t) \to 0$ (respectively $u(x,t) \to 0$) for $t \to T$ then $u_t(x,t) \to -\infty$ (resp. $v_t(x,t) \to -\infty$).*

The quenching phenomenon has been widely studied in recent years by several authors, see for example [1], [4], [5], [6] and references therein. These works deal with the occurrence of quenching, the behaviour close to the quenching time and the possibility of continuation after quenching.

Our interest is completely different. A priori there is no reason why both functions, u and v, should reach the level zero simultaneously at time T. Indeed, as we will show, for certain choices of the parameters p and q there are initial data for which one of the components of the system remains bounded away from zero while the other vanishes at some point. We denote this phenomenon as *non-simultaneous quenching*. Non-simultaneous formation of singularities in parabolic systems has already been observed in blow-up problems, [7], [8], but, up to our knowledge not for quenching problems. Here we characterize the range of parameters for which non-simultaneous quenching occurs (for suitable initial data). Our main results are contained in the following theorems.

Theorem 1 *If $p < 1$, then for any initial data u_0 there exists an open set (in the C^2 topology) of initial data v_0 such that v quenches (reaches zero) while u is strictly positive.*

The reciprocal also holds at least if we restrict ourselves to certain classes of solutions. Thus, from now on we assume that u and v are non decreasing in space and strictly decreasing in time. This will follow for instance if the initial values are non decreasing and satisfy $u_0'' < v_0^{-p}$, $v_0'' < u_0^{-q}$.

Theorem 2 *If $u \geq c > 0$ for $0 \leq t \leq T$, then $p < 1$.*

These results say that if both p and q are greater or equal than one then quenching is simultaneous for every initial data. On the other hand if p or q are less than one, there exists a wide class of initial data with non-simultaneous quenching. We improve this last assertion by proving that quenching is always non-simultaneous for every initial data if $p < 1 \leq q$ or $q < 1 \leq p$.

Theorem 3 *If $p < 1 \leq q$ then $u \geq c > 0$ for $0 \leq t \leq T$.*

In every case we find the quenching rates, which are different in the simultaneous or non-simultaneous cases. The notation $f \sim g$ means that there exist finite positive constants c_1, c_2 such that $c_1\, g \leq f \leq c_2\, g$.

Theorem 4 *a) If quenching is non-simultaneous and, for instance v is the quenching variable, then for t close to T, $v(0,t) \sim (T-t)$.*

b) If quenching is simultaneous then for t close to T:

i) $u(0,t) \sim (T-t)^{(p-1)/(pq-1)}$, $v(0,t) \sim (T-t)^{(q-1)/(pq-1)}$ if $p, q > 1$ or $p, q < 1$.

ii) $u(0,t), v(0,t) \sim (T-t)^{1/2}$ if $p = q = 1$.

iii) $u(0,t) \sim (T-t)|\log(T-t)|^{p/(p-1)}$, $v(0,t) \sim |\log(T-t)|^{-1/(p-1)}$ if $p > q = 1$.

If the initial data are constants, problem (1.1)–(1.2) reduces to $u_t = v^{-p}$, $v_t = u^{-q}$. For this ODE system our results are straightforward. However, if the initial data are not flat, as may happen under our hypotheses, PDE techniques are needed and the proofs become more involved.

2 Proofs.

Proof of Proposition 1. Since u and v are subsolutions of the heat equation, then $u \leq M = \|u_0\|_\infty$, $v \leq N = \|v_0\|_\infty$. Hence, integration of (1.1) in space gives the following mass estimates,

$$\int_0^1 u(x,t)\,dx \leq M - \frac{1}{N^p}t, \qquad \int_0^1 v(x,t)\,dx \leq N - \frac{1}{M^q}t, \tag{2.1}$$

which yield a contradiction if u and v are positive for all times. $\square$

Now we prove a lemma that gives control of the time derivatives, which turns out to be crucial in our arguments.

Lemma 1 *Let (u,v) be a strictly decreasing solution of (1.1)–(1.3), then there exists $\delta > 0$ such that*

$$u_t(x,t) \leq -\delta v^{-p}(x,t), \qquad v_t(x,t) \leq -\delta u^{-q}(x,t), \tag{2.2}$$

for every $0 < t < T$, $0 \leq x \leq 1$.

Proof. We adapt an idea from [3] and define $w = v_t + \delta u^{-q}$, $z = u_t + \delta v^{-q}$. Now we calculate,

$$w_t - w_{xx} = v_{tt} - \delta q u^{-q-1} u_t - v_{txx} + \delta q u^{-q-1} u_{xx} - \delta q(q+1) u^{-q-2} (u_x)^2 \leq q u^{-q-1} z.$$

A similar computation holds for z, and we arrive to

$$\begin{cases} w_t \leq w_{xx} + q u^{-q-1} z \\ z_t \leq z_{xx} + p v^{-p-1} w \end{cases} \quad \text{in } (0,1) \times (0,T), \tag{2.3}$$

with boundary conditions

$$\begin{cases} w_x(0,t) = z_x(0,t) = 0 \\ w_x(1,t) = z_x(1,t) = 0 \end{cases} \quad \text{in } (0,T). \tag{2.4}$$

As u and v are strictly decreasing, choosing δ small enough we get $w(x,0), z(x,0) \leq -a < 0$. Assume that there exists a first time $t_1 < T$ such that $\max_x \{w(x,t_1), z(x,t_1)\} = -a/2$. Suppose for instance $w(x_1,t_1) = -a/2$. Hence

$$0 \leq w_t(x_1,t_1) \leq w_{xx}(x_1,t_1) + q u^{-q-1} z(x_1,t_1) \leq -q u^{-q-1} a/2 < 0,$$

a contradiction. Therefore $w(x,t) \leq -a$ and $z(x,t) \leq -a$, as we wanted to show. □

Proof of Proposition 2. Lemma 1 shows that $u_t(x,t) \leq -\delta v^{-p}(x,t) \to -\infty$. □

Proof of Theorem 1. Let $u_0 > 0$ be fixed. The idea is to choose v_0 small in order that the quenching time T is so small that u does not have time to vanish.

We take $v_0(x) = \varepsilon$. Formula (2.1) shows that $T \leq \varepsilon M^q$. We define $w = v_t$, $z = u_t$. These functions satisfy (2.3), in fact with equalities ($\delta = 0$), and the boundary conditions (2.4). Moreover, from (1.1) we have $w(x,0) \leq -M^{-q}$ and, taking $\varepsilon > 0$ small, also that $z(x,0) < 0$. Hence the above comparison argument shows that $v_t(x,t) = w(x,t) \leq -M^{-q}$. If we integrate this inequality in (t,T) we get

$$v(x,t) \geq v(x,T) + M^{-q}(T-t) \geq M^{-q}(T-t). \tag{2.5}$$

We put this estimate in the equation for u and obtain $u_t \geq u_{xx} - M^{-q}(T-t)^{-p}$. Comparison with a solution of $\underline{u}_t = -M^{-q}(T-t)^{-p}$ with initial data $\underline{u}(0) = \min_x u_0(x)$ gives

$$u(x,T) \geq \min_x u_0(x) - CT^{1-p},$$

where C does not depend on ε. As $p < 1$, the right hand side is strictly positive if ε, and hence T, are small enough.

To end the proof we just observe that the above argument also works with any initial data v_0 which is close to $v_0 = \varepsilon$ in the C^2 topology. □

Proof of Theorem 2. Since $v_{xx}(0,t) \geq 0$, we have $v_t(0,t) = v_{xx}(0,t) - u(0,t)^{-q} \geq -C$, which integrated in (t,T) gives

$$v(0,t) \leq C(T-t). \tag{2.6}$$

Using (2.2) we have $u_t(0,t) \le -\delta v^{-p}(0,t) \le -C(T-t)^{-p}$. Integrating in $(0,T)$ we get

$$u(0,0) - u(0,T) \ge C \int_0^T (T-s)^{-p}\, ds.$$

We arrive at a contradiction if $p \ge 1$, because this integral diverges. $\square$

In order to prove theorems 3 and 4, we collect here the estimates of the time derivatives obtained before. At $x = 0$ we have

$$\begin{aligned} -v^{-p}(0,t) &\le u_t(0,t) \le -\delta v^{-p}(0,t), \\ -u^{-q}(0,t) &\le v_t(0,t) \le -\delta u^{-q}(0,t). \end{aligned} \tag{2.7}$$

Therefore, $(\hat{u}(t), \hat{v}(t)) = (u(0,t), v(0,t))$ behaves as a solution of the ODE system

$$\begin{cases} \hat{u}' = -\hat{v}^{-p}, \\ \hat{v}' = -\hat{u}^{-q}, \end{cases}$$

which is the main idea to derive the following proofs.

Proof of Theorem 3. From (2.7) we have that there exist two positive constants, c_1, c_2 such that

$$c_1 \hat{v}^{-p}\hat{v}' \le \hat{u}^{-q}\hat{u}' \le c_2 \hat{v}^{-p}\hat{v}'. \tag{2.8}$$

Assume first $p < 1 < q$. Integration of the first inequality in (2.8) gives that there exist positive constants C_1, C_2 such that $\hat{v}^{-p+1} \le C_1 - C_2\hat{u}^{-q+1}$. We obtain a contradiction if u quenches. The case $p < 1 = q$ can be handled in an analogous way. $\square$

Proof of Theorem 4. The proof of the first assertion is a consequence of (2.5) and (2.6).

Now assume that we have simultaneous quenching and that $p, q > 1$. Integration of (2.8) yields

$$c_1 \hat{v}^{-p+1}(t) + c_2 \le \hat{u}^{-q+1}(t) \le C_1 \hat{v}^{-p+1}(t) + C_2.$$

For t close to T the additive constants become negligible. Hence, introducing this in (2.7) we get

$$\hat{u}'(t) \sim -\hat{u}^{-p(q-1)/(p-1)}(t), \qquad \hat{v}'(t) \sim -\hat{v}^{-q(p-1)/(q-1)}(t).$$

The rates are straightforward.

If $p > q = 1$ after integrating (2.8) we get

$$(T-t) \sim \int_0^{\hat{v}(t)} \exp\left(-s^{1-p}/(p-1)\right) ds = C\,\Gamma\left(\frac{1}{1-p}, \frac{\hat{v}^{1-p}(t)}{p-1}\right),$$

where $\Gamma(a,z)$ is the incomplete Gamma function. The asymptotic behaviour of this latter function gives the desired result for v. The behaviour for u is now immediate.

If $p, q < 1$ we deduce the quenching rate by a bootstrap argument. First of all, from (2.7) we get $\hat{u}' \le -\delta N^{-p}$, which means $\hat{u}(t) \ge \delta N^{-p}(T-t)$. Introducing this again in (2.7) implies $\hat{v}' \ge -(\delta N^{-p})^{-q}(T-t)^{-q}$, i.e., $\hat{v}(t) \le (\delta N^{-p})^{-q}(T-t)^{1-q}/(1-q)$. Iterating this procedure, we get $\hat{u}(t) \geq c_n(T-t)^{\alpha_n}$, $\hat{v}(t) \leq C_n(T-t)^{\beta_n}$, where α_n, β_n satisfy $\alpha_0 = 1$, $\beta_0 = 1 - q$ and

$$\alpha_{n+1} = 1 - p\beta_n, \qquad \beta_{n+1} = 1 - q\alpha_n.$$

One can check that $\alpha_n \to (1-p)/(1-pq)$, $\beta_n \to (1-q)/(1-pq)$ and that $C_n \leq C < \infty$ and $c_n \geq c > 0$. Therefore, passing to the limit we get $\hat{u}(t) \geq c(T-t)^{(1-p)/(1-pq)}$ and $\hat{v}(t) \leq C(T-t)^{(1-q)/(1-pq)}$. The reverse inequalities can be obtained in the same way.

If $p = q = 1$, we use a different approach. Let $w = Cu - v$. Taking C big enough we get $w(x, 0) > 0$. On the other hand, w satisfies the equation

$$w_t = w_{xx} - \frac{w}{uv}$$

and the boundary conditions $w_x(0,t) = w_x(1,t) = 0$. It is obvious that w can not attain an interior negative minimum. Thus $w \geq 0$, i.e., $Cu \geq v$. A similar argument shows that $Cv \geq u$, hence $\hat{u}(t) \sim \hat{v}(t)$. Introducing this in (2.7) the rates follow. $\square$

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