Asymptotic simplification for a reaction-diffusion problem with a nonlinear boundary condition

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Abstract

We study nonnegative solutions of the porous medium equation with a source and a nonlinear flux boundary condition,

$$\begin{cases} u_t = (u^m)_{xx} + u^p & \text{in } (0,\infty)\times(0,T), \\ -(u^m)_x(0,t) = u^q(0,t) & t\in(0,T), \\ u(x,0) = u_0(x) & \text{in } (0,\infty), \end{cases}$$

where $m > 1$, $p, q > 0$ are parameters. For every fixed m we prove that there are two critical curves in the (p,q) plane: (i) the critical existence curve, separating the region where every solution is global from the region where there exist blowing up solutions, and (ii) the Fujita curve, separating a region of parameters in which all solutions blow up from a region where both global in time solutions and blowing up solutions exist. In the case of blow-up we find the blow-up rates, the blow-up sets and the blow-up profiles, showing that there is a phenomenon of asymptotic simplification. If $2q < p + m$ the asymptotics are governed by the source term. On the other hand, if $2q > p + m$ the evolution close to blow-up is ruled by the boundary flux. If $2q = p + m$ both terms are of the same order.

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1 Introduction and description of results.

In this paper we deal with the following problem,

$$\left\{\begin{array}{ll} u_t = (u^m)_{xx} + u^p & \text{in } \mathbb{R}_+ \times (0,T), \\ -(u^m)_x(0,t) = u^q(0,t) & t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}_+, \end{array}\right. \tag{1.1}$$

where $m > 1$, $p, q > 0$ are parameters. We assume that u_0 is bounded, continuous, nonnegative and nontrivial in $\mathbb{R}_+ = (0,\infty)$. The time T is the maximal existence time for the solution. It may be finite or infinite (see below). The problem can be thought of as a model to describe heat propagation in a medium with chemical reaction, [P].

If $m > 1$ the function u is in general a solution only in a weak sense, i.e., u and $(u^m)_x$ are absolutely continuous functions and the equation is understood in its standard integral version. However, u is C^∞ in its positivity set.

In certain cases the solution of (1.1) can only be defined for a finite period of time, $T < \infty$, and u becomes unbounded as $t \nearrow T$. We say that u *blows up*. In other cases u is well defined for every time $t > 0$, $(T = \infty)$. Then we say that u is a *global solution.* We are mainly interested in studying the blow-up phenomenon. There are two terms contributing to the growth of the solution, the nonlinear flux at the boundary and the reaction term in the equation. The actual behaviour of blowing-up solutions is determined by which one of the terms is dominant. We find a critical line, namely,

$$2q = m + p. \tag{1.2}$$

The main goal of this paper is to show, in a precise way, the critical role of the line.

For p large, $2q < m + p$, the reaction term u^p dominates, and blowing up solutions of problem (1.1) behave as solutions of the *reduced problem without flux*,

$$\left\{\begin{array}{ll} u_t = (u^m)_{xx} + u^p & \text{in } \mathbb{R}_+ \times (0,T), \\ -(u^m)_x(0,t) = 0 & t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}_+. \end{array}\right. \tag{1.3}$$

For q large, $2q > m + p$, the nonlinear boundary flux dominates and blowing up solutions of problem (1.1) behave as solutions of the *reduced problem without source*,

$$\left\{\begin{array}{ll} u_t = (u^m)_{xx} & \text{in } \mathbb{R}_+ \times (0,T), \\ -(u^m)_x(0,t) = u^q(0,t) & t \in (0,T), \\ u(x,0) = u_0(x) & \text{in } \mathbb{R}_+. \end{array}\right. \tag{1.4}$$

On the critical line both terms are of the same order, and we have to take into account both effects.

Problems (1.3) and (1.4) show a lot of similarities. For each of them two critical exponents appear. The first one, called *global existence exponent*, separates the region where every solution is global from the one where there exist blowing up solutions. The second critical exponent, the so-called *Fujita exponent*, [F], separates the region where every solution blows up from the region where there exist both global solutions and blowing up solutions, [L]. For problem (1.3) the critical global existence exponent is $p = 1$ and the Fujita exponent is $p = m + 2$. Thus, for $p \leq 1$ every solution is global, for $1 < p \leq m + 2$ every nontrivial solution blows up and for $p > m + 2$ there exist global solutions (with u_0 small) and blowing up solutions (with u_0 large). For problem (1.4) the global existence exponent is $q = (m + 1)/2$ and the Fujita exponent is $q = m + 1$. For proofs of these facts, as well as for existence, comparison and other blow-up results, we refer the reader to [GKMS], [PV1], [SGKM] for problem (1.3) and to [GL], [QR] for problem (1.4) (see also the references in those papers).

Problem (1.1) combines both effects, the nonlinear source and the nonlinear boundary condition. Therefore, we expect the existence of similar phenomena. Our first theorem shows, for every fixed $m > 1$, the existence of a critical *global existence curve* in the (p, q) plane (see Fig. 1).

Theorem 1.1 *(i) In the region of parameters*

$$\mathcal{B} = \{0 < p \leq 1,\ 0 < 2q \leq m + 1\}$$

every nonnegative solution is global in time.
(ii) In the region

$$\mathcal{U} = \{p > 1\} \cup \{2q > m + 1\}$$

there exist solutions that blow up in finite time.

Let us look at this result from a more inspiring point of view. If (p, q) is above the critical line, i.e. $2q \geq m + p$, then the global existence curve is given by $q = (m + 1)/2$. This is exactly the critical exponent for the reduced problem (1.4). If (p, q) is below, the critical curve corresponds to the critical exponent for (1.3). This is the first result showing the critical role played by the line (1.2).

Our second result shows the existence of a *Fujita curve*.

Theorem 1.2 *(i) In the region $\mathcal{F}_u \subset \mathcal{U}$,*

$$\mathcal{F}_u = \{p > 1,\ q < m + 1\} \cup \{p < m + 2,\ 2q > m + 1\},$$

every solution blows up in finite time.
(ii) In the region $\mathcal{F}_b \subset \mathcal{U}$,

$$\mathcal{F}_b = \{p \geq m + 2,\ q \geq m + 1\},$$

there exist solutions that blow up in finite time and also global solutions.

Observe that the Fujita curve is given by $p = m+2$ when (p, q) is above the critical line (1.2) and by $q = m+1$ when (p, q) is below that line (see again Fig. 1). Thus, as far as the Fujita curve is concerned, the boundary dominates for p large, while the source dominates for q large. This does not contradict our claim about which term matters on each side of the line (1.2). Indeed, we should keep in mind that the Fujita curve is mainly related to the existence of global solutions. The dominant term for u large becomes unimportant for u small.

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Figure 1. Critical curves.

We remark that the region $\mathcal{F}_u$ is not only the union of the regions where all solutions blow up for the reduced problems (1.3), (1.4), the central cross

$$\mathcal{F}_{u1} = \{(m+1)/2 < q \le m+1\} \cup \{1 < p \le m+2\}.$$

Indeed, it includes two new regions,

$$\mathcal{F}_{u2} = \{p \le 1,\ q > m+1\} \quad \text{and} \quad \mathcal{F}_{u3} = \{p > m+2,\ 2q \le m+1\},$$

where the influence of one term (flux or reaction) makes the solution large enough so that the other term makes it blow up.

As a direct precedent we have [ZCM], where the case of linear diffusion $m = 1$ is considered. We remark that the regions $\mathcal{F}_{u2}$ and $\mathcal{F}_{u3}$ were not taken into account in that paper. In Theorem 3.1 we extend Theorem 1.2 to the case $m = 1$, thus covering this gap.

Now we turn our attention to the asymptotic behaviour of the blowing up solutions of problem (1.1). As mentioned above, the different terms in that problem have a different influence depending on the values of the parameters. This influence affects not only the question about the possibility of blow-up, but also the way solutions blow up.

In order to study this asymptotic behaviour we need to make further assumptions on the data. In the sequel we assume that the initial data u_0:

(i) are nonincreasing;
(ii) satisfy a compatibility condition, $-(u_0^m)'(0) = u_0^q(0)$;
(iii) satisfy $(u_0^m)'' + u_0^p \geq 0$ in $\mathcal{D}'(\mathbb{R}_+)$.

We denote this set of hypotheses on the initial data by H_0.

Let us explain the consequences of H_0 on solutions for positive times: (i) implies that $u(\cdot, t)$ is nonincreasing for every $0 < t < T$. To see this just study the equation satisfied by u_x. Condition (ii) provides smoothness up to the parabolic boundary. Finally, hypothesis (iii) guarantees that the solution is nondecreasing in t (see [BC, Lemma 1.1]).

We first study the *blow-up rate*, that is, the speed at which u grows as $t \nearrow T$. Let us introduce the number

$$\alpha = \begin{cases} 1/(2q-m-1) & \text{if } \ 2q \geq m+p, \\ \\ 1/(p-1) & \text{if } \ 2q \leq m+p. \end{cases} \tag{1.5}$$

Theorem 1.3 *Let* $(p, q) \in \mathcal{U}$ *and let* u *be a solution to problem* (1.1) *that blows up at a finite time* $t = T$*, with initial data such that* H_0 *holds. Then there exist positive constants* $0 < c \leq C < \infty$ *such that for every* $t < T$*,*

$$c \leq \max_{x \geq 0} u(x, t) \ (T-t)^{\alpha} \leq C. \tag{1.6}$$

This theorem throws additional light (further evidence below) upon the critical role of the line (1.2). Indeed, if $2q > m + p$ the blow-up rate is given by the boundary term (it depends on q and not on p) and is the same one as for problem (1.4), see [QR], while in the case $2q < m + p$ the blow-up rate is given by the source term (it depends only on p) and is the same as for (1.3). Note that on the line $2q = m + p$ both exponents in the theorem coincide, $1/(2q - m - 1) = 1/(p - 1)$, i.e., the two terms in the problem are of the same order.

The blow-up rate for the case $m = 1$ was included in the paper [ZCM]. The semilinear structure, which allows to use a representation formula, is widely used in the proofs therein. In the quasilinear case, $m > 1$, there is no such a representation formula, and the proofs become more involved.

Our next aim is to study, for compactly supported blowing up solutions, whether their supports remain uniformly bounded for every time $t < T$. In this case we say that the solution is *localized.*

When u_0 has compact support so does the solution u for every time $t > 0$ if $p \geq 1$ (cf. [OKC]). If $p < 1$ this property is not true in general. For instance, if $m + p < 2$ every solution is strictly positive, and if $m + p \geq 2$ the support can be prescribed almost arbitrarily for every initial data (cf. [PV2]). Therefore we restrict ourselves from now on to the case $p \geq 1$. If u_0 is compactly supported we will say that it satisfies hypothesis H_1.

We are considering a solution u to problem (1.1) which blows up at a certain time $T < \infty$. Let

$$s(t) = \sup\{\, x > 0 \,:\, u(x,t) > 0 \,\}$$

be its interface. Since the solutions to problem (1.1) are supersolutions to the porous medium equation, we get that $s(t)$ is a nondecreasing curve (see [A]). Also $s(t)$ is continuous and $s(t) < \infty$ for every $t \in [0,T)$. Next theorem describes the behaviour of $s(t)$ as t approaches the blow-up time T. We introduce the number

$$\beta = \begin{cases} (q-m)/(2q-m-1) & \text{if } 2q \geq m+p, \\ \\ (p-m)/(2(p-1)) & \text{if } 2q \leq m+p. \end{cases} \tag{1.7}$$

Theorem 1.4 *Let $p \geq 1$ and u_0 satisfy H_0, H_1.*

(i) If $(p,q) \in \mathcal{L} = \{p \geq m\} \cup \{q \geq m\} \subset \mathcal{U}$, i.e. $\beta \geq 0$, then $\lim_{t \nearrow T} s(t) < \infty$.

(ii) If $(p,q) \in \mathcal{P} = \mathcal{U} \setminus \mathcal{L}$, i.e. $\beta < 0$, then $\lim_{t \nearrow T} s(t) = \infty$. Moreover, in this case there are positive constants c, C, such that

$$c(T-t)^{\beta} \leq s(t) \leq C(T-t)^{\beta}. \tag{1.8}$$

This theorem shows once more that the line (1.2) is critical, as β depends on whether we are above or below it, see (1.7). Above it (for q large) localization is proved under the condition $q \geq m$, the same exponent as for the reduced problem (1.4), [QR]. Moreover, in the absence of localization the asymptotic behaviour of the interface is given by a power that does not depend on p. On the other hand the source gives the localization properties of the solution if we are below the critical line.

Once we know the growth rate of the solution (1.6) and its support (1.8), we rescale u accordingly. Thus, we define, for $y \geq 0$, $\tau \geq \tau_0 \equiv -\log T$, the function

$$f(y,\tau) = (T-t)^{\alpha} u(x,t), \qquad y = x(T-t)^{-\beta}, \qquad \tau = -\log(T-t), \tag{1.9}$$

where the exponents satisfy (1.5) and (1.7). Observe that the relation

$$\alpha(m-1) + 2\beta = 1$$

holds. In similarity variables we arrive at the following problem for f

$$\begin{cases} f_{\tau} = (f^m)_{yy} - \beta y f_y - \alpha f + e^{-\delta\tau} f^p, \\ -(f^m)_y(0,\tau) = e^{-\epsilon\tau} f^q(0,\tau), \\ f(y,\tau_0) = f_0(y) \equiv T^{\alpha} u_0(yT^{\beta}), \end{cases} \tag{1.10}$$

where

$$\delta = 1 - (p-1)\alpha, \quad \epsilon = (1 - (2q-m-1)\alpha)/2. \tag{1.11}$$

The initial value f_0 inherits the properties of u_0. Hence it is nonincreasing and has compact support.

Let us look with more care at the above equation for f. First of all, the behaviour of u as $t \nearrow T$ is translated into the behaviour of f as $\tau \to \infty$. With the aid of the blow-up rate (1.6) we have that f remains bounded. Above the critical line $\delta > 0$ and $\epsilon = 0$, so the source term vanishes as τ goes to infinity. On the other hand, below the critical line we have $\delta = 0$ and $\epsilon > 0$, so the nonlinear flux disappears as $\tau \to \infty$. On the critical line $\delta = \epsilon = 0$ and both terms have importance for the evolution of f. This allows us to characterize the blow-up profile in self-similar variables. The limit profile will be given by the solution of the corresponding stationary limit equation. The particular stationary equation will depend on the parameters. In some cases the uniqueness of the limit profiles has not been established yet. Hence, for every continuous initial value $f_0(y)$, we must consider the so called *ω-limit set*,

$$\begin{aligned}\omega(f_0) = \ & \{g \in C([0,\infty)),\ g \geq 0 \mid \exists\ \tau_j \to \infty \ \text{ such that} \\ & \lim_{\tau_j \to \infty} f(\cdot, \tau_j) = g(\cdot) \ \text{ uniformly in compacts sets of } \ [0,\infty)\,\}.\end{aligned}$$

In the particular case $m + p = 2q$ and $1 < p < m$, we need to impose an additional technical hypothesis on the geometrical structure of the initial data as in [Ga1], that we call H_2. As this condition is rather cumbersome, we do not state it here, and postpone it to Section 6.

Theorem 1.5 *Assume $p \geq 1$ and that the initial data u_0 verify H_0, H_1. Let $f_0(y) = T^{\alpha} u_0(yT^{\beta})$ and $\omega(f_0)$ be its $\omega-$limit.*

(i) If $2q < m + p$, then $\omega(f_0)$ is contained in the set of nontrivial solutions of

$$\begin{cases} (g^m)'' - \beta y g' - \alpha g + g^p = 0 & \text{in } \mathbb{R}_+, \\ (g^m)'(0) = 0. \end{cases} \tag{1.12}$$

(ii) If $2q = m + p$, (and u_0 satisfies also H_2 if $1 < p < m$), then $\omega(f_0)$ is contained in the set of nontrivial solutions of

$$\begin{cases} (g^m)'' - \beta y g' - \alpha g + g^p = 0 & \text{in } \mathbb{R}_+, \\ -(g^m)'(0) = g^q(0). \end{cases} \tag{1.13}$$

(iii) If $2q > m+p$, then $\omega(f_0) = \{\widetilde{g}\}$, where $\widetilde{g}$ is the unique positive bounded solution of

$$\begin{cases} (g^m)'' - \beta y g' - \alpha g = 0 & \text{in } \mathbb{R}_+, \\ -(g^m)'(0) = g^q(0). \end{cases} \tag{1.14}$$

Remark: Solutions of (1.12), (1.13) and (1.14) provide us with self-similar solutions for problems (1.3), (1.1) and (1.4) respectively. The existence of a profile solving problem (1.13) is obtained in the course of the proof via dynamical arguments as the nontrivial limit of $f(y,\tau)$. In case $p < m$ it can also be obtained following ODE techniques as the ones used in [SGKM] for problem (1.12).

Remark: Uniqueness of solutions to problem (1.12) is only known in the case $p < m$ (see [GKMS]). It is still an open problem both for (1.12) if $p \geq m$, and for (1.13) in all cases. In case we have uniqueness, the above result implies, for the unique profile $\widetilde{g}$,

$$\lim_{t \nearrow T} (T-t)^{\alpha} u(y(T-t)^{\beta}, t) = \widetilde{g}(y),$$

uniformly in compact sets of $[0, \infty)$.

Remark: In the course of the proof of the above theorem we also obtain the asymptotic behaviour of compactly supported solutions of the reduced problem (1.4). For the case $q = m$ see [V]. In the case of linear diffusion this was proved in [FQ].

One final question arises. At which points does u blow up? Those points form the *blow-up set* for u, $B(u)$, which is defined as

$$B(u) = \{\, x \geq 0 \,:\, \lim_{t \nearrow T} u(x,t) = \infty \,\}.$$

Since $u_x \leq 0$, there are only three possibilities for the blow-up set: global blow-up, $B(u) = [0, \infty)$; regional blow-up, $B(u)$ is a bounded interval; or single-point blow-up, $B(u) = \{0\}$.

Theorem 1.6 *Let $p \geq 1$ and u_0 verify H_0, H_1. Let β satisfy* (1.7). *We have*
(i) global blow-up if $\beta < 0$;
(ii) regional blow-up if $\beta = 0$;
(iii) single-point blow-up if $\beta > 0$.

Again this result coincides with the results obtained for the reduced problems (1.3) and (1.4), see [SGKM], [QR].

Organization of the paper: All the results that we will use (sometimes without explicit reference) in what follows are collected together in Section 2. In Section 3 we prove the blow-up vs. global existence results, Theorems 1.1 and 1.2. In Section 4 we prove the blow-up rate, Theorem 1.3. Section 5 contains our localization result, Theorem 1.4. We devote the large Section 6 to prove Theorem 1.5 concerning the asymptotic profiles. Finally, Section 7 contains the description of the blow-up sets, Theorem 1.6.

2 Preliminaries.

We first write down what we understand by a supersolution and a subsolution to our problem.

Definition (i) We call $\overline{u}$ a *supersolution* of (1.1) if it satisfies

$$\begin{cases} \overline{u}_t \geq (\overline{u}^m)_{xx} + \overline{u}^p & \text{in } \mathbb{R}_+ \times (0,T), \\ -\overline{u}_x(0,t) \geq \overline{u}^q(0,t) & t \in (0,T), \\ \overline{u}(x,0) \geq u_0(x) & \text{in } \mathbb{R}_+. \end{cases} \tag{2.1}$$

(ii) We call $\underline{u}$ a *subsolution* of (1.1) if it satisfies (2.1) with the opposite inequalities.

In the case $p < 1$, there is no general comparison principle (see [SGKM], [PV1]), the difficulty arising from the non Lipschitz character of the reaction term. In fact there is no uniqueness. Thus we need an extra definition.

Definition (i) We call $\overline{u}$ a *strict supersolution* of (1.1) if $\overline{u} = \lim\limits_{\varepsilon\to 0} u^{(\varepsilon)}$, where $u^{(\varepsilon)}$ satisfies

$$\begin{cases} u_t^{(\varepsilon)} \geq ((u^{(\varepsilon)})^m)_{xx} + (u^{(\varepsilon)})^p & \text{in } \mathbb{R}_+ \times (0,T), \\ -u_x^{(\varepsilon)}(0,t) \geq (u^{(\varepsilon)})^q(0,t) & t \in (0,T), \\ u^{(\varepsilon)}(x,0) \geq u_0(x) + \varepsilon & \text{in } \mathbb{R}_+. \end{cases}$$

(ii) We call $\underline{u}$ a *strict subsolution* of (1.1) if $\underline{u} = \lim\limits_{\varepsilon\to 0} u_{(\varepsilon)}$, where $u_{(\varepsilon)}$ satisfies

$$\begin{cases} (u_{(\varepsilon)})_t \leq (u^m_{(\varepsilon)})_{xx} + f_\varepsilon(u_{(\varepsilon)}) & \text{in } \mathbb{R}_+ \times (0,T), \\ -(u_{(\varepsilon)})_x(0,t) \leq u^q_{(\varepsilon)}(0,t) & t \in (0,T), \\ 0 \leq u_{(\varepsilon)}(x,0) \leq u_0(x) & \text{in } \mathbb{R}_+, \end{cases}$$

for any approximating from below Lipschitz reaction $0 \leq f_\varepsilon(s) \leq s^p$.

These definitions are of course compatible in the case $p \geq 1$, where a solution is at the same time a strict supersolution and a strict subsolution. This is not the case for $p < 1$, where there may appear an infinity of distinct solutions (see ([PV1]). As to the applications we have that minimal solutions of the equation in (1.1) with outward flux at the border are strict subsolutions. The corresponding result for maximal solutions also holds. Even more, the solutions to problems (1.3) and (1.4) are subsolutions to our problem.

With these definitions we state the following comparison lemma.

Lemma 2.1 (i) *Let $p \geq 1$, and let $\overline{u}$ and $\underline{u}$ be, respectively, a supersolution and a subsolution to problem* (1.1). *Then*

$$\overline{u}_0(0) > \underline{u}_0(0) \Longrightarrow \overline{u} \geq \underline{u} \qquad \text{in } \mathbb{R}_+ \times [0,T).$$

(ii) *Let $0 < p < 1$, and let $\overline{u}$, u and $\underline{u}$ be, respectively, a strict supersolution, a solution and a strict subsolution to problem* (1.1). *Then*

$$\overline{u}_0(0) > u_0(0) > \underline{u}_0(0) \Longrightarrow \overline{u} \geq u \geq \underline{u} \qquad \text{in } \mathbb{R}_+ \times [0,T).$$

Proof: The techniques of [PV1] allow us to obtain the comparison at least while the solutions are strictly ordered at $x = 0$. To prove that we have indeed this strict ordering for all $t \in (0,T)$ we use Hopf's Lemma as in [QR]. □

In several places we will use some special self-similar solutions to problems related to (1.1). The profiles of these self-similar solutions verify some elliptic equation that has been studied in [G], [GP1] and [GP2]. The precise results of those papers that we will need are summarized in the following lemma.

Lemma 2.2 *Consider the problem*

$$\begin{cases} (\varphi^m)''(\xi) - a\xi\varphi'(\xi) = b\varphi(\xi), \\ \varphi(0) = A > 0, \\ -(\varphi^m)'(0) = B \geq 0. \end{cases} \tag{2.2}$$

(i) If $b < 0$ and $a + b \geq 0$, there exists a unique $V_* \geq 0$ and a unique solution with $B = A^{\frac{m+1}{2}} V_*$. This solution has compact support. Moreover, if $a + b = 0$ then $V_* = 0$, while if $a + b > 0$ then $V_* > 0$.
(ii) If $b > 0$, there exists a unique $V_* > 0$ and a unique solution with $B = A^{\frac{m+1}{2}} V_*$. Moreover, if $a \geq 0$ the solution has compact support, while if $a < 0$ it is positive.

We will use an argument based on counting the number of intersections of two different solutions of our problem and the fact that this number cannot increase with time (the Sturmian argument, see e.g. [SGKM] and the references therein).

Proposition 2.1 *(i) Let u_1 and u_2 be two solutions of problem* (1.1) *with the same blow-up time T. Assume that $u_1(0,t) - u_2(0,t)$ does not change sign for $t_0 \leq t < T$. Thus, if $N(t) < \infty$ is the number of sign changes of $u_1(\cdot,t) - u_2(\cdot,t)$ in $\mathbb{R}_+$, then $N(t)$ is nonincreasing for $t_0 < t < T$.*
(ii) The same statement is valid if the problem is defined only in some subset $(a(t), b(t)) \times (t_0, T)$ and we impose that $u_1(x,t) - u_2(x,t)$ does not change sign for $x = a(t)$ and $x = b(t)$.

We also make use of the following easy fact, which means that two solutions of the equation in (1.1) with the same blow-up time must intersect. Even more, the supersolutions to problem (1.1) cannot be larger than the subsolutions if they blow up at the same time.

Proposition 2.2 *Let u_1 and u_2 be, respectively, a subsolution and a supersolution to problem* (1.1), *with $u_1 \leq u_2$ ($u_1 \not\equiv u_2$) in $\mathbb{R}_+ \times (0,T)$, where T is the blow-up time of u_2. Then u_1 remains bounded up to time $T + \varepsilon$ for some $\varepsilon > 0$.*

For the proofs of Propositions 2.1 and 2.2 we refer to [SGKM].

3 Blow-up versus global existence.

In this section we prove Theorems 1.1 and 1.2. The proofs rely completely on comparison arguments.

Proof of Theorem 1.1: (*i*) To see that in $\mathcal{B}$ all solutions are globally defined in time, we use a comparison argument. We look for a globally defined in time strict supersolution of self-similar form

$$\overline{u}(x,t) = e^{Lt}\varphi(\xi), \qquad \xi = xe^{-L\frac{m-1}{2}t}, \quad L > 0.$$

For $\overline{u}$ to be a supersolution we need

$$\begin{aligned}
&\overline{u}_t - (\overline{u}^m)_{xx} - \overline{u}^p = e^{Lt}\left(L\varphi - \tfrac{1}{2}(m-1)L\xi\varphi' - (\varphi^m)'' - e^{L(p-1)t}\varphi^p\right) \geq 0,\\
&-(\overline{u}^m)_x(0,t) - \overline{u}^q(0,t) = e^{Lmt}\left(\tfrac{1}{2}L(m-1)(\varphi^m)'(0) - e^{L(q-\frac{m+1}{2})t}\varphi^q(0)\right) \geq 0,\\
&\overline{u}(x,0) \geq u_0(x).
\end{aligned}$$

The function

$$\varphi(\xi) = (K + e^{-(K+1)\xi})^{1/m},$$

will do the job if we choose

$$L = \frac{(K+1)^2 + (K+1)^{\frac{p}{m}}}{K^{\frac{1}{m}}}, \qquad K^{\frac{1}{m}} \geq \|u_0\|_\infty = u_0(0).$$

Since $\overline{u}$ is strictly positive in $\mathbb{R}_+$, it is in fact a strict supersolution.

(ii) To show that in the region $\mathcal{U}$ there are solutions that blow up in finite time, we consider a blowing up solution of problem (1.3), if $p > 1$ (for the existence see [SGKM]), or a blowing up solution of problem (1.4), if $2q > m + 1$ (see [GL]). Such functions are subsolutions of (1.1). Any solution which is above one of these functions at time $t = 0$ blows up in finite time. □

Proof of Theorem 1.2: (i) In $\mathcal{F}_u$ we distinguish three regions: the central cross $\mathcal{F}_{u1}$, where just one of the terms (flux or reaction) in our problem is capable to provoque the blow-up, and the regions $\mathcal{F}_{u2}$ and $\mathcal{F}_{u3}$ mentioned in the introduction, where the combined effect of both terms is needed to make every solution blow up.

REGION $\mathcal{F}_{u1}$. Here the result is immediate, since we can use as blowing up subsolutions:
- the solution of problem (1.3) with the same initial data, if $1 < p < m + 2$ (it blows up, see [SGKM]);
- the solution of problem (1.4) with nontrivial initial data below u_0 and strictly below at the origin, if $(m + 1)/2 < q < m + 1$ (it also blows up, see [GL]).

REGION $\mathcal{F}_{u2}$. Here we do comparison with particular subsolutions in two steps, and prove that the reaction term makes the solution large enough to be in the set of initial data for which the flux condition alone is enough to cause blow-up.

We first consider, if $p < 1$, the absolute minimal solution to problem (1.3)

$$U(x,t) = t^\gamma \varphi(xt^{-\delta}) \tag{3.1}$$

with exponents $\gamma = 1/(1-p)$, $\delta = (m-p)/(2(1-p))$ (see [PV1]). If $m+p \geq 2$, the profile is a continuous, bounded, symmetric, compactly supported, nonincreasing function, and if $m+p < 2$ the profile is constant $\varphi(\xi) = \gamma^{-\gamma}$. Since it begins with zero initial value and has zero flux at the boundary, it can be used as a strict subsolution with respect to every nontrivial initial data (we always assume $u_0(0) > 0$).

If $p = 1$ we use a compactly supported self-similar solution to problem (1.3) of exponential type,

$$U(x,t) = e^{\gamma(t-\tau)}\varphi(xe^{-\delta(t-\tau)}), \tag{3.2}$$

with exponents $\gamma = 2/(m+1)$ and $\delta = (m-1)/(m+1)$ (see Lemma 2.2). We choose $\tau \geq 0$ such that $U(x,0) \leq u_0(x)$. Therefore U is a strict subsolution. Now comparison implies a minimal growth rate for the solution and its support. Hence there exists a finite time t_0 such that the solution is larger than the initial value of the blowing up subsolution

$$V(x,t) = (T-t)^{-\gamma}\psi(x(T-t)^{-\delta})$$

with exponents $\gamma = 1/(2q-m-1)$, $\delta = (q-m)/(2q-m-1)$ and profile

$$\psi(\xi) = A(a-\xi)_+^{\frac{1}{m-1}},$$

A and a being some positive constants (see [GL]). Observe that V is a subsolution to problem (1.4), and thus it is a strict subsolution to problem (1.1). Hence our solution blows up in a finite time which is not larger than $T + t_0$.

REGION $\mathcal{F}_{u3}$. We apply the same argument as in the previous case. As a subsolution for the first step we consider, if $2q < m+1$, the self-similar solution of problem (1.4), a function like (3.1), where now the exponents are $\gamma = 1/(m+1-2q)$, $\delta = (m-q)/(m+1-2q)$. We just use Lemma 2.2 choosing $A = V_*^{-2\gamma}$ in order to fulfill the boundary condition. It has compact support and zero initial value ($\gamma > 0$). If $2q = m+1$, the compactly supported subsolution used is a solution to problem (1.4) of exponential type (3.2), see again Lemma 2.2 for the existence. Once we have that our solution is large enough the second step follows from the results in [GKMS], [SGKM], where a blowing up subsolution is constructed.

(ii) In the case $2q < p+m$ and $q > m+1$ we construct bounded supersolutions again using solutions in self-similar form to the problem without the reaction term (1.4). The global solutions obtained for this range of parameters can be used as supersolutions to cover the whole range $q > m+1$ and $p > m+2$.

CASE $2q < m+p$ AND $q > m+1$. We consider the function in self-similar form

$$w(x,t) = (t+t_0)^{-\gamma}\varphi(x(t+t_0)^{-\delta})$$

with $\gamma = 1/(2q-m-1)$, $\delta = (q-m)/(2q-m-1)$. Let $\tilde{\gamma} > \gamma$. The profile $\varphi(\xi)$ is given by the unique nontrivial solution of (2.2), where

$$a = \delta, \quad b = -\tilde{\gamma}, \quad A = V_*^{2\gamma}.$$

It exists from Lemma 2.2 if $\gamma < \tilde{\gamma} < (q-m)\gamma$ (recall that $q-m > 1$). The condition to be a supersolution is

$$L(w) \equiv w_t - (w^m)_{xx} - w^p = (t+t_0)^{-\gamma-1}[-\gamma\varphi - \delta\xi\varphi' - (\varphi^m)'' - (t+t_0)^{-\gamma p+\gamma-1}\varphi^p] > 0,$$

since at the boundary we have

$$B(w) \equiv -(w^m)_x(0,t) - w^q(0,t) = (t+t_0)^{-\gamma m-\delta}[-(\varphi^m)'(0) - \varphi^q(0)] = 0.$$

As $-\gamma p + \gamma - 1 < 0$ it is not hard to choose t_0 large enough to get the above inequality. Thus w is a supersolution and any solution with initial data $u_0(x)$ lying below $w(x,0) = t_0^{-\gamma}\varphi(xt_0^{-\delta})$ is globally defined in time. Even more, it decays to zero as $t \to \infty$.

CASE $2q \geq m + p$ AND $p > m + 2$. First we observe that, as the global solution that we have found for $2q < m + p$ with $q > m + 1$ goes to zero, we can assume that it is less than one. Therefore it is a supersolution for every pair of exponents $(\tilde{p}, \tilde{q})$ such that $\tilde{p} \geq p$ and $\tilde{q} \geq q$. Thus we have a global nontrivial supersolution for every $p > m + 2$ with $q > m + 1$. We remark that the line $p + m = 2q$ contains the point $p = m + 2$, $q = m + 1$. □

The above arguments also work for linear diffusion.

Theorem 3.1 *If* $m = 1$ *and the parameters* (p,q) *lie on the regions* $\mathcal{F}_{u2}$ *or* $\mathcal{F}_{u3}$*, i.e.,* $0 < p \leq 1$, $q > 2$ *or* $0 < q \leq 1$, $p > 3$*, then every solution to problem* (1.1) *blows up in finite time.*

Proof: (i) In $\mathcal{F}_{u2}$ we use as subsolution in the first step the flat solution to problem (1.3), $U(x,t) = ((1-p)t)^{\frac{1}{1-p}}$. Thus there exists a time such that u satisfies the size condition for blow-up obtained in [GL].

(ii) In $\mathcal{F}_{u3}$ it is easy to check that the self-similar function

$$V(x,t) = t^{\gamma}\varphi(x/\sqrt{t})$$

with $\gamma = 1/(2(1-q))$ and profile

$$\varphi(\xi) = A(e^{c(\xi_0 - \xi)} - 1)_+$$

is a subsolution to our problem if we choose the constants properly: $c > 0$ is large and A, ξ_0 are small. We conclude as before using the size condition for blow-up given in [F]. Though in that paper the blow-up question for problem (1.3) is studied only in bounded domains, every solution in a bounded domain contained in $\mathbb{R}_+$ is a subsolution to the problem in all of $\mathbb{R}_+$. □

4 Blow-up rate.

In this section we study the speed at which the solutions to problem (1.1) blow up. For the blow-up rate for linear diffusion with a similar boundary condition we refer to [ZCM] and [HY].

Proof of Theorem 1.3: Let u be a solution of (1.1) with finite blow-up time T. We define

$$\varphi_M(y,s) = \frac{1}{M}u(ay, bs + t^*), \qquad s \in (-t^*/b, 0),\ y \in \mathbb{R}_+$$

where, for each $0 < t^* < T$, we put

$$M = M(t^*) = u(0,t^*) = \max_{x \geq 0} u(x,t^*)$$

(u is nonincreasing in x). If we choose $a^2 = M^{m-1}b$, this function satisfies (u is nondecreasing in t)

$$0 \le \varphi_M \le 1, \quad \varphi_M(0,0) = 1, \quad (\varphi_M)_s \ge 0, \tag{4.1}$$

and

$$\begin{cases} (\varphi_M)_s = (\varphi_M^m)_{yy} + bM^{p-1}(\varphi_M)^p, \\ -(\varphi_M^m)_y(0,s) = aM^{q-m}(\varphi_M)^q(0,s). \end{cases}$$

Assume first that $m+p<2q$ and choose $a = M^{m-q}$. This determines $b = M^{m+1-2q}$. The reaction coefficient in the equation is then $\lambda_M = M^{m+p-2q}$. Observe that λ_M goes to 0 as $M \to \infty$. We claim that there exist constants C_1, C_2 such that for every large M

$$0 < C_1 \le (\varphi_M)_s(0,0) \le C_2. \tag{4.2}$$

From the results for bounded solutions of porous medium equations (see [D]), we have that the family $\{\varphi_M\}$ is equicontinuous. Therefore, passing to a subsequence $M_j \to \infty$ if necessary, we have that

$$\varphi_{M_j} \to \varphi$$

as $j \to \infty$, uniformly on compact sets of $\{y \ge 0,\ s_* \le s \le 0\}$. This function φ is continuous and satisfies $\varphi(0,0) = 1$. Hence, there exists a small neighbourhood U of $(0,0)$ such that $\varphi > 1/2$ in U. As we have uniform convergence over $\overline{U}$, for j large enough we have that $1/4 \le \varphi_{M_j} \le 1$. Therefore, each φ_{M_j} is a solution of a uniformly parabolic equation in $\overline{U}$. Thus, from the fact that the sequence $\{\varphi_{M_j}\}$ is uniformly bounded, we obtain, using well known Schauder estimates (see [Li]),

$$\|\varphi_{M_j}\|_{C^{2+\varepsilon,1+\frac{\varepsilon}{2}}} \le C \qquad \text{in } \overline{U},$$

for some $0 < \varepsilon < 1$ and every j large enough. We conclude that for M large enough

$$(\varphi_M)_s(0,0) \le C,$$

proving the first half of our claim.

It remains to prove that $(\varphi_M)_s(0,0) \not\to 0$. Assume that it is false. That is, there is a sequence $M_j \to \infty$ such that $(\varphi_{M_j})_s(0,0) \to 0$. We proceed just as before to obtain that

$$\varphi_{M_j} \to \varphi, \qquad \text{and} \qquad \|\varphi_{M_j}\|_{C^{2+\varepsilon,1+\frac{\varepsilon}{2}}} \le C \qquad \text{in } \overline{U}.$$

Using that $C^{2+\varepsilon,1+\frac{\varepsilon}{2}}(\overline{U})$ is compactly included in $C^{2+\varepsilon',1+\frac{\varepsilon'}{2}}(\overline{U})$, $0 < \varepsilon' < \varepsilon$, we can conclude, refining the sequence if necessary, that $\varphi_{M_j} \to \varphi$, in $C^{2+\varepsilon',1+\frac{\varepsilon'}{2}}(\overline{U})$ and hence

$$\varphi_s(0,0) = 0.$$

But we observe that φ satisfies (4.1) and also

$$\begin{cases} \varphi_s = (\varphi^m)_{yy}, \\ -(\varphi^m)_y(0,s) = \varphi^q(0,s) \end{cases} \tag{4.3}$$

in $\mathbb{R}_+ \times (-\infty, 0]$. We set now $w = \varphi_s$. Since w solves the problem

$$\begin{cases} w_s = (m\varphi^{m-1}w)_{yy}, \\ -m\varphi^{m-1}w_y(0,s) = -(m\varphi^{m-1}w)_y(0,s) = q\varphi^{q-1}w(0,s), \\ w(0,0) = 0, \ w \geq 0, \end{cases}$$

we conclude, using Hopf's Lemma (recall that $\varphi(0,0) = 1$), that $w \equiv 0$. We get that φ does not depend on s, and thus, from (4.3), it satisfies

$$\begin{cases} (\varphi^m)'' = 0, \\ \varphi(0) = 1, \ -\varphi'(0) = 1, \\ 0 \leq \varphi \leq 1, \end{cases}$$

which is impossible, and the claim is proved.

In terms of u estimates (4.2) read

$$C_1 \leq \frac{bu_t(0,t^*)}{M(t^*)} \leq C_2$$

for every large $M(t^*)$. Recalling what b and M are, and using that $\lim\limits_{t^* \nearrow T} M(t^*) = \infty$, there exists $0 < T_0 < T$ such that

$$C_1 \leq M^{m-2q}(t^*)M'(t^*) \leq C_2 \qquad \text{for every } T_0 < t^* < T.$$

Now we integrate between t and T for each $T_0 < t < T$, and obtain

$$C_1(T-t) \leq \int_t^T M^{m-2q}(t^*)M'(t^*)\,dt^* = \int_{M(t)}^\infty \eta^{m-2q}\,d\eta = \frac{M^{-2q+m+1}(t)}{2q-m-1} \leq C_2(T-t),$$

which implies (1.6).

Suppose now that $m + p > 2q$. We proceed as before but in this case we choose $b = M^{1-p}$ (this means $a = M^{\frac{m-p}{2}}$). Therefore the flux coefficient $\lambda_M = M^{\frac{2q-m-p}{2}}$ tends to 0 as $M \to \infty$. The rest of the proof is exactly the same, once we establish estimate (4.2), and again we only need to prove the lower estimate. Assuming the contrary and passing to the limit (in a subsequence $M_j \to \infty$ if necessary), we use Hopf's Lemma to obtain a nontrivial solution of

$$\begin{cases} (\varphi^m)'' + \varphi^p = 0, \\ \varphi(0) = 1, \ \varphi'(0) = 0, \\ 0 \leq \varphi \leq 1. \end{cases}$$

This gives a contradiction.

In the case $m + p = 2q$ we choose $a = M^{m-q}$, $b = M^{1-p}$. We conclude by observing that the problem

$$\begin{cases} (\varphi^m)'' + \varphi^p = 0, \\ \varphi(0) = 1, \ \varphi'(0) = -1/m, \\ 0 \leq \varphi \leq 1, \end{cases}$$

has no solution. □

5 Localization of the support.

In this section we study whether unbounded compactly supported solutions to problem (1.1) are localized or not.

Proof of Theorem 1.4: (i) We have $(p,q)\in\mathcal{L}$, i.e. $\beta\geq 0$. The proof will be different for the cases $q\geq m>p$ and $p\geq m$.

Case $q\geq m>p$. We consider the rescaled function f given in (1.9) but with different exponents, namely $\beta=0$ and $\alpha=1/(m-1)$. Thus we get the equation for f (see (1.10))

$$f_\tau=(f^m)_{yy}-\frac{1}{m-1}f+e^{-\delta\tau}f^p,$$

where $\delta=(m-p)/(m-1)>0$. Observe that the supports of f and u are the same. By (1.6) we have

$$f(y,\tau)\leq ce^{-\frac{2(q-m)}{2q-m-1}\tau}\leq c$$

for every $\tau\geq\tau_0$. On the other hand, for every $0<\overline{\alpha}<1/(m-1)$ there exists $\tau_1>\tau_0$ such that f is a subsolution to the problem

$$\left\{\begin{array}{l}\varphi_\tau=(\varphi^m)_{yy}-\overline{\alpha}\varphi,\\ \varphi(0,\tau)=c,\end{array}\right.$$

for $\tau\geq\tau_1$. By a comparison argument, we get $f\leq\varphi$. We conclude with the observation that any stationary solution to that problem has compact support.

Case $p\geq m$. We will use intersection comparison with a function of the form

$$v(x,t)=(T-t)^{-\alpha}\varphi(\xi)$$

with $\xi=(x_0-x)(T-t)^{-\beta}$, $\alpha=1/(p-1)$ and $\beta=(p-m)/(2(p-1))$, and φ solving

$$\left\{\begin{array}{ll}(\varphi^m)''-\beta\xi\varphi'-\alpha\varphi+\varphi^p=0 & 0<\xi<\xi_*,\\ \varphi(0)=(\varphi^m)'(0)=0, & \\ \varphi(\xi_*)=0. & \end{array}\right.$$

Such a function φ is constructed in [SGKM]. If $\beta=0$ this function is explicit and $(\varphi^m)'(\xi_*)=0$, i.e. it can be extended by zero for $\xi>\xi_*$. If $\beta>0$ it satisfies $(\varphi^m)'(\xi_*)\neq 0$. Thus, in general v is a solution in the set $S=(a(t),\infty)\times(0,T)$, where $a(t)=x_0-\xi_*(T-t)^\beta$, and a subsolution to our problem in $\mathbb{R}_+\times(0,T)$. Now take x_0 large enough to get that the supports of $u(\cdot,0)$ and $v(\cdot,0)$ do not intersect (it suffices to choose $x_0>L+\xi_*T^\beta$, where $\mathrm{supp}(u_0)=[0,L]$). Proposition 2.1 implies that the number of intersections of $u(\cdot,t)$ and $v(\cdot,t)$ in $(a(t),\infty)$ is at most one. Suppose that $s(t_0)>x_0$ for any $t_0\in(0,T)$. As the number of intersections is at most one, then $u(x,t_0)\geq v(x,t_0)$. Hence Proposition 2.2 implies that u blows up before $t=T$, a contradiction.

(ii) We now prove no localization for $(p,q) \in \mathcal{P} = \mathcal{U} - \mathcal{L}$, i.e. $\beta < 0$. We combine the blow-up rate with an argument from [GH]. Multiplying the equation in (1.1) by x and integrating in $\mathbb{R}_+ \times (0,t)$, we get

$$\begin{aligned} \int_0^\infty xu(x,t)\,dx = & \int_0^\infty xu_0(x)\,dx + \int_0^t u^m(0,s)\,ds + \int_0^t \int_0^\infty xu^p(x,s)\,dx\,ds \\ \geq & \int_0^t u^m(0,s)\,ds. \end{aligned}$$

Then, using (1.6), we obtain

$$\frac{1}{2} C(T-t)^{-\alpha} s^2(t) \geq c \int_0^t (T-s)^{-\alpha m}\,ds.$$

Therefore we have the claimed bound from below, $s(t) \geq c(T-t)^\beta$, in both cases, $2q \geq m+p$ and $2q \leq m+p$. This also proves that $s(t)$ grows without bound as t approaches T.

We divide the proof of the upper bound for $s(t)$ in two cases.

CASE $1 < p < m$ WITH $p \geq 2q-m$. We use the solution of the equation $u_t = (u^m)_{xx} + u^p$ in self-similar form

$$U(x,t) = (T-t)^{-\alpha}\varphi(\xi), \quad \xi = (x-x_0)/(T-t)^\beta$$

constructed by [SGKM]. The support of φ is a compact interval. We can assume that x_0 is such that the supports of $U(x,0)$ and $u(x,0)$ do not intersect. Hence the number of intersections of $u(x,0)$ and $U(x,0)$ is equal to one. We claim that this intersection number cannot change as the solutions evolve. In fact, by Proposition 2.1 it cannot increase unless $u(0,t) - U(0,t)$ changes sign. To rule out this possibility, take the first time such that $u(0,t_0) = U(0,t_0)$. Since $(u^m)_x(0,t_0) < 0$ and $(U^m)_x(0,t_0) > 0$ we arrive to a contradiction. To end the proof of the claim we just observe that the number of intersections cannot decrease. Indeed, U is a strict subsolution of (1.1) with the same blow-up time as u and Proposition 2.2 excludes the possibility of $u(x,t) \geq U(x,t)$. This proves that $s(t)$ is less than the right interface of U and hence

$$s(t) \leq C(T-t)^\beta.$$

CASE $(m+1)/2 < q < m$ WITH $p < 2q-m$. Here we use a different approach. We use the self-similar variables (1.9) and arrive at problem (1.10) with $\alpha = 1/(2q-m-1)$, $\beta = (q-m)/(2q-m-1)$ and $\delta = 1-(p-1)\alpha$. Recall that the function f inherits the properties of u, so that it is nonincreasing and has compact support.

Now we observe that as a consequence of the blow-up rate (Theorem 1.3) we have that $0 \leq f(y,\tau) \leq C$. We choose $\alpha > \tilde{\alpha} > 0$, and we observe that there exists a time τ_0 such that

$$-\alpha f + e^{-\delta\tau} f^p \leq -\tilde{\alpha} f$$

for every $y > 0$ and $\tau \geq \tau_0$. Hence f is a solution of

$$f_\tau = (f^m)_{yy} - \beta y f_y - \alpha f + e^{-\delta\tau} f^p \leq (f^m)_{yy} - \beta y f_y - \tilde{\alpha} f.$$

The function f has compact support, $[0, y_0]$, at time τ_0.

Let $g(y)$ be a compactly supported solution of

$$0 = (g^m)_{yy} - \beta y g_y - \tilde{\alpha} g, \qquad g(y_0) \geq C,$$

in $\{y \geq y_0\}$. Such a function g exists by Lemma 2.2. Using a comparison argument for the Dirichlet problem in $[y_0, \infty)$ we obtain that

$$f(y,\tau) \leq g(y) \qquad \text{for } y > y_0,\ \tau > \tau_0.$$

Therefore the support of f is uniformly bounded in τ and hence the support of u, $s(t)$, verifies

$$s(t) \leq C(T-t)^\beta.$$

This ends the proof of Theorem 1.4. □

6 Blow-up profiles.

We devote this section to the proof of the asymptotic self-similar behaviour, Theorem 1.5. We first summarize the ideas used to consider the different cases of that theorem. In particular the techniques will be different depending on which term in problem (1.1) is dominant, i.e., depending on the region of parameters where (p,q) lies in relation to the critical line (1.2), and also depending on the sign of the exponent β given by (1.7).

Let u be a solution to problem (1.1) that blows up at a time $t = T$, and consider the problem (1.10) for the rescaled function (1.9), which we write down here again for convenience

$$\left\{ \begin{array}{l} f_\tau = (f^m)_{yy} - \beta y f_y - \alpha f + e^{-\delta\tau} f^p, \\ -(f^m)_y(0,\tau) = e^{-\epsilon\tau} f^q(0,\tau), \\ f(y,\tau_0) = f_0(y) \equiv T^\alpha u_0(yT^\beta), \end{array} \right. \tag{6.1}$$

where

$$\delta = 1 - (p-1)\alpha, \qquad \epsilon = (1 - (2q - m - 1)\alpha)/2.$$

If we are below the critical line, i.e., $m + p > 2q$, then the flux term in (6.1) becomes a small perturbation. Thus the idea is, by an easy double limit technique (see below), to reduce problem (6.1) to the corresponding problem without flux and apply to this reduced problem the known results on asymptotic behaviour for compactly supported solutions, [Ga2], [Ga3], [Ga4]. This works only for $\beta \leq 0$, since for $\beta > 0$ the support of the rescaled function (1.9) grows without bound (see Theorem 1.4), and we loose, in the first limit step, the compact support property. As for the region above the critical line, $m + p < 2q$, the same idea applies, getting in the first limit the corresponding reduced problem without reaction, and with a compactly supported initial data if $\beta \leq 0$. Since the asymptotic behaviour for this reduced problem seems not to be obtained anywhere else, we include a short proof. It is based on the construction of a Lyapunov function as is done in [Ga2] for the problem without flux, taking care of the boundary flux condition. In the cases $\beta > 0$, both above and below the critical line, we consider as a Lyapunov function for problem (6.1) the above Lyapunov function for the reduced problem. In this situation it will not be positive nor decreasing, but asymptotically this two properties are true. This method also works on the critical line $m + p = 2q$, but for the case $\beta < 0$, where we have to impose, as in [Ga1], an extra hypothesis on the initial data, to get this function well defined. We remark that this hypothesis (H_2) is not very natural, and for the reaction problem it has been avoided in [Ga3] by means of a different, and fairly simpler, geometric proof. This proof cannot be used for nonsymmetric problems as uses that the problem is invariant under space translations.

We now pass to the proofs in the different regions.

6.1 Reaction dominating region.

CASE $1 < p \leq m$. Here $\alpha = 1/(p-1)$, and $\beta = (p-m)/(2(p-1)) \leq 0$, $\delta = 0$, $\epsilon = (m+p-2q)/(2(p-1)) > 0$. The flux term becomes a small perturbation. Using the boundedness of f, (1.6) and general regularity results [D], we have that there exists a sequence $s_k \to \infty$ such that

$$\lim_{k\to\infty} f(y, \tau + s_k) = g(y,\tau)$$

uniformly in compact sets of $\mathbb{R}_+$, where g satisfies the reduced problem (no flux at the boundary)

$$\begin{cases} g_\tau = (g^m)_{yy} - \beta y g_y - \alpha g + g^p \\ -(g^m)_y(0,\tau) = 0 \\ g(y,\tau_0) = g_0(y) \equiv \lim\limits_{k\to\infty} f(y, \tau_0 + s_k) \end{cases} \tag{6.2}$$

The initial value g_0 has compact support by Theorem 1.4. Extending g by reflection to the whole line, we have that there exists a function $\tilde{g}$, stationary solution to problem (6.2), and a sequence $\tau_j \to \infty$ such that

$$\lim_{j\to\infty} g(y,\tau_j) = \tilde{g}(y)$$

(cf. [Ga3] for $\beta < 0$ and [Ga4] for $\beta = 0$). We now write

$$|f(y,\tau_j + s_k) - \tilde{g}(y)| \leq |f(y,\tau_j + s_k) - g(y,\tau_j)| + |g(y,\tau_j) - \tilde{g}(y)| \to 0,$$

uniformly in compact sets of $\mathbb{R}_+$ as j, $k \to \infty$. Thus, the ω–limit set $\omega(f_0)$ is nonempty and contained in the set of stationary solutions of (6.2).

CASE $p > m$. Putting $h = f^m$, we get the problem

$$\begin{cases} h_\tau = a(h)[h_{yy} + b(y, h, h_y)], \\ -h_y(0,\tau) = e^{-\epsilon\tau} h^{\frac{q}{m}}(0,\tau), \end{cases} \tag{6.3}$$

where

$$\begin{aligned} &a(h) = m h^{\frac{m-1}{m}}, \\ &b(y,h,z) = -\frac{\beta}{m} y h^{\frac{1-m}{m}} z - \alpha h^{\frac{1}{m}} + e^{-\delta\tau} h^{\frac{p}{m}}. \end{aligned}$$

We construct a Lyapunov function for h following the ideas of [Z] and [Ga2], taking note of the boundary condition. The reader is referred to those papers for the details, specially for the regularizations needed to justify the formal calculations.

Consider the function

$$L_h(\tau) = \int_0^\infty \Phi(y, h(y,\tau), h_y(y,\tau))\, dy - \frac{m}{m+q} e^{-\epsilon\tau} h^{\frac{m+q}{m}}(0,\tau). \tag{6.4}$$

Differentiating and integrating by parts, we get

$$\begin{aligned} \frac{d}{d\tau} L_h(\tau) &= -(e^{-\epsilon\tau} h^{\frac{q}{m}}(0,\tau) + \Phi_z(0,h,h_y)) h_\tau(0,\tau) + \frac{\epsilon m}{m+q} e^{-\epsilon\tau} h^{\frac{m+q}{m}}(0,\tau) \\ &\quad - \int_0^\infty \frac{1}{a} \Phi_{zz} (h_\tau)^2\, dy + \int_0^\infty (\Phi_h - \Phi_{yz} - \Phi_{hz} h_y + b\Phi_{zz}) h_\tau\, dy. \end{aligned}$$

We can eliminate the last integral by choosing appropriately the function Φ using the method of characteristics. It is given by

$$\Phi(y,h,z) = \int_0^z (z-s)\rho(y,h,s)\, ds - \int_0^h b(y,s,0)\rho(y,s,0)\, ds, \tag{6.5}$$

where $\rho(y,h,z) = \exp[\int_0^y b_z(\eta) d\eta]$, and b_z is evaluated along the characteristic $\phi(\eta, y, h, z)$, solution to

$$\begin{cases} \phi'' + b(\eta, \phi, \phi') = 0, \\ \phi(y) = h,\ \phi'(y) = z, \end{cases} \tag{6.6}$$

(see [Ga2]). Thus

$$\rho(y,h,z) = \exp\left[-\frac{\beta}{m} \int_0^y \eta \phi^{\frac{1-m}{m}}(\eta, y, h, z)\, d\eta \right]. \tag{6.7}$$

We now calculate

$$\Phi_z(0,h,h_y) = \int_0^{h_y} \rho(0,h,s)\, ds = h_y(0,\tau) = -e^{-\epsilon\tau} h^{\frac{q}{m}}(0,\tau).$$

Putting all together we get

$$\frac{d}{d\tau} L_h(\tau) = -\int_0^\infty \frac{1}{a(h)} \rho(y,h,h_y)(h_\tau)^2\, dy + \frac{\epsilon m}{m+q} e^{-\epsilon\tau} h^{\frac{m+q}{m}}(0,\tau).$$

Using [Ga2] and (1.6) we get that $|L_h(\tau)|$ is bounded. This implies

$$\frac{4m}{(m+1)^2}\int_{\tau_0}^{\tau_2}\int_0^\infty \left|(h^{\frac{m+1}{2m}})_\tau\right|^2 \rho(y,h,h_y)\,dy\,d\tau$$

$$= L_h(\tau_0) - L_h(\tau_2) + \int_{\tau_0}^{\tau_2} \frac{\epsilon m}{m+q} e^{-\epsilon\tau} h^{\frac{m+q}{m}}(0,\tau)\,d\tau \leq C.$$

We must estimate now ρ from below. The function h at which we evaluate ρ satisfies $h(0,\tau) \geq c > 0$ for every $\tau \geq \tau_0$ (see (1.6)), and it is Hölder continuous in y uniformly in τ (see [D]). This implies that there exists $y_1 > 0$ small enough such that $h(y,\tau) \geq c/2 > 0$ for every $0 \leq y \leq y_1$ and every $\tau \geq \tau_0$. Also $-c \leq h_y(y,\tau) \leq 0$. This implies (see [Ga2]) that for every $0 < y \leq y_1$, $0 < \eta < y$, the solution ϕ of (6.6) satisfies $\phi(\eta,y,h,z) \geq c_1$ along the trajectory h, which leads to

$$\rho(y,h,z) \geq e^{-Cy^2}. \tag{6.8}$$

Therefore

$$\int_{\tau_0}^{\tau_2}\int_0^{y_1} \left|(g^{\frac{m+1}{2}})_\tau\right|^2 dy\,d\tau \leq C. \tag{6.9}$$

We conclude in a rather standard way (see for instance [ACP], [Ga2]), the convergence of the orbits to a stationary solution to the limit problem (6.2) with zero flux. With (6.9) we have

$$\left\|g^{\frac{m+1}{2}}(\cdot,\tau_j+\tau) - g^{\frac{m+1}{2}}(\cdot,\tau_j)\right\|^2_{L^2([0,y_1])} = \int_0^{y_1} |g^{\frac{m+1}{2}}(y,\tau_j+\tau) - g^{\frac{m+1}{2}}(y,\tau_j)|^2\,dy$$

$$\leq \tau \int_0^{y_1}\int_{\tau_j}^{\tau_j+\tau} \left|(g^{\frac{m+1}{2}})_\tau(x,s)\right|^2 ds\,dy \to 0$$

as $\tau_j \to \infty$ uniformly for $\tau > \tau_0$. Then the sequence $g^{\frac{m+1}{2}}(\cdot,\tau_j+\tau)$ converges in the space $L^\infty([\tau_0,\tau_*]\,;\,L^2([0,y_1]))$ for every $\tau_* > \tau_0$. The limit does not depend on τ, and is a stationary solution in $(0,y_1)$ to the limit problem. We end with a continuation argument from y_1 (see [Ga2] and Lemma 6.3 below).

6.2 Flux dominating region.

Now $\alpha = 1/(2q-m-1)$, $\beta = (q-m)/(2q-m-1)$, $\delta = (2q-m-p)/(2q-m-1) > 0$ and $\epsilon = 0$. Thus reaction becomes a small perturbation. Our aim is to prove convergence to a stationary solution to the reduced problem in self-similar variables, i.e., a solution to problem (1.14). We observe that from Lemma 2.2 we have that this problem has a unique nontrivial stationary solution $\widetilde{g}$. Moreover $\widetilde{g}$ has compact support if $\beta \leq 0$, while it is positive if $\beta > 0$.

CASE $(m+1)/2 < q \leq m$. As before, along a sequence $s_k \to \infty$, we have that

$$\lim_{k\to\infty} f(y, \tau + s_k) = g(y, \tau),$$

uniformly in compact sets of $\mathbb{R}_+$, where g satisfies the reduced problem (reaction is absent)

$$\begin{cases} g_\tau = (g^m)_{yy} - \beta y g_y - \alpha g, \\ -(g^m)_y(0,\tau) = g^q(0,\tau), \\ g(y,\tau_0) = g_0(y) \equiv \lim\limits_{k\to\infty} f(y, \tau_0 + s_k). \end{cases} \tag{6.10}$$

We now proceed to define the Lyapunov function corresponding to the present case. We consider problem (6.3) for $h = g^m$, where now

$$b(y,h,z) = -\frac{\beta}{m} y h^{\frac{1-m}{m}} - \alpha h^{\frac{1}{m}}. \tag{6.11}$$

We consider also the same Lyapunov function (6.4), defined in terms of Φ from (6.5) and ρ from (6.7). Observe that here $\epsilon = 0$ and also that ρ is evaluated along the characteristic $\phi(\eta, y, h, z)$, solution to a different problem (problem (6.6) with a different b),

$$\begin{cases} \phi'' - \dfrac{\beta}{m} y \phi^{\frac{1-m}{m}} \phi' - \alpha \phi^{\frac{1}{m}} = 0, \\ \phi(y) = h, \ \phi'(y) = z. \end{cases} \tag{6.12}$$

Lemma 6.1 *For every $y > 0$, $h > 0$ and $z \leq 0$, problem* (6.12) *has a unique solution ϕ defined in* $[0, y]$*. Moreover $\phi(\eta) \geq h$, and for $\eta \approx y$ we have*

$$\phi(\eta, y, h, z) = h - z(y-\eta) + \left[\frac{|\beta|(m-1)}{m} y(y-\eta)\right]^{\frac{m}{m-1}} + o(|y-\eta|). \tag{6.13}$$

The fact that $\phi(\eta) \geq h$ is immediate. The behaviour (6.13) can be proved exactly as in [Ga1].

We now deduce the estimates on $|L_h(\tau)|$ and $\frac{d}{d\tau} L_h(\tau)$ necessary to obtain the estimate analogous to (6.9). First of all, since $b(y, s, 0) \geq 0$, we have $\Phi \geq 0$ and then $L_h(\tau) \geq -C$. To estimate $L_h(\tau)$ from above, we state the following bounds for the function ρ. Since $\beta \leq 0$ we have $\rho \geq 1$. We claim that along the solution h of problem (6.3), $z = h_y$, we have for τ large,

$$\rho(y,h,z) \leq C h^{-\frac{1}{m}}. \tag{6.14}$$

This estimate is obtained in [Ga1] for the reaction problem, using the behaviour near the interface of solutions to degenerate porous medium equations (see e.g. [A]). It is also true in our situation. In fact, near the interface $y \approx s(\tau)$ where h vanishes, we have, for every $\tau > \tau_*$,

$$h(y,\tau) \approx c(s(\tau) - y)^{\frac{m}{m-1}}, \quad h_y(y,\tau) \approx c(s(\tau)-y)^{\frac{1}{m-1}}, \tag{6.15}$$

where τ_* is the waiting-time for the evolution of the interface [A]. From Theorem 1.4, $\tau_* < T$. Using Lemma 6.1 we obtain, in first approximation,

$$\rho(y,h,z) \le \exp\left[C\int_0^y \eta h^{\frac{1-m}{m}}\,d\eta\right],$$

which from (6.15) diverges. But using (6.13) and that $zh^{-\frac{1}{m}}$ is bounded from (6.15), we have

$$\begin{aligned}\frac{|\beta|}{m}\int_{y-\varepsilon}^{y}\eta\phi^{\frac{1-m}{m}}d\eta \ &\approx \frac{1}{m}\int_0^{M_\varepsilon} w^{-\frac{1}{m}}\left(1 - Nzh^{-\frac{1}{m}}w^{\frac{m-1}{m}} + w\right)^{\frac{1-m}{m}}\,dw \\ &\le \frac{1}{m}\int_0^{M_\varepsilon}\frac{1}{w}\,dw = \frac{1}{m}\log(M_\varepsilon),\end{aligned}$$

where

$$M_\varepsilon = \frac{1}{h}\left(\frac{m-1}{m}|\beta|\varepsilon y\right)^{\frac{m}{m-1}}, \quad N = \frac{m}{(m-1)|\beta|y}.$$

Therefore, we get

$$\rho(y,h,z) \le \exp[C + \frac{1}{m}\log(M_\varepsilon)] = Ch^{-\frac{1}{m}},$$

and the claim is proved.

With estimate (6.14) we obtain

$$\Phi(y,h,z) \le \frac{1}{2}z^2h^{-\frac{1}{m}} + ch^{\frac{m-1}{m}},$$

and therefore, using again (6.15),

$$L_h(\tau) \le C \qquad \text{for every } \tau > \tau_*.$$

Once we have the main estimate (6.9) for every $y_1 > 0$, we end exactly as for the reaction case.

CASE $q > m$. We want to prove the convergence of the rescaled function f to the unique stationary solution $\tilde{g}$ of problem (6.10). As we have said, we cannot perform in this case the double limit technique. Thus we define a Lyapunov function for problem (6.1) directly. It will be the same $L_h(\tau)$ for $h = f^m$ as for the reduced problem, i.e., function (6.4) for the solutions to problem (6.3), and Φ from (6.5). But here we write problem (6.3) in the following form

$$\begin{cases} h_\tau = a(h)[h_{yy} + b(y,h,h_y) + e^{-\delta\tau}h^{\frac{p}{m}}], \\ -h_y(0,\tau) = e^{-\epsilon\tau}h^{\frac{q}{m}}(0,\tau), \end{cases}$$

where b is defined in (6.11). Finally we define the function ρ in (6.5) along the solutions to problem (6.12).

Next we deduce the estimates required to obtain (6.9). First of all, $\rho \le 1$ since $\beta > 0$. Also ρ is locally bounded from below, since we have $\phi \ge h$ and $h \ge c$ for $0 \le y \le y_1$, which implies again (6.8). As to the derivative of $L_h(\tau)$, we have here

$$\frac{d}{d\tau}L_h(\tau) = -\int_0^\infty \frac{1}{a(h)}\rho(y,h,h_y)(h_\tau)^2\,dy + \int_0^\infty e^{-\delta\tau}\rho(y,h,h_y)h^{\frac{p}{m}}h_\tau(y,\tau)\,dy.$$

In the last term we have that $|h_\tau|$ is bounded. It suffices to look at the equation satisfied by h_τ, observe that $h_\tau(y,\tau_0) \le c$ and apply the maximum principle. Therefore we are reduced to estimate the integral

$$I(\tau) = e^{-\delta\tau}\int_0^\infty \rho(y,h,h_y)h^{\frac{p}{m}}(y,\tau)\,dy.$$

If $q > p$ we use Theorem 1.4 to get

$$\mathrm{supp}(h(\cdot,\tau)) \subset [0, s(\tau)],$$

where $s(\tau) \le ce^{\beta\tau}$. Therefore,

$$I(\tau) \le Ce^{(\beta-\delta)\tau}. \tag{6.16}$$

Observe that $\beta < \delta$ in the range of parameters under consideration. If now $q \le p$ we have that $\beta - \delta$ is no longer negative, and we have to handle the above integral more carefully. We first use the following estimate on the solutions of problem (1.1).

Lemma 6.2 *If $p > m$, the solutions to problem* (1.1) *satisfy*

$$u(x,t) \le Cx^{-\frac{2}{p-m}}. \tag{6.17}$$

Proof: The proof uses ideas from [FM]. Consider the function

$$J(x,t) = (u^m)_x + \lambda x u^p.$$

Since $J(0,t) < 0$ we can prove as in [Ga1] that $J(x,t) \le 0$ for λ small enough. Hence,

$$-(u^m)_x(x,t) \ge \lambda x u^p(x,t).$$

Integrating in space we get, for $x > 0$,

$$\frac{m}{p-m}u^{m-p}(x,t) \ge \frac{\lambda}{2}x^2,$$

and we conclude (6.17). □

As to the rescaled function h, estimate (6.17) means

$$h(y,\tau) \le Ce^{(-\alpha+\frac{2\beta}{p-m})m\tau}y^{-\frac{2m}{p-m}},$$

and we get

$$I(\tau) \le Ce^{(-\delta+(-\alpha+\frac{2\beta}{p-m})p)\tau} \int_0^{s(\tau)} y^{-\frac{2p}{p-m}} dy \le Ce^{-q\alpha\tau}. \tag{6.18}$$

With (6.16) and (6.18) we conclude

$$\int_{\tau_0}^{\tau_2} \int_0^{\infty} e^{-\delta\tau} \rho(y, h, h_y) h^{\frac{p}{m}} h_\tau(y, \tau)\, dy \le C.$$

This gives, with the estimates previously obtained, the desired main estimate (6.9) for the case $m + p < 2q$.

The local asymptotic result is now obtained again like for the case $m + p > 2q$,

$$\lim_{\tau\to\infty} f(y, \tau) = w(y) \tag{6.19}$$

uniformly for $0 \le y \le y_1$, where w is a stationary solution to problem (6.10). Thus, from uniqueness, $w = \tilde{g}$ in that interval. The only thing still remaining is the continuation result.

Lemma 6.3 *In the above hypotheses we have*

$$y^* = \sup\{\, y_1 > 0 \,:\, (6.19)\ \textit{holds in}\ [0, y_1] \,\} = \infty.$$

Proof: The proof given in [Ga2] for the reaction case can be adapted in our situation with minor changes. We divide the proof into three steps. We claim that under the assumption $y^* < \infty$, it holds

(i) $h(y^*, \tau) \ge c > 0$ for every τ large;
(ii) $\rho^*(\tau; y^*) \equiv \inf_{0\le y\le y^*} \rho(y, h(y, \tau), h_y(y, \tau))$ satisfies $\liminf_{\tau\to\infty} \rho^*(\tau; y^*) > 0$;
(iii) there exists $\varepsilon_0 > 0$ such that $\lim_{\tau\to\infty} \rho^*(\tau; y^* + \varepsilon_0) > 0$.

Having proved claim (iii) we obtain (6.9) with y_1 replaced by $y^* + \varepsilon_0$, which gives (6.19) with y_1 replaced by $y^* + \varepsilon_0$, contradicting the definition of y^*. We now prove the claims.

Claim (i). Assuming it is false, there exist sequences $\{y_k\} \subset [0, y^*]$, $y_k \to y_0$, $\tau_k > \tau_0$, $\tau_k \to \infty$, such that

$$\tilde{g}(y_0) = \lim_{k\to\infty} h(y_k, \tau_k) = 0,$$

contradicting the fact that $\tilde{g} > 0$, (Lemma 2.2).

Claim (ii). If it is false, we have, for the above sequences,

$$\begin{aligned}
&\lim_{k\to\infty} h(y_k, \tau_k) = \tilde{g}(y_0) > 0,\\
&\lim_{k\to\infty} h_y(y_k, \tau_k) = \tilde{g}'(y_0),\\
&\rho(y_0, \tilde{g}(y_0), \tilde{g}'(y_0)) = \lim_{k\to\infty} \rho(y_k, h(y_k, \tau_k), h_y(y_k, \tau_k)) = 0.
\end{aligned}$$

But this last limit contradicts (6.8).

CLAIM (iii). If it is false, there exist sequences $\varepsilon_k > 0$, $\varepsilon_k \to 0$, $\{y_k\} \subset [y^*, y^* + \varepsilon_k]$, $\tau_k > \tau_0$, $\tau_k \to \infty$, such that

$$\rho(y^*, \tilde{g}(y^*), \tilde{g}'(y^*)) = \lim_{k\to\infty} \rho(y_k, h(y_k, \tau_k), h_y(y_k, \tau_k)) = 0,$$

which is again a contradiction. $\square$

6.3 The critical line.

In this case we have, with the same α and β as in the previous subsection, the exponents $\epsilon = \delta = 0$. It can be checked that all the arguments used in the construction of the Lyapunov function (6.4) work as well for the present case, but for $\beta < 0$ ($p < m$). In this case we have to impose an extra condition on the initial data to get $L_h(\tau)$ bounded (see [Ga1]). The difficulty arises in estimating ρ from above. In fact, the integral defining ρ could diverge not only at the points $y = s(\tau)$ of the interface of h where $\phi = 0$, but also at any point $0 \le \eta < y$, where the function ϕ solution to problem (6.6) could also vanish. Since ϕ can only vanish at $\eta = 0$, and there it is like $\eta^{\frac{2m}{m-1}}$, (see [Ga1]), this possibility is eliminated with the following hypothesis on the initial function u_0.

Let v be any solution to problem

$$\begin{cases} v'' - \beta\eta(v^{\frac{1}{m}})' - \alpha v^{\frac{1}{m}} + v^{\frac{p}{m}} = 0 & \eta > 0, \\ v(0) = 0, & \\ v(\eta) \le c\eta^{\frac{2m}{m-1}} & \eta \approx 0, \end{cases}$$

and define $\overline{v}(x,t) = (T-t)^{-\alpha} v(x(T-t)^{-\beta})$. We say that u_0 satisfies hypothesis H_2 if the number of intersections of u_0 with $\overline{v}(\cdot, 0)$ is exactly one for every profile v as above.

This hypothesis implies that the number of intersections of $u(\cdot,t)$ with $\overline{v}(\cdot,t)$ is always one, and also the number of intersections of h with the functions v^m is always one. In particular we cannot have at any point $\eta > 0$ both $h(\eta,\tau) = v^m(\eta)$ and $h_y(\eta,\tau) = (v^m)'(\eta)$. Therefore, along the trajectory h we have ϕ strictly positive on $[0, s(\tau))$.

7 Blow-up sets.

In this section we describe the blow-up sets for blowing up solutions to problem (1.1).

Proof of Theorem 1.6: (i) Global blow-up in the case $\beta < 0$ is a consequence of the no localization result, Theorem 1.4. In fact, if there exists some $x_0 > 0$ such that $u(x_0, t) \le K$ for every $t \le T$ then u is a subsolution to the problem

$$\begin{cases} u_t = (u^m)_{xx} + u^p & x > x_0,\ 0 < t < T, \\ u(x_0, t) = K & 0 < t < T, \\ u(x, 0) = u_0(x) & x > x_0. \end{cases}$$

As $u_0(x)$ is decreasing, so is $u(\cdot,t)$. Hence the solution of this problem has a uniformly bounded compact support for $t \in (0,T)$, see [OKC], contradicting Theorem 1.4.

(ii) Theorem 1.4 gives in the case $\beta = 0$ that the support of $u(\cdot,t)$ is bounded for every $t \leq T$, $s(t) \leq C$. This implies $B(u) \subset [0,C]$. Using Theorem 1.5, we have that $B(u)$ contains some interval, the support of the nontrivial self-similar limit $\widetilde{g}$, and thus the blow-up is regional.

(iii) In this case we have, as before, $B(u) \subset [0,C]$. This excludes global blow-up. We must prove that $B(u)$ is a single point, $B(u) = \{0\}$. The proof is different depending on the values of the parameters.

Case $p > m$. Formula (6.17) proves that the blow-up set for u is a single point, $x = 0$.

Case $m + p < 2q$ with $p \leq m$. Let

$$v(x,s) = (T-t)^{\frac{1}{m-1}} u(x,t), \quad s = -\log(T-t).$$

This function v verifies

$$v_s = (v^m)_{xx} - \frac{1}{m-1} v + e^{\frac{p-m}{m-1}s} v^p. \tag{7.1}$$

Using the blow-up rate, we get

$$v(x,s) \leq C e^{\frac{1}{m-1}s} e^{-\frac{1}{2q-m-1}s} \to 0 \qquad \text{as } s \to \infty.$$

Hence in every compact interval, K, included in $\mathbb{R}_+$, v is a function that goes to 0 as $s \to \infty$ that satisfies (7.1). We can proceed as in [CDE] to conclude an exponential decay of the form

$$v(x,s) \leq C e^{-\frac{1}{m-1}s}$$

for $x \in K$. From this we have that

$$u(x,t) = e^{\frac{1}{m-1}s} v(x,s) \leq C.$$

This proves that $B(u) = \{0\}$. $\square$

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References

[A] D. G. Aronson. *The porous medium equation*, in "Some problems in Nonlinear Diffusion", (A. Fassano and E. Primicerio eds.), Lecture Notes in Math., Springer-Verlag, 1986.

[ACP] D. G. Aronson, M. G. Crandall and L. A. Peletier. *Stabilization of solutions of a degenerate nonlinear diffusion problem.* Nonlinear Anal. Vol. 6, (1982), 1001–1022.

[BC] P. Baras and L. Cohen. *Complete blow-up after* T_{max} *for the solution of a semilinear heat equation.* J. Funct. Anal. Vol. 71, (1987), 142–174.

[CDE] C. Cortázar, M. del Pino and M. Elgueta. *On the blow-up set for* $u_t = \Delta u^m + u^m$, $m > 1$. Indiana Univ. Math. J. Vol. 47, (1998), 541–561.

[D] E. Dibenedetto. *Continuity of weak solutions to a general porous medium equation.* Indiana Univ. Math. J. Vol. 32(1), (1983), 83–118.

[FQ] M. Fila and P. Quittner. *The blow-up rate for the heat equation with a nonlinear boundary condition.* Math. Methods Appl. Sci. Vol. 14, (1991), 197–205.

[FM] A. Friedman and B. McLeod. *Blow-up of positive solutions of semilinear heat equations.* Indiana Univ. Math. J. Vol. 34, (1985), 425–447.

[F] H. Fujita. *On the blowing up of solutions of the Cauchy problem for* $u_t = \Delta u + u^{1+\alpha}$. J. Fac. Sci. Univ. Tokyo Sec. IA Math. Vol. 16, (1966), 105–113.

[Ga1] V. A. Galaktionov. *On asymptotic self-similar behaviour for quasilinear heat equation: global blow-up.* Report No. AM-91-10, School of Math., Univ. of Bristol, (1991).

[Ga2] V. A. Galaktionov. *On asymptotic self-similar behaviour for a quasilinear heat equation. Single point blow-up.* SIAM J. Math. Anal. Vol. 26, (1995), 675–693.

[Ga3] V. A. Galaktionov. *Asymptotic self-similar global blow-up for a quasilinear heat equation.* Differential Integral Equations Vol. 10, (1997), 487–497.

[Ga4] V. A. Galaktionov. *Asymptotic behaviour of unbounded solutions of the nonlinear parabolic equation* $u_t = (u^\sigma u_x)_x + u^{\sigma+1}$. J. Differential Equations Vol. 21, (1985), 751–758.

[GKMS] V. A. Galaktionov, S. P. Kurdyumov, A. P. Mikhailov and A. A. Samarskii. *On blowing-up solutions to the Cauchy problem for the parabolic equation* $u_t = \nabla(u^\sigma \nabla u) + u^\beta$. Dokl. Akad. Nauk SSSR Ser. Math. Phys. Vol. 252, (1980), 1362–1364 (in Russian). English transl.: Soviet Phys. Dokl. Vol. 25, (1980), 458–459.

[GL] V. A. Galaktionov and H. A. Levine. *On critical Fujita exponents for heat equations with nonlinear flux boundary conditions on the boundary.* Israel J. Math. Vol. 94, (1996), 125–146.

[G] B. H. Gilding. *On a class of similarity solutions of the porous media equation III.* J. Math. Anal. Appl. Vol. 77, (1980), 381–402.

[GH] B. H. Gilding and M. A. Herrero. *Localization and blow-up of thermal waves in nonlinear heat conduction with peaking.* Math. Ann. Vol. 282, (1988), 223–242.

[GP1] B. H. Gilding and L. A. Peletier. *On a class of similarity solutions of the porous media equation.* J. Math. Anal. Appl. Vol. 55, (1976), 351–364.

[GP2] B. H. Gilding and L. A. Peletier. *On a class of similarity solutions of the porous media equation II.* J. Math. Anal. Appl. Vol. 57, (1977), 522–538.

[HY] B. Hu and H. M. Yin. *The profile near blowup time for solution of the heat equation with a nonlinear boundary condition.* Trans. Amer. Math. Soc. Vol. 346(1), (1994), 117–135.

[L] H. A. Levine. *The role of critical exponents in blow up theorems.* SIAM Rev. Vol. 32, (1990), 262–288.

[Li] G. M. Liberman. "Second Order Parabolic Differential Equations". World Scientific, River Edge, 1996.

[OKC] O. A. Oleinik, A. S. Kalashnikov and C. Yui-Lin'. *The Cauchy problem and boundary–value problems for equations of unsteady filtration type.* Izv. Akad. Nauk SSSR Ser. Mat. Vol. 22 (1958), 667–704.

[PV1] A. de Pablo and J. L. Vázquez. *Travelling waves and finite propagation in a reaction–diffusion equation.* J. Differential Equations Vol. 93(1), (1991), 19–61.

[PV2] A. de Pablo and J. L. Vázquez. *An overdetermined initial and boundary–value problem for a reaction–diffusion equation.* Nonlinear Anal. Vol. 19, (1992), 259–269.

[P] C. V. Pao. "Nonlinear parabolic and elliptic equations". Plenum Press, New York, 1992.

[QR] F. Quirós and J. D. Rossi. *Blow-up sets and Fujita type curves for a degenerate parabolic system with nonlinear boundary conditions.* Preprint, (1999).

[SGKM] A. A. Samarskii, V. A. Galaktionov, S. P. Kurdyumov and A. P. Mikhailov. "Blow-up in problems for quasilinear parabolic equations". Nauka, Moscow, 1987 (in Russian). English transl.: Walter de Gruyter, Berlin, 1995.

[V] O. Venegas. *On the blow-up set for $u_t = (u^m)_{xx}$, $m > 1$, with nonlinear boundary conditions.* M. Sc. Thesis, Pontificia Univ. Católica de Chile, (1999) (in Spanish).

[Z] T. I. Zelenyak, *Stabilization of solution of boundary value problems for a second-order parabolic equation with one space variable.* Differ. Uravn. Vol. 4 (1986) 34–45 (in Russian). English transl.: Differential Equations Vol. 4, (1986) 17–22.

[ZCM] L. Zhigui, X. Chunhong and W. Mingxin. *On critical exponents for semilinear heat equations with nonlinear boundary conditions.* Appl. Math. J. Chinese Univ. Ser. B Vol. 13, (1998), 363–372.

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