ASYMPTOTIC BEHAVIOR FOR A SEMILINEAR NONLOCAL EQUATION

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Abstract. We study the semilinear nonlocal equation $u_t = J*u-u-u^p$ in the whole $\mathbb{R}^N$. First, we prove the global well-posedness for initial conditions $u(x,0) = u_0(x) \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Next, we obtain the long time behavior of the solutions. We show that different behaviours are possible depending on the exponent p and the kernel J: finite time extinction for $p < 1$, faster than exponential decay for the linear case $p = 1$, a weakly nonlinear behaviour for p large enough and a decay governed by the nonlinear term when p is greater than one but not so large.

1. Introduction

Nonlocal evolution equations of the form $v_t(x,t) = J * v - v(x,t)$ and variations of it, have been recently widely used to model diffusion processes, see, for instance, [1], [2], [3], [4], [6], [7], [8], [9], [14], [16] and [17]. As stated in [8], if $v(x,t)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution (and hence we assume that $\int_{\mathbb{R}^N} J(r)\,dr = 1$) of jumping from location y to location x, then $(J * v)(x,t) = \int_{\mathbb{R}^N} J(y-x)v(y,t)\,dy$ is the rate at which individuals are arriving to position x from all other places and $-v(x,t) = -\int_{\mathbb{R}^N} J(y-x)v(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads to the equation $v_t(x,t) = J * v - v(x,t)$. This equation shares many properties with the classical heat equation, $v_t = \Delta v$, such as: a maximum principle holds for both of them, perturbations propagate with infinite speed and both equations have the same asymptotic behaviour ([5], [8], [13]). However, there is no regularizing effect in general.

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The aim of this paper is to study the asymptotic behavior of solutions to a semilinear nonlocal equation with absorption, namely,

$$
\begin{aligned}
u_t(x,t) &= J * u - u(x,t) - u^p(x,t) \\
&= \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t) - u^p(x,t), \\
u(x,0) &= u_0(x).
\end{aligned} \tag{1.1}
$$

We assume that $J:\mathbb{R}^N \to \mathbb{R}$ is a nonnegative radial function with $\int_{\mathbb{R}^N} J(r)dr = 1$ and that the initial condition u_0 is nonnegative and belongs to $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Our main interest in this work is to see how the nonlinear term affects the asymptotic behaviour of the solutions. Roughly speaking, our result says that the behaviour of the solution depends on p and the behaviour of the Fourier transform of J near the origin. In fact, if $p<1$ the absorption is so strong that there is finite time extinction, when $p=1$ we get a faster than exponential decay rate with a precise asymptotic profile, while if p is very large we have a weakly nonlinear behaviour, i.e., solutions have the same decay and the same profile as the linear part of the equation. In the intermediate range, that is, for p greater than one but not very large, we obtain that solutions decay faster than the linear part.

More precisely, our main result reads as follows:

Theorem 1.1. *Given $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ there exists a unique function $u \in C^0([0,+\infty); L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$, global solution to (1.1).*

Let J be such that its Fourier transform $\widehat{J}$ verifies that there exist $A>0$ and $0<\alpha\leqslant 2$ with

$$\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \quad \text{as } \xi \to 0.$$

The asymptotic behaviour of the solutions is described as follows: Let $G_A(y)$ be determined by $\widehat{G_A}(\xi) = e^{-A|\xi|^\alpha}$, then

(1) If $p<1$ there is extinction in finite time, that is, there exists $T<\infty$ with $u(x,T)\equiv 0$, for all $x\in\mathbb{R}^N$.

(2) If $p=1$, the following holds

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{\alpha}} e^t\, u(yt^{\frac{1}{\alpha}},t) - \|u_0\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0.$$

(3) If $p>(N+\alpha)/N$, then

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{\alpha}} u(yt^{\frac{1}{\alpha}},t) - \left(\|u_0\|_{L^1(\mathbb{R}^N)} - \int_0^\infty \int_{\mathbb{R}^N} u^p \right) G_A(y) \right| = 0.$$

(4) If $1 < p < (N+\alpha)/N$*, then the solution satisfies*

$$\lim_{t\to+\infty} \max_{x} \ t^{\frac{N}{\alpha}} u(x,t) = 0.$$

Moreover,

$$\lim_{t\to+\infty} \int_{\mathbb{R}^N} u(x,t)\,dx = 0.$$

Let us briefly comment on the ideas used to prove our results. The global well-posedness is obtained combining a fixed point argument and a priori estimates. In order to obtain finite time extinction, *(1)* of Theorem 1.1, we use comparison arguments. For *(2)*, *(3)* and *(4)* we combine super and subsolution arguments and the variations of constants formula with precise estimates on the semigroup governed by the linear part obtained in [5]. As there is no conservation of mass, due to the presence of the nonlinear term, we need to multiply the profile by an adequate constant when $p > (N+\alpha)/N$.

The critical case $p = (N+\alpha)/N$ remains open. We conjecture that the conclusion of *(4)* also holds in this case.

Now, let us comment on related literature. For the analogous local semilinear problem $u_t = \Delta u - u^p$, we refer to [10] and [11], [12], where similar results are proved. To obtain the results in those papers scaling techniques are used, taking advantage of the invariance of the Laplace operator. As J is non necessarily homogeneous, these scaling techniques are not appropriate to deal with (1.1).

On the other hand, the asymptotic behaviour of the linear part of the problem $v_t = J * v - v$ was recently addressed in [5] (see also [13]). In these works, the main tool used to obtain the results is the explicit formula for the solution obtained via the Fourier transform. However, the use of Fourier variables seems not to be appropriate in our case due to the presence of a nonlinear term.

As far as we are concerned the problem we address here has not been treated in the literature yet and the existing results on nonlocal problems do not give an immediate answer to it. In fact, this is the first time that the influence of an absorption nonlinear term in the asymptotic behaviour is considered.

Before closing this section we observe that we can also consider sign changing initial conditions just taking $-|u|^{p-1}u$ as the absorption term. Our results remain valid with this modification. However, we prefer to deal with nonnegative solutions to simplify some formulas.

The rest of the paper is organized as follows: in Section 2 we prove existence and uniqueness of solutions and provide a comparison lemma; in Section 3 we show finite time extinction for $p < 1$; Section 4 is devoted to the linear case $p = 1$; in Section 5 we study the case $p > (N+\alpha)/N$ and, finally, in Section 6 we treat the case $1 < p < (N + \alpha)/N$.

2. Existence and uniqueness

In this section we prove existence and uniqueness of solutions to (1.1).

Theorem 2.1. *Let $u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Then there exists a unique solution $u \in C^0([0,\infty); L^1(\mathbb{R}^N)) \cap L^\infty(\mathbb{R}^N)$ to (1.1). Moreover, we have the estimates,*

$$\|u(t)\|_{L^1(\mathbb{R}^N)} \leqslant \|u_0\|_{L^1(\mathbb{R}^N)} \quad and \quad \|u(t)\|_{L^\infty(\mathbb{R}^N)} \leqslant \|u_0\|_{L^\infty(\mathbb{R}^N)}, \tag{2.1}$$

for every $t \geqslant 0$.

Proof. First, let us prove the existence and uniqueness of a local solution. To this end we use a fixed point argument.

Let us consider the space

$$X = C^0([0,T]; L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N))$$

with the norm

$$\|u\|_X = \max_{t\in[0,T]} \left\{ \|u(t)\|_{L^1(\mathbb{R}^N)} + \|u(t)\|_{L^\infty(\mathbb{R}^N)} \right\}$$

and the operator Φ

$$\Phi(u)(x,t) = u_0(x) + \int_0^t (J * u - u - u^p)(x,s)\, ds.$$

Note that a fixed point of Φ is a solution to (1.1).

Let

$$R = 2(\|u_0\|_{L^1(\mathbb{R}^N)} + \|u_0\|_{L^\infty(\mathbb{R}^N)})$$

and take $B(0,R)$ the ball of radius R in X.

We want to show that $\Phi : B(0,R) \mapsto B(0,R)$ is a strict contraction. To this end, given $u, v \in B(0,R)$ we compute

$$\begin{aligned}\|\Phi(u) - \Phi(v)\|_{L^\infty(\mathbb{R}^N)}(t) \leqslant & \int_0^T \|J * (u-v)\|_{L^\infty(\mathbb{R}^N)} \\ & + \|u - v\|_{L^\infty(\mathbb{R}^N)} + \|u^p - v^p\|_{L^\infty(\mathbb{R}^N)}\, ds.\end{aligned}$$

Since

$$\|J * (u-v)\|_{L^\infty(\mathbb{R}^N)}(t) \leqslant \|J\|_{L^1(\mathbb{R}^N)} \|(u-v)\|_{L^\infty(\mathbb{R}^N)}(t)$$

and, by the mean value theorem,

$$|u^p - v^p| = p|\xi|^{p-1}|u-v| \leqslant pR^{p-1}|u-v|,$$

we have,

$$\|\Phi(u) - \Phi(v)\|_{L^\infty(\mathbb{R}^N)}(t) \leqslant (2T + pR^{p-1}T)\|u-v\|_X,$$

for any $t \in [0,T]$.

In an analogous way, we obtain that there exists a positive constant $C = C(J)$ satisfying

$$\|\Phi(u) - \Phi(v)\|_{L^1(\mathbb{R}^N)}(t) \leqslant (C(J)T + pR^{p-1}T)\|u-v\|_X.$$

Choosing T small (depending on J and R) we obtain the local existence result. In order to prove that the solution is global we need some a priori estimates.

Integrating the equation in $\mathbb{R}^N$ and using Fubini's theorem we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_{\mathbb{R}^N} u(x,t)\,dx = & \int_{\mathbb{R}^N} u(y,t)\int_{\mathbb{R}^N} J(x-y)\,dx\,dy \\ & -\int_{\mathbb{R}^N} u(x,t)\,dx - \int_{\mathbb{R}^N} u^p(x,t)\,dx \leqslant 0,\end{aligned}$$

from where it follows that

$$\|u(t)\|_{L^1(\mathbb{R}^N)} \leqslant \|u_0\|_{L^1(\mathbb{R}^N)}. \tag{2.2}$$

Now, multiplying the equation by $(u-M)_+$, where $M = \|u_0\|_{L^\infty(\mathbb{R}^N)}$, and integrating in $\mathbb{R}^N$ we get

$$\begin{aligned}&\frac{\partial}{\partial t}\int_{\mathbb{R}^N} \frac{(u(x,t)-M)_+^2}{2}\,dx = \\ &\quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))(u(x,t)-M)_+\,dx\,dy \\ &\quad -\int_{\mathbb{R}^N} u^p(x,t)(u(x,t)-M)_+\,dx.\end{aligned}$$

Using the formula below (which takes into account the symmetry of J)

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(\varphi(y)-\varphi(x))\psi(x)\,dx\,dy = \\ &\quad -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(\varphi(y)-\varphi(x))(\psi(y)-\psi(x))\,dx\,dy\end{aligned}$$

it follows that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_{\mathbb{R}^N}\frac{(u(x,t)-M)_+^2}{2}\,dx \leqslant \\
&\qquad -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|(u(y,t)-M)-(u(x,t)-M)|^2\,dx\,dy \\
&\qquad -\int_{\mathbb{R}^N} u^p(x,t)(u(x,t)-M)_+\,dx \leqslant 0.
\end{aligned}
$$

Therefore,

$$\int_{\mathbb{R}^N}\frac{(u(x,t)-M)_+^2}{2}\,dx = 0$$

and we obtain

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leqslant \|u_0\|_{L^\infty(\mathbb{R}^N)}. \tag{2.3}$$

From (2.2) and (2.3), we conclude the result. □

We will also need the following comparison result. First, let us define super and subsolutions.

Definition 2.1. *A function* $\overline{u} \in C([0,+\infty); L^1(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N))$ *is a supersolution of*

$$u_t = J * u - u - u^p$$

if

$$\overline{u}_t(x,t) \geqslant \int_{\mathbb{R}^N} J(x-y)\overline{u}(y,t)\,dy - \overline{u}(x,t) - \overline{u}^p(x,t).$$

Subsolutions are defined analogously by reversing the inequalities.

Lemma 2.1. *Let* $\overline{u}$ *be a supersolution and* $\underline{u}$ *be a subsolution to* (1.1). *If* $\overline{u}(x,0) \leqslant \underline{u}(x,0)$, *then*

$$\overline{u}(x,t) \leqslant \underline{u}(x,t) \qquad \forall (x,t) \in \mathbb{R}^N \times [0,+\infty).$$

Proof. The argument is similar to the one used to obtain the $L^\infty(\mathbb{R}^N)$ bound. Indeed, we have

$$(\underline{u}-\overline{u})_t \leqslant J * (\underline{u}-\overline{u}) - (\underline{u}-\overline{u}) - (\underline{u}^p - \overline{u}^p).$$

We multiply the above inequality by $(\underline{u}-\overline{u})_+$ and use the properties of the convolution operator and the monotonicity of the nonlinearity to obtain that

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^N}\frac{(\underline{u}(x,t)-\overline{u}(x,t))_+^2}{2}\,dx \leqslant 0,$$

from which we conclude the result. □

Let us close this section with some estimates of the semigroup associated to the linear part (see [5] for complete proofs).

Let v be the solution to

$$\begin{aligned} v_t(x,t) &= J * v - v(x,t), \\ v(x,0) &= v_0(x). \end{aligned} \tag{2.4}$$

We have an explicit form for the solution in Fourier variables,

$$\widehat{v}(\xi,t) = e^{(\widehat{J}(\xi)-1)t}\widehat{v_0}(\xi).$$

From this explicit formula the following result is obtained. Recall that we assume that the Fourier transform of J has an asymptotic expansion of the form $\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$ for $\xi \to 0$ (with $A > 0$ and $0 < \alpha \leqslant 2$).

Theorem I *(See* [5]*). For every $1 \leqslant q \leqslant \infty$, we have*

$$\|v(\cdot,t)\|_{L^q(\mathbb{R}^N)} \leqslant C\, t^{-\frac{N}{\alpha}(1-\frac{1}{q})},$$

and the asymptotic profile is given by

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{\alpha}} v(yt^{\frac{1}{\alpha}},t) - \|v_0\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\widehat{G_A}(\xi) = e^{-A|\xi|^\alpha}$.

In the special case $\alpha = 2$, the decay rate is $t^{-\frac{N}{2}}$ and the asymptotic profile is a gaussian $G_A(y) = (4\pi A)^{\frac{N}{2}} \exp(-A|y|^2/4)$ with $A \cdot \mathrm{Id} = -(1/2)D^2\hat{J}(0)$. Note that in this case (that occurs, for example, when J is compactly supported) the asymptotic behavior is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a gaussian.

3. Finite time extintion for $p < 1$

In this section we prove item *(1)* of Theorem 1.1, that is, that solutions to (1.1) vanish in finite time when $p < 1$.

Lemma 3.1. *Let $p < 1$ and u a solution to* (1.1)*, then there exists a finite time T such that*

$$u(x,T) \equiv 0.$$

Moreover, the exitinction time T verifies

$$T \leqslant \frac{\|u_0\|_{L^\infty(\mathbb{R}^N)}^{1-p}}{1-p}.$$

Proof. Let $z(t)$ be the solution to the following ODE,

$$\begin{aligned} z_t(t) &= -z^p(t) \\ z(0) &= \|u_0\|_{L^\infty(\mathbb{R}^N)}. \end{aligned} \tag{3.1}$$

A comparison argument using Lemma 2.1 shows that

$$u(x,t) \leqslant z(t),$$

and an explicit integration of (3.1) gives

$$z(t) = \left((z(0))^{1-p} - (1-p)t\right)^{\frac{1}{1-p}}.$$

Since z vanishes at the finite time

$$t_0 = \frac{z(0)^{1-p}}{1-p},$$

the result follows. □

4. The linear case $p = 1$

In this section we consider the linear case $p = 1$, that is,

$$u_t = J * u - u - u.$$

Proof of (2). If we consider

$$v(x,t) = e^t u(x,t)$$

we get a solution to the nonlocal diffusion equation

$$v_t = J * v - v,$$

that was studied in [5] (see Theorem I).

As we have mentioned, the asymptotic behaviour of such v is given by

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{\alpha}} v(yt^{\frac{1}{\alpha}}, t) - \|u_0\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\widehat{G_A}(\xi) = e^{-A|\xi|^\alpha}$. Therefore, we conclude that

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{\alpha}} e^t u(yt^{\frac{1}{\alpha}}, t) - \|u_0\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0.$$

This ends the proof. □

5. Asymptotic behaviour for $p > (N+\alpha)/N$

In this section we obtain the asymptotic bahaviour for $p > (N+\alpha)/N$ which is governed by the linear part.

First, let us state two simple lemmas that provide us with upper bounds for the solutions.

Lemma 5.1. *Let u be the solution to* (1.1)*, then*

$$u(x,t) \leqslant \frac{1}{\left(\|u_0\|^{1-p}_{L^\infty(\mathbb{R}^N)} + (p-1)t\right)^{\frac{1}{p-1}}}.$$

Proof. Just use a comparison argument with the explicit solution

$$z(t) = \frac{1}{\left(\|u_0\|^{1-p}_{L^\infty(\mathbb{R}^N)} + (p-1)t\right)^{\frac{1}{p-1}}},$$

recalling that this is the solution to the ODE $z_t = -z^p$ with initial datum $z(0) = \|u_0\|_{L^\infty(\mathbb{R}^N)}$. □

Lemma 5.2. *Let u be the solution to* (1.1) *and v the solution to*

$$v_t = J * v - v$$

with the same initial condition $v(x,0) = u_0(x)$. Then,

$$u(x,t) \leqslant v(x,t).$$

Proof. Again it is just a comparison argument, since v is a supersolution to (1.1), see Lemma 2.1. □

It is important to observe that from these two lemmas we get the decay estimate

$$\|u(t)\|_{L^\infty(\mathbb{R}^N)} \leqslant C \min\left\{ \frac{1}{t^{\frac{1}{p-1}}}, \ \frac{1}{t^{\frac{N}{\alpha}}} \right\}.$$

Then, when $p > (N+\alpha)/N$, we have that

$$\frac{1}{p-1} < \frac{N}{\alpha}.$$

Therefore, we expect that the decay rate will be given by the linear part of the equation.

Proof of (3). We use the variation of constants formula to rewrite (1.1) as

$$u(x,t) = S(t)u_0 - \int_0^t S(t-s)u^p(s)\, ds,$$

where by $S(\cdot)$ we have denoted the semigroup associated with the linear part of the equation.

First, let us prove that the integral

$$C_0 = \int_0^\infty \int_{\mathbb{R}^N} u^p(x,s)\,dx\,ds$$

is finite. In fact, by Lemma 5.2 and the decay estimates obtained for the linear part (see Theorem I) we have

$$\begin{aligned}
\int_0^\infty \int_{\mathbb{R}^N} u^p(x,s)\,dx\,ds &\leqslant \int_0^\infty \int_{\mathbb{R}^N} v^p(x,s)\,dx\,ds \\
&= \int_0^\infty \int_{\mathbb{R}^N} |S(s)u_0|^p(x,s)\,dx\,ds \\
&= \int_0^\infty \|S(s)u_0\|^p_{L^p(\mathbb{R}^N)}\,ds \\
&\leqslant C\int_0^\infty \frac{1}{(1+s)^{\frac{N(p-1)}{\alpha}}}\,ds < \infty
\end{aligned}$$

since $-\frac{N(p-1)}{\alpha} + 1 < 0$.

Take t_0 large such that

$$\left| \int_0^\infty \int_{\mathbb{R}^N} u^p(x,s)\,dx\,ds - \int_0^t \int_{\mathbb{R}^N} u^p(x,s)\,dx\,ds \right| < \varepsilon, \tag{5.1}$$

for every $t \geqslant t_0$.

For this t_0 introduced above we take the solution to the linear part of the equation with initial datum (at $t = t_0$) $v(x,t_0) = u(x,t_0)$, that is,

$$\begin{aligned}
&v_t(x,t) = J * v - v(x,t),\\
&v(x,t_0) = u(x,t_0).
\end{aligned}$$

Now, our next task is to estimate the difference $u - v$ in $L^\infty(\mathbb{R}^N)$-norm. We have,

$$(u-v)(t) = -\int_{t_0}^t S(t-s)u^p(s)\,ds.$$

Therefore, using the properties of the linear semigroup (recall that we have the uniform a priori bound (2.1)),

$$\begin{aligned} t^{\frac{N}{\alpha}}\|u-v\|_{L^\infty(\mathbb{R}^N)}(t) &\leqslant t^{\frac{N}{\alpha}}\int_{t_0}^{t}\|S(t-s)u^p(s)\|_{L^\infty(\mathbb{R}^N)}\,ds \\ &\leqslant Ct^{\frac{N}{\alpha}}\int_{t_0}^{t}\frac{1}{(1+(t-s))^{\frac{N}{\alpha}}}\|u(s)\|^p_{L^p(\mathbb{R}^N)}\,ds \\ &\leqslant Ct^{\frac{N}{\alpha}}\int_{t_0}^{t}\frac{1}{(1+(t-s))^{\frac{N}{\alpha}}}\frac{1}{(1+s)^{\frac{N(p-1)}{\alpha}}}\,ds. \end{aligned}$$

Letting $s = Az + t_0$, with $A = 1+t_0$, and using a lemma due to Strauss, [15], we get

$$\begin{aligned} t^{\frac{N}{\alpha}}\|u-v\|_{L^\infty(\mathbb{R}^N)}(t) &\leqslant \frac{Ct^{\frac{N}{\alpha}}}{A^{\frac{Np}{\alpha}-1}}\int_{0}^{\frac{t}{A}}\frac{1}{(1+\frac{t}{A}-z)^{\frac{N}{\alpha}}}\frac{1}{(1+z)^{\frac{N(p-1)}{\alpha}}}\,dz \\ &\leqslant \frac{Ct^{\frac{N}{\alpha}}}{A^{\frac{Np}{\alpha}-1}}\frac{1}{(1+\frac{t}{A})^{\frac{N}{\alpha}}}. \end{aligned}$$

This estimate allows to conclude that

$$t^{\frac{N}{\alpha}}\|u-v\|_{L^\infty(\mathbb{R}^N)}(t) < \varepsilon$$

for t large enough.

Let $\tilde{v}$ be the solution to

$$\begin{aligned} &\tilde{v}_t(x,t) = J*\tilde{v} - \tilde{v}(x,t), \\ &\tilde{v}(x,0) = \tilde{v}_0(x), \end{aligned}$$

where the initial condition $\tilde{v}_0(x)$ has total mass $\int_{\mathbb{R}^N} u_0 - C_0$. Remark that, by Theorem I, two solutions of $\tilde{v}_t = J*\tilde{v}-\tilde{v}$ with the same total mass have the same asymptotic behaviour.

Collecting our previous bounds, we get

$$t^{\frac{N}{\alpha}}\|u-\tilde{v}\|_{L^\infty(\mathbb{R}^N)}(t) \leqslant t^{\frac{N}{\alpha}}\|u-v\|_{L^\infty(\mathbb{R}^N)}(t) + t^{\frac{N}{\alpha}}\|v-\tilde{v}\|_{L^\infty(\mathbb{R}^N)}(t) < C\varepsilon$$

if t is large enough. Indeed, by (5.1), the total mass of v and $\tilde{v}$ satisfy

$$\left|\int_{\mathbb{R}^N} v - \int_{\mathbb{R}^N}\tilde{v}\right| < \varepsilon,$$

and the results in [5] imply

$$\begin{aligned} t^{\frac{N}{\alpha}}\|v-\tilde{v}\|_{L^\infty(\mathbb{R}^N)}(t) &\leqslant t^{\frac{N}{\alpha}}\|v-t^{-\frac{N}{\alpha}}\|v_0\|_{L^1(\mathbb{R}^N)}G_A\|_{L^\infty(\mathbb{R}^N)}(t) \\ &\quad +t^{\frac{N}{\alpha}}\|\tilde{v}-t^{-\frac{N}{\alpha}}\|\tilde{v}_0\|_{L^1(\mathbb{R}^N)}G_A\|_{L^\infty(\mathbb{R}^N)}(t) \\ &\quad +\Big|\|v_0\|_{L^1(\mathbb{R}^N)}-\|\tilde{v}_0\|_{L^1(\mathbb{R}^N)}\Big|\|G_A\|_{L^\infty(\mathbb{R}^N)}(t) \\ &< C\varepsilon \end{aligned}$$

for t large enough. Therefore,

$$\lim_{t\to\infty} t^{\frac{N}{\alpha}}\|u-\tilde{v}\|_{L^\infty(\mathbb{R}^N)}(t)=0$$

and from Theorem I the result follows. □

Remark 5.1. *It is important to point out that the above result says that the asymptotic behaviour of solutions to* (1.1) *is governed by the linear part of the equation. Indeed, the decay rate is* $t^{-\frac{N}{\alpha}}$ *and the asymptotic profile is the same* G_A*. However, since the problem* (1.1) *does not conserve the total mass due to the presence of the nonlinear term, we have to introduce the constant* $C_0=\int_0^\infty\int_{\mathbb{R}^N}u^p$*.*

Remark 5.2. *In* [10] *the authors use the scaling invariance of the Laplacian to study the asymptotic behaviour of the solutions to* $u_t=\Delta u+u^p$*. In our case we can not use that kind of approach since the kernel* J *is not necessarily homogeneous.*

Remark 5.3. *In the critical case* $p=(N+\alpha)/N$ *the lemma due to Strauss is not applicable and hence the integral estimated above is not necessarily small.*

We end this section obtaining the decay rate of the solutions in $L^q(\mathbb{R}^N)$.

Proposition 5.1. *Let* $p>(N+\alpha)/N$*, then for every* $1\leqslant q\leqslant\infty$*, we have*

$$\|u(\cdot,t)\|_{L^q(\mathbb{R}^N)}\leqslant C\,t^{-\frac{N}{\alpha}(1-\frac{1}{q})}.$$

Proof. By Lemma 5.2 and Theorem I we get

$$\|u(t)\|_{L^q(\mathbb{R}^N)}\leqslant\|v(t)\|_{L^q(\mathbb{R}^N)}\leqslant C\,t^{-\frac{N}{\alpha}(1-\frac{1}{q})},$$

as we wanted to prove. □

6. Nonlinear behaviour for $1 < p < (N+\alpha)/N$

In this section we prove that if $1 < p < (N+\alpha)/N$ the solutions decay faster than $t^{-\frac{N}{\alpha}}$, i.e., the decay of the linear part. Hence the nonlinear term strongly influences the asymptotic behaviour.

Proof of (4). From Lemma 5.1 we have

$$t^{\frac{N}{\alpha}} u(x,t) \leqslant \frac{t^{\frac{N}{\alpha}}}{\Big(\|u_0\|^{1-p}_{L^\infty(\mathbb{R}^N)} + (p-1)t \Big)^{\frac{1}{p-1}}} \to 0,$$

as $t \to \infty$ since $\frac{N}{\alpha} < \frac{1}{p-1}$.

Now, let us prove that

$$\lim_{t\to+\infty} \int_{\mathbb{R}^N} u(x,t)\, dx = 0.$$

Indeed, using Lemmas 5.1 and 5.2, we have

$$\begin{aligned} \int_{\mathbb{R}^N} u(x,t)\, dx &= \int_{|x|\geqslant Kt^{\frac{1}{\alpha}}} u(x,t)\, dx + \int_{|x|\leqslant Kt^{\frac{1}{\alpha}}} u(x,t)\, dx \\ &\leqslant t^{\frac{N}{\alpha}} \int_{|y|\geqslant K} v(t^{\frac{1}{\alpha}} y, t)\, dy + t^{\frac{N}{\alpha}} \int_{|y|\leqslant K} u(t^{\frac{1}{\alpha}} y, t)\, dy \\ &\leqslant t^{\frac{N}{\alpha}} \int_{|y|\geqslant K} v(t^{\frac{1}{\alpha}} y, t)\, dy + C(K)\, t^{\frac{N}{\alpha} - \frac{1}{p-1}}. \end{aligned}$$

On the other hand, from [13], we have the convergence, as $t \to +\infty$, of $t^{\frac{N}{\alpha}} v(t^{\frac{1}{\alpha}} y, t)$ to $\|u_0\|_{L^1(\mathbb{R}^N)} G_A(y)$ in $L^q(\mathbb{R}^N)$-norm. Therefore,

$$\lim_{t\to+\infty} t^{\frac{N}{\alpha}} \int_{|y|\geqslant K_0} v(t^{\frac{1}{\alpha}} y, t)\, dx < \varepsilon$$

for some K_0 large. Once K_0 is fixed we only have to observe that

$$\lim_{t\to+\infty} C(K_0)\, t^{\frac{N}{\alpha} - \frac{1}{p-1}} = 0$$

since $\frac{N}{\alpha} - \frac{1}{p-1} < 0$. $\square$

Remark 6.1. *In the critical case $p = (N+\alpha)/N$ the previous computation only shows that $t^{\frac{N}{\alpha}} u(x,t)$ is bounded.*

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