

AN OBSTACLE PROBLEM FOR TUG-OF-WAR GAMES

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ABSTRACT. We consider the obstacle problem for the infinity Laplace equation. Given a Lipschitz boundary function and a Lipschitz obstacle we prove the existence and uniqueness of a super infinity-harmonic function constrained to lie above the obstacle which is infinity harmonic where it lies strictly above the obstacle. Moreover, we show that this function is the limit of value functions of a game we call obstacle tug-of-war.

1. INTRODUCTION

Recently, Peres, Schram, Sheffield and Wilson, [14], discovered the relationship between limits of value functions of Tug-of-War games and solutions to the infinity Laplacian. Also, Peres and Sheffield, [15], found a game whose values approximate solutions to the p -Laplacian, see also the work by Manfredi, Parvianen and Rossi, [9], [10], [11], [12], Bjorland, Caffarelli and Figalli, [5], by Armstrong, Smart and Somersille, [2], by Peres, Peté and Somersille, [13], and by Antunović, Peres, Sheffield and Somersille, [1].

Our main goal in this work is to study the obstacle problem in this context. That is, we propose a game that involves a function Ψ , (the obstacle), and is such that the value function of the game is above it. We prove existence, uniqueness and some properties of the value functions of this game and we find that a certain limit of these functions is a viscosity solution of the obstacle problem for the infinity Laplacian.

Next, let us describe briefly the game in which we are interested. The Tug-of-War game in [14] and described in detail below is a two player zero sum game. In our case it is played in a bounded domain $\Omega \subset \mathbb{R}^N$ with a given “boundary” function $F : \Gamma \mapsto \mathbb{R}$ (here Γ is a neighbourhood of $\partial\Omega$ in $\mathbb{R}^N \setminus \Omega$). In our modification we also have an obstacle $\Psi : \mathbb{R}^N \mapsto \mathbb{R}$ such that $\Psi \leq F$ in Γ . As in ordinary Tug-of-War if the boundary is reached at $x_n \in \Gamma$ then Player I receives $F(x_n)$. However, in our case, Player I can opt to stop the game at any position $x_n \in \Omega$ and receive the payoff $\Psi(x_n)$.

This is much like the case in American options where investors can exercise the option at any time up to expiry and accept a payoff equal to the intrinsic

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value which in our case is the obstacle. Or they may wait (continue to play) if the expected benefit of waiting is greater than the intrinsic value. Tug of war and the infinity Laplacian has applications to mass transport problems, control theory and economic modeling among others. Specifically, it is our belief that our results may have applications to the pricing of American options.

Our game is similar to investing in American Options in that optimal strategies are to stop where the value function agrees with the intrinsic value. Underlying our proof is the idea that Player I will choose to stop where his value function is equal to the obstacle. However, that he does so is not a requirement for any of our proofs.

We have the following results concerning properties of u^ϵ the value function of this game where ϵ indicates the maximum size of each move:

Theorem 1. *There exists a unique value of the game. This value is the solution to the discrete obstacle problem; that is, it satisfies*

$$u^\epsilon(x) \geq \frac{1}{2} \sup_{y \in B_\epsilon(x)} u^\epsilon(y) + \frac{1}{2} \inf_{y \in B_\epsilon(x)} u^\epsilon(y),$$

lies above the obstacle in Ω , coincides with F in Γ , and is such that the above inequality is an equality where u^ϵ lies strictly above the obstacle.

In addition, a comparison principle holds.

Lemma 2. *Let $u_1^\epsilon, u_2^\epsilon$ be values of the ϵ games with boundary functions F_1, F_2 and obstacles Ψ_1, Ψ_2 respectively. If $F_1 \geq F_2$ and $\Psi_1 \geq \Psi_2$, then $u_1^\epsilon \geq u_2^\epsilon$.*

Moreover, the value function of the game satisfies the following Lewy-Stampacchia inequalities

Lemma 3. *We have*

$$0 \leq u^\epsilon(x) - \frac{1}{2} \left(\sup_{B_\epsilon(x)} u^\epsilon + \inf_{B_\epsilon(x)} u^\epsilon \right) \leq \left[\Psi(x) - \frac{1}{2} \left(\sup_{B_\epsilon(x)} \Psi + \inf_{B_\epsilon(x)} \Psi \right) \right]_+.$$

Here we use the notation $[A(x)]_+ = \max\{A(x), 0\}$.

Finally, if the obstacle Ψ is Lipschitz, then the value function is Lipschitz with respect to the discrete distance $d_\epsilon(x, y) = \epsilon \lceil \frac{|x-y|}{\epsilon} \rceil + 1$ (by $\lceil \cdot \rceil$ we denote the integer part).

Lemma 4. *If the obstacle Ψ Lipschitz, then there exists a constant C , independent of ϵ , such that the value function u^ϵ satisfies*

$$|u^\epsilon(x) - u^\epsilon(y)| \leq C d_\epsilon(x, y).$$

Concerning the limit as $\epsilon \rightarrow 0$ of these value functions we have the following result:

Theorem 5. *We have that, as $\epsilon \rightarrow 0$,*

$$u^\epsilon \rightarrow u$$

uniformly. The limit u is the unique viscosity solution to the obstacle problem for the infinity Laplacian, that is, it is the unique super-infinity harmonic function; i.e., functions that satisfy

$$-\Delta_\infty u = -\left\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \right\rangle \geq 0,$$

in the viscosity sense, is above the obstacle Ψ in Ω , takes the boundary value, F on Γ and is infinity harmonic where it lies strictly above the obstacle.

There is a close connection between p -harmonic functions and infinity harmonic functions. Indeed, if we pass to the limit, as $p \rightarrow \infty$, in a sequence (u_p) of p -harmonic functions (in the viscosity sense, see [7]), that is, solutions of $\Delta_p u_p = 0$, with given boundary values, the limit exists (in the uniform topology) and is a solution of the infinity Laplace equation (see [4])

$$-\Delta'_\infty u = -\langle D^2 u Du, Du \rangle = -\sum_{i,j=1}^d \frac{\partial u}{\partial x_i} \frac{\partial u}{\partial x_j} \frac{\partial^2 u}{\partial x_i \partial x_j} = 0.$$

The infinity Laplacian is connected with the optimal Lipschitz extension problem [3], and arises also in the context of mass transportation problems and several other applications, such as image reconstruction and enhancement [6]. Note that here we have normalized the operator and consider $-\Delta_\infty u = -\langle D^2 u \frac{Du}{|Du|}, \frac{Du}{|Du|} \rangle$. Both equations turns out to be equivalent, in the sense that they have the same viscosity solutions, see [14].

On the other hand, the obstacle problem for elliptic operators has been extensively studied. In the classical approach one seeks to minimize the energy $E(u) = \int_\Omega |Du|^2$ among the functions that coincide with a given function F at the boundary of $\Omega \subset \mathbb{R}^d$ and remain above a prescribed obstacle Ψ . Such a problem is motivated by the description of the equilibrium position of a membrane (the graph of the solution) attached at level F along the boundary of Ω and that is forced to remain above the obstacle in the interior of Ω . Many of the results obtained for the Laplacian were generalized for the p -Laplacian whose energy functional is given by $E(u) = \int_\Omega |Du|^p$.

However, the infinity Laplacian is not variational (hence no energy methods are directly available). One may rely on methods from potential theory (Perron method), on limit procedures like the ones described here, or one can take the limit as $p \rightarrow \infty$ in the obstacle problem for the p -Laplacian and obtain a solution to the obstacle problem for the infinity Laplacian, see [16] for example.

The paper is organized as follows: in Section 2 we describe the Tug-of-War game; in Section 3 we collect some properties of the value function of the

game; finally, in Section 4 we deal with the limit as $\epsilon \rightarrow 0$ and find a proof of existence of a solution to the obstacle problem for the infinity Laplacian based on Tug-of-War games. We also discuss the convergence of the contact sets. (The sets where the value function equals the obstacle).

2. DESCRIPTION OF THE GAME

2.1. Description of the game. The game that we describe below is called a leavable game. Some leavable games are described in [8] Chapter 7. Tug-of-War is developed in [14].

Tug-of-War is a two-person, zero-sum game, in other words, two players are in contest and the total earnings of one are the losses of the other. Hence, one of them, whom we call Player I, plays trying to maximize his expected outcome, while the other, Player II, is trying to minimize Player I's outcome (or, since the game is zero-sum, to maximize his own outcome). In this Tug-of-War leavable game Player I can decide to end the game before the boundary is reached i.e. his strategy includes stopping rule.

Now, let us describe the game more precisely. Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. For a fixed $\gamma > 0$, consider a strip around the boundary $\Gamma \subset \mathbb{R}^N \setminus \Omega$ given by

$$\Gamma = \{x \in \mathbb{R}^N \setminus \Omega : \text{dist}(x, \partial\Omega) \leq \gamma\}.$$

Let $F : \Gamma \rightarrow \mathbb{R}$ be a Lipschitz continuous function (the final payoff). In addition we have a function $\Psi : \mathbb{R}^N \mapsto \mathbb{R}$ (the obstacle) such that

$$\Psi \leq F \quad \text{in } \Gamma.$$

Note that any neighborhood of $\partial\Omega$ in $\mathbb{R}^N \setminus \Omega$ contains a strip of this form.

The rules of the game are as follows: At an initial time a token is placed at a point $x_0 \in \Omega$ and we fix $\epsilon \in (0, \gamma]$. Then, a (fair) coin is tossed and the winner of the toss is allowed to move the game position to any $x_1 \in B_\epsilon(x_0)$. At each turn, the coin is tossed again, and the winner of the toss chooses a new game state $x_k \in B_\epsilon(x_{k-1})$. Once the token has reached some $x_\tau \in \Gamma$, the game ends and Player I earns $F(x_\tau)$ (while Player II earns $-F(x_\tau)$). This is the reason why we will refer to F as the *final payoff function*. In addition, at every position x_n , Player I is allowed choose to end the game earning $\Psi(x_n)$ (while Player II earns $-\Psi(x_n)$). We will call Ψ the *obstacle function*. This procedure yields a sequence of game states $x_0, x_1, x_2, \dots, x_\tau$, where every x_k except x_0 are random variables, depending on the coin tosses, the strategies (defined below) adopted by the players and the stopping rule chosen by Player I.

Note that the relevant values of F are those taken in the set

$$\Gamma_\epsilon = \bigcup_{x \in \Omega} B_\epsilon(x) \cap \Gamma$$

since those are the the points at which the game could end.

Next, we give a precise definition of the *value of the game*. To this end we have to introduce some notation and put the game into its normal or strategic form (see [15]). The initial state $x_0 \in \Omega$ is known to both players (public knowledge). Each player i chooses an *action* $a_0^i \in B_\epsilon(x_0)$ which is announced to the other player; this defines an action profile $a_0 = \{a_0^1, a_0^2\} \in B_\epsilon(x_0) \times B_\epsilon(x_0)$. Then, the new state $x_1 \in B_\epsilon(x_0)$ is selected according to a probability distribution $p(\cdot|x_0, a_0)$ in Ω which, in our case, is given by the fair coin toss. In addition, Player I, chooses a stop rule τ (here τ takes values in \mathbb{N} , and determines that the game ends at step τ). At stage k , knowing the history $h_k = (x_0, a_0, x_1, a_1, \dots, a_{k-1}, x_k)$, (the sequence of states and actions up to that stage), Player I chooses to end the game or to continue according to the stopping rule τ (that is a mapping from histories h_k to \mathbb{N} , if $\tau \neq k$ the game continues, while, if $\tau = k$ the game ends), if she decides to continue, each player i chooses an action a_k^i . If the game ends at time j (the game ends if the position x_j belongs to Γ or if the stopping rule for the first player applies), we set $x_m = x_j$ and $a_m = x_j$ for $j \leq m$.

Denote $H_k = (\Omega \cup \Gamma)^k = ((\Omega \cup \Gamma) \times (\Omega \cup \Gamma) \times \dots \times (\Omega \cup \Gamma))$, the set of *histories up to stage k* , and by $H_\infty = \bigcup_{k \geq 1} H_k$ the set of all histories. Notice that H_k , as a product space, has a measurable structure. The *complete history space* H_∞ is the set of plays defined as infinite sequences $(x_0, a_0, \dots, a_{k-1}, x_k, \dots)$ endowed with the product topology. Then, the final payoff for Player I, defined by

$$\tilde{F}(x) = \begin{cases} F(x), & x \in \Gamma, \\ \Psi(x), & x \in \Omega, \end{cases}$$

induces a Borel-measurable function on H_∞ . A *strategy* $S_i = \{S_i^k\}_k$ for Player i , is a sequence of mappings from histories to actions, such that S_i^k is a Borel-measurable mapping that maps histories ending with x_k to elements of $B_\epsilon(x_k)$ (roughly speaking, at every stage the strategy gives the next movement for the player, provided he win the coin toss, as a function of the current state and the past history). A *stopping rule* for Player I is a stopping time, τ , from histories H to \mathbb{N} that is a finite everywhere, Borel-measurable, such that for every k the set $\{\tau = k\}$ belongs to the sigma field generated by the coordinate functions X_1, \dots, X_k of H (if $\tau \neq k$ the game continues, while if $\tau = k$ the game ends).

The initial state x_0 , a stopping rule τ and a profile of strategies $\{S_I, S_{II}\}$ define (by Kolmogorov's extension theorem) a unique probability $\mathbb{P}_{\tau, S_I, S_{II}}^{x_0}$ on the space of plays H_∞ . We denote by $\mathbb{E}_{\tau, S_I, S_{II}}^{x_0}$ the corresponding expectation.

Then, if τ denotes the stopping rule for Player I and S_I and S_{II} denote the strategies adopted by Player I and Player II respectively, we define the

expected payoff for Player I as

$$V_{x_0, I}(\tau, S_I, S_{II}) = \begin{cases} \mathbb{E}_{\tau, S_I, S_{II}}^{x_0}[\tilde{F}(x_\tau)], & \text{if the game terminates a.s.} \\ -\infty, & \text{otherwise.} \end{cases}$$

Analogously, we define the expected payout for Player II as

$$V_{x_0, II}(\tau, S_I, S_{II}) = \begin{cases} \mathbb{E}_{\tau, S_I, S_{II}}^{x_0}[\tilde{F}(x_\tau)], & \text{if the game terminates a.s.} \\ +\infty, & \text{otherwise.} \end{cases}$$

Finally, we can define the ϵ -value of the game for Player I as

$$u_I^\epsilon(x_0) = \sup_{\tau, S_I} \inf_{S_{II}} V_{x_0, I}(\tau, S_I, S_{II}),$$

while the ϵ -value of the game for Player II is defined as

$$u_{II}^\epsilon(x_0) = \inf_{S_{II}} \sup_{\tau, S_I} V_{x_0, II}(\tau, S_I, S_{II}).$$

In some sense, $u_I^\epsilon(x_0)$, $u_{II}^\epsilon(x_0)$ are the least possible outcomes that each player expects when the ϵ -game starts at x_0 . Notice that, as in [14], we penalize severely the games that never end.

In [8] it is shown that, under very general hypotheses that are fulfilled in the present setting, $u_I^\epsilon = u_{II}^\epsilon := u^\epsilon$. The function u^ϵ is called the value of the ϵ -Tug-of-War game.

2.2. Dynamic Programming Principle. For $x \in \Omega$, looking at the stopping strategy and the outcome of the first coin toss, we immediately get the following lemma, that says that the values of the game satisfy a Dynamic Programming Principle (DPP) formula (see also [8] for similar Dynamic Programming Principles).

Lemma 6. (DPP) *The value functions u_I^ϵ and u_{II}^ϵ satisfy*

$$u^\epsilon(x) = \max \left\{ \Psi(x), \frac{1}{2} \sup_{y \in B_\epsilon(x)} u^\epsilon(y) + \frac{1}{2} \inf_{y \in B_\epsilon(x)} u^\epsilon(y) \right\} \quad \forall x \in \Omega,$$

and

$$u^\epsilon(x) = F(x), \quad \forall x \in \Gamma.$$

This immediately implies that u_I^ϵ and u_{II}^ϵ satisfy the following formulation of the discrete obstacle problem:

$$\begin{cases} u^\epsilon(x) = F(x), & \text{in } \Gamma, \\ u^\epsilon(x) \geq \Psi(x), & \text{in } \Omega, \\ u^\epsilon(x) \geq \frac{1}{2} \left(\sup_{y \in B_\epsilon(x)} u^\epsilon(y) + \inf_{y \in B_\epsilon(x)} u^\epsilon(y) \right), & \text{in } \Omega, \\ u^\epsilon(x) = \frac{1}{2} \left(\sup_{y \in B_\epsilon(x)} u^\epsilon(y) + \inf_{y \in B_\epsilon(x)} u^\epsilon(y) \right), & \text{in } \Omega \setminus A^{u^\epsilon}. \end{cases}$$

Here A^{u^ϵ} is the coincidence set, that is, the set where $u^\epsilon = \Psi$ in Ω .

We call the last equality discrete ϵ infinity harmonic. We call the last inequality discrete ϵ infinity super harmonic. Discrete ϵ infinity subharmonic is defined analogously.

3. PROPERTIES OF THE GAME VALUE FUNCTIONS

Here we prove the existence of a value function for the game, Theorem 1, and our comparison principle Lemma 2.

To prove these we need some lemmas. We will show that u_I^ϵ is the smallest supersolutions that satisfies our conditions and u_{II}^ϵ is, in some sense, the largest subsolution.

Remark 7. Note that u_I^ϵ and u_{II}^ϵ are at least as large as the corresponding ordinary tug of war game with “boundary” $Y = \Gamma \cup A^u$ where A^u is the corresponding contact set i.e. $A^u = A^{u_I}$ or $A^u = A^{u_{II}}$. More precisely, let A^u be the contact set of u and let $Y = \Gamma \cup A^u$. Let $\hat{F} : Y \rightarrow \mathbb{R}$ be the Lipschitz function

$$\hat{F}(x) = \begin{cases} F(x), & x \in \Gamma, \\ \Psi(x), & x \in A^u. \end{cases}$$

Notice \hat{F} is well defined because if $\Gamma \cap A^u$ is nonempty then $F = \Psi$ there. Then we have the inequality $w^\epsilon \leq u$ where w^ϵ is the value function for the ordinary tug-of-war game in this setting.

We have $w^\epsilon \leq u$ because Player I could always play as if he were in this ordinary tug-of-war situation so he can do at least as well in the obstacle game.

Lemma 8. *Let v be a supersolution to the DPP, that is, a function that satisfies*

$$\begin{cases} v(x) \geq F(x), & \text{in } \Gamma, \\ v(x) \geq \Psi(x), & \text{in } \Omega, \\ v(x) \geq \frac{1}{2} \left(\sup_{B_\epsilon(x)} v(y) + \inf_{B_\epsilon(x)} v(y) \right), & \text{in } \Omega, \end{cases}$$

then we have

$$u_I^\epsilon(x) \leq v(x).$$

Proof. If $x_0 \in A^{u_I^\epsilon}$ then $u_I^\epsilon(x_0) = \Psi(x_0) \leq v(x_0)$. So we assume $x_0 \notin A^{u_I^\epsilon}$. In $\Omega \setminus A^{u_I^\epsilon}$ we have

$$u_I^\epsilon(x) = \frac{1}{2} \sup_{B_\epsilon(x)} u_I^\epsilon(y) + \frac{1}{2} \inf_{B_\epsilon(x)} u_I^\epsilon(y) > \Psi(x).$$

Let w^ϵ be the tug-of-war game, without obstacle, described in remark 7 (with $A^u = A^{u_I^\epsilon}$). Thus, since u_I^ϵ is ϵ discrete infinity harmonic in $\Omega \setminus A^{u_I^\epsilon}$, and w^ϵ is the unique such function by [14], we have $u_I^\epsilon = w^\epsilon \leq v$. \square

Lemma 9. *Let v be a subsolution away from the obstacle which also lies above the obstacle. That is, a function that satisfies*

$$\begin{cases} v(x) \leq F(x), & \text{in } \partial\Omega, \\ v(x) \geq \Psi(x), & \text{in } \Omega, \\ v(x) \leq \frac{1}{2} \left(\sup_{B_\epsilon(x)} v(y) + \inf_{B_\epsilon(x)} v(y) \right), & \text{in } \Omega \setminus A^v, \end{cases}$$

then we have

$$v(x) \leq u_{II}^\epsilon(x).$$

Proof. For $x_0 \in A^v$ we have $v(x_0) = \Psi(x_0) \leq u_{II}^\epsilon(x_0)$.

Assume $x_0 \in \Omega \setminus A^v$. Let w^ϵ be the value of discrete ϵ tug of war (without obstacle) on Ω as described in the remark 7 with $A = A^v$ and \hat{F} a Lipschitz extension of F to A^v such that $v \leq \hat{F} \leq u_{II}^\epsilon$.

Since $v \leq \hat{F}$ and v is subharmonic in $\Omega \setminus A^v$, by [14] we have that

$$v(x_0) \leq w^\epsilon(x_0).$$

We have

$$w^\epsilon \leq u_{II}^\epsilon$$

on all of Ω by the remark. Therefore

$$v \leq u_{II}^\epsilon$$

on $\Omega \setminus A^v$ as well. \square

Now we are ready to prove existence of a unique value of the game.

Proof of Theorem 1. We always have $u_I^\epsilon \leq u_{II}^\epsilon$. For $x_0 \in A^{u_I^\epsilon}$ we have

$$u_{II}^\epsilon(x_0) = \Psi(x_0) \leq u_I^\epsilon(x_0).$$

Assume $x_0 \in \Omega \setminus A^{u_I^\epsilon}$. Let w^ϵ be as in the remark with $A = A^{u_I^\epsilon}$. We have that u_{II}^ϵ is ϵ game harmonic on $\Omega \setminus A^{u_I^\epsilon}$ therefore, by [14], $u_{II}^\epsilon = w^\epsilon$. And, by the remark, we have

$$w^\epsilon(x_0) \leq u_I^\epsilon(x_0).$$

\square

We will now drop the subscripts and let $u^\epsilon = u_I^\epsilon = u_{II}^\epsilon$.

We now prove a small lemma that will be needed for the proof of Lemma 3, the Lewy-Stampaccia Lemma.

Lemma 10. *We have that Ψ is discrete ϵ infinity superharmonic on the coincidence set, i.e.*

$$\Psi(x) - \frac{1}{2} \left(\sup_{B_\epsilon(x)} \Psi(y) + \inf_{B_\epsilon(x)} \Psi(y) \right) \geq 0, \quad x \in A^{u^\epsilon}.$$

Proof. In the set A^{u^ϵ} we have

$$\begin{aligned} \Psi(x) = u^\epsilon(x) &= \max \left\{ \Psi(x), \frac{1}{2} \left(\sup_{B_\epsilon(x)} u^\epsilon(y) + \inf_{B_\epsilon(x)} u^\epsilon(y) \right) \right\} \\ &\geq \frac{1}{2} \left(\sup_{B_\epsilon(x)} u^\epsilon(y) + \inf_{B_\epsilon(x)} u^\epsilon(y) \right) \\ &\geq \frac{1}{2} \left(\sup_{B_\epsilon(x)} \Psi(y) + \inf_{B_\epsilon(x)} \Psi(y) \right). \end{aligned}$$

The last inequality holds because $u^\epsilon \geq \Psi$ on Ω . \square

Proof of Lemma 3. The first inequality is immediate from the dynamic programming principle. If $x \notin A^{u^\epsilon}$ then the second inequality is clear since u^ϵ is discrete ϵ infinity harmonic there. (Also but the dynamic programming principle). If $x \in A^{u^\epsilon}$, then, from the fact that $u^\epsilon \geq \Psi$ on Ω and $u^\epsilon(x) = \Psi(x)$ for $x \in A^{u^\epsilon}$, we get the last inequality. \square

4. LIMIT AS $\epsilon \rightarrow 0$.

In this section we prove our main result Theorem 5 regarding the limit of the game value functions.

Recall the discrete distance is given by $d_\epsilon(x, y) = \epsilon \lceil \frac{|x-y|}{\epsilon} \rceil$. The proof of Lemma 4, that the game value function is Lipschitz with respect to the d_ϵ metric, is now immediate from [14]. The proof of the Uniform Lipschitz Lemma 3.5 from [14] shows that the Lipschitz constant depends only on the Lipschitz constants of F and Ψ .

We are now in a position to apply the following variant of the Arzela-Ascoli Lemma. For its proof we refer the reader to [11] Lemma 4.2.

Lemma 11. *Fix $\delta > 0$. Let $\{u^\epsilon : \bar{\Omega} \rightarrow \mathbb{R}, \delta \geq \epsilon > 0\}$ be a set of functions such that*

- (1) *there exists $C > 0$ so that $|u^\epsilon(x)| < C$ for every $\delta \geq \epsilon > 0$ and every $x \in \bar{\Omega}$,*
- (2) *given $\eta > 0$ there are constants r_0 and ϵ_0 such that for every $\epsilon < \epsilon_0$ and any $x, y \in \bar{\Omega}$ with $|x - y| < r_0$ it holds*

$$|u^\epsilon(x) - u^\epsilon(y)| < \eta.$$

Then, there exists a uniformly continuous function $u : \bar{\Omega} \rightarrow \mathbb{R}$ and a subsequence denoted by $\{u^{\epsilon_j}\}$ such that

$$u^{\epsilon_j} \rightarrow u \quad \text{uniformly in } \bar{\Omega},$$

as $j \rightarrow \infty$.

Theorem 12. *If F and Ψ are Lipschitz continuous functions then there exists a subsequence of the values of the game u^{ϵ_j} that converges uniformly to a continuous function u in $\bar{\Omega}$,*

$$\lim_{\epsilon_j \rightarrow 0} u^{\epsilon_j} = u.$$

Proof. Lemma 11 can be applied since condition 1 holds with $C = \max\{F(x), \psi(x)\}$ and for condition 2 we can take, for instance, $\epsilon_0 = \delta$ and $r_0 = \frac{\delta}{L}$, where L is the Lipschitz constant of u^δ with respect to d_δ which does not depend on δ . \square

Remark 13. If we assume that Ψ is C^2 the Lewy-Stampacchia estimate gives that there exists $K > 0$ such that

$$0 \leq u^\epsilon(x) - \frac{1}{2} \left(\sup_{B_\epsilon(x)} u^\epsilon(y) + \inf_{B_\epsilon(x)} u^\epsilon(y) \right) \leq K\epsilon^2.$$

From here it follows that we can get estimates on the values u^ϵ that allow us to pass to the limit using the variant of the well-known Arzela-Ascoli type result from [11], Lemma 11.

Next we prove that this uniform limit of the values of the game is the viscosity solution of the obstacle problem for the infinity Laplacian. We are now ready to prove our main result, Theorem 5

We prove this theorem by comparing the u^{ϵ_j} to appropriately defined discrete harmonic v^{ϵ_j} with fixed boundary conditions which we know converge to an infinity harmonic function. We prove uniqueness by proving that our limit is the least super harmonic function that lies above the obstacle.

Proof of Theorem 5. We first prove that our limit is infinity harmonic where it lies strictly above the obstacle. Passing to a subsequence if necessary we let $u = \lim u^{\epsilon_j}$. Fix $x_0 \in \Omega \setminus A^u$, and choose r such that $B_r(x_0)$ is included in the set $\Omega \setminus A^u$ (which is open). Given δ , for ϵ_j small enough, we have that $|u - u^{\epsilon_j}| < \delta$ on $B_r(x_0)$.

Define v^{ϵ_j} to be the discrete ϵ_j harmonic function that agrees with u on $\partial B_r(x_0)$. Then we have $v^{\epsilon_j} - \delta < u^{\epsilon_j} < v^{\epsilon_j} + \delta$ on $\partial B_r(x_0)$. Since $v^{\epsilon_j} - \delta$ and $v^{\epsilon_j} + \delta$ are also discrete harmonic on $B_r(x_0)$ with lower and higher boundary values respectively than u^{ϵ_j} , with the help of the comparison principle for discrete harmonic functions we get the inequalities on all of $B_r(x_0)$. By [11] and [14], we know that v^{ϵ_j} converges uniformly to an infinity harmonic function v on $B_r(x_0)$. Therefore, by the sandwich lemma and sending $\delta \rightarrow 0$,

we have that $u = v$ thus u is infinity harmonic on $B_r(x_0)$ and therefore u is infinity harmonic on $\Omega \setminus A^u$.

Now let $x_0 \in A^u$. We have that Ψ is C^2 and for ϵ small enough satisfies

$$\Psi(x_0) \geq \frac{1}{2} \sup_{y \in B_\epsilon(x_0)} \Psi(y) + \frac{1}{2} \inf_{y \in B_\epsilon(x_0)} \Psi(y),$$

therefore, in A^u we have that

$$-\Delta_\infty \Psi = -\langle D^2 u \frac{D\Psi}{|D\Psi|}, \frac{D\Psi}{|D\Psi|} \rangle \geq 0.$$

From this and the proof of Lemma 3 we have that u also satisfies this inequality on A^u .

Therefore we have that the uniform limit of a subsequence of the values of the game, u , satisfies

$$-\Delta_\infty u = 0, \quad \text{in } \Omega \setminus A^u, \quad \text{and} \quad -\Delta_\infty u \geq 0, \quad \text{in } \Omega.$$

We now prove uniqueness. We define u_∞ to be the least infinity superharmonic function that is above the obstacle and the boundary function.

Since u is infinity superharmonic and above the obstacle and boundary function we get the inequalities $u \geq u_\infty \geq \Psi$ from which we have

$$A^u \subset A^{u_\infty}.$$

so on A^u we have $u = u_\infty$.

Now, in the set $\Omega \setminus A^u$, u is a solution to $-\Delta_\infty u = 0$ and u_∞ is a supersolution with the same boundary values ($u_\infty = u = F$ on Γ and $u_\infty = u = \Psi$ on A^u). Therefore, the comparison principle for Δ_∞ implies that

$$u_\infty \geq u$$

in $\Omega \setminus A^u$. And then we conclude that

$$u_\infty = u.$$

Since we have uniqueness of the limit, the whole sequence u^ϵ converges uniformly. \square

4.1. Convergence of the contact sets. We now simplify notation slightly and let $A^{\epsilon_j} := A^{u^{\epsilon_j}}$ and we discuss the convergence of the contact sets of the u^{ϵ_j} to the contact set of the limit function u .

We define

$$\limsup_{j \rightarrow \infty} A^{\epsilon_j} = \bigcap_{p=1}^{\infty} \bigcup_{j=p}^{\infty} A^{\epsilon_j} \quad \text{and} \quad \liminf_{j \rightarrow \infty} A^{\epsilon_j} = \bigcup_{p=1}^{\infty} \bigcap_{j=p}^{\infty} A^{\epsilon_j}.$$

Now, let us define $\limsup_{\epsilon \rightarrow 0} A^\epsilon$ and $\liminf_{\epsilon \rightarrow 0} A^\epsilon$ as

$$\limsup_{\epsilon \rightarrow 0} A^\epsilon = \bigcup_{\epsilon_j \rightarrow 0} \limsup_{j \rightarrow \infty} A^{\epsilon_j},$$

that is, the smallest set that contains all possible limits along subsequences, and

$$\liminf_{\epsilon \rightarrow 0} A^\epsilon = \bigcap_{\epsilon_j \rightarrow 0} \liminf_{j \rightarrow \infty} A^{\epsilon_j},$$

that is, the largest set that is included in every possible sequential limit.

We have an upper bound for $\limsup_{\epsilon \rightarrow 0} A^\epsilon$.

Lemma 14. *It holds that*

$$\limsup_{\epsilon \rightarrow 0} A^\epsilon \subset A^u.$$

Proof. Let $K \subset \subset \Omega \setminus A^u$ and so $V = \Omega \setminus K$ is a neighborhood of A^u . There exists an η such that $u - \Psi > \eta$ in K . By the uniform convergence there exists an ϵ_0 depending on K such that $u^\epsilon - \Psi > \eta/2$ for $\epsilon < \epsilon_0$. Then we have for every $\epsilon < \epsilon_0$,

$$A^\epsilon \subset V.$$

Thus we have

$$\limsup_{\epsilon_j \rightarrow 0} A^{\epsilon_j} \subset V$$

for any sequence $\epsilon_j \rightarrow 0$ and for any neighborhood V of A^u . Therefore

$$\limsup_{\epsilon \rightarrow 0} A^\epsilon \subset A^u.$$

□

To obtain a lower bound for $\liminf_{\epsilon \rightarrow 0} A^\epsilon$ we need to assume an extra condition on the obstacle.

Lemma 15. *Assume that Ψ satisfies $-\Delta_\infty \Psi(x_0) > 0$ in the viscosity sense in $(A^u)^o$ then we have*

$$\overline{(A^u)^o} \subset \liminf_{\epsilon \rightarrow 0} A^\epsilon.$$

Proof. Fix $x_0 \in (A^u)^o$ and choose any δ such that $B_\delta(x_0) \subset (A^u)^o$. If

$$u^{\epsilon_j}(x) > \Psi(x) \text{ for all } x \in B_\delta(x_0) \text{ and some sequence } \epsilon_j \rightarrow 0$$

then

$$-\Delta_\infty^{\epsilon_j} u^{\epsilon_j}(x) = 0 \text{ for all } x \in B_\delta(x_0) \text{ and every } \epsilon_j$$

therefore, by the argument in the proof of Theorem 5,

$$-\Delta_\infty u(x) = 0 \text{ for } x \in B_{\delta/2}(x_0)$$

in the viscosity sense. As $B_{\delta/2}(x_0) \subset (A^u)^o$ we have that $u = \Psi$ there and hence we have

$$-\Delta_\infty \Psi(x_0) = 0$$

a contradiction with our hypothesis. Therefore, for any sequence $\epsilon_j \rightarrow 0$ there exists $x_j \in A^{\epsilon_j}$ such that $x_j \rightarrow x_0$. Hence, $(A^u)^o \subset \liminf_{\epsilon_j \rightarrow 0} A^{\epsilon_j}$, and since $\liminf_{\epsilon_j \rightarrow 0} A^{\epsilon_j}$ is a closed set we get $\overline{(A^u)^o} \subset \liminf_{\epsilon_j \rightarrow 0} A^{\epsilon_j}$ for every sequence $\epsilon_j \rightarrow 0$. Therefore $\overline{(A^u)^o} \subset \liminf_{\epsilon \rightarrow 0} A^\epsilon$. \square

An immediate consequence of the previous two lemmas is the following result.

Theorem 16. *Assume that Ψ satisfies $-\Delta_\infty \Psi(x_0) > 0$ in the viscosity sense in $(A^u)^o$ and also assume that the contact set satisfies $\overline{(A^u)^o} = A^u$, then we have*

$$\lim_{\epsilon \rightarrow 0} A^\epsilon = \liminf_{\epsilon \rightarrow 0} A^\epsilon = \limsup_{\epsilon \rightarrow 0} A^\epsilon = A^u.$$

Remark 17. For a Lipschitz obstacle, it may happen that $(A^u)^o = \emptyset$. In fact, take in $\Omega = B_1(0)$ and boundary function $F(x) = 0$ and obstacle $\Psi(x) = -3|x| + 1$. The solution to the obstacle problem for the infinity Laplacian is given by the cone $u(x) = -|x| + 1$ and the contact set A^u is just a single point $A^u = \{0\}$.

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