THE BEST SOBOLEV TRACE CONSTANT IN A DOMAIN
WITH OSCILLATING BOUNDARY

JULIÁN FERNÁNDEZ BONDER, RAFAEL ORIVE AND JULIO D. ROSSI

Abstract. In this paper we study homogenization problems for the best constant in the Sobolev trace embedding $W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega)$ in a bounded smooth domain when the boundary is perturbed by adding an oscillation. We find that there exists a critical size of the amplitude of the oscillations for which the limit problem has a weight on the boundary. For sizes larger than critical the best trace constant goes to zero and for sizes smaller than critical it converges to the best constant in the domain without perturbations.

1. Introduction.

In this paper we consider homogenization problems for the best Sobolev trace constant when a periodic oscillation is added on the boundary.

Sobolev inequalities have been studied by many authors and is by now a classical subject. Relevant for the study of boundary value problems for differential operators is the Sobolev trace inequality that has been intensively studied, see for example, [3, 8, 9, 12] and references therein.

Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$, we deal with the best constant of the Sobolev trace embedding $W^{1,p}(\Omega_\varepsilon) \hookrightarrow L^q(\partial \Omega_\varepsilon)$ where $\Omega_\varepsilon$ is given by adding an oscillating perturbation to the boundary of a fixed domain, $\Omega$. When $q = p$ this leads to an eigenvalue problem of the Steklov type, [19].

To be more concrete, let us describe the domains, $\Omega_\varepsilon$. For simplicity, we consider only perturbations in a region of the boundary $\partial \Omega$ but is clear that the same kind of analysis can be done if the boundary is perturbed everywhere. First, we identify the region of the boundary of $\Omega \subset \mathbb{R}^N$ where the perturbation is localized. We assume that there exits a smooth function $\Phi : U' \subset \mathbb{R}^{N-1} \rightarrow \mathbb{R}$, where $U'$ is a connected open neighborhood such that parameterizes a region $\Gamma_1$ of $\partial \Omega$.

\[
\{(x_1, x') \in \mathbb{R}^N \mid x' \in U', x_1 = \Phi(x')\} = \Gamma_1 \subset \partial \Omega.
\]

Without loss of generality, we can assume that $\Omega$ is under $\Gamma_1$. We consider a connected tube open neighborhood $U = (\delta_1, \delta_2) \times U' \subset \mathbb{R}^N$ such that $U \cap \partial \Omega = \Gamma_1$ and

\[
\Omega \cap U = \{(x_1, x') \in U \mid x' \in U', x_1 < \Phi(x')\}.
\]

Now, let $f : \mathbb{R}^{N-1} \rightarrow \mathbb{R}$ be a smooth ($C^1$ is enough) periodic function with period $Y' := [0, 1]^{N-1}$ with $f(0) = 0$. We denote the translate cells as $\varepsilon Y'' = \varepsilon n + \varepsilon Y'$.
with \( n \in \mathbb{Z}^{N-1} \). We then define the perturbed domain \( \Omega_\varepsilon \) as follows:

\[
\Omega_\varepsilon \cap U := \{(x_1, x') \in U \mid x' \in U'_\varepsilon, \ x_1 < \Phi(x') + \varepsilon^a f(x'/\varepsilon)\chi_{U'_\varepsilon}(x')\},
\]

where \( U'_\varepsilon = \bigcup \{\varepsilon Y_n' \mid \text{such that } \varepsilon Y_n' \subset U', \ n \in \mathbb{Z}^N\} \),

\[
\Gamma^*_\varepsilon = \{(x_1, x') \in \mathbb{R}^N \mid x' \in U', \ x_1 = \Phi(x') + \varepsilon^a f(x'/\varepsilon)\chi_{U'_\varepsilon}(x')\},
\]

\[
\Omega_\varepsilon \cap U^c := \Omega \cap U^c,
\]

where \( \chi_{U'_\varepsilon} \) is the characteristic function of \( U'_\varepsilon \). Therefore, we are considering oscillations of period \( \varepsilon \) with size \( \varepsilon^a \). Let us remark that the oscillations are not located at the boundary of \( U' \).

For any \( 1 < p < \infty \) and for every subcritical exponent,

\[
1 \leq q < p_* := \frac{p(N-1)}{(N-p)_+},
\]

we consider the Sobolev trace inequality, \( S(\varepsilon) \|v\|_{L^p(\partial\Omega_\varepsilon)} \leq \|v\|_{W^{1,p}(\Omega_\varepsilon)} \), for all \( v \in W^{1,p}(\Omega_\varepsilon) \). The best Sobolev trace constant is the largest \( S(\varepsilon) \) such that the above inequality holds,

\[
S(\varepsilon) := \inf_{v \in W^{1,p}(\Omega_\varepsilon) \setminus W^{1,p}_0(\Omega_\varepsilon)} \frac{\int_{\Omega_\varepsilon} |\nabla v|^p + |v|^p \, dx}{\left( \int_{\partial\Omega_\varepsilon} |v|^q \, dS \right)^{p/q}}.
\]

For subcritical exponents, \( 1 \leq q < p_* \), the embedding \( W^{1,p}(\Omega_\varepsilon) \hookrightarrow L^q(\partial\Omega_\varepsilon) \) is compact, so we have existence of extremals, i.e., functions where the infimum is attained. These extremals are strictly positive in \( \Omega_\varepsilon \) (see [11]) and \( C^{1,\alpha}_{\text{loc}}(\Omega) \cap C^{\alpha}(\overline{\Omega}) \) (see [20, 16]). When one normalize the extremals with

\[
\int_{\partial\Omega_\varepsilon} |u_\varepsilon|^q \, dS = 1,
\]

they are weak solutions of the following problem

\[
\begin{aligned}
\Delta_p u_\varepsilon &= |u_\varepsilon|^{p-2} u_\varepsilon & \quad & \text{in } \Omega_\varepsilon, \\
|\nabla u_\varepsilon|^{p-2} \frac{\partial u_\varepsilon}{\partial \nu_\varepsilon} &= S(\varepsilon) |u_\varepsilon|^{p-2} u_\varepsilon & \quad & \text{on } \partial\Omega_\varepsilon,
\end{aligned}
\]

where \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \) is the usual \( p \)-laplacian operator and \( \nu_\varepsilon \) is the unit outward normal vector, see [11]. In the rest of this article we will assume that the extremals are normalized according to (1.2) and hence solutions of (1.3). Note that when \( p = 2 \) the equation becomes linear, even in this case our results are new. Of special importance is the case \( q = p \). In this case (1.3) is an eigenvalue problem of Steklov type, see [11, 13, 17, 19], etc.

Our concern in this article is the study of the limit of \( S(\varepsilon) \) and of the corresponding extremals as \( \varepsilon \) goes to zero. We find that there is a critical size of the amplitudes for the oscillations such that the extremals for this embedding converges as the oscillations go to infinity to a solution of an homogenized limit problem and the best trace constant converges to a homogenized best trace constant. For amplitudes larger than the critical one, the size of the boundary becomes too large and so the Sobolev trace constant goes to zero. For amplitudes smaller that the critical
one, the perturbation is too small, so the Sobolev trace constant converges to the one of the unperturbed domain.

The precise statement of our result is as follows:

**Theorem 1.** Let $S(\epsilon)$ be the best Sobolev trace constant given by (1.1).

1. If $a < 1$, then $S(\epsilon)$ goes to zero as $\epsilon \to 0$. Moreover, it holds

\[
S(\epsilon) \leq C\epsilon^{(1-a)p/q} \to 0 \quad \text{as } \epsilon \to 0.
\]

2. If $a > 1$, then $S(\epsilon)$ converges as $\epsilon \to 0$ to $S(0)$ the best Sobolev trace constant of the unperturbed domain $\Omega$. The corresponding normalized extremals, rescaled to $\Omega$, converge (along subsequences) strongly in $W^{1,p}(\Omega)$ to an extremal of the unperturbed domain.

3. If $a = 1$, then $S(\epsilon)$ converges as $\epsilon \to 0$ to $S^*$ the best Sobolev trace constant of the original domain with a weight on the boundary,

\[
S^* = \inf_{v \in W^{1,p}(\Omega) \setminus W^{1,p}_0(\Omega)} \frac{\int_\Omega |\nabla v|^p + v^p \, dx}{\left(\int_{\partial\Omega} m(x)|v|^q \, dS\right)^{p/q}},
\]

where

\[
m(x) := \begin{cases} 
\int_Y \frac{\sqrt{1 + |\nabla \Phi(x') + \nabla f(y)|^2} \, dy}{\sqrt{1 + |\nabla \Phi(x')|^2}} & \text{for } x \in \partial \Omega \cap U, \\
1 & \text{elsewhere}
\end{cases}
\]

Moreover, the normalized extremals (rescaled to $\Omega$ in a suitable way) converge (along subsequences) weakly in $W^{1,p}$ to an extremal of (1.5).

Let us end the introduction with some bibliographical discussion. The interest in problems with oscillating boundary appears in the influence of micro-structures of surfaces (porous medium, composites, micro-materials) over the large scale behavior. The mathematical analysis of problems with oscillating boundary was presented in [18].

Now, we describe briefly some related results for problems with oscillating boundary for a second order elliptic equation. In [4] and [15], the asymptotic behavior of solutions to the Neumann boundary value problem with respect to the oscillating boundary shows a limiting macrostructure. In [1] the authors study the behavior of the Laplace equation in as oscillating domain imposing non-homogeneous Dirichlet boundary conditions on the oscillating part of the boundary. In [5] and [14] a rapidly oscillating boundary with unlimited growth and inhomogeneous Fourier boundary condition is studied. The limiting problem can involve Dirichlet, Fourier or Neumann boundary conditions depending on the structure. There exists references that deal with quasilinear operators and oscillating boundaries, see [2] and [7]. On the other hand, the homogenization problem for the best Sobolev trace constant with a fixed domain for periodic media or in domains with holes was recently studied in [10].
2. Proofs of the results

2.1. Subcritical case. $a < 1$. This is the easiest case. The result follows just by taking $v \equiv 1$ as a test in the variational characterization of $S(\varepsilon)$, (1.1). In fact what we get is the following inequality

$$S(\varepsilon) \leq \frac{\left| \Omega_{\varepsilon} \right|}{\left| \partial \Omega_{\varepsilon} \right|^{p/q}}.$$ 

It is clear that

$$\lim_{\varepsilon \to 0} \left| \Omega_{\varepsilon} \right| = \left| \Omega \right|.$$

Let us estimate $\left| \partial \Omega_{\varepsilon} \right|$. We have

$$\left| \partial \Omega_{\varepsilon} \right| \geq \left| \partial \Omega_{\varepsilon} \cap U \right| \geq \int_{U_{\varepsilon}} \sqrt{1 + |\nabla \Phi(x') + \varepsilon^{a-1} \nabla f(x'/\varepsilon)|^2} \, dx' = \varepsilon^{a-1} \int_{U_{\varepsilon}} \sqrt{\varepsilon^{2(1-a)} + |\varepsilon^{1-a} \nabla \Phi(x') + \nabla f(x'/\varepsilon)|^2} \, dx'.$$

As $a < 1$, it is easy to see that

$$\int_{U_{\varepsilon}} \sqrt{\varepsilon^{2(1-a)} + |\varepsilon^{1-a} \nabla \Phi(x') + \nabla f(x'/\varepsilon)|^2} \, dx' \to m(|\nabla f|) := \int_{Y'} |\nabla f(y')| \, dy' > 0.$$

Hence, $|\partial \Omega_{\varepsilon}| \geq c \varepsilon^{a-1}$. From where it follows that

$$S(\varepsilon) \leq C \varepsilon^{(1-a)p/q} \to 0 \quad \text{as} \quad \varepsilon \to 0,$$

as we wanted to show.

Remark 2.1. It is clear from the previous proof that the constant $C$ in (1.4) can be any constant larger than $|\Omega|/(m(|\nabla f|))^{p/q}$.

2.2. Supercritical case. $a > 1$. Let us introduce the following notation, for any $u \in W^{1,p}(\Omega_{\varepsilon})$ and $v \in W^{1,p}(\Omega)$, we denote

$$Q_{\varepsilon}(u) := \frac{\int_{\Omega_{\varepsilon}} |\nabla u|^p + u^p \, dx}{\left( \int_{\partial \Omega_{\varepsilon}} |u|^q \, dS \right)^{p/q}} \quad \text{and} \quad Q_{0}(v) := \frac{\int_{\Omega} |\nabla v|^p + v^p \, dx}{\left( \int_{\partial \Omega} |v|^q \, dS \right)^{p/q}}.$$

To transform integrals in $\Omega_{\varepsilon}$ into integrals in $\Omega$, let us perform the following change of variables

$$x' = x', \quad \bar{x}_1 = x_1 - \varepsilon^a f(x'/\varepsilon) \varphi(x_1, x'),$$

where $\varphi$ is a smooth cut-off function with bounded derivatives that vanishes outside $U$. For every $u \in C^4(\bar{\Omega}_{\varepsilon})$ consider

$$u(x_1, x') = v(\bar{x}_1, \bar{x}').$$

We obtain that $v \in C^4(\bar{\Omega})$. In order to change variables in $Q_{\varepsilon}(u)$, let us compute the jacobian of the change of variables

$$J^{-1} = 1 - \varepsilon^a f(x'/\varepsilon) \varphi_{x_1}.$$

The derivatives of $v$ and $u$ are related by

$$u_{x_1} = v_{x_1} (1 - \varepsilon^a f(x'/\varepsilon) \varphi_{x_1})$$
and
\[ \nabla_{x'} u = -v \varepsilon \varphi + \varepsilon f(x'/\varepsilon) \nabla_{x'} \varphi + \nabla_{x'} v. \]

Therefore, as \( \varphi \) and \( f \) have bounded derivatives,
\[
\int_{\Omega} |\nabla u|^p \, dx = (1 + O(\varepsilon^{a-1})) \int_{\Omega} |\nabla v|^p \, d\bar{x},
\]
\[
\int_{\Omega} u^p \, dx = (1 + O(\varepsilon^a)) \int_{\Omega} |v|^p \, d\bar{x},
\]
and
\[
\int_{\partial \Omega} |u|^q \, dS = (1 + O(\varepsilon^{a-1})) \int_{\partial \Omega} |v|^q \, dS.
\]

Hence, we obtain, as \( a > 1 \),
\[
Q(\varepsilon) = Q_0(v) + \delta \varepsilon, \quad \text{with } \delta \varepsilon \to 0, \quad \text{as } \varepsilon \to 0.
\]

As \( \varphi \) and \( f \) have bounded derivatives, it can be checked that \( \delta \varepsilon \) can be chosen uniformly on bounded sets of \( W^{1,p}(\Omega) \). Then,
\[
Q(\varepsilon) \geq S(0) + \delta \varepsilon.
\]

Now let \( u \) be an extremal for (1.1), that is a minimizer of \( Q_\varepsilon \), normalized by (1.2). Taking \( u \equiv 1 \) in (1.1) we get
\[
\|u\|_{W^{1,p}(\Omega)}^p = \int_{\Omega} |\nabla u|^p + u_p^p \, dx \leq \frac{\|\Omega\|}{[\partial \Omega]^{p/q}} \leq C.
\]

Hence when we change variables we get that \( v \) is bounded in \( W^{1,p}(\Omega) \) independently of \( \varepsilon \), from where it follows that
\[
\liminf_{\varepsilon \to 0} S(\varepsilon) \geq S(0).
\]

To obtain the upper bound, given \( \rho > 0 \) we take a \( C^1(\Omega) \) function \( v \) such that
\[
Q_0(v) \leq S(0) + \rho,
\]

hence, from (2.1) we obtain
\[
S(\varepsilon) \leq Q_\varepsilon(u) = Q_0(v) + \delta \varepsilon \leq S(0) + \rho + \delta \varepsilon.
\]

Therefore
\[
\limsup_{\varepsilon \to 0} S(\varepsilon) \leq S(0) + \rho.
\]

As this inequality holds for every \( \rho > 0 \) we get
\[
\limsup_{\varepsilon \to 0} S(\varepsilon) \leq S(0).
\]

Combining (2.2) and (2.3) we conclude
\[
\lim_{\varepsilon \to 0} S(\varepsilon) = S(0).
\]

Now we deal with the convergence of the extremals. Let \( u \) be an extremal for \( Q_\varepsilon \). From our previous arguments we have that the rescaled functions \( v \) are bounded in \( W^{1,p}(\Omega) \). Therefore we can extract a subsequence (that we still call \( v \)) such that \( v \to v \) weakly in \( W^{1,p}(\Omega) \). We have
\[
1 = \int_{\partial \Omega} |v|^q \, dS = (1 + O(\varepsilon^{a-1})) \int_{\partial \Omega} |v|^q \, dS.
\]

SOBOLEV TRACE CONSTANT

and
\[ \nabla_{x'} u = -v \varepsilon \varphi + \varepsilon f(x'/\varepsilon) \nabla_{x'} \varphi + \nabla_{x'} v. \]
Hence, by the compactness of the embedding \( W^{1,p}(\Omega) \hookrightarrow L^q(\partial \Omega) \),
\[
\int_{\partial \Omega} |v|^q \, dS = 1.
\]
Moreover,
\[
\|v\|_{W^{1,p}(\Omega)}^p \leq \liminf_{\varepsilon \to 0} \|v_\varepsilon\|_{W^{1,p}(\Omega)}^p = S(0) \leq \|v\|_{W^{1,p}(\Omega)}^p.
\]
Therefore,
\[
\lim_{\varepsilon \to 0} \|v_\varepsilon\|_{W^{1,p}(\Omega)}^p = \|v\|_{W^{1,p}(\Omega)}^p,
\]
and we conclude that the sequence \( v_\varepsilon \) converges strongly to \( v \), an extremal of \( S(0) \).

2.3. Critical case. \( \alpha = 1 \). Since our perturbations are of size \( \varepsilon \) the perturbations of \( \Omega \) are contained in a small neighborhood of the perturbed portion of the boundary. In fact, let
\[
A_\varepsilon = \{ x \in U \cap \Omega : \text{dist}(x, \partial \Omega) < \varepsilon \}.
\]
Observe that \( |A_\varepsilon| \sim \varepsilon \).

Now, we observe that, as before, taking \( u \equiv 1 \) in (1.1) we get that \( S(\varepsilon) \) is bounded independently of \( \varepsilon \). Thus, as before, the \( W^{1,p}(\Omega_\varepsilon) \) norm of the normalized extremals \( u_\varepsilon \) bounded independently of \( \varepsilon \).

The key point to handle this case is to perform a change of variables like the one that is used in the supercritical case, but now with a cut-off function \( \varphi_\varepsilon \) depending on \( \varepsilon \) such that \( \varphi_\varepsilon \equiv 1 \) on \( \partial \Omega_\varepsilon \cap A_\varepsilon \). Let
\[
\bar{x}' = x', \quad \bar{x}_1 = x_1 - \varepsilon f(x'/\varepsilon) \varphi_\varepsilon(x_1, x'),
\]
where \( \varphi_\varepsilon \) is a smooth cut-off function supported in \( A_\varepsilon \). For \( u \in C^1(\overline{\Omega}_\varepsilon) \) consider
\[
u(x_1, x') = v(\bar{x}_1, \bar{x}').
\]
The derivatives of \( v \) and \( u \) are related by
\[
u_{x_1} = v_{x_1} (1 - \varepsilon f(x'/\varepsilon)(\varphi_\varepsilon)_{x_1})
\]
and
\[
\nabla_{x'} u = -v_{x_1} (\nabla_{x'} f(x'/\varepsilon) \varphi_\varepsilon + \varepsilon f(x'/\varepsilon) \nabla_{x'} \varphi_\varepsilon) + \nabla u \cdot v.
\]
We obtain that \( v \in C^1(\overline{\Omega}) \). Moreover, the \( W^{1,p}(\Omega) \) norm of the rescaled extremals \( v_\varepsilon \) is bounded independently of \( \varepsilon \). Hence we may assume, taking a subsequence if necessary, that \( v_\varepsilon \to v \) weakly in \( W^{1,p}(\Omega) \).

Since the derivatives of \( \varphi_\varepsilon \) are bounded by \( C/\varepsilon \), the jacobian of the change of variables verifies \( J^{-1} = 1 \) in \( \Omega \setminus A_\varepsilon \) and \( J^{-1} \leq C \) in \( A_\varepsilon \). Therefore, as \( f \) has bounded derivatives, the derivatives of \( \varphi_\varepsilon \) are bounded by \( C/\varepsilon \) and the measure of \( A_\varepsilon \) is of order \( \varepsilon \),
\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \theta \, dx = \int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \theta \, d\bar{x}
\]
and
\[
\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |u_\varepsilon|^{p-2} u_\varepsilon \theta \, dx = \int_{\Omega} |v|^{p-2} v \theta \, d\bar{x}.
\]
For the boundary term we have
\[
\int_{\partial\Omega_\varepsilon \cap U} |u_\varepsilon|^{q-2} u_\varepsilon \theta \, dS = \int_{U \setminus U_\varepsilon} |v_\varepsilon|^{q-2} v_\varepsilon \theta \sqrt{1 + |\nabla \Phi(x')|^2} \, dx' + \int_{U_\varepsilon} |v_\varepsilon|^{q-2} v_\varepsilon \theta \sqrt{1 + |\nabla \Phi(x') + \nabla f(\frac{x'}{\varepsilon})|^2} \, dx'.
\]

When \( \varepsilon \to 0 \), we get that \( U_\varepsilon' \to U' \) and
\[
\lim_{\varepsilon \to 0} \int_{\partial\Omega_\varepsilon \cap U'} |u_\varepsilon|^{q-2} u_\varepsilon \theta \, dS = \int_{U'} |v|^{q-2} v \theta m(x') \sqrt{1 + |\nabla \Phi(x')|^2} \, dx',
\]
considered \( m \) defined in (1.6). Hence, we get
\[
(2.6) \quad \int_{\partial\Omega_{\varepsilon}} |u_\varepsilon|^{q-2} u_\varepsilon \theta \, dS \to \int_{\partial\Omega} |v|^{q-2} v \theta m(x) \, dS.
\]

Since the extremals \( u_\varepsilon \) are solutions to (1.3), they satisfy, for every \( \theta \in C^\infty(\mathbb{R}^N) \)
\[
(2.7) \quad \int_{\Omega_{\varepsilon}} |\nabla u_\varepsilon|^{p-2} \nabla u_\varepsilon \nabla \theta \, dx + \int_{\Omega_{\varepsilon}} |u_\varepsilon|^{p-2} u_\varepsilon \theta \, dx = S(\varepsilon) \int_{\partial\Omega_{\varepsilon}} |u_\varepsilon|^{q-2} u_\varepsilon \theta \, dS.
\]

Using (2.4), (2.5) and (2.6) we obtain that a weak limit of the sequence \( v_\varepsilon \) in \( W^{1,p}(\Omega) \) satisfies
\[
\int_{\Omega} |\nabla v|^{p-2} \nabla v \nabla \theta \, dx + \int_{\Omega} |v|^{p-2} v \theta \, dx = \bar{S} \int_{\partial\Omega} |v|^{q-2} v \theta \, dS.
\]

That is to say that \( v \) is a weak solution of
\[
\begin{cases}
\Delta_p v = |v|^{p-2} v & \text{in } \Omega, \\
|\nabla v|^{p-2} \frac{\partial v}{\partial \nu} = S(m(x)|v|^{q-2} v & \text{on } \partial\Omega.
\end{cases}
\]

Therefore
\[
\int_{\Omega} |\nabla v|^p + v^p \, dx = \bar{S} \int_{\partial\Omega} m(x)|v|^q \, dS.
\]

Moreover, from our previous calculations in (2.6), we have
\[
1 = \lim_{\varepsilon \to 0} \int_{\partial\Omega_{\varepsilon}} |u_\varepsilon|^q \, dS = \int_{\partial\Omega} m(x)|v|^q \, dS.
\]

Now, for every \( w \in W^{1,p}(\Omega) \) we can define \( w_\varepsilon \in W^{1,p}(\Omega_{\varepsilon}) \) by the change of variables \( w_\varepsilon(x) = w(\varepsilon x) \). Thus, these \( w_\varepsilon \) verify
\[
S(\varepsilon) \int_{\partial\Omega_{\varepsilon}} |w_\varepsilon|^q \, dS \leq \int_{\Omega_{\varepsilon}} |\nabla w_\varepsilon|^p + |w_\varepsilon|^p \, dx.
\]
Taking limits in the above inequality, we arrive at
\[
S \int_{\partial\Omega} m(x)|w|^q \, dS \leq \int_{\Omega} |\nabla w|^p + |w|^p \, dx.
\]
But, \( v \) is an extremal for this inequality. We conclude that \( \bar{S} = S^* \) given by (1.5) and that \( v \) is an extremal. This proves that
\[
\lim_{\varepsilon \to 0} S(\varepsilon) = S^*.
\]
Concerning the convergence of the extremals we have proved that the rescaled extremals \( v_\varepsilon \) converge weakly to \( v \) in \( W^{1,p}(\Omega) \).

**Acknowledgements.** The first and third authors are supported by UBA X066, Fundacion Antorchas, CONICET and ANPCyT PICT 05009 and 10608. The second author is partially supported by the grants MTM2005-00715 and MTM2005-05980 of the MEC (SPAIN) and S-0505/ESP/0158 of the CAM (SPAIN). This work was started while the first author was visiting CSIC in Madrid, Spain. He wants to thank the institution for its hospitality and the provided facilities.

**References**


**Julién Fernández Bonder**  
Departamento de Matemática, FCEyN, Universidad de Buenos Aires, Pabellón I, Ciudad Universitaria (1428), Buenos Aires, Argentina.  
E-mail address: jfbonder@dm.uba.ar  
Web page: http://mate.dm.uba.ar/~jfbonder

**Rafael Orive**  
Departamento de Matemáticas  
Facultad de Ciencias  
Universidad Autónoma de Madrid  
CITA. Colmenar Viejo km. 15, 28049 Madrid, Spain.  
E-mail address: rafael.orive@uam.es  
Web page: http://www.uam.es/rafael.orive

**Julio D. Rossi**  
Instituto de Matemática y Física Fundamental  
Consejo Superior de Investigaciones Científicas  
Serrano 123, Madrid, Spain,  
on leave from Departamento de Matemática, FCEyN UBA (1428)  
Buenos Aires, Argentina.  
E-mail address: jrossi@dm.uba.ar  
Web page: http://mate.dm.uba.ar/~jrossi