A DIFFUSION EQUATION WITH A VARIABLE ORDER REACTION

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ABSTRACT. This paper deals with the problem,

$$\begin{cases} -\Delta u = \lambda u^{q(x)} & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{P}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain, $\lambda > 0$ is a parameter and the reaction order $q(x)$ is a Hölder continuous positive function satisfying $q(x) > 1$ for all $x \in \Omega$. The relevant feature here is that q is assumed to achieve the value one on $\partial\Omega$. Assuming that q is subcritical our main result states the existence of a positive solution for all $\lambda > 0$. We also study its asymptotic behavior as $\lambda \to 0$ and as $\lambda \to \infty$. It should be noticed that the fact that $q = 1$ somewhere in $\partial\Omega$ gives rise to serious difficulties when looking for critical points of the functional associated with (P). This work is a continuation of [13] where q is assumed to take values both greater and smaller than one in Ω but is constrained to satisfy $q(x) > 1$ on $\partial\Omega$.

1. Introduction

This work is devoted to the analysis of positive solutions to the semilinear boundary value problem:

$$\begin{cases} -\Delta u = \lambda u^{q(x)} & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded smooth domain and $\lambda > 0$ is a bifurcation parameter. The exponent $q(x)$ is assumed to be a positive function $q \in C^\alpha(\overline{\Omega})$, $0 < \alpha < 1$, such that

$$q(x) > 1 \qquad x \in \Omega. \tag{1.2}$$

Thus, the reaction term in problem (1.2) exhibits a variable order and a convex profile (since q is greater than one in Ω). However, we are also assuming that q achieves the value one somewhere on the boundary. Specifically, and to simplify the exposition we will assume that

$$q(x) = 1 \qquad x \in \partial\Omega. \tag{1.3}$$

As will be seen later, this behavior of q turns out to be critical with respect to several technical points (for instance, the applicability of the method of sub and supersolutions, checking Palais–Smale type conditions, the moving planes method or Pohozaev-type relations). More importantly, the standard

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rescaling technique in [14] can not be employed to obtain a priori estimates for (1.1). Indeed, a critical case arises when the points where a possible blowing-up sequence of solutions attain their maxima, accumulates at a point x_0 of the boundary $\partial\Omega$ where $q(x_0) = 1$ (see [13]).

The study of problem (1.1) under the limiting case (1.3) of the exponent q remained open in [13] where q was allowed to take values both greater and smaller than one, but q was restricted to satisfy $q(x) > 1$ in those common components (if any) of $\partial\Omega$ and $\partial\Omega_+$, where $\Omega_+ = \{x \in \Omega : q(x) > 1\}$. In addition, the growth of q was constrained in [13] to fulfill $q(x) < \dfrac{N}{N-2}$. We are also relaxing this seemingly "technical" restriction to permit the more natural subcritical growth

$$q(x) < \frac{N+2}{N-2}. \tag{1.4}$$

In fact we are assuming henceforth that $N \geq 3$. An analysis similar as the one developed here can be used to handle the case $N = 2$ without further restriction in the size of q.

The subject of reaction–diffusion equations with constant order reactions has been widely studied (see [9], [17], [26], [22] and [21], to quote some few standards in the topic). However, the variable exponent case is not yet completely understood. We refer to [10], [11] on large solutions, [12], [18] dealing with population dynamics models, [7], [19] on diffusion through a porous medium and the already mentioned [13] on a variable exponent problem of concave–convex nature. Problem (1.1) under the more restrictive assumption $q(x) \geq q_0 > 1$ in $\overline{\Omega}$ was analyzed in [19] (see further comments in Section 2). Finally, see [25] for an updated review on problems in the spirit of [13]. This is just a minimal sample of references rather than an exhaustive account on the subject.

Our main result is the following:

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, $q \in C^\alpha(\overline{\Omega})$ satisfying $q(x) > 1$ in Ω, $q \equiv 1$ on $\partial\Omega$ and the growth condition* (1.4)*. Assume moreover that there exist a small $\eta > 0$ and a constant $C_0 > 0$ such that*

$$q(x) \geq 1 + C_0 d(x) \qquad \text{if } d(x) < \eta, \tag{1.5}$$

where $d(x) = \operatorname{dist}(x, \partial\Omega)$. Then the boundary value problem

$$\begin{cases} -\Delta u = \lambda |u|^{q(x)-1} u & x \in \Omega \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.6}$$

exhibits the following features:

i) *For each $\lambda > 0$,* (1.6) *possesses a positive solution $u_\lambda \in C^{2,\beta}(\overline{\Omega})$.*
ii) *If $\Omega = B$ is an open ball and q is a radially symmetric function then for every $\lambda > 0$ the positive solution given by i) can be chosen to be radially symmetric.*

iii) *Any family* u_λ *of positive solutions satisfies* $\|u_\lambda\|_\infty \to \infty$ *as* $\lambda \to 0+$. *Moreover,*

$$\varliminf_{\lambda\to 0+} \lambda^{\frac{1}{q_+-1}} \|u_\lambda\|_\infty > 0, \tag{1.7}$$

where $q_+ = \max_\Omega q$.

iv) *Let* u_λ *be either of the families of positive solutions to* (1.6) *introduced in i) and ii). Then,*

$$\lim_{\lambda\to\infty} \|u_\lambda\|_{C^{2,\beta}(\overline{\Omega})} = 0. \tag{1.8}$$

Moreover, $\|u_\lambda\|_{C^{2,\beta}(\overline{\Omega})}$ *decays exponentially to zero as* $\lambda \to \infty$. *More precisely, there exist* $C_1, C_2, \lambda_0 > 0$ *such that*

$$\|u_\lambda\|_{C^{2,\beta}(\overline{\Omega})} \leq C_1 e^{-C_2 \sqrt[3]{\lambda}}, \qquad \text{for } \lambda \geq \lambda_0.$$

Remarks 1.

a) When $q(x) = q_+ > 1$ is a constant, a scaling argument shows that any family u_λ of nontrivial solutions can be written as

$$u_\lambda = \lambda^{-\frac{1}{q_+-1}} u_1, \tag{1.9}$$

u_1 being a solution corresponding to $\lambda = 1$. Thus the limit in (1.7) achieves a finite value in this case. This may be false when dealing with variable exponents (see Remark 2).

b) Relation (1.9) shows that in the case $q(x) = q_+$, all families of positive solutions u_λ to (1.6) decay to zero as a negative power of λ as $\lambda \to \infty$. This has to be contrasted with the variable exponent case where the convergence to zero is exponential, as shown in part iv) of Theorem 1 (see Section 4 for details).

The rest of the paper is organized as follows: Section 2 introduces the proper variational tools required to handle our problem, i. e. in the critical framework where $q = 1$ on $\partial\Omega$ (here the results in [16] deserve a special mention). The proof of Theorem 1 is contained in Section 3 while some key uniform estimates are postponed to Section 4.

2. Background results

By a solution to problem (1.6) it will be understood a function $u \in H_0^1(\Omega)$ solving (1.6) in the weak sense. Since the growth condition (1.4) leads to the estimate

$$|u|^{q(x)} \leq C(1 + |u|^{q_+}), \tag{2.1}$$

which holds for every $u \in \mathbb{R}$, $x \in \overline{\Omega}$, with $q_+ = \max_{\overline{\Omega}} q$ satisfying

$$q_+ < \frac{N+2}{N-2},$$

then, every weak solution $u \in H_0^1(\Omega)$ is indeed a classical solution (see [13] and [27]). Moreover, since $q \in C^\alpha(\overline{\Omega})$ then such solution lies in $C^{2,\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

Solutions $u \in H_0^1(\Omega)$ to (1.6) are characterized as the critical points of the functional

$$J_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \lambda \mathcal{P}(u) = \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \lambda\mathcal{P}(u), \tag{2.2}$$

with

$$\mathcal{P}(u) = \int_\Omega \frac{|u|^{q(x)+1}}{q(x)+1}\, dx, \tag{2.3}$$

$\mathcal{P}$ referring to "potential". Here we are following a variational approach to get the existence of nontrivial solutions to (1.6).

At this point, it is worth to mention that it is not possible to find a positive solution u to (1.6) by the method of sub and supersolutions (as in the case where q is constant). Otherwise this approach would imply the existence of a minimal positive solution $u_+ \in H_0^1(\Omega)$ satisfying

$$0 < u_+ \le u.$$

However, if $\sigma = \sigma_1(-\Delta - \lambda q {u_+}^{q-1})$ stands for the first eigenvalue of the linearization of problem (1.6) at u_+, i. e.,

$$\begin{cases} -\Delta u - \lambda q {u_+}^{q-1} u = \sigma u & x \in \Omega \\ \ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.4}$$

then it is well–known that $\sigma_1(-\Delta - \lambda q {u_+}^{q-1}) \ge 0$. We refer to [13] and [1] for a proof of this fact and further properties of the eigenvalue problem (2.4).

On the other hand, that u_+ solves (1.6) means that the principal eigenvalue $\sigma_1(-\Delta - \lambda {u_+}^{q-1}) = 0$ (here and above $\sigma_1(-\Delta + V(x))$ stands for the first eigenvalue of the operator $-\Delta + V(x)$ in $H_0^1(\Omega)$). Since

$$\sigma_1(-\Delta - \lambda q {u_+}^{q-1}) < \sigma_1(-\Delta - \lambda {u_+}^{q-1}) = 0,$$

we get a contradiction. So, the sub and supersolutions method does not provide positive solutions to (1.6).

Let us check now some variational properties of J_λ. First, $J_\lambda : H_0^1(\Omega) \to \mathbb{R}$ is a C^1 functional. Moreover, $\mathcal{P} : H_0^1(\Omega) \to \mathbb{R}$ is C^1 and its derivative

$$D\mathcal{P} : H_0^1(\Omega) \to H^{-1}(\Omega),$$

($H^{-1}(\Omega)$ stands for the dual space of $H_0^1(\Omega)$) defines a completely continuous operator. In fact, for $\eta > 0$ satisfying

$$q_+ \le \frac{2^* - \eta}{2^*}(2^* - 1) \qquad 2^* = \frac{2N}{N-2},$$

the Nemytskii operator $\mathcal{N} : H_0^1(\Omega) \to L^{\frac{2^*-\eta}{q_++1}}(\Omega)$, given by

$$\mathcal{N}(u) = \frac{|u|^{q(x)+1}}{q(x)+1},$$

and its derivative $D\mathcal{N}(u) : H_0^1(\Omega) \to L^{\frac{2^*-\eta}{q_+}}(\Omega)$, $D\mathcal{N}(u)v = |u|^{q(x)-1}uv$, are completely continuous operators. Finally,

$$D\mathcal{P}(u)v = \int_\Omega D\mathcal{N}(u)v\, dx.$$

To deal with the existence of nontrivial solutions of (1.6) some further ingredients of critical point theory are necessary. Let X be a Banach space and $J : X \to \mathbb{R}$ a C^1 functional. It is said that J exhibits a Mountain Pass (MP for short) geometry near $u = 0$ if there exist $R, \eta > 0$, $\psi \in X \setminus \{0\}$ with $\|\psi\|_X > R$ satisfying

$$J(u) \geq \eta > \max\{J(0), J(\psi)\}, \tag{2.5}$$

for all $u \in X$, $\|u\|_X = R$. In that case the number

$$c = \inf_\Gamma \max_{t\in[0,1]} J \circ \gamma(t),$$

with $\Gamma = \{\gamma \in C[0, 1] : \gamma(0) = 0,\ \gamma(1) = \psi\}$, is called a MP level. On the other hand, a sequence $\{u_n\} \subset X$ is a Palais–Smale (PS) sequence at level c_0 if $J(u_n) \to c_0$ and the sequence of derivatives $DJ(u_n) \to 0$ in X^* (the dual space of X). Finally, J is said to verify the PS condition at level c_0 if it is possible to extract a convergent subsequence from every PS sequence $\{u_n\}$ at level c_0. If such condition is satisfied regardless the value of c_0 we say that J satisfies the PS condition.

The MP theorem ([2],[24],[27]), asserts that every C^1 functional J having a MP geometry near $u = 0$ and satisfying the PS condition possesses a critical point u at the MP level c given by (2.5).

However, MP theorem can not be directly applied to problem (1.6) due to the behavior (1.3) of q on the boundary. While it will be shown in Section 3 that the functional J_λ defined in (2.2) has a MP geometry near $u = 0$, to check the PS condition is not an easy task. A weaker statement, whose proof is standard, and therefore omitted (we refer to [24] and [27]), is the following:

Lemma 2. *Every* bounded *PS sequence $\{u_n\} \subset H_0^1(\Omega)$ of J_λ admits a convergent subsequence in $H_0^1(\Omega)$.*

In the case where

$$q(x) \geq q_0 > 1 \qquad x \in \overline{\Omega}, \tag{2.6}$$

the so-called Ambrosetti-Rabinowitz relation ([2]):

$$\frac{1}{q(x)+1}|u|^{q(x)+1} \leq \theta|u|^{q(x)+1} \qquad u \in \mathbb{R}, \tag{2.7}$$

holds for a certain $\theta \in [0, \frac{1}{2})$. Based upon (2.7) it can be shown that every PS sequence is indeed a bounded PS sequence (BPS in the sequel). See Lemma 3.6 in [2] and [24], [27]. Therefore, J_λ verifies the PS condition provided (2.6) is satisfied and problem (1.1) admits, for each $\lambda > 0$, a positive solution under this restrictive condition on q. This argument provides an alternative proof of Theorem 2.1 in [19].

However, (2.7) fails near $\partial\Omega$ in our case. To circumvent the problem we are employing an alternative approach using ideas from [16]. To this purpose consider a family $J_\lambda : X \to \mathbb{R}$, $\lambda \in I$ (where I is a real interval) of C^1 functionals. We say that J_λ has a MP geometry at $u = 0$ which is *uniform* with respect to $\lambda \in I$, if $R, \eta > 0$ and $\psi \in X \setminus \{0\}$ can be chosen independent of $\lambda \in I$ in the previous definition. In that case, the reference MP level will be designated as

$$c_\lambda = \inf_\Gamma \max J_\lambda \circ \gamma, \tag{2.8}$$

$\Gamma = \{\gamma \in C[0,1] : \gamma(0) = 0,\ \gamma(1) = \psi\}$.

The next result is a shortened version of Theorem 1.1 and Corollary 1.2 in [16]. They are stated there under less restrictive hypotheses. However, we have narrowed the scope of the assertions to confine ourselves to the setting of problem (1.6).

Theorem 3 ([16])**.** *Let I be a real interval, and $J_\lambda : X \to \mathbb{R}$, $\lambda \in I$, be a family of C^1 functionals of the form,*

$$J_\lambda(u) = A(u) - \lambda \mathcal{P}(u),$$

where $\mathcal{P}(u) \geq 0$ for all $u \in X$ and $A(u) \to \infty$ as $\|u\|_X \to \infty$. Assume that J_λ has a uniform MP geometry at $u = 0$ when $\lambda \in I$. Then the following features hold.

1) *For almost all $\lambda \in I$ there exists a BPS at level c_λ, with c_λ defined by (2.8).*
2) *Assume that both $\mathcal{P}$ and $D\mathcal{P}$ keep bounded on bounded sets of X, that any BPS sequence at the level c_λ admits a convergent subsequence in X, and let λ_0 belong to the interior of I. Then there exist an increasing sequence λ_n with $\lambda_n \to \lambda_0$, and $\{u_n\} \subset X$ such that*

$$J_{\lambda_n}(u_n) = c_{\lambda_n}, \qquad c_{\lambda_n} \to c_{\lambda_0}, \qquad \text{and} \quad DJ_{\lambda_n}(u_n) = 0, \tag{2.9}$$

for all n. Moreover, provided *that u_n is* bounded *it becomes a PS sequence for J_{λ_0} at the level c_{λ_0}.*

Theorem 3 is our main tool to establish the existence of nontrivial solutions to problem (1.6).

3. Proof of Theorem 1.

We are first discussing the geometry of J_λ near $u = 0$. A preliminary fact is

$$\int_\Omega \frac{1}{q(x)+1} |u|^{q(x)+1}\, dx = o(\|u\|^2_{H^1_0(\Omega)}),$$

as $\|u\|_{H^1_0(\Omega)} \to 0$. In fact, let $u_n \in H^1_0(\Omega)$ be any sequence such that $t_n = \|u_n\|_{H^1_0(\Omega)} \to 0$. By setting $u_n = t_n v_n$, a subsequence $v_{n'}$ of v_n can be extracted such that (we write v_n instead $v_{n'}$ in what follows) $v_n \to v$ weakly in $H^1_0(\Omega)$ and strongly in $L^{q_+ +1}(\Omega)$. In addition, $v_n \to v$ a. e. in Ω while, by the results in [3], there exists $h \in L^{q_+ +1}(\Omega)$ such that $|v_n(x)| \le h(x)$, a. e. in Ω. By using (2.1) we obtain

$$\lim_{n\to\infty} \frac{1}{\|u_n\|^2_{H^1_0(\Omega)}} \int_\Omega \frac{1}{q(x)+1} |u_n|^{q(x)+1}\, dx = \lim_{n\to\infty} \int_\Omega \frac{t_n^{q(x)-1}}{q(x)+1} |v_n|^{q(x)+1}\, dx = 0.$$

This proves the assertion. As a consequence, given any bounded interval $I \subset \mathbb{R}^+$ there exists $\varepsilon > 0$ and $C > 0$, both independent of $\lambda \in I$, such that

$$J_\lambda(u) \ge C\|u\|^2_{H^1_0(\Omega)} \tag{3.10}$$

for all $u \in H^1_0(\Omega)$ satisfying $\|u\|_{H^1_0(\Omega)} \le \varepsilon$.

To check that the functional J_λ exhibits a MP geometry at $u = 0$ which is "uniform" when λ varies in bounded intervals $I \subset \mathbb{R}^+$, we look for a function $\psi \in H^1_0(\Omega)$ so that

$$J_\lambda(\psi) < 0, \tag{3.11}$$

for all $\lambda \in I$, where ψ does not depend on $\lambda \in I$. Set $\psi = t\phi_1$, where $\phi_1 > 0$ is an eigenfunction associated to the first Dirichlet eigenvalue λ_1, and $t > 0$ is to be determined. Then,

$$J_\lambda(t\phi_1) \le \int_{d(x)\le d_0} \Big(B(t\phi_1)^2 - E(t\phi_1)^{q(x)+1} \Big)\, dx + \int_{d(x)> d_0} \Big(B(t\phi_1)^2 - E(t\phi_1)^{q(x)+1} \Big)\, dx,$$

where $d(x) = \mathrm{dist}(x, \partial\Omega)$, $d_0 > 0$ is a suitably small constant,

$$E = \frac{\lambda}{q_+ + 1}$$

and

$$B = \frac{\lambda_1}{2}.$$

Now choose $\bar{x} \in \Omega$ so that $q(\bar{x}) = q_+ = \max_{\overline{\Omega}} q$, and take a small δ with the property that $B(\bar{x}, \delta) \subset \{d(x) > d_0\}$ and $q(x) > q_+ - \eta > 1$ in $B(\bar{x}, \delta)$. We have

$$J_\lambda(t\phi_1) \le Bt^2 \int_{d(x)\le d_0} \phi_1^2\, dx - \int_{B(\bar{x},\delta)} \left(E(t\phi_1)^{q_+ - \eta + 1} - B(t\phi_1)^2 \right) dx, \tag{3.12}$$

if $t > 0$ is taken so large as to have $(t\phi_1)^{q(x)-1} \ge B/E$ in $d(x) > d_0$. Since $q_+ - \eta + 1 > 2$ it follows from (3.12) that a value of $t_0 > 0$ can be found (independent on $\lambda \in I$) so that (3.11) holds for $\psi = t_0\phi_1$.

Set now $A(u) = \frac{1}{2}\|u\|^2_{H^1_0(\Omega)}$, $\mathcal{P}(u)$ as defined in (2.3) and fix a bounded interval $I \subset \mathbb{R}^+$. Then, Theorem 3–1) can be applied to show the existence, for almost all $\lambda \in I$, of a BPS sequence $v_n \in H^1_0(\Omega)$ of J_λ at the MP level c_λ. Since Lemma 2 permits to extract a convergent subsequence $v_{n'} \to v$ in $H^1_0(\Omega)$ and $c_\lambda > 0$ (after a proper choice of $0 < R \le \varepsilon$ in (3.10)) then we get a nontrivial solution v to (1.6). Moreover, this argument proves the existence of a nontrivial solution to (1.6) for almost all $\lambda > 0$.

Let us show next that existence actually holds for *all* $\lambda > 0$. First, notice that (2.1) implies that both

$$N(u) = \frac{|u|^{q(x)}}{q(x)+1} \qquad \text{and} \qquad DN(u) = |u|^{q(x)-1}u$$

remain bounded on bounded sets of $H_0^1(\Omega)$, and so are both $\mathcal{P}$ and $D\mathcal{P}$. On the other hand, Lemma 2 says that every BPS sequence of J_λ admits a convergent subsequence in $H_0^1(\Omega)$. Thus, by applying Theorem 3–2) at a fixed $\lambda_0 > 0$ we obtain sequences $\lambda_n \to \lambda_0$, λ_n increasing, and $u_n \in H_0^1(\Omega)$ such that

$$J_{\lambda_n}(u_n) = c_{\lambda_n}, \qquad DJ_{\lambda_n}(u_n) = 0 \qquad \text{and} \quad c_{\lambda_n} \to c_{\lambda_0}.$$

We will prove in Section 4 that such sequence is bounded in $H_0^1(\Omega)$ (see Lemma 5). Therefore, Theorem 3–2) allows us to conclude that u_n is a BPS sequence of J_{λ_0} at the level c_{λ_0} and, as above, it furnishes a nontrivial solution $u \in H_0^1(\Omega)$ to (1.6) satisfying $J_{\lambda_0}(u) = c_{\lambda_0}$.

Observe also that the entire analysis of this section and the corresponding one in Section 4, can still be carried out if J_λ is replaced by the functional

$$J_{\lambda,+}(u) = \frac{1}{2}\|u\|^2_{H_0^1(\Omega)} - \lambda \int_\Omega \frac{u_+^{q(x)+1}}{q(x)+1}\, dx, \tag{3.13}$$

where $u_+ = \max\{u, 0\}$. In fact, it can be checked that a nontrivial critical point $u \in H_0^1(\Omega)$ of J_λ^+ defines a positive solution to (1.6). This concludes the proof of i).

Assertion ii) is readily attained if the full analysis is performed in the subspace of $H_0^1(\Omega)$ which consists of radially symmetric functions.

The statement in iii) is a consequence of standard L^p estimates. Indeed, if $u_n := u_{\lambda_n}$ is a sequence of positive solutions with $\lambda_n \to 0$ and $\lambda_n t_n^{q_+-1} \to 0$, (here $t_n = \|u_n\|_\infty$), then $v_n = u_n/t_n$ solves $-\Delta v_n = \lambda_n t_n^{q(x)-1} v_n$. Since the right hand side converges to zero in $L^\infty(\Omega)$, a subsequence $v_{n'}$ can be found so that $v_{n'} \to 0$ in $C^1(\overline{\Omega})$, what is not possible.

A proof of iv) is postponed until the end of Section 4.

Remark 2. The limit in (1.7) could be infinite. Indeed, let $\Omega = B$ be a ball centered at $x = 0$ with radius $R > 0$ and assume that $q > 1$ is radially symmetric and such that $\{r : q(r) = q_+\}$ is a set of measure zero. Suppose that $u_n := u_{\lambda_n}$ is a family of positive radial solutions with $\lambda_n \to 0$. As already shown, $t_n := \|u_n\|_\infty = u_n(0) \to \infty$. Setting $u_n = t_n v_n$, a straightforward computation yields the relation,

$$t_n = \frac{\lambda_n}{N-2} \int_0^R s \left(1 - \left(\frac{s}{R}\right)^{N-2}\right) v_n(s)^{q(s)} t_n^{q(s)}\, ds.$$

Thus,

$$1 = \frac{\lambda_n t_n^{q_+-1}}{N-2} \int_0^R s \left(1 - \left(\frac{s}{R}\right)^{N-2}\right) v_n(s)^{q(s)} t_n^{-(q_+-q(s))}\, ds,$$

whence $\lambda_n t_n^{q_+-1} \to \infty$ since the integral goes to zero.

4. Uniform bounds.

This section is dedicated to show the a priori bounds required in the proof of Theorem 1 i) and also the statement in Theorem 1 iv). We will assume throughout that a "reference" sequence $(\lambda_n, u_n) \in (0,+\infty) \times H_0^1(\Omega)$ satisfies the following hypothesis:

(H)$_J$ The sequence λ_n is increasing and $\lambda_n \to \lambda_0$ for some positive λ_0 while $J_{\lambda_n}(u_n) = c_{\lambda_n}$ with

$$c_{\lambda_n} = \inf_{\gamma\in\Gamma} \max J_{\lambda_n} \circ \gamma,$$

where $\Gamma = \{\gamma \in C([0,1], H_0^1(\Omega)) : \gamma(0) = 0,\ \gamma(1) = \psi\}$, $\psi \in H_0^1(\Omega)$ does not depend on n and $c_{\lambda_n} > \max\{0, J_{\lambda_n}(\psi)\}$. In addition,

$$DJ_{\lambda_n}(u_n) = 0,$$

for all $n \in \mathbb{N}$.

Through a series of steps we will show that under condition (H)$_J$, the sequence u_n is bounded in $H_0^1(\Omega)$. Notice that, since J_λ is non increasing in λ, then c_{λ_n} is bounded.

In what follows C will designate different constants whose explicit values are irrelevant for the purposes of the proofs.

Lemma 4. *Let $\{(\lambda_n, u_n)\} \subset (0,+\infty) \times H_0^1(\Omega)$ be a sequence satisfying (H)$_J$. Then there exists $C > 0$ such that*

$$\|\, |u_n|^{q(x)+1} \|_{L^1(\Omega,d\mu)} \le C, \tag{4.1}$$

where $d\mu = \operatorname{dist}(x, \partial\Omega)\, dx$. Furthermore, for every $1 \le p < \dfrac{N}{N-1}$ there exists another constant C such that

$$\|u_n\|_{L^p(\Omega)} \le C. \tag{4.2}$$

Proof. First observe that $J_{\lambda_n}(u_n)$ is bounded from above since c_{λ_n} is non increasing. On the other hand

$$\|u_n\|^2_{H_0^1(\Omega)} - \lambda_n \int_\Omega \frac{1}{q(x)+1} |u_n|^{q(x)+1}\, dx = 0,$$

for all n. This implies that

$$\lambda_n \int_\Omega \frac{q(x)-1}{q(x)+1} |u_n|^{q(x)+1}\, dx = O(1), \tag{4.3}$$

as $n \to \infty$. Using hypothesis (1.5) we conclude that

$$\int_\Omega d(x) |u_n|^{q(x)+1}\, dx \le C,$$

what proves (4.1). In particular $\lambda_n |u_n|^{q(x)}$ is bounded in $L^1(\Omega, d\mu)$ since λ_n remains bounded. We use now the crucial fact that (λ_n, u_n) solves (1.6) together with the estimates in Lemma 2.1 of [5] (see also [23]) to achieve (4.2). $\square$

Lemma 5. *Assume that* $\{(\lambda_n, u_n)\} \subset (0, +\infty) \times H_0^1(\Omega)$ *satisfies the hypothesis (H)$_J$. Then, the sequence u_n is bounded in $H_0^1(\Omega)$.*

Proof. Using once again that (λ_n, u_n) solves (1.6) we get

$$\int_\Omega |\nabla u_n|^2 \, dx = \lambda_n \int_\Omega |u_n|^{q(x)+1} \, dx,$$

that is, to get a bound for $\|u_n\|_{H_0^1(\Omega)}$ we need to obtain a uniform bound for the integral in the right hand side of the above equality. So set $\Omega_\delta = \{x \in \Omega : d(x) < \delta\}$, with $\delta > 0$ and suitably small, $Q_\delta = \Omega \setminus \Omega_\delta$. Writing

$$\int_\Omega |u_n|^{q(x)+1} \, dx = \int_{\Omega_\delta} |u_n|^{q(x)+1} \, dx + \int_{Q_\delta} |u_n|^{q(x)+1} \, dx,$$

we observe that (4.1) entails that the third integral in the equality is uniformly bounded. To estimate the second integral we select a value q_1 verifying

$$1 < q_1 < \frac{N}{N-1}.$$

Then we obtain

$$|u|^{q(x)} \leq 1 + |u|^{q_1}$$

for all $x \in \overline{\Omega}_\delta$, if δ is chosen small enough.

On the other hand, using Lemma 4, we obtain:

$$\int_{\Omega_\delta} |u_n|^{q(x)+1} \, dx \leq \int_{\Omega_\delta} |u_n| \, dx + \int_{\Omega_\delta} |u_n|^{q_1+1} \, dx \leq C + \int_{\Omega_\delta} |u_n|^{q_1+1} \, dx.$$

We now borrow ideas from [4] to estimate the last integral. First, observe that

$$\int_{\Omega_\delta} |u_n|^{q_1+1} \, dx = \int_{\Omega_\delta} (d|u_n|^{q_1})^\theta (|u_n|^{q_1})^{1-\theta} \frac{|u_n|}{d^\theta} \, dx,$$

where,

$$\theta = \frac{2}{N+1} < 1.$$

By using Hölder's inequality, we get

$$\begin{aligned} \int_{\Omega_\delta} |u_n|^{q_1+1} \, dx &\leq \left(\int_{\Omega_\delta} d|u_n|^{q_1} \, dx \right)^\theta \left(\int_{\Omega_\delta} |u_n|^{q_1} \frac{|u_n|^{\frac{1}{1-\theta}}}{d^{\frac{\theta}{1-\theta}}} \, dx \right)^{1-\theta} \\ &\leq C \left(\int_{\Omega_\delta} |u_n|^{q_1} \frac{|u_n|^{\frac{1}{1-\theta}}}{d^{\frac{\theta}{1-\theta}}} \, dx \right)^{1-\theta}, \end{aligned} \tag{4.4}$$

where Lemma 4 has been used to estimate the second integral in the first line, since $q_1 < \frac{N}{N-1}$. We next observe that for arbitrarily small $\varepsilon > 0$ a positive constant C_ε can be chosen so that

$$|u|^{q_1} \leq \varepsilon |u|^{q_{BT}} + C_\varepsilon,$$

for all $u \geq 0$, where $q_{BT} = \frac{N+1}{N-1}$ is the well-known Brezis-Turner exponent (cf. [4]). Thus, the last integral in (4.4) can be estimated as,

$$\begin{aligned}
\left(\int_{\Omega_\delta} |u_n|^{q_1} \frac{|u_n|^{\frac{1}{1-\theta}}}{d^{\frac{\theta}{1-\theta}}}\, dx \right)^{1-\theta} &\leq \varepsilon^{1-\theta} \left(\int_{\Omega_\delta} \frac{|u_n|^{\frac{2}{1-\theta}}}{d^{\frac{\theta}{1-\theta}}}\, dx \right)^{1-\theta} \\
&\quad + C_\varepsilon \left(\int_{\Omega_\delta} \frac{|u_n|^{\frac{1}{1-\theta}}}{d^{\frac{\theta}{1-\theta}}}\, dx \right)^{1-\theta} \\
&\leq \varepsilon^{1-\theta} \left\| \frac{u_n}{d^{\frac{\theta}{2}}} \right\|^2_{L^{\frac{2}{1-\theta}}(\Omega)} + C_\varepsilon \left\| \frac{u_n}{d^\theta} \right\|_{L^{\frac{1}{1-\theta}}(\Omega)}, \qquad (4.5)
\end{aligned}$$

where it has been used that

$$q_{BT} = \frac{1}{1-\theta}.$$

We now recall the next variant of Hardy inequality (Lemma 2.2 in [4]). It states that for every $u \in H_0^1(\Omega)$ and $0 \leq s \leq 1$, a positive constant C exists so that the inequality

$$\left\| \frac{v}{d^s} \right\|_{L^p(\Omega)} \leq C \|v\|_{H_0^1(\Omega)},$$

holds true for every $v \in H_0^1(\Omega)$, provided that

$$\frac{1}{p} = \frac{1}{2} - \frac{1-s}{N}.$$

Taking into account that $p = \dfrac{2}{1-\theta}$ for $s = \dfrac{\theta}{2}$ while the corresponding p associated to $s = \theta$ satisfies $p > \dfrac{1}{1-\theta}$, we conclude from (4.4) and (4.5) (after possibly diminishing ε) that

$$\int_{\Omega_\delta} |u_n|^{q(x)+1}\, dx \leq \varepsilon^{1-\theta} \|u\|^2_{H_0^1(\Omega)} + C_\varepsilon \|u\|_{H_0^1(\Omega)} + C.$$

Thus,

$$\|u_n\|^2_{H_0^1(\Omega)} \leq \varepsilon^{1-\theta} \|u_n\|^2_{H_0^1(\Omega)} + C_\varepsilon \|u_n\|_{H_0^1(\Omega)} + C,$$

which certainly implies that $\|u_n\|_{H_0^1(\Omega)}$ is bounded if ε is properly chosen. $\square$

Proof of assertion iv) *in Theorem 1.* We first show that the MP level c_λ involved in Section 3 and associated to the solution u_λ satisfies

$$\lim_{\lambda \to \infty} c_\lambda = 0.$$

Let γ_λ be the path $\gamma_\lambda(\tau) = \tau\psi$, $0 \leq \tau \leq 1$, which joins $u = 0$ with the function $\psi = t_0 \phi_1$ computed in Section 3 (notice that now t_0 depends on λ since $\lambda \to \infty$). Set

$$\tilde{c}_\lambda = \max J_\lambda \circ \gamma_\lambda = \max_{0 \leq t \leq t_0} J_\lambda(t\phi_1).$$

Since $0 < c_\lambda \leq \tilde{c}_\lambda$ then it suffices to show that

$$\lim_{\lambda \to \infty} \tilde{c}_\lambda = 0. \qquad (4.6)$$

Take $h(t) = J_\lambda(t\phi_1)$. It is clear that h achieves its maximum at a value $t = t_\lambda$ satisfying

$$\int_\Omega \lambda_1 \phi_1^2 \, dx = \lambda \int_\Omega t_\lambda^{q(x)-1} \phi_1^{q(x)+1} \, dx. \tag{4.7}$$

Hence, $t_\lambda \to 0$ as $\lambda \to \infty$. From

$$\tilde{c}_\lambda = J_\lambda(t_\lambda \phi_1) \le \frac{\lambda_1 {t_\lambda}^2}{2} \int_\Omega \phi_1^2 \, dx, \tag{4.8}$$

(4.6) follows.

Next, we use both (4.7) and (4.8) to provide a better estimate of $\tilde{c}_\lambda$. For a small $\delta > 0$ there exists $k_0 > 0$ such that

$$\phi_1(x) \ge k_0 d(x),$$

for all $x \in \Omega_\delta$. For the sake of simplicity we assume that q is Lipschitz in $\overline{\Omega}_\delta$ and so a constant k_1 exists so that q satisfies

$$1 < q(x) < 1 + k_1 d(x),$$

for all $x \in \overline{\Omega}_\delta$ (the case $q \in C^\alpha$ can be handled analogously). Denoting temporarily $\varepsilon = t_\lambda$, we obtain

$$\int_\Omega \varepsilon^{q(x)-1} \phi_1^{q(x)+1} \, dx \ge \int_{\Omega_\delta} \varepsilon^{q(x)-1} \phi_1^{q(x)+1} \, dx \ge \int_{\Omega_\delta} \varepsilon^{k_1 d} (k_0 d)^{2+k_1 d} \, dx,$$

while

$$\int_{\Omega_\delta} \varepsilon^{k_1 d} (k_0 d)^{2+k_1 d} \, dx \ge |\partial\Omega| \int_0^\delta \varepsilon^{k_1 s} (k_0 s)^{2+k_1 s} \, ds.$$

On the other hand,

$$\int_0^\delta \varepsilon^{k_1 s} (k_0 s)^{2+k_1 s} \, ds \ge (1-\eta) \int_0^\delta \varepsilon^{k_1 s} (k_0 s)^2 \, ds,$$

where $0 < \eta < 1$ provided that δ is chosen suitably small. A direct computation reveals that

$$\int_0^\delta \varepsilon^{k_1 s} (k_0 s)^2 \, ds = -\frac{C}{\ln^3 \varepsilon}(1 + o(1)), \tag{4.9}$$

as $\varepsilon \to 0+$, for a certain positive constant C. Going back to (4.7) and using (4.9) we arrive at

$$\int_\Omega \lambda_1 \phi^2 \, dx \ge -C \frac{\lambda}{\ln^3 \varepsilon},$$

that yields, after restoring t_λ instead of ε

$$t_\lambda = O(e^{-C\sqrt[3]{\lambda}}),$$

as $\lambda \to \infty$.

Using (4.8) and taking into account that $c_\lambda \le \tilde{c}_\lambda$ we finally get that

$$c_\lambda = O(e^{-C\sqrt[3]{\lambda}}), \tag{4.10}$$

as $\lambda \to \infty$.

Now we proceed to obtain exponentially small uniform bounds of u_λ as $\lambda \to \infty$. Just as before the relation

$$\lambda \int_\Omega \frac{q(x)-1}{q(x)+1} |u_\lambda|^{q(x)+1}\ dx = O(e^{-C\sqrt[3]{\lambda}}), \tag{4.11}$$

holds as $\lambda \to \infty$ and so

$$\int_\Omega |u_\lambda|^{q(x)+1} d\mu = o(e^{-C\sqrt[3]{\lambda}}) \qquad \text{as } \lambda \to \infty, \tag{4.12}$$

where $d\mu = d(x)\ dx$.

Some few facts from the theory of $L^{p(x)}$ spaces are now required. For $p \in L^\infty(\Omega)$, $p \geq 1$ a. e. in Ω, the *modular* functional ρ_p is defined on the set $M(\Omega)$ of measurable functions u in Ω as

$$\rho_p(u) = \int_\Omega |u|^{p(x)}\ d\mu \qquad d\mu = d(x)\ dx.$$

Then the class $L^{p(x)}(\Omega) := \{u \in M(\Omega) : \rho_p(u) < \infty\}$ becomes a Banach space under the norm

$$\|u\|_{p(x)} = \inf \left\{\lambda > 0 : \rho_p\left(\frac{u}{\lambda}\right) \leq 1\right\}.$$

This is, of course, the so–called Luxemburg norm. We refer to [20] for a comprehensive account on this and further abstract classes of generalized Orlicz–type spaces. Some well–known features are the following (see for instance [8] for an expeditious overview):

(a) $\|u\|_{p(x)} < 1$ if and only if $\rho_p(u) < 1$.
(b) $\|u\|_{p(x)} < 1$ implies $\|u\|_{p(x)}^{p_+} \leq \rho_p(u) \leq \|u\|_{p(x)}^{p_-}$ where $p_- = \text{ess inf}\, u$, $p_+ = \text{ess sup}\, u$.
(c) The embedding $L^{r(x)} \subset L^{p(x)}$ is continuous provided that $r \in L^\infty(\Omega)$ fulfills $r \geq p$ a. e. in Ω.

Hence, according to (a) and (4.12),

$$\|u\|_{q(x)+1}^{q_+ + 1} \leq \rho_{q+1}(u) = o(e^{-C\sqrt[3]{\lambda}}) \qquad \text{as } \lambda \to \infty,$$

while (c) implies that

$$\|u\|_{q(x)} = o(e^{-C\sqrt[3]{\lambda}}) \qquad \text{as } \lambda \to \infty,$$

since $q_- = 1$. Thus, it follows from (b) that

$$\int_\Omega \lambda |u_\lambda|^{q(x)}\ d\mu = o(e^{-C\sqrt[3]{\lambda}}) \qquad \text{as } \lambda \to \infty. \tag{4.13}$$

Recall that the constant C is not the same in each particular appearance.

Using the same argument as in Lemma 4 we conclude that

$$\|u_\lambda\|_p = o(e^{-C\sqrt[3]{\lambda}}) \qquad \text{as } \lambda \to \infty, \tag{4.14}$$

for every $1 \leq p < \dfrac{N}{N-1}$.

To estimate $\|u_\lambda\|_{H_0^1(\Omega)}$ we follow the approach in Lemma 5 and arrive at the estimate

$$\int_\Omega \|\nabla u_\lambda\|^2 \, dx \le \lambda \int_{\Omega_\delta} |u_\lambda| \, dx + \lambda \int_{\Omega_\delta} |u_\lambda|^{q_1+1} \, dx + O(e^{-C\sqrt[3]{\lambda}}). \tag{4.15}$$

Here, $\delta > 0$ and q_1 are the numbers introduced in the proof of Lemma 5 and (4.13) has been employed to estimate the integral

$$\lambda \int_{Q_\delta} |u_\lambda|^{q(x)+1} \, dx, \qquad \text{where } Q_\delta = \Omega \setminus \Omega_\delta.$$

The second integral in (4.15) is $O(e^{-C\sqrt[3]{\lambda}})$ thanks to (4.14). By using the computation in (4.5) we obtain

$$\lambda \int_{\Omega_\delta} |u_\lambda|^{q_1+1} \, dx \le \left(\int_{\Omega_\delta} \lambda^{\frac{1}{\theta}} |u_\lambda|^{q_1} \, dx \right)^\theta \left\{ \varepsilon^{1-\theta} \|u_\lambda\|^2_{H_0^1(\Omega)} + C_\varepsilon \|u_\lambda\|_{H_0^1(\Omega)} \right\}.$$

Thus,

$$\|u_\lambda\|^2_{H_0^1(\Omega)} \le O(e^{-C\sqrt[3]{\lambda}}) \{ \varepsilon^{1-\theta} \|u_\lambda\|^2_{H_0^1(\Omega)} + C_\varepsilon \|u_\lambda\|_{H_0^1(\Omega)} \} + O(e^{-C\sqrt[3]{\lambda}}), \tag{4.16}$$

as $\lambda \to \infty$. This allows us to conclude first that $\|u_\lambda\|_{H_0^1(\Omega)} = O(1)$ as $\lambda \to \infty$. In a second instance it implies that $\|u_\lambda\|_{H_0^1(\Omega)} = O(e^{-C\sqrt[3]{\lambda}})$ as $\lambda \to \infty$ (interchange between "O" and "o" is obtained by modifying C). Finally, estimate (4.16) leads in a standard way to a corresponding one in $C^{2,\beta}(\overline{\Omega})$ for a certain $0 < \beta < 1$.

This completes the proof of Theorem 1. □

Remark 3. When studying the asymptotic behavior as $\lambda \to \infty$ of a family u_λ of solutions through a scaling method (cf. [14]), a critical issue is the concentration of maxima of solutions on $\partial\Omega$. In the case where Ω is a convex domain this possibility can be ruled out in some problems by means of the moving planes technique as in [6]. However, this approach cannot be used here since our nonlinearity $u^{q(x)}$ lacks the right monotonicity properties near the boundary. This is due, of course, to the fact that $q = 1$ on $\partial\Omega$.

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