ELLIPTIC SYSTEMS WITH BOUNDARY BLOW-UP: EXISTENCE, UNIQUENESS AND APPLICATIONS TO REMOVABILITY OF SINGULARITIES

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ABSTRACT. In this paper we consider the elliptic system $\Delta u = u^p - v^q$, $\Delta v = -u^r + v^s$ in Ω, where the exponents verify $p, s > 1$, $q, r > 0$ and $ps > qr$ and Ω is a smooth bounded domain of $\mathbb{R}^N$. First, we show existence and uniqueness of boundary blow-up solutions, that is, solutions (u, v) verifying $u = v = +\infty$ on $\partial\Omega$. Then, we use them to analyze the removability of singularities of positive solutions of the system in the particular case $qr \leq 1$, where comparison is available.

1. Introduction and description of the main results.

The main purpose of the present paper is to perform an analysis of existence and uniqueness of positive solutions for the nonlinear elliptic system

$$\begin{cases} \Delta u = u^p - v^q \\ \Delta v = -u^r + v^s, \end{cases} \quad \text{in } \Omega, \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $p, s > 1$ and $q, r > 0$. We will be mainly dealing with the so-called "large" or "boundary blow-up" solutions, that is, functions $u, v \in C^2(\Omega)$ with the property that

$$u = v = +\infty \qquad \text{on } \partial\Omega.$$

This "boundary condition" is understood in the sense that $u(x), v(x) \to +\infty$ as x approaches the boundary $\partial\Omega$.

Equations with boundary blow-up have been largely studied in the literature in the last years. It is not our intention to give a complete list of references on the subject, but we prefer to refer the reader to the survey [22]. However, we remark that most of the works in this topic have been restricted to scalar equations, and, at the best of our knowledge, not very much is known for elliptic systems. Actually, there is no real understanding of the general picture for systems of the form

$$\begin{cases} \Delta u = f(u, v) \\ \Delta v = g(u, v) \end{cases} \quad \text{in } \Omega.$$

However, we quote [8, 9, 12, 13, 15, 16, 17, 18, 19, 21], for systems of Lotka-Volterra type (cooperative, competitive, predator-prey) and for special systems involving power nonlinearities. We also quote [10] for results on large solutions for a system with a nonlinearity in the gradient.

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In the present paper, our aim is to understand the features of (1.1). Throughout the paper we will always assume a *weakly coupling* condition, which in our context is given by the inequality

$$ps > qr. \tag{1.2}$$

First, we note that the existence and the boundary behavior of solutions depend on several relations between the exponents. We begin by assuming that p, q, r, s verify

$$\frac{p}{p-1} > \frac{q}{s-1} \qquad \text{and} \qquad \frac{s}{s-1} > \frac{r}{p-1}. \tag{1.3}$$

These inequalities ensure that the dominating terms in the right-hand side of both equations in (1.1) near the boundary are u^p and v^s, respectively. This makes the existence of large solutions somewhat simpler.

Theorem 1. *Let Ω be a bounded C^2 domain of $\mathbb{R}^N$ and assume $p, s > 1$ and $q, r > 0$ verify* (1.2) *and* (1.3)*. Then, there exists a positive large solution of the system* (1.1)*. Moreover, if $qr \le 1$, then the solution is unique and verifies,*

$$\begin{aligned} \lim_{x\to\partial\Omega} d(x)^{\alpha} u(x) &= (\alpha(\alpha+1))^{\frac{1}{p-1}} \\ \lim_{x\to\partial\Omega} d(x)^{\beta} v(x) &= (\beta(\beta+1))^{\frac{1}{s-1}}, \end{aligned} \tag{1.4}$$

where $\alpha = \frac{2}{p-1}$ and $\beta = \frac{2}{s-1}$.

Next, we note that condition (1.3) is not necessary for the existence of large solutions of (1.1). When one of the inequalities in (1.3) does not hold, it is still possible to have positive large solutions (as long as condition (1.2) is not violated). The only difference is that now both terms in the right-hand side of one of the equations in (1.1) have the same growth near $\partial\Omega$.

Thus we assume next that (1.3) does not hold. It is to be noted that condition (1.2) prevents both inequalities in (1.3) to fail, therefore one of them will always hold. Due to the symmetry of the problem we can always assume that

$$\frac{p}{p-1} \le \frac{q}{s-1} \qquad \text{and} \qquad \frac{s}{s-1} > \frac{r}{p-1}. \tag{1.5}$$

In this case, our results for system (1.1) read as follows:

Theorem 2. *Let Ω be a bounded C^2 domain of $\mathbb{R}^N$ and assume $p, s > 1$ and $q, r > 0$ verify* (1.2) *and* (1.5)*. Then, there exists a positive large solution of the system* (1.1)*. Moreover, if $qr \le 1$, then the solution is unique and it verifies:*

$$\begin{aligned} \lim_{x\to\partial\Omega} d(x)^{\gamma} u(x) &= A_0 \\ \lim_{x\to\partial\Omega} d(x)^{\beta} v(x) &= (\beta(\beta+1))^{\frac{1}{s-1}}. \end{aligned} \tag{1.6}$$

where $\gamma = \frac{2q}{p(s-1)}$, $\beta = \frac{2}{s-1}$ and A_0 is the unique positive root of the equation $\gamma(\gamma+1)A = A^p - (\beta(\beta+1))^{\frac{q}{s-1}}$ when $\frac{p}{p-1} = \frac{q}{s-1}$ while $A_0 = (\beta(\beta+1))^{\frac{q}{p(s-1)}}$ when the first inequality in (1.5) *is strict.*

The existence proofs in Theorems 1 and 2 follow by the use of the method of sub and supersolutions. We refer to the appendix in [18] for an instance of the method when applied to a particular competitive system, which is however easily extended to deal with (1.1). As for the uniqueness, it follows by means of a sweeping argument, once the boundary behavior of all possible solutions (equations (1.4) and (1.6)) is obtained. In this regard, it is to be remarked that the condition $qr \le 1$ is essential in our approach not only to have uniqueness, but also to obtain the boundary behavior, since comparison is used there (see in particular Lemma 7 in Section 3).

Let us also mention in passing that with the same ideas it is possible to study solutions of (1.1) with different boundary conditions, for instance $u = \lambda$, $v = \mu$ on $\partial\Omega$, where $\lambda, \mu > 0$ or $u = +\infty$, $v = \mu$ on $\partial\Omega$, in the same spirit as in [18].

Another interesting question regarding system (1.1) and closely connected to the existence of large solutions is the analysis of positive solutions with an *isolated singularity*, that is, solutions solving the equation in the whole domain except at one point. There is no loss of generality in assuming that the equation holds in $B \setminus \{0\}$, where B is the unit ball in $\mathbb{R}^N$, and we will always do so from now on. Then, we consider the system

$$\begin{cases} \Delta u = u^p - v^q \\ \Delta v = -u^r + v^s \end{cases} \quad \text{in } B \setminus \{0\}. \tag{1.7}$$

Our interest now is to obtain conditions which ensure that a positive solution (u, v) of (1.7) has a *removable singularity*, in the sense that (u, v) is actually smooth in B and it solves the problem in the whole B.

The topic of removability of isolated singularities has been largely studied, beginning with the pioneering work [6]. Numerous works have dealt with this question for different types of elliptic equations (see for instance [7] for further references). However, the study of elliptic system is not so developed. We refer to [2, 3, 4, 5, 14] for systems which are of Hamiltonian type. As far as we know, the study of removable singularities for general elliptic systems involving powers but which are not of Hamiltonian type (as (1.7)) is still open.

We state next our results for system (1.7) which depend on whether the exponents verify (1.3) or not. It is to be remarked again that, since comparison is needed in our proofs, we will assume that $qr \le 1$.

Theorem 3. *Assume $p, s > 1$ and $q, r > 0$ are such that $qr \le 1$ and (1.3) holds. Let (u, v) be a positive classical solution of (1.7). Then:*

(a) *If $p \ge \frac{N}{N-2}$, $s \ge \frac{N}{N-2}$, then the singularity at $x = 0$ is removable, so that $u, v \in C^\infty(B)$ and they are a solution of the system in B;*
(b) *If $p \ge \frac{N}{N-2}$, $s < \frac{N}{N-2}$ and $q < s - 1$, then $u \in L^\infty_{\text{loc}}(B)$.*
(c) *If $p < \frac{N}{N-2}$, $s \ge \frac{N}{N-2}$ and $r < p - 1$, then $v \in L^\infty_{\text{loc}}(B)$.*

It is worthy of mention that these results are optimal in the following sense: when $p < \frac{N}{N-2}$, $s < \frac{N}{N-2}$ there are solutions to the system which have both components unbounded. In the case $p \ge \frac{N}{N-2}$, $s < \frac{N}{N-2}$, $q \ge s - 1$ (and similarly in the symmetric situation $p < \frac{N}{N-2}$ $s \ge \frac{N}{N-2}$, $r \ge p - 1$), there are solutions (u, v) where u is bounded but v is not (see Remark 1 in Section 4). On the other hand, in all cases, and according to Theorem 1, there are always solutions where both components are bounded in B.

Theorem 4. *Assume $p, s > 1$ and $q, r > 0$ are such that $qr \leq 1$ and* (1.5) *holds. Suppose further that $s \geq \frac{N}{N-2}$. Let (u, v) be a positive classical solution of* (1.7). *Then:*

(a) *If $p \geq \frac{N}{N-2}$, then the singularity at $x = 0$ is removable, so that $u, v \in C^\infty(B)$ and they are a solution of the system in B;*
(b) *If $p < \frac{N}{N-2}$ and $qr < p(s-1)$, then $v \in L^\infty_{\text{loc}}(B)$.*

As in the previous theorem, it is possible to show that, whenever $s < \frac{N}{N-2}$, solutions which are unbounded in both components can be constructed. As for case (b), we do not know if the condition $qr < p(s-1)$ is necessary or not for the removability of singularities.

The rest of the paper is organized as follows: in Section 2 we will analyze some scalar equations with power-type nonlinearities which will be useful when considering the existence of large solutions of (1.1). Section 3 will be devoted to the proof of the existence and uniqueness of large solutions (Theorems 1 and 2) while in Section 4 we will study the removability of singularities for system (1.7).

2. Some scalar equations.

In this section, we consider some scalar equations related to the system (1.1). Their solutions will be used as instrument when proving existence for (1.1). We begin by considering the well-known problem

$$\begin{cases} \Delta u = u^p & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $p > 1$. This problem has been deeply analyzed in several papers, therefore we only state its most important properties without proofs (see for instance [1]).

Theorem 5. *Assume Ω is a C^2 bounded domain and let $p > 1$. Then problem* (2.1) *admits a unique positive solution, which in addition verifies*

$$\lim_{x \to \partial\Omega} d(x)^\alpha u(x) = (\alpha(\alpha+1))^{\frac{1}{p-1}},$$

where $\alpha = \frac{2}{p-1}$.

Each of the equations in (1.1) can be regarded as a perturbation of (2.1) with a non-homogeneous singular term. Hence it is natural to consider the problem

$$\begin{cases} \Delta u = u^p - C_0 d(x)^{-\gamma} & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega \end{cases} \tag{2.2}$$

with $C_0 > 0$ and $\gamma > 0$. Some instances of this problem have been already studied, actually with more general perturbation terms, but only the case $\gamma < \frac{2p}{p-1}$ seems to have been considered up to now (see [23] and [11]). We mention in passing that the term $C_0 d(x)^{-\gamma}$ can be replaced by a continuous, nonnegative function h which has the same behavior near the boundary, and the same results can be obtained. However, we are sticking to model (2.2), since it will be enough for our purposes.

Theorem 6. *Assume Ω is a C^2 bounded domain and let $p > 1$, $C_0 > 0$ and $\gamma > 0$. Then, problem (2.2) admits a unique positive solution. In addition, we always have*

$$\lim_{x \to \partial\Omega} d(x)^\alpha u(x) = A_0, \tag{2.3}$$

where:

(a) *If $\gamma < \frac{2p}{p-1}$, then $\alpha = \frac{2}{p-1}$ and $A_0 = (\alpha(\alpha+1))^{\frac{1}{p-1}}$.*

(b) *If $\gamma = \frac{2p}{p-1}$, then $\alpha = \frac{2}{p-1}$ and A_0 is the unique positive root of the equation $\alpha(\alpha+1)A - A^p = -C_0$.*

(c) *If $\gamma > \frac{2p}{p-1}$, then $\alpha = \frac{\gamma}{p}$ and $A_0 = C_0^{\frac{1}{p}}$.*

Proof. We can restrict our attention to $\gamma \geq \frac{2}{p-1}$ and refer to [23] and [11] for the case $\gamma < \frac{2p}{p-1}$. For $n \in \mathbb{N}$, consider the problem

$$\begin{cases} \Delta u = u^p - C_0 \left(d(x) + \frac{1}{n}\right)^{-\gamma} & \text{in } \Omega \\ u = +\infty & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

According to Theorem 1 in [23], problem (2.4) admits a unique positive solution, which we will denote by u_n. It is a simple consequence of the uniqueness that the sequence $\{u_n\}$ is increasing in n. Let us see that $\{u_n\}$ converges to a solution of (2.2). Let $\Omega' \subset\subset \Omega$, and set $d_0 = \inf_{\Omega'} d(x)$. Then $\Delta u_n \geq u_n^p - C_0 d_0^{-\gamma}$ in Ω', so that again by uniqueness we obtain that $u_n \leq \bar{v}$ in Ω', where $\bar{v}$ is the unique positive solution of

$$\begin{cases} \Delta v = v^p - C_0 d_0^{-\gamma} & \text{in } \Omega' \\ v = +\infty & \text{on } \partial\Omega'. \end{cases}$$

This implies that the sequence $\{u_n\}$ is locally uniformly bounded. Now we can use the standard interior estimates in [20] to obtain first that $\{u_n\}$ is locally bounded in $C^{1,\alpha}$ for every $\alpha \in (0,1)$ and then in $C^{2,\alpha}$ for every $\alpha \in (0,1)$. Therefore, with the use of Ascoli-Arzelá's theorem and a diagonal procedure, we have $u_n \to u$ in $C^2_{\text{loc}}(\Omega)$, where u is a classical solution to (2.2).

Next, let us show (2.3). It is well known (cf. for instance the Appendix in Chapter 14 of [20]) that the distance function $d(x)$ is C^2 in a neighborhood of the boundary of the form $\Omega_{\delta_0} = \{x \in \Omega : \ d(x) < \delta_0\}$, where it also verifies $|\nabla d| = 1$. For $\delta \in (0, \delta_0)$, $\theta \in (0, \delta)$ and A, B to be chosen, let $\bar{u} = A(d-\theta)^{-\alpha} + B$, where $\alpha = \frac{\gamma}{p}$ (remember that we are assuming $\gamma \geq \frac{2p}{p-1}$). Let us show that $\bar{u}$ is a supersolution of $\Delta u = u^p - C_0 d^{-\gamma}$ in the set $\theta < d < \delta$ if $\delta < \delta_0$ is chosen small enough. This is equivalent to the following inequality:

$$A\alpha(\alpha+1)(d-\theta)^{\alpha(p-1)-2} - A\alpha(d-\theta)^{\alpha(p-1)-1}\Delta d \leq A^p - C_0$$

if $\theta < d < \delta$. Since $\alpha(p-1) - 1 \geq 1$, it would be enough to have

$$A\alpha(\alpha+1)(d-\theta)^{\alpha(p-1)-2} - A\alpha \sup_{\Omega_{\delta_0}} |\Delta d| \; \delta \leq A^p - C_0. \tag{2.5}$$

Next, let $\gamma = \frac{2p}{p-1}$, so that $\alpha(p-1)-2=0$. For $\varepsilon > 0$ small enough, let $A = A_0+\varepsilon$, where A_0 is as in part (b) of the statement. Since $(A_0+\varepsilon)\alpha(\alpha+1) < (A_0+\varepsilon)^p - C_0$, we can certainly obtain (2.5) if δ is small enough. When $\gamma > \frac{2p}{p-1}$, it is sufficient to have $0 < (A_0+\varepsilon)^p - C_0$ and then choose δ small enough. Thus, in either case, $\bar{u}$ is a supersolution in $\theta < d < \delta$ for δ small enough, depending on ε but not on θ, and where $B > 0$ is arbitrary. Now choose B so that $\bar{u} > u$ on $d = \delta$. Then since $\bar{u} = +\infty$ on $d = \theta$ while u is finite there, we may apply the comparison principle to deduce that $u < \bar{u}$ in $\theta < d < \delta$. Letting $\theta \to 0$, we arrive at $d(x)^\alpha u(x) \le (A_0+\varepsilon) + Bd(x)^\alpha$ if $0 < d < \delta$. Thus we can let $x \to \partial\Omega$ and then ε go to zero to arrive at

$$\limsup_{x\to\partial\Omega} d(x)^\alpha u(x) \le A_0. \tag{2.6}$$

In a similar way, it can be shown that the function $(A_0-\varepsilon)(d+\theta)^{-\alpha} - B$ is a subsolution of (2.2) in $0 < d < \delta$ as long as it is nonnegative. Then taking $\underline{u} = \max\{(A_0-\varepsilon)(d+\theta)^{-\alpha} - B, 0\}$, we have a nonnegative subsolution and we can use a comparison as before to obtain that $u \ge \underline{u}$ in $0 < d < \delta$. Letting $x \to \partial\Omega$ and then $\varepsilon \to 0$, we have

$$\liminf_{x\to\partial\Omega} d(x)^\alpha u(x) \ge A_0.$$

which together with (2.6) shows (2.3).

To conclude the proof we consider uniqueness. Assume u_1 and u_2 are positive solutions of (2.2). Then, according to (2.3), we have

$$\lim_{x\to\partial\Omega} \frac{u_1(x)}{u_2(x)} = 1$$

uniformly. Hence for every $\varepsilon > 0$, there exists $\delta > 0$ such that $u_1 \le (1+\varepsilon)u_2$ in Ω_δ. Now denote $\Omega^\delta = \{x \in \Omega : \ d(x) > \delta\}$ and consider the problem

$$\begin{cases} \Delta v = v^p - C_0 d(x)^{-\gamma} & \text{in } \Omega^\delta \\ v = u_1 & \text{on } \partial\Omega^\delta, \end{cases}$$

whose unique solution is $v = u_1$. It is clear that $(1+\varepsilon)u_2$ is a supersolution of this problem, so that by comparison $u_1 \le (1+\varepsilon)u_2$ in Ω^δ, and consequently in Ω. Letting $\varepsilon \to 0$ we have $u_1 \le u_2$, and reversing the roles of both solutions we finally achieve uniqueness. This finishes the proof. $\square$

3. Existence and uniqueness of blow-up solutions.

The goal of this section is to prove Theorems 1 and 2. The proofs of existence rely on the method of sub and supersolutions (cf. the appendix in [18] for a proof of the method for a different system, which is readily extended to deal with (1.1)). Those of uniqueness are based on a sweeping argument, complemented with the obtention of the boundary rates (1.4) and (1.6).

Proof of Theorem 1 (existence part). Denote by U the unique solution of (2.1) and V the unique solution of the same problem when p is replaced by s. Then there exist positive constants C_1, C_2 such that

$$\begin{aligned} C_1 d(x)^{-\alpha} \le U(x) \le C_2 d(x)^{-\alpha} \\ C_1 d(x)^{-\beta} \le V(x) \le C_2 d(x)^{-\beta}, \end{aligned} \qquad x \in \Omega, \tag{3.1}$$

where $\alpha = \frac{2}{p-1}$ and $\beta = \frac{2}{s-1}$. We claim that the pair $(MU, M^\eta V)$ is a supersolution of (1.1), for some suitable $\eta > 0$ and large enough positive M. To see this, it suffices to check that

$$\begin{aligned} MU^p &\le M^p U^p - M^{\eta q} V^q \\ M^\eta V &\le -M^r U^r + M^{\eta s} V^s \end{aligned}$$

which is a consequence of

$$\begin{aligned} M^{\eta q} C_2 d(x)^{-\beta q} &\le (M^p - M) C_1 d(x)^{-\alpha p} \\ M^r C_2 d(x)^{-\alpha r} &\le (M^{\eta s} - M^\eta) C_1 d(x)^{-\beta r}. \end{aligned}$$

Now, taking into account that $\alpha p > \beta q$ and $\beta s > \alpha r$, these inequalities are possible if we have, for some positive constant C (depending only on Ω):

$$\begin{aligned} M^{\eta q} &\le C(M^p - M) \\ M^r &\le C(M^{\eta s} - M^\eta). \end{aligned} \tag{3.2}$$

The choice

$$\frac{r}{s} < \eta < \frac{p}{q}, \tag{3.3}$$

which is possible by assumption (1.2), implies that (3.2) holds if M is large enough. Therefore $(MU, M^\eta V)$ is a supersolution of (1.1).

On the other hand, (U, V) is easily seen to be a subsolution. Thus the method of sub and supersolutions implies the existence of a positive solution (u, v) of (1.1) which verifies $u = v = +\infty$ on $\partial\Omega$. $\square$

For the proof of uniqueness, we will use the following comparison lemma, which is interesting in its own right. We do not know if the condition $qr \le 1$, which is used in our proofs, is necessary or not.

Lemma 7. *Assume $p, s > 1$ and $qr \le 1$. Let $(u_1, v_1) \in C^2(\Omega)^2$ (respectively (u_2, v_2)) be a positive subsolution (resp. supersolution) of the system* (1.1) *such that*

$$\limsup_{x \to \partial\Omega} \frac{u_1(x)}{u_2(x)} \le 1, \quad \limsup_{x \to \partial\Omega} \frac{v_1(x)}{v_2(x)} \le 1. \tag{3.4}$$

Then $u_1 \le u_2$ and $v_1 \le v_2$ in Ω.

Proof. We claim first that $(tu_2, t^\nu v_2)$ is a supersolution for $t > 1$, provided $\nu > 0$ is chosen in a suitable way. To see this we need to check the inequalities

$$\begin{cases} (t^{\nu q} - t) v_2^q \le (t^p - t) u_2^p \\ (t^r - t^\nu) u_2^r \le (t^{\nu s} - t^\nu) v^s \end{cases} \quad \text{in } \Omega. \tag{3.5}$$

Observe that $p, s > 1$ imply that the right-hand side in both equations in (3.5) is positive, since $t > 1$. Thus the inequalities will hold provided $\nu q \le 1$ and $r \le \nu$, which will make the left-hand sides in (3.5) negative. This is possible if

$$r \le \nu \le \frac{1}{q},$$

which can be obtained since $qr \le 1$. Thus $(tu_2, t^\nu v_2)$ is a supersolution of (1.1).

As we have mentioned, we want to use a sweeping argument. Since u_2 and v_2 are positive, and according to (3.4), we have $u_1 \le tu_2$ and $v_1 \le t^\nu v_2$ in Ω for large t. Define:

$$t_0 = \inf\{t > 1 : \ tu_2 > u_1, \ t^\nu v_2 > v_1 \text{ in } \Omega\}.$$

We claim that $t_0 = 1$, so that $u_2 \ge u_1$ and $v_2 \ge v_1$, and the proof is finished.

To prove the claim, assume that $t_0 > 1$. We have $t_0 u_2 \ge u_1$ and $t_0^\nu v_2 \ge v_1$ in Ω. Moreover,

$$-\Delta(t_0 u_2) + (t_0 u_2)^p \ge (t_0^\nu v_2)^q \ge v_1^q \ge -\Delta u_1 + u_1^p$$

so that the strong maximum principle implies that either $t_0 u_2 \equiv u_1$ in $\overline{\Omega}$ or $t_0 u_2 > u_1$ in Ω (recall that $p > 1$). Of course, the first option is ruled out by (3.4), since $t_0 > 1$, hence $t_0 u_2 > u_1$ in Ω.

Similarly $-\Delta(t_0^\nu v_2) + (t_0^\nu v_2)^s \ge -\Delta v_1 + v_1^s$, so that again $t_0^\nu v_2 > v_1$ in Ω. Using again (3.4), it follows that for small enough ε, $(t_0 - \varepsilon)u_2 > u_1$ and $(t_0 - \varepsilon)^\nu v_2 > v_1$, contradicting the minimality of t_0. Therefore our assumption that $t_0 > 1$ is false, and we deduce $t_0 = 1$, as we wanted to show. □

Proof of Theorem 1 (boundary behavior and uniqueness part). The proof of (1.4) follows in a similar way as that of (2.3) in Theorem 6. We begin by constructing a supersolution in a neighborhood of the boundary Ω_δ for small enough δ (such that $d \in C^2(\Omega_\delta)$ and $|\nabla d| = 1$ in Ω_δ). Choose $\theta \in (0, \delta)$ and let

$$\begin{aligned} \bar{u} &= A(d-\theta)^{-\alpha} + K \\ \bar{v} &= B(d-\theta)^{-\beta} + M \end{aligned}$$

where $K, M > 0$ are to be chosen. In order that $(\bar{u}, \bar{v})$ is a supersolution in $\theta < d < \delta$, we need the two inequalities:

$$\begin{aligned} A\alpha(\alpha+1)\ &(d-\theta)^{-\alpha-2} - A\alpha(d-\theta)^{-\alpha-1}\Delta d \\ &\le (A(d-\theta)^{-\alpha} + K)^p - (B(d-\theta)^{-\beta} + M)^q \\ &\text{and} \\ B\beta(\beta+1)\ &(d-\theta)^{-\beta-2} - B\beta(d-\theta)^{-\beta-1}\Delta d \\ &\le -(A(d-\theta)^{-\alpha} + K)^r + (B(d-\theta)^{-\beta} + M)^s. \end{aligned} \tag{3.6}$$

Since $p, s > 1$, we have $(a+b)^p \ge a^p + b^p$, $(a+b)^s \ge a^s + b^s$ for every $a, b > 0$. Moreover, for every $\varepsilon > 0$, there exists a constant $C = C(\varepsilon)$ such that $(a+b)^q \le (1+\varepsilon)a^q + Cb^q$, $(a+b)^r \le (1+\varepsilon)a^r + Cb^r$ for every $a, b > 0$. Therefore (3.6) is implied by

$$\begin{aligned} A\alpha(\alpha+1)\ &(d-\theta)^{-\alpha-2} - A\alpha(d-\theta)^{-\alpha-1}\Delta d \\ &\le A^p(d-\theta)^{-\alpha p} + K^p - (1+\varepsilon)B^q(d-\theta)^{-\beta q} - CM^q \\ &\quad \text{and} \\ B\beta(\beta+1)\ &(d-\theta)^{-\beta-2} - B\beta(d-\theta)^{-\beta-1}\Delta d \\ &\le -(1+\varepsilon)A^r(d-\theta)^{-\alpha r} - CK^r + B^s(d-\theta)^{-\beta s} + M^s. \end{aligned}$$

Taking into account that $\alpha+2=\alpha p>\beta q$, $\beta+2=\beta s>\alpha r$, these inequalities can be achieved for $\theta<d<\delta$ provided that

$$\begin{aligned}
&(A\alpha\ (\alpha+1)-A^p)-A\alpha\delta\sup\nolimits_{\Omega_{\delta_0}}|\Delta d|\\
&\qquad\le-(1+\varepsilon)B^q\delta^{\alpha p-\beta q}+(K^p-CM^q)(d-\theta)^{\alpha p}\\
&\qquad\text{and}\\
&(B\beta\ (\beta+1)-B^s)-B\beta\delta\sup\nolimits_{\Omega_{\delta_0}}|\Delta d|\\
&\qquad\le-(1+\varepsilon)A^r\delta^{\beta s-\alpha r}+(M^s-CK^r)(d-\theta)^{\beta s}.
\end{aligned}\tag{3.7}$$

If we fix $A>(\alpha(\alpha+1))^{\frac{1}{p-1}}$ and $B>(\beta(\beta+1))^{\frac{1}{s-1}}$, we can then choose δ small enough to ensure that (3.7) holds. It suffices to choose $K^p>CM^q$, $M^s>CK^r$, which is certainly possible for large K and M, since $ps>qr$. Thus $(\bar u,\bar v)$ is a supersolution of (1.1) in $\theta<d<\delta$, and we can choose large enough K and M so that $u\le\bar u$, $v\le\bar v$ on $d=\delta$.

Applying Lemma 7 in $\theta<d<\delta$, we see that $u\le\bar u$, $v\le\bar v$ there, so that letting $\theta\to0$, then $x\to\partial\Omega$, and finally $A\to(\alpha(\alpha+1))^{\frac{1}{p-1}}$, $B\to(\beta(\beta+1))^{\frac{1}{s-1}}$ we obtain

$$\limsup_{x\to\partial\Omega} d(x)^\alpha u(x)\le(\alpha(\alpha+1))^{\frac{1}{p-1}}$$
$$\limsup_{x\to\partial\Omega} d(x)^\beta v(x)\le(\beta(\beta+1))^{\frac{1}{s-1}}.$$

This establishes the upper bound in (1.4). The lower bound is immediate, since for every solution (u,v) we always have $\Delta u\le u^p$ in Ω, so by comparison $u\ge U$, and from Theorem 5 we get

$$\liminf_{x\to\partial\Omega} d(x)^\alpha u(x)\ge(\alpha(\alpha+1))^{\frac{1}{p-1}}.$$

A lower bound for v is obtained in the similar way; thus the proof of the boundary behavior of solutions when $qr\le1$ is finished.

The proof of uniqueness is immediate, since from (1.4) we have that every two pairs of positive solutions (u_1,v_1), (u_2,v_2) verify

$$\lim_{x\to\partial\Omega}\frac{u_1(x)}{u_2(x)}=\lim_{x\to\partial\Omega}\frac{v_1(x)}{v_2(x)}=1\tag{3.8}$$

Therefore we can apply Lemma 7 to obtain that $u_1=u_2$, $v_1=v_2$. This concludes the proof of Theorem 1. □

Proof of Theorem 2. The proof of this theorem runs along the same lines as the previous one, so we only remark the relevant differences. To begin with the existence, consider the unique solutions of the problems

$$\begin{cases}\Delta u=u^p-d(x)^{-\frac{2q}{s-1}} & \text{in }\Omega\\ u=+\infty & \text{on }\partial\Omega\end{cases}$$

and

$$\begin{cases}\Delta v=v^s-d(x)^{-\frac{2qr}{p(s-1)}} & \text{in }\Omega\\ v=+\infty & \text{on }\partial\Omega\end{cases}$$

(cf. Theorem 6), which will be denoted by $\bar{u}$ and $\bar{v}$, respectively. By our hypotheses, and owing to Theorem 6, there exist two positive constants K_1, K_2 such that

$$\begin{cases} K_1 d(x)^{-\frac{2q}{p(s-1)}} \le \bar{u}(x) \le K_2 d(x)^{-\frac{2q}{p(s-1)}} \\ K_1 d(x)^{-\frac{2}{s-1}} \le \bar{v}(x) \le K_2 d(x)^{-\frac{2}{s-1}}, \end{cases} \quad x \in \Omega. \tag{3.9}$$

We claim that, choosing η as in (3.3), the pair $(M\bar{u}, M^\eta \bar{v})$ is a supersolution of (1.1) if $M > 1$ is large enough. Indeed, it suffices to have

$$\begin{cases} M^{\eta q}\bar{v}^q - M d^{-\frac{2q}{s-1}} \le (M^p - M)\bar{u}^p \\ M^r \bar{u}^r - M^\eta d^{-\frac{2qr}{p(s-1)}} \le (M^{\eta s} - M^\eta)\bar{v}^s \end{cases} \quad \text{in } \Omega.$$

Using (3.9), these inequalities are implied by

$$\begin{cases} (M^{\eta q} K_2^q - M) \le K_1^p (M^p - M) \\ (M^r K_2^r - M^\eta) d^{-\frac{2qr}{p(s-1)}} \le K_1^s (M^{\eta s} - M^\eta) d^{-\frac{2s}{s-1}} \end{cases} \quad \text{in } \Omega.$$

Observing that $\frac{2s}{s-1} > \frac{2qr}{p(s-1)}$ and due to the choice of η, it follows that the previous inequality can be achieved in Ω choosing M large enough. Hence $(M\bar{u}, M^\eta \bar{v})$ is a supersolution of (1.1).

As before, a subsolution is given by (U, V). Moreover, using (3.1) and (3.9), we have

$$U(x) \le C_2 d(x)^{-\frac{2}{p-1}} \le C_2 K d(x)^{-\frac{2q}{p(s-1)}} \le \frac{C_2 K}{K_1}\bar{u} \le M\bar{u},$$

if M is taken large enough, where $K = (\sup_\Omega d)^{\frac{2q}{p(s-1)} - \frac{2}{p-1}}$. Similarly, $V \le M^\eta \bar{v}$ if M is large enough, so that we are in a position to apply the method of sub and supersolutions and obtain a positive solution (u, v) of problem (1.1) with $u = v = +\infty$ on $\partial\Omega$.

The boundary behavior (1.6) also follows by constructing suitable sub and supersolutions near the boundary of Ω. Again, we take them of the form $\bar{u} = A(d-\theta)^{-\gamma} + K$, $\bar{v} = B(d-\theta)^{-\beta} + M$, where as before $\beta = \frac{2}{s-1}$, but now $\gamma = \frac{2q}{p(s-1)}$. A simple computation shows that $(\bar{u}, \bar{v})$ will be a supersolution in $\theta < d < \delta$ if we have

$$\begin{aligned} A\gamma(\gamma+1) - A\gamma(d-\theta)\Delta d \le{}& (A^p - (1+\varepsilon)B^q)(d-\theta)^{\gamma+2-\gamma p} \\ &+ (K^p - CM^q)(d-\theta)^{\gamma+2} \end{aligned}$$

and

$$\begin{aligned} B\beta(\beta+1) - B\beta(d-\theta)\Delta d \le{}& B^s - (1+\varepsilon)A^r(d-\theta)^{\beta+2-\gamma r} \\ &+ (M^s - CK^r)(d-\theta)^{\beta+2}, \end{aligned}$$

where $C = C(\varepsilon) > 0$. First of all, we choose M and K verifying $K^p - CM^q, M^s - CK^r > 0$, which is possible by the condition $ps > qr$. Also, we always have $\beta + 2 - \gamma r > 0$, so that the second inequality above will be verified if we choose δ small enough and $B > (\beta(\beta+1))^{\frac{1}{s-1}}$. As for the first inequality, it depends on whether $\gamma+2-\gamma p = 0$ or $\gamma+2-\gamma p < 0$. In the first case, it suffices to take δ small and A verifying $A\gamma(\gamma+1) < A^p - (1+\varepsilon)B^q$, while in the second $A^p - (1+\varepsilon)B^q > 0$ would suffice. In either case we obtain a supersolution $(\bar{u}, \bar{v})$ and by using Lemma 7 we would arrive at $u \leq \bar{u}$ and $v \leq \bar{v}$ in $\theta < d < \delta$. Letting $\theta \to 0$, then $x \to \partial\Omega$, we would obtain

$$\limsup_{x \to \partial\Omega} d(x)^{\gamma} u(x) \leq A, \qquad \limsup_{x \to \partial\Omega} d(x)^{\beta} v(x) \leq B.$$

Letting $B \to (\beta(\beta+1))^{\frac{1}{s-1}}$ and then $A \to A_0$, we obtain the desired upper bound. The lower bound is shown in a completely similar way.

Finally, uniqueness is obtained exactly as in Theorem 1, since (1.6) implies (3.8) for every two pairs of positive solutions (u_1, v_1), (u_2, v_2), and Lemma 7 can be used in the same way. □

4. Removability results.

In this section, we will consider the removability results stated in the Introduction. Our proofs are based on the classical paper [6], together with the use of the boundary blow-up solutions of system (1.1) to obtain suitable upper bounds for the singularities. Since the proof of both Theorem 3 and Theorem 4 are very similar, we only give the first one.

We begin with the proof of the upper bounds.

Lemma 8. *Assume $p, s > 1$ and $qr \leq 1$. Let (u, v) be a positive classical solution of*

$$\begin{cases} \Delta u = u^p - v^q \\ \Delta v = -u^r + v^s \end{cases} \quad \text{in } B \setminus \{0\}.$$

Then:

(a) *If $\frac{p}{p-1} > \frac{q}{s-1}$ and $\frac{s}{s-1} > \frac{r}{p-1}$, then there exist positive constants C_1 and C_2 such that:*

$$u(x) \leq C_1 |x|^{-\frac{2}{p-1}}, \qquad v(x) \leq C_2 |x|^{-\frac{2}{s-1}}, \quad x \in B_{1/2} \setminus \{0\}.$$

(b) *If $\frac{p}{p-1} \leq \frac{q}{s-1}$ and $\frac{s}{s-1} > \frac{r}{p-1}$, then there exist positive constants C_1 and C_2 such that:*

$$u(x) \leq C_1 |x|^{-\frac{2q}{p(s-1)}}, \qquad v(x) \leq C_2 |x|^{-\frac{2}{s-1}}, \quad x \in B_{1/2} \setminus \{0\}.$$

Proof. We only show part (a), since part (b) can be accomplished in a completely similar way. Choose x with $0 < |x| < \frac{1}{2}$ and let

$$\begin{array}{l} z(y) = |x|^{\alpha} u(x + |x| y) \\ w(y) = |x|^{\beta} v(x + |x| y) \end{array} \qquad y \in B_{1/2},$$

where $\alpha = \frac{2}{p-1}$ and $\beta = \frac{2}{s-1}$. It is easy to see that

$$\begin{cases} \Delta z = z^p - |x|^{\alpha p - \beta q} w^q \geq z^p - w^q \\ \Delta w = -|x|^{\beta s - \alpha r} z^r + w^s \geq -z^r + w^s \end{cases} \quad \text{in } B_{1/2},$$

since $\alpha p > \beta q$ and $\beta s > \alpha r$, according to our hypotheses. Let (Z, W) be the unique solution of the problem

$$\begin{cases} \Delta z = z^p - w^q & \\ \Delta w = -z^r + w^s & \text{in } B_{1/2} \\ z = w = +\infty & \text{on } \partial B_{1/2} \end{cases}$$

given by Theorem 1. Since $qr \le 1$, we can compare as in Lemma 7 to obtain that actually $z \le Z$, $w \le W$ in $B_{1/2}$. In particular, setting $y = 0$, we have $u(x) \le Z(0)|x|^{-\alpha}$, $v(x) \le W(0)|x|^{-\beta}$, as was to be shown. $\square$

The next lemma is a slight generalization of the results in [6], to deal with right-hand sides which are not bounded. From now on, we will use the letter C to denote positive constants, which may vary from line to line.

Lemma 9. *Assume $u \in C^2(B \setminus \{0\})$ is positive and verifies*

$$-\Delta u + u^p \le C_0|x|^{-\omega} \qquad \text{in } B \setminus \{0\}, \tag{4.1}$$

where $p \ge \frac{N}{N-2}$, $C_0 > 0$ and $\omega < N$. Suppose in addition that $u(x) \le C|x|^{2-N}$ in $B_{1/2} \setminus \{0\}$. Then,

$$u(x) \le C|x|^{2-\omega} \qquad \text{in } B_{1/2} \setminus \{0\},$$

if $\omega > 2$,

$$u(x) \le C(1 - |x|^{2-\omega}) \qquad \text{in } B_{1/2} \setminus \{0\},$$

if $\omega < 2$, while

$$u(x) \le -C \log |x| \qquad \text{in } B_{1/2} \setminus \{0\}.$$

for $\omega = 2$.

In particular, $u \in L^\infty_{\text{loc}}(B)$ provided that $\omega < 2$.

Proof. We only prove the case $\omega \ne 2$ (minor adjustments are needed when $\omega = 2$). We will show first that $u \in L^p_{\text{loc}}(B)$. For this aim, choose a family of functions $\xi_n \in C^\infty(\mathbb{R}^N)$ with the properties that $\xi_n(x) = 0$ for $|x| < \frac{1}{2n}$, $\xi(x) = 1$ if $|x| > \frac{1}{n}$ and $0 \le \xi_n \le 1$. Then there exists a positive constant C such that $|\nabla \xi_n| \le Cn$, $|\Delta \xi_n| \le Cn^2$.

Taking $\varphi \in C_0^\infty(B)$ with $\varphi \ge 0$ and multiplying the equation in (4.1) by $\varphi \xi_n$ we obtain, after integration by parts:

$$\int u^p \varphi \xi_n \le C_0 \int |x|^{-\omega} \varphi + \int u \Delta(\xi_n \varphi). \tag{4.2}$$

For the last integral, we have the inequality:

$$\left| \int u \Delta(\xi_n \varphi) \right| \le C \int_{\{\frac{1}{2n} < |x| < \frac{1}{n}\}} n^2 u + C \int_{\{\frac{1}{2n} < |x| < \frac{1}{n}\}} nu + C \int_{\text{supp } \varphi} u.$$

Using that $u \le C|x|^{2-N}$, we see that all the above integrals are bounded by a positive constant independent of n. Taking into account that $\omega < N$ implies $|x|^{-\omega} \in L^1(B)$, we have from (4.2)

$$\int u^p \varphi \xi_n \le C,$$

where C does not depend on n. Letting $n \to +\infty$ we see that $u \in L^p_{\text{loc}}(B)$.

Now, let $w = K|x|^{2-\omega}$ if $\omega > 2$, $w = K(1 - |x|^{2-\omega})$ if $\omega < 2$. Since $\omega < N$, it is not hard to check that in either case w verifies

$$-\Delta w \ge C_0|x|^{-\omega} \qquad \text{in } B \setminus \{0\},$$

if K is chosen large enough. Enlarging K if necessary we can also assume $u \le w$ on $\partial B_{1/2}$.

Next, we use Lemma 1 in [6] to obtain

$$\Delta(u-w)^+ \geq \Delta(u-w)\,\mathrm{sign}^+(u-w) \geq 0$$

in $\mathcal{D}'(B \setminus \{0\})$, where

$$\mathrm{sign}^+ t = \begin{cases} 1 & t > 0 \\ \dfrac{1}{2} & t = 0 \\ 0 & t < 0. \end{cases}$$

Since $p \geq \frac{N}{N-2}$ and $u \in L^p_{\mathrm{loc}}(B)$, we also have $u - w \in L^{\frac{N}{N-2}}_{\mathrm{loc}}(B)$. Thus we can apply Lemma 2 in [6] to obtain that u verifies $\Delta(u-w)^+ \geq 0$ in $\mathcal{D}'(B)$. Hence, by the maximum principle, $(u-w)^+ \equiv 0$ in $B_{1/2}$, which completes the proof. □

We finally consider the proof of our main result concerning removability of singularities.

Proof of Theorem 3. By Lemma 8, we have $u(x) \leq C|x|^{-\alpha}$, $v(x) \leq C|x|^{-\beta}$ in $0 < |x| < 1/2$, for some positive constant C, where $\alpha = \frac{2}{p-1}$, $\beta = \frac{2}{s-1}$.

Then, using the first equation in (1.7), we have

$$-\Delta u + u^p = v^q \leq C|x|^{-\beta q} \qquad \text{in } B_{1/2} \setminus \{0\}.$$

Obverve that $p \geq \frac{N}{N-2}$ implies on one hand that $\alpha \leq N-2$ therefore $u(x) \leq C|x|^{2-N}$ in $B_{1/2} \setminus \{0\}$ and on the other hand $\beta q = \frac{2q}{s-1} < \frac{2p}{p-1} \leq N$. Therefore we can apply Lemma 9 to obtain that

$$u(x) \leq \begin{cases} C|x|^{2-\beta q} & \beta q > 2 \\ -C \log|x| & \beta q = 2, \end{cases}$$

while $u \in L^\infty_{\mathrm{loc}}(B)$ if $\beta q < 2$. The latter case amounts to $q < s-1$ and just proves part (b). In this regard, part (c) is the symmetric statement, thus it follows in the same way.

To prove part (a) we may assume first that $\beta q \geq 2$. Since $s \geq \frac{N}{N-2}$, we can use a similar argument as above in the equation involving v to obtain that

$$v(x) \leq \begin{cases} C|x|^{2-\alpha r} & \alpha r > 2 \\ -C \log|x| & \alpha r = 2, \end{cases}$$

being $v \in L^\infty_{\mathrm{loc}}(B)$ if $\alpha r < 2$. Therefore, using the first equation in (1.7) we deduce that $-\Delta v + v^p \leq C$ in $B \setminus \{0\}$, and we obtain from Theorem 1 in [6] that $u \in L^\infty_{\mathrm{loc}}(B)$. Now it is standard to conclude: since $v \in L^\infty(B_{1/2})$, the problem

$$\begin{cases} -\Delta z + z^p = v^q & \text{in } B_{1/2} \\ \quad z = u & \text{on } \partial B_{1/2} \end{cases}$$

admits a unique solution $z \in C^{1,\alpha}(\overline{B_{1/2}})$. Arguing as in Lemma 9, since u solves the same equation in the sense of $\mathcal{D}'(B_{1/2})$, we obtain that $u \equiv z$. Therefore $u \in C^1(B)$. The same argument shows that $v \in C^1(B)$ and by bootstrapping we obtain $u, v \in C^\infty(B)$. Observe that when $\beta q < 2$, so that $u \in L^\infty_{\mathrm{loc}}(B)$, we deduce in the same way that $u, v \in C^\infty(B)$.

Thus only the case $\beta q \geq 2$, $\alpha r \geq 2$ remains to be proved. Set $\alpha_1 = \beta q - 2$, $\beta_1 = \alpha r - 2$, and suppose that, say, $\alpha_1 = \min\{\alpha_1, \beta_1\} \leq 2$. If $\alpha_1 < 2$ and by using Lemma 9 we obtain that $u \in L^\infty_{\rm loc}(B)$. The previous argument then shows that $u, v \in C^\infty(B)$. If $\alpha_1 = 2$, Lemma 9 implies that

$$u(x) \leq -C \ln|x| \leq C|x|^{-\delta} \qquad x \in B_{1/2},$$

for some $\delta < 2$. By the same token we achieve again $u, v \in C^\infty(B)$.

If $\min\{\alpha_1, \beta_1\} > 2$ then $u(x) \leq C|x|^{-\alpha_1}$, $v(x) \leq C|x|^{-\beta_1}$ in $B_{1/2}$. By iteration we construct sequences $\{\alpha_k\}$ and $\{\beta_k\}$ given by

$$\begin{cases} \alpha_k = \beta_{k-1} q - 2 \\ \beta_k = \alpha_{k-1} r - 2, \end{cases}$$

so that $u(x) \leq C|x|^{-\alpha_k}$, $v(x) \leq C|x|^{-\beta_k}$ in $B_{1/2}$ provided that $\min\{\alpha_l, \beta_l\} > 2$ for $l = 1, \ldots, k$. However, it is easily seen that $\alpha_k, \beta_k \to -\infty$ as $k \to +\infty$ when $qr = 1$, while $\alpha_k \to -\frac{2(q+1)}{1-qr}$, $\beta_k \to -\frac{2(r+1)}{1-qr}$ as $k \to +\infty$ when $qr < 1$. In either case, $\min\{\alpha_k, \beta_k\}$ becomes eventually less than 2 after finitely many steps. The proof is concluded. $\square$

Remark 1. The results contained in Theorem 3 are optimal. In all remaining cases a solution of (1.7) which is singular at zero can be constructed. For instance, when $p, s < \frac{N}{N-2}$, the solution is easily found by considering as a subsolution the pair $(\varepsilon|x|^{-\alpha}, \delta|x|^{-\beta})$ for small ε and δ, and as a supersolution $(A|x|^{-\alpha}, B|x|^{-\beta})$, for large A and B.

When $q > s - 1$, the exponent α above has to be replaced by $\theta = \beta q - 2$, while for $q = s - 1$, the power in the first component of both the sub and the supersolution has to be substituted by $-\log|x|$.

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