OPTIMAL REGULARITY FOR THE PSEUDO INFINITY LAPLACIAN

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Abstract. In this paper we find the optimal regularity for viscosity solutions of the pseudo infinity Laplacian. We prove that the solutions are locally Lipschitz and show an example that proves that this result is optimal. We also show existence and uniqueness for the Dirichlet problem.

1. Introduction.

The main goal of this article is to study the optimal regularity of viscosity solutions to the pseudo infinity Laplacian. We find that the solutions are Lipschitz but not necessarily C^1.

The pseudo infinity Laplacian is the second order nonlinear operator given by

$$\tilde{\Delta}_\infty u = \sum_{i \in I(\nabla u)} u_{x_i x_i} |u_{x_i}|^2, \tag{1.1}$$

where the sum is taken over the indexes in $I(\nabla u) = \{i \,:\, |u_{x_i}| = \max_j |u_{x_j}|\}$. This operator appears naturally as a limit of $p-$Laplace type problems. In fact, let u_p be a sequence of solutions to

$$\tilde{\Delta}_p u = \sum_{i=1}^{N} (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = 0. \tag{1.2}$$

Suppose that u_0 is a uniform limit of the sequence u_p, then u_0 is a viscosity solution to

$$\tilde{\Delta}_\infty u = 0.$$

A proof of this fact is contained in this paper for completeness, the main arguments being taken from [4].

Eigenvalue problems for this operator were studied in [4]. Belloni and Kawohl point out that this operator arises in the problem of finding an optimal Lipschitz extension of boundary data when the Euclidean norm is substituted by the l^1-norm. The Lipschitz extension problem is a relevant question that has been studied by several other authors, see for example [1, 2, 7, 10]. Since, the Lipschitz constant is dependent on the norm used to measure distances in the domain, it is natural to expect that different elliptic operators will arise from different norms. In our case, the operator

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(1.1) arises by considering the l^1-norm (for details see [4]). When the standard Euclidean norm is considered the corresponding equation is

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_j \partial x_i} \frac{\partial u}{\partial x_i} = (Du)^T D^2 u (Du) = 0. \tag{1.3}$$

This widely studied operator is known in the literature as the infinity Laplacian. A comprehensive survey in the subject of Lipschitz extensions and the infinity Laplacian can be found in [2]. Limits of $p-$Laplacians are also relevant in mass transfer problems, see [6], [9].

Concerning regularity results for these operators the best known result is contained in [11] where Savin proved that a solution to the standard infinity Laplacian (1.3) in two space dimensions is C^1. This regularity result contrasts with our main result. More specifically, we show:

Theorem 1. *Let $u : \Omega \to \mathbb{R}$ be a viscosity solution to*

$$\tilde{\Delta}_\infty u = 0, \tag{1.4}$$

where $\Omega \subset \mathbb{R}^N$. Then u is locally Lipschitz.

Moreover, this result is optimal for $N \geq 2$, since

$$u(x, y) = x + \frac{1}{2}|y|, \tag{1.5}$$

is viscosity solution to (1.4) *that has no further regularity than Lipschitz.*

Given that for $N = 1$ the infinity Laplacian and pseudo infinity Laplacian coincide and the similar motivation for these two operators, at first one could expect a similar regularity for solutions to (1.3) and (1.4). Nevertheless, the control over second derivatives of solutions to (1.3) is better, since for a solution to (1.4), the equation may not give any control over the second derivatives of some of the variables (like y in our example (1.5)).

Further discussion about regularity and the Proof of Theorem 1 can be found in Section 2. The main ingredient of the proof is that solutions to (1.4) verify a comparison with l^1-cones property. This property analogous to the one satisfied by solutions to the usual infinity Laplacian with l^2-cones, see [2] and [5].

For sake of completeness, we also show existence and uniqueness to the Dirichlet problem.

Theorem 2. *Given a bounded smooth domain $\Omega \subset \mathbb{R}^N$, for any Lipschitz boundary data $g(x)$, there exists a unique viscosity solution to*

$$\begin{aligned} \tilde{\Delta}_\infty u &= 0 && \text{in } \Omega, \\ u &= g && \text{on } \partial\Omega. \end{aligned}$$

The proof of Theorem 2 can be found in Section 3. The existence result is proved using arguments from [4]. The strategy is to take limits (along subsequences) of variational solutions to (1.2) as $p \to \infty$. Uniqueness follows by adapting results in [3].

2. Optimal regularity. Proof of Theorem 1

First, let us recall the standard definition of viscosity solution, see [8].

Definition 2.1. *Consider*

$$F(x, Du, D^2u) = 0 \qquad \text{in } \Omega.$$

(1) *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict maximum at the point $x_0 \in \Omega$ with $u(x_0) = \phi(x_0)$ we have:*

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

(2) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \Omega$ with $u(x_0) = \phi(x_0)$ we have:*

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

(3) *Finally, u is a viscosity solution if it is a super and a subsolution.*

We will use this definition with

$$F(x, \xi, M) = -\sum_{i \in I(\xi)} M_{ii}|\xi_i|^2,$$

where $I(\xi) = \{i \,:\, |\xi_i| = \max_j |\xi_j|\}$.

Now we prove that viscosity solutions enjoy comparison with l^1-cones. These cones are defined by

$$C_{x_0}(x) = a + b\sum_{i=1}^{N} |x_i - (x_0)_i|.$$

We denote the l^1 ball by

$$\mathcal{B}_r(x_0) = \left\{ x \;:\; \sum_{i=1}^{N} |x_i - (x_0)_i| \leq r \right\}.$$

Lemma 2.2. *Let u be a viscosity solution to* (1.4)*. If $u(x) \geq C_{x_0}(x)$ for $x \in \partial(\mathcal{B}_r(x_0) \setminus \{x_0\})$, then $u(x) \geq C_{x_0}(x)$ for $x \in \mathcal{B}_r(x_0)$.*

Proof. We follow [7] and argue by contradiction. Suppose that $u(y) < C_{x_0}(y)$ for some $y \in \mathcal{B}_r(x_0)$ and consider the perturbation of the cone,

$$w(x) = \tilde{C}_{x_0}(x) - \varepsilon\left(R^2 - \sum_{i=1}^{N} |x_i - (x_0)_i|^2\right),$$

where $\tilde{C}_{x_0}$ is a smooth approximation of C_{x_0} (one can consider $\tilde{C}_{x_0}(x) = a + b\sum_{i=1}^{N} |x_i - (x_0)_i|^a$ with $a > 1$ close to 1). If R is large enough and ε is small enough we obtain $w(x) \leq u(x)$ for $x \in \partial(\mathcal{B}_r(x_0) \setminus \{x_0\})$ and $\max(w - u) = w(z) - u(z) > 0$. A direct computation shows $w_{x_i}(z) \neq 0$ and $w_{x_ix_i}(z) > 0$, for ε small, which contradicts the fact that u is a viscosity solution. □

Similarly we have that

Lemma 2.3. *Let u be a viscosity solution to* (1.4)*. If $u(x) \le C_{x_0}(x)$ for $x \in \partial(\mathcal{B}_r(x_0) \setminus \{x_0\})$, then $u(x) \le C_{x_0}(x)$ for $x \in \mathcal{B}_r(x_0)$.*

Now we are ready to prove Theorem 1.

Proof of Theorem 1. First we show, following [7], that every viscosity solution is locally Lipschitz. Let

$$S_r(x_0) = \max_{\sum_{i=1}^N |x_i-(x_0)_i|=r} \left\{ \frac{u(x)-u(x_0)}{r} \right\}.$$

Consider the cones centered at x_0,

$$C^b_{x_0}(x) = u(x_0) + b\sum_{i=1}^N |x_i - (x_0)_i|$$

and let

$$b_r = \inf\left\{ b \ : \ u(x) \le C^b_{x_0}(x) \text{ for } x \in \partial(\mathcal{B}_r(x_0) \setminus \{x_0\}) \right\}.$$

This number b_r is well defined since u is continuos and hence bounded on $\partial\mathcal{B}_r(x_0)$.

Taking $b = b_r$ we have

$$\frac{u(x)-u(x_0)}{r} \le \frac{C^b_{x_0}(x)-u(x_0)}{r} = b_r.$$

Hence

$$S_r(x_0) \le b_r.$$

From Lemma 2.3 we get that b_r is nondecreasing, if $r' < r$ then $b_{r'} \le b_r$. Therefore

$$S_{r'} \le b_r \qquad \text{or all } r' \le r. \tag{2.1}$$

A similar argument using Lemma 2.2 proves that

$$T_r(x_0) = \min_{\sum_{i=1}^N |x_i-(x_0)_i|=r} \left\{ \frac{u(x)-u(x_0)}{r} \right\} \tag{2.2}$$

is bounded below.

From (2.1) and (2.2) we obtain that there exists a constant C such that

$$\max_{\sum_{i=1}^N |x_i-(x_0)_i|\le r} \left\{ \frac{|u(x)-u(x_0)|}{r} \right\} \le C, \qquad \text{for all } r \text{ small.}$$

Or equivalently, u is locally Lipschitz.

To finish the proof of the theorem we have to show that

$$u(x,y) = x + \frac{1}{2}|y| \tag{2.3}$$

is a viscosity solution of (1.4). To see this fact, we need to check Definition 2.1. First, let us verify (1) in 2.1. Assume that $u-\phi$ has a maximum at (x_0,y_0) with $y_0 \ne 0$. Then, since $u-\phi$ is smooth and satisfies

$$(u-\phi)_x(x_0,y_0) = 0, \qquad (u-\phi)_y(x_0,y_0) = 0$$

and

$$(u-\phi)_{xx}(x_0,y_0) \le 0.$$

Since $u_x(x_0, y_0) = 1$, $|u_y(x_0, y_0)| = 1/2$ and $u_{xx}(x_0, y_0) \leq 0$ we get

$$\phi_x(x_0, y_0) = 1 > |\phi_y(x_0, y_0)| = 1/2 \text{ and } -\phi_{xx}(x_0, y_0) \leq 0.$$

Hence, we obtain

$$-\tilde{\Delta}_\infty \phi(x_0, y_0) \leq 0.$$

Analogously if $u - \phi$ has a minimum at (x_0, y_0) with $y_0 \neq 0$ we have

$$-\tilde{\Delta}_\infty \phi(x_0, y_0) \geq 0.$$

Now, if $y_0 = 0$ and ϕ is smooth, $u - \phi$ cannot have a minimum at (x_0, y_0). On the other hand, if it has a maximum at this point we obtain

$$\phi_x(x_0, y_0) = 1 > |\phi_y(x_0, y_0)| \leq 1/2 \text{ and } -\phi_{xx}(x_0, y_0) \leq 0.$$

Therefore

$$-\tilde{\Delta}_\infty \phi(x_0, y_0) \leq 0.$$

Combining the previous inequalities, we conclude that u is a viscosity solution to (1.4), finishing the proof of Theorem 1. □

Remark 2.4. *Note that l^1-cones are not differentiable along the axes (they are only Lipschitz), whereas the l^2-cones are differentiable everywhere away from the vertex. Here l^1-cones play the same role as the one played by the l^2-cones in the theory for the usual infinity Laplacian, Δ_∞, see* [2]*. This is a good argument to explain why solutions to the pseudo infinity Laplacian, $\tilde{\Delta}_\infty$, are Lipschitz but not C^1.*

Remark 2.5. *In the definition of the pseudo infinity Laplacian the sum is taken over all indexes where the l^∞-norm of the gradient is attained. One may think that the lack of C^1 regularity comes from the fact that the indexes of the sum may change from one point to another. This is not always the case, as in our example where $|u_x| = \max\{|u_x|, |u_y|\}$.*

Remark 2.6. *As trivial examples of solutions we may consider bilinear functions, that is, $u(x, y) = a_1xy + a_2x + a_3y + a_4$. With these examples it is easy to find solutions in which the indexes of the maximum of the derivatives depend on the point. Also, by the above proof, $u(x, y) = a(y)x + b(y)$ is a solution if $a(y) > a'(y)x + b'(y)$.*

3. Existence and uniqueness. Proof of Theorem 2

We will obtain a solution of (1.4) taking limit as $p \to \infty$ of solutions to

$$\tilde{\Delta}_p u = 0 \qquad \text{in } \Omega \tag{3.1}$$

$$u(x) = g(x) \qquad \text{for } x \in \partial\Omega. \tag{3.2}$$

Solutions to (3.1)-(3.2) can be obtained by variational arguments. Notice first that since g is Lipschitz it can be extended in Ω to a function in $W^{1,\infty}(\Omega)$, that we still denote as g. Hence, we can define

$$E_p(g) = \inf_{v \in W^{1,p}(\Omega) \,:\, u - g \in W_0^{1,p}(\Omega)} \int_\Omega \sum_{i=1}^N |u_{x_i}|^p. \tag{3.3}$$

Let us denote by

$$|||u||| = \left(\int_\Omega \sum_{i=1}^N |u_{x_i}|^p \right)^{1/p}.$$

This is a seminorm in $W^{1,p}(\Omega)$ equivalent to the usual one.

By a standard compactness argument, we can show that there is a $u_p \in W^{1,p}(\Omega)$ that realizes $E_p(g)$. This function u_p is a weak solution to (1.2).

From definition (3.3), using g as a test function, we can easily obtain a constant C, independent of p, such that

$$\|u_p\|_{W^{1,p}(\Omega)} \leq C\|g\|_{W^{1,\infty}(\Omega)}.$$

Therefore, fixing $q < p$ and using Holder's inequality, we get

$$\|u_p\|_{W^{1,q}(\Omega)} \leq C. \tag{3.4}$$

By a diagonal procedure we obtain a sequence $p_i \to \infty$ such that

$$u_{p_i} \to u \qquad \text{weakly in } W^{1,q}(\Omega).$$

Again by compactness, the limit verifies $u = g$ on the boundary. Using (3.4) we conclude $u \in W^{1,\infty}(\Omega)$.

Now, we want to prove that u is a viscosity solution to (1.4). To this end we need a lemma that shows that u_p are also viscosity solutions. Then, we pass to the limit in the viscosity sense.

Lemma 3.1. *Every weak solution u_p to* (3.1) *is also viscosity solution to* (3.1).

Proof. We will follow the ideas in [4]. Let us prove first that u_p is a viscosity subsolution. Fix $x_0 \in \Omega$ and ϕ smooth such that $u - \phi$ has a strict maximum at x_0 with $u(x_0) = \phi(x_0)$. Assume, arguing by contradiction, that there exists $r > 0$ such that

$$-\tilde{\Delta}_p \phi(z) > 0, \qquad z \in B_r(x_0).$$

Set $M = \sup\{(\phi - u_p)(y) \,:\, y \in \partial B_r(x_0)\}$ and $\Phi = \phi - M/2$. Hence we have $\Phi > u$ on $\partial B_r(x_0)$, $\Phi(x_0) < u(x_0)$ and

$$-\tilde{\Delta}_p \Phi(z) > 0, \qquad z \in B_r(x_0).$$

We multiply by $(u_p - \Phi)_+$ and integrate by parts to obtain

$$\int_{\{u>\Phi\}} \sum_{i=1}^N |\Phi_{x_i}|^{p-2} \Phi_{x_i} (u_p - \Phi)_{x_i} \, dx > 0.$$

Now, we use the fact that u_p is a weak solution to obtain

$$\int_{\{u>\Phi\}} \sum_{i=1}^N |u_{x_i}|^{p-2} u_{x_i} (u_p - \Phi)_{x_i} \, dx = 0.$$

Therefore

$$0 \leq \int_{\{u>\Phi\}} \sum_{i=1}^N \left(|u_{x_i}|^{p-2} u_{x_i} - |\Phi_{x_i}|^{p-2} \Phi_{x_i} \right) (u_p - \Phi)_{x_i} \, dx < 0,$$

which is a contradiction.

The proof of u_p being a viscosity supersolution is completely analogous. This finishes the proof of the Lemma. □

Now we are ready to prove our second result.

Proof of Theorem 2. We will divide the proof in two steps. In the first step we prove that the uniform limit of u_p is a viscosity solution to (1.4). Then, in the second step, we finish by proving uniqueness of viscosity solutions.

Step 1:

We prove that u is subsolution. As before, we skip the proof of u being a supersolution, since it is analogous.

Let ϕ be a smooth function such that $u-\phi$ has a strict maximum at $x_0 \in \Omega$. By uniform convergence of u_{p_i}, we have that there is sequence of points $x_{p_i} \to x_0$ such that $u_{p_i} - \phi$ attains a maximum at x_{p_i}. By Lemma 3.1 we know that

$$-\sum_{j=1}^{N} |\phi_{x_j}|^{p_i-2}\phi_{x_jx_j}(x_{p_i}) \leq 0.$$

If the $\max_j |\phi_{x_j}|(x_0) = 0$ the result is trivial. Otherwise, since ϕ is smooth, dividing by $\max_j |\phi_{x_j}|^{p_i-4}(x_{p_i})$ and taking limits we obtain that

$$-\sum_{j\in I(\nabla\phi)(x_0)} |\phi_{x_j}|^2\phi_{x_jx_j}(x_0) \leq 0.$$

This shows that the limit is a viscosity subsolution, finishing step 1.

Step 2:

We will follow closely the arguments in [3]. The main point of the proof is to obtain an equivalent result to their Lemma 3.2, that is,

Lemma 3.2 (Hopf's Lemma)**.** *Assume that w is a viscosity supersolution to equation* (1.4) *with a local minimum at y_0. Then w is constant in a neighborhood of y_0.*

Proof. Let w_β be the inf-convolution of w, that is

$$w_\beta(y) = \inf_z \left(w(z) + \frac{|y-z|^2}{\beta^2} \right).$$

It is possible to show that w_β is also a viscosity supersolution. Moreover, w_β is semi-convex.

We will prove the result by contradiction. Since w_β is a semi-convex function, by translating it we can assume that there is an Euclidean ball B_R such that $w_\beta > 0$ on $\partial B_{\frac{R}{2}}$ and $w_\beta(z_0) = 0$ for some $z_0 \in B_R \setminus \overline{B_{\frac{R}{2}}}$.

We consider

$$\chi(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2}.$$

A straightforward computation shows that

$$-\tilde{\Delta}_\infty \chi(x) = -\sum_{\{i:\, |x_i| = \max_j |x_j|\}} 8\alpha^3 e^{-3\alpha|x|^2}|x_i|^2(2\alpha|x_i|^2 - 1)$$

Given that for $x \in B_R \setminus \overline{B_{\frac{R}{2}}}$ it holds $|x_i| = \max_j |x_j| \geq \frac{R}{2\sqrt{N}}$, for α large enough we obtain

$$-\tilde{\Delta}_\infty \chi(x) < 0. \tag{3.5}$$

Now we show, that for a large enough α, it holds that $w_\beta \geq \chi$. In fact, $0 = \chi \leq w_\beta$ on ∂B_R and for α large also $\chi \leq w_\beta$ on $\partial B_{\frac{R}{2}}$. If the $\max_{B_R \setminus \bar{B}_{\frac{R}{2}}} (\chi - w_\beta) > 0$, we have that the maximum is attained for some z that satisfies $\chi(z) - w_\beta(z) > 0$ and, since w_β is a viscosity supersolution, it must also hold

$$-\tilde{\Delta}_\infty \chi(z) \geq 0,$$

which contradicts (3.5). Hence, for every $x \in B_R \setminus \overline{B_{\frac{R}{2}}}$ it holds

$$w_\beta \geq \chi,$$

which contradicts that $w_\beta(z_0) = 0$.

This implies that w_β is locally constant in a neighborhood of z_0. Hence, following the proof in [3], we obtain that w is constant in some neighborhood of every minimum. □

By using the sup-convolution we can prove a similar statement to Lemma 3.2 when a viscosity subsolution has a local maximum.

Once these Lemmas are established, the rest of the proof of uniqueness is contained in the comparison principle proved in [3]. We briefly sketch the proof here.

Let u and v be sub and supersolution to (1.4) respectively, such that $u \leq v$ on $\partial\Omega$. By regularizing u and v by sup and inf convolution and taking $u - \eta$ instead of u we can assume u is semi-convex, v is semi-concave and $u < v$ on $\partial\Omega$. We need to show $u \leq v$ in Ω.

We establish the comparison principle by contradiction. Suppose that

$$\max_{\overline{\Omega}} (u - v) > 0. \tag{3.6}$$

Define for h small

$$\mathcal{M}(h) = \max_{x \in \overline{\Omega}_h} (u(x+h) - v(x)),$$

where $\Omega_h = \{x \in \Omega : d(x, \partial\Omega) > h\}$.

Notice that (3.6) implies that $\mathcal{M}(0) > 0$. Hence, if (3.6) holds, we must have for h small enough that $\mathcal{M}(h) > 0$.

Before finding the contradiction we will show that necessarily one of the following holds:

(1) There is a sequence $h_n \to 0$ such that at any maximum point x_{h_n} of $u(\cdot + h_n) - v(\cdot)$ holds $Du(x_{h_n} + h_n) = Dv(x_{h_n}) \neq 0$ for every n. or
(2) There is a neighborhood of 0 such that for every h in this neighborhood $\mathcal{M}(h) = \mathcal{M}(0)$.

If (1) holds we are going to reach the contradiction by proving that necessarily for n large enough $\mathcal{M}(h_n) \leq 0$, which contradicts (3.6). On the other

hand, if (2) holds, we show that Lemma 3.2 implies that the set where $\mathcal{M}(0)$ is achieved is open and closed, hence equal to $\overline{\Omega}$, contradicting that $u < v$ on $\partial\Omega$ (when $\mathcal{M}(0) > 0$).

Let us start by showing that either (1) or (2) holds. Suppose that (1) does not occur. Notice that $\mathcal{M}$ is a maximum of semi-convex functions which implies that $\mathcal{M}$ is semi-convex in a neighborhood of 0. Let $x_h \in \Omega$ be a maximum point of $u(\cdot + h) - v(\cdot)$. By general properties of semiconvex functions (see (DMP) in [3]) $u(\cdot + h)$ and $v(\cdot)$ are differentiable at x_h and $Du(x_h + h) = Dv(x_h)$.

The semiconvexity of u implies that, if $Du(x_h+h) = Dv(x_h) = 0$ for some h, then there is a constant r, small enough, such that for every $h' \in B_r(h)$

$$\begin{aligned}\mathcal{M}(h') \geq & u(x_h + h') - v(x_h)\\ \geq & u(x_h + h) - v(x_h) - C|h - h'|^2.\end{aligned}$$

It follows that

$$\mathcal{M}(h') \geq \mathcal{M}(h) - C|h - h'|^2.$$

Which implies $0 \in \partial\mathcal{M}(h)$.

Since (1) does not hold, necessarily for any h in a neighborhood of 0 must hold that $0 \in \partial\mathcal{M}(h)$, or equivalently $\mathcal{M}(h) = \mathcal{M}(0)$ for h in some neighborhood of 0. That is (2) holds.

Now we are left to show that both of these alternatives lead us to contradiction.

(a) If (2) holds: Then for every h in a neighborhood from 0

$$u(x_0) - v(x_0) = \mathcal{M}(0) = \mathcal{M}(h) \geq u(x_0 + h) - v(x_0).$$

That is x_0 is a local maximum for u. Lemma 3.2 implies that u is constant in a neighborhood of x_0. Since $u - v$ attains a local maximum at x_0 and u is constant, v must attain a local minimum at x_0. Using once more Lemma 3.2, we conclude that v must also be constant in a neighborhood of x_0. It follows that the set where $\mathcal{M}(0)$ is attained is open. By continuity of u and v it must be also closed, hence it must equal $\overline{\Omega}$, which contradicts that $u < v$ on $\partial\Omega$.

(b) If (1) holds: Let φ_ε such that

$$\varphi'_\epsilon(t) = \exp\left(\int_0^t \exp\left(-\varepsilon^{-1}(s + \varepsilon^{-1})\right)ds\right) \tag{3.7}$$

and $\psi_\varepsilon(t)$ its inverse. Define

$$G_\varepsilon(w, q, N) = -\sum_{i\in I(q)} |q_i|^2 \left(\varphi'_\epsilon(w)N_{ii} + \varphi''_\varepsilon(w)|q_i|^2\right),$$

where $I(q) = \{i : |q_i| = \max_j |q_j|\}$.

We denote as

$$\begin{aligned}U_\varepsilon(x) =& \psi_\varepsilon(u),\\ V_\varepsilon(x) =& \psi_\varepsilon(v).\end{aligned}$$

Then, given that

$$G_\varepsilon\left(U_\varepsilon, \frac{\partial U_\varepsilon}{\partial x_i}, D^2U_\varepsilon\right) = -(\varphi_\varepsilon'(u))^2\tilde{\Delta}_\infty u,$$

we have that U_ε is a subsolution to

$$G_\varepsilon\left(U_\varepsilon, \frac{\partial U_\varepsilon}{\partial x_i}, D^2U_\varepsilon\right) = 0.$$

Similarly, V_ε is a supersolution.

By definition of φ_ε we have that U_ε and V_ε are, respectively, semi-convex and semi-concave. It holds that $U_\varepsilon(\cdot + h_n) - V_\varepsilon(\cdot)$ attains its maximum at some interior point $x_\varepsilon \in \Omega$. Since U_ε and V_ε are semiconvex and semiconcave (property (DMP) in [3]), both of them are differentiable at x_ε and $|DU_\varepsilon| = |DV_\varepsilon|$. Notice that $U_\varepsilon \to u$ and $V_\varepsilon \to v$ as $\varepsilon \to 0$. Hence, we can find a sequence $\varepsilon_k \to 0$ such that $x_{\varepsilon_k} \to \overline{x}$, where $\overline{x}$ is a maximum of $u_\varepsilon(\cdot + h_n) - v_\varepsilon(\cdot)$. Since (1) holds, we have $|Du| = |Dv| \geq \delta(n) > 0$. By general properties of semi-convex and semi-concave functions (property (PGC) in [3]) and the definition of φ_ε, we have for ε small enough $|DU_\varepsilon| = |DV_\varepsilon| \geq \frac{\delta(n)}{2}$.

Note that one can construct a sequence of points p_m and a sequence functions

$$f_m(x) = U_\varepsilon(x + h_n) - V_\varepsilon(x) - \langle p_m, x\rangle,$$

such that f_m has a strict maxima at x_ε^m and $x_\varepsilon^m \to x_\varepsilon$ as $m \to \infty$. Lemma A.3 in [8] shows that if $r > 0$ is small enough, there is a $\overline{\rho} > 0$ such that the set of maximum points in $B_r(x_\varepsilon^m)$ of

$$g_m(x) = U_\varepsilon(x + h_n) - V_\varepsilon(x) - \langle p_m, x\rangle - \langle q, x\rangle$$

with $q \in B_\rho(0)$ ($\rho \leq \overline{\rho}$), contains a set of positive Lebesgue measure. By Alexansandrov's result, $U_\varepsilon(\cdot + h_n)$ and $V_\varepsilon(\cdot)$ are twice differentiable a.e. Therefore for r small and $\rho \leq \overline{\rho}$, there is a $z \in B_r(x_\varepsilon^m)$ and $q \in B_\rho(0)$ such that z is a maximum of g_m and U_ε and V_ε are twice differentiable at z. Since z is a maximum it holds $DU_\varepsilon = DV_\varepsilon + p_m + q$. As before, for q, ρ small and m large

$$|DU_\varepsilon| = |DV_\varepsilon + p_m + q| \geq \frac{\delta(n)}{4}.$$

Moreover, since U_ε is semi-convex and V_ε semi-concave, it holds

$$-C \cdot Id \leq D^2U_\varepsilon(z) \leq D^2V_\varepsilon(z) \leq C \cdot Id,$$

for some constant $C > 0$ independent of ρ, r and m. Evaluating at z we have by the definition of G

$$\begin{aligned} G_\varepsilon(U_\varepsilon(z + h_n), DV_\varepsilon(z) + p_m + q, D^2V_\varepsilon(z)) &\leq 0 \\ \leq G_\varepsilon(V_\varepsilon(z), DV_\varepsilon(z), D^2V_\varepsilon(z)). \end{aligned} \tag{3.8}$$

Since $DV_\varepsilon(z)$ and $D^2V_\varepsilon(z)$ are bounded, by taking a subsequence when $\rho, r \to 0$ and $m \to \infty$ we can find $\overline{P} \geq \frac{\delta(n)}{4}$ and $\overline{X}$ such that $DV_\varepsilon(z) \to \overline{P}$ and $D^2V_\varepsilon(z) \to \overline{X}$.

Taking limits in (3.8) we obtain

$$G_\varepsilon(U_\varepsilon(z+h_n),\overline{P},\overline{X}) \le 0 \le G_\varepsilon(V_\varepsilon(z),\overline{P},\overline{X}). \tag{3.9}$$

On the other hand, it is easy to see by the definition of G_ε and φ_ε that

$$\begin{aligned}\frac{\partial G_\varepsilon(w,q,N)}{\partial w} &= -\sum_{i\in I(q)} |q_i|^2 \left(\varphi_\epsilon''(w)N_{ii} + \varphi_\varepsilon'''(w)|q_i|^2\right)\\ &= -\sum_{i\in I(q)} |q_i|^2 \exp(-\varepsilon^{-1}(w+\varepsilon^{-1}))\\ &\quad\times \exp\left(\int_0^w \exp(-\varepsilon^{-1}(s+\varepsilon^{-1}))ds\right)\\ &\quad\times \left[\left(\frac{1}{\varepsilon} - \exp(-\varepsilon^{-1}(s+\varepsilon^{-1}))\right)|q_i|^2 - N_{ii}\right]\end{aligned}$$

Hence, for any $\delta > 0$, ε small enough, $|w| \le \delta^{-1}$, $\delta \le |q| \le \delta^{-1}$ and $|N| \le \delta^{-1}$ it holds

$$\frac{\partial G_\varepsilon(w,q,N)}{\partial w} > 0, \tag{3.10}$$

which contradicts (3.9), finishing the proof of uniqueness. □

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