

Nonlocal higher order evolution equations.

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Abstract. In this paper we study the asymptotic behavior of solutions to the nonlocal operator $u_t(x, t) = (-1)^{n-1} (J * Id - 1)^n (u(x, t))$, $x \in \mathbb{R}^N$ which is the nonlocal analogous to the higher order local evolution equation $v_t = (-1)^{n-1} (\Delta)^n v$. We prove that solutions to both equations have the same asymptotic decay rate as t goes to infinity. Moreover, we prove that the solutions of the nonlocal problem converge to the solution of the higher order problem with right hand side given by powers of the Laplacian when the kernel J is rescaled in an appropriate way.

Keywords. Nonlocal diffusion, higher order, asymptotic behavior.
 2000 *Mathematics Subject Classification.* 35B40, 45A07, 45G10.

1. Introduction

Our main concern in this paper is the asymptotic behavior of solutions of a nonlocal diffusion operator of higher order in the whole $\mathbb{R}^N$, $N \geq 1$.

Let us consider the following nonlocal evolution problem:

$$\begin{cases} u_t(x, t) = (-1)^{n-1} (J * Id - 1)^n (u(x, t)) \\ \quad = (-1)^{n-1} \left(\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (J*)^k(u) \right) (x, t), \\ u(x, 0) = u_0(x), \end{cases} \quad (1.1)$$

for $x \in \mathbb{R}^N$ and $t > 0$. Here $(J * u)(x, t) = \int_{\mathbb{R}^N} J(x - y)u(y, t) dy$ is the usual convolution of J and u and $(J*)^k(u)$ denotes the convolution with J iterated k times. $J \in C(\mathbb{R}^N, \mathbb{R})$ is a nonnegative, radial function with $\int_{\mathbb{R}^N} J(x) dx = 1$ and $u_0 \in L^1(\mathbb{R}^N)$ denotes the initial condition for (1.1). We call equation (1.1) a higher-order nonlocal diffusion equation since the diffusion of the density u at a point x and time t not only depends on u and its derivatives at the point (x, t) , but on all the values of u in a fixed neighborhood of x through the convolution term $J * u$.

Nonlocal problems like (1.1) with $n = 1$, i.e.,

$$\begin{cases} u_t(x, t) = (J * u)(x, t) - u(x, t), & x \in \mathbb{R}^N, t > 0, \\ u(x, 0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \quad (1.2)$$

have been recently widely used to model diffusion processes. In this case, $u(x, t)$ is interpreted as the density of a single population at the point x at time t and $J(x-y)$ is the probability of “jumping” from location y to location x . The convolution $(J * u)(x)$ is then the rate at which individuals arrive to position x from all other positions, while $-u(x, t) = -\int_{\mathbb{R}^N} J(y-x)u(y, t) dy$ is the rate at which they leave position x to reach any other position. Hence, in absence of external sources, we obtain the evolution problem (1.2), see [23]. For more references concerning these evolution problems and their stationary counterparts we quote for instance [4], [8], [16], [17], [18], [19], [20] and [35], devoted to travelling front type solutions. The problem (1.2) is considered in [10], [27], while the “Neumann” boundary condition for the same problem is treated in [1], [14] and [15]. See also [28] for the appearance of convective terms and [12], [13] for other interesting features in related nonlocal problems.

Recently nonlocal equations like (1.2) also found applications in image processing. The main advantage of nonlocal operators in image processing is the ability to process both structures (like edges) but also textures within the same framework. In [5] a nonlocal filter, referred to as nonlocal means, was suggested for image denoising. A variational understanding of this filter was first presented in [29] as a nonconvex functional and later in [25] as a convex quadratic functional. In the latter reference the authors investigated the functional

$$J(u) = \frac{1}{2} \int_{\Omega \times \Omega} |u(x) - u(y)|^2 w(x, y) dx dy,$$

where the weight function $w(x, y) \in \Omega \times \Omega$ is positive and symmetric, i.e. $w(x, y) = w(y, x)$. The proposed flow for minimizing the energy $J(u)$ was then defined as

$$\begin{cases} u_t(x) = \int_{\Omega} (u(y) - u(x)) w(x, y) dy, & x \in \Omega, \\ u(x, 0) = u_0(x), \end{cases} \quad (1.3)$$

taking the given (noisy) image u_0 as the initial condition. With $w(x, y) = J(x-y)$ equation (1.3) has the same structure as the nonlocal equation (1.2).

Note that in our problem (1.1) we just have the iteration k -times of the nonlocal operator $J * u - u$ as right hand side of the equation. This can be seen as a nonlocal generalization of higher order equations of the form

$$v_t(x, t) = -A^n (-\Delta)^{\frac{\alpha n}{2}} v(x, t), \quad (1.4)$$

with A and α are positive constants specified later in this section. Note that when $\alpha = 2$ (1.4) is just $v_t(x, t) = -A^n (-\Delta)^n v(x, t)$. Higher order diffusions of this type appear in various applications. The Cahn-Hilliard equation, for instance, is a

fourth order reaction diffusion equation which models phase separation and coarsening of binary alloys, see [22] for more details and references. A modified Cahn-Hilliard equation was further proposed in [6] and [7] for inpainting (i.e., image interpolation) of binary images. Another fourth order example is the Kuramoto-Sivashinsky equation (cf. e.g. [30]), used in the study of spatiotemporal chaos (cf. [21]). In both equations a linear fourth order diffusion as in (1.4) for $n = \alpha = 2$ is involved. Nonlocal higher order problems have been, for instance, proposed as models for periodic phase separation. Here the nonlocal character of the problem is associated with long-range interactions of "particles" in the system. An example is the nonlocal Cahn-Hilliard equation (cf. e.g. [26], [31], [32]).

Here we propose (1.1) as a model for higher order nonlocal evolution. For this model, we first prove existence and uniqueness of a solution, but our main aim is to study the asymptotic behaviour as $t \rightarrow \infty$ of solutions to (1.1). Moreover, we prove that solutions to (1.1) converge to the solution to (1.4) when the problem is rescaled in an appropriate way.

Now, let us proceed with the precise description of our main results.

Statement of the results. For a function f we denote by $\hat{f}$ the Fourier transform of f and by $\check{f}$ the inverse Fourier transform of f . Our hypotheses on the convolution kernel J that we will assume throughout the paper are:

The kernel $J \in C(\mathbb{R}^N, \mathbb{R})$ is a nonnegative, radial function with total mass equals one, $\int_{\mathbb{R}^N} J(x) dx = 1$. This means that J is a radial density probability which implies that its Fourier transform verifies $|\hat{J}(\xi)| \leq 1$ with $\hat{J}(0) = 1$. Moreover, we assume that

$$\hat{J}(\xi) = 1 - A |\xi|^\alpha + o(|\xi|^\alpha) \quad \text{for } \xi \rightarrow 0, \quad (1.5)$$

for some $A > 0$ and $\alpha > 0$.

Under these conditions on J we have the following results. First, we show existence and uniqueness of a solution

Theorem 1.1. *Let $u_0 \in L^1(\mathbb{R}^N)$ such that $\hat{u}_0 \in L^1(\mathbb{R}^N)$. There exists a unique solution $u \in C^0([0, \infty); L^1(\mathbb{R}^N)^N)$ of (1.1) that, in Fourier variables, is given by the explicit formula,*

$$\hat{u}(\xi, t) = e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} \hat{u}_0(\xi).$$

Next, we deal with the asymptotic behavior as $t \rightarrow \infty$.

Theorem 1.2. *Let u be a solution of (1.1) with $u_0, \hat{u}_0 \in L^1(\mathbb{R}^N)$. Then the asymptotic behavior of $u(x, t)$ is given by*

$$\lim_{t \rightarrow +\infty} t^{\frac{N}{\alpha n}} \max_x |u(x, t) - v(x, t)| = 0,$$

where v is the solution of $v_t(x, t) = -A^n (-\Delta)^{\frac{\alpha n}{2}} v(x, t)$ with initial condition $v(x, 0) = u_0(x)$ and A and α as in (1.5). Moreover, we have that there exists a constant $C > 0$ such that

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq C t^{-\frac{N}{\alpha n}}$$

and the asymptotic profile is given by

$$\lim_{t \rightarrow +\infty} \max_y \left| t^{\frac{N}{\alpha n}} u(yt^{\frac{1}{\alpha n}}, t) - \|u_0\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G}_A(\xi) = e^{-A^n |\xi|^{\alpha n}}$.

Finally, we show that solutions to nonlocal problems like (1.1), rescaled in an appropriate way, converge to a solution to the problem (1.4) as the scaling parameter tends to 0.

Theorem 1.3. *Let u_ε be the unique solution to*

$$\begin{cases} (u_\varepsilon)_t(x, t) = (-1)^{n-1} \frac{(J_\varepsilon * Id - 1)^n}{\varepsilon^{\alpha n}} (u_\varepsilon(x, t)), \\ u(x, 0) = u_0(x), \end{cases} \quad (1.6)$$

where $J_\varepsilon(s) = \varepsilon^{-N} J(\frac{s}{\varepsilon})$. Then, for every $T > 0$, we have

$$\lim_{\varepsilon \rightarrow 0} \|u_\varepsilon - v\|_{L^\infty(\mathbb{R}^N \times (0, T))} = 0,$$

where v is the solution to the local problem $v_t(x, t) = -A^n (-\Delta)^{\frac{\alpha n}{2}} v(x, t)$ with the same initial condition $v(x, 0) = u_0(x)$.

Organization of the paper. The rest of the paper is organized as follows: in Section 2 we prove existence and uniqueness of a solution; in Section 3 we deal with the asymptotic behavior and, finally, in Section 4 we approximate the usual higher order problem by nonlocal ones.

2. Existence and uniqueness. Proof of Theorem 1.1

To prove existence and uniqueness of solutions we make use of the Fourier transform.

Proof of Theorem 1.1. We have

$$u_t(x, t) = (-1)^{n-1} \left(\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (J*)^k(u) \right) (x, t).$$

Applying the Fourier transform to this equation we obtain

$$\begin{aligned} \hat{u}_t(\xi, t) &= (-1)^{n-1} \sum_{k=0}^n \binom{n}{k} (-1)^{n-k} (\hat{J}(\xi))^k \hat{u}(\xi, t) \\ &= (-1)^{n-1} (\hat{J}(\xi) - 1)^n \hat{u}(\xi, t). \end{aligned}$$

Hence

$$\hat{u}(\xi, t) = e^{(-1)^{n-1} (\hat{J}(\xi) - 1)^n t} \hat{u}_0(\xi).$$

Since $\hat{u}_0(\xi) \in L^1(\mathbb{R}^N)$ and $e^{(-1)^{n-1} (\hat{J}(\xi) - 1)^n t}$ is continuous and bounded, $\hat{u}(\cdot, t) \in L^1(\mathbb{R}^N)$ and the result follows by taking the inverse Fourier transform. $\square$

Now we prove a lemma concerning the fundamental solution of (1.1).

Lemma 2.1. *The fundamental solution w of (1.1), that is the solution of the equation with initial condition $u_0 = \delta_0$, can be decomposed as*

$$w(x, t) = e^{-t} \delta_0(x) + v(x, t), \quad (2.1)$$

with $v(x, t)$ smooth. Moreover, if u is a solution of (1.1) it can be written as

$$u(x, t) = (w * u_0)(x, t) = \int_{\mathbb{R}^N} w(x - z, t) u_0(z) dz.$$

Proof. By the previous result we have

$$\hat{w}_t(\xi, t) = (-1)^{n-1} (\hat{J}(\xi) - 1)^n \hat{w}(\xi, t).$$

Hence, as the initial datum verifies $\hat{w}_0 = \hat{\delta}_0 = 1$, we get

$$\hat{w}(\xi, t) = e^{(-1)^{n-1} (\hat{J}(\xi) - 1)^n t} = e^{-t} + e^{-t} \left(e^{[(-1)^{n-1} (\hat{J}(\xi) - 1)^n + 1]t} - 1 \right).$$

The first part of the lemma follows applying the inverse Fourier transform.

To finish the proof we just observe that $w * u_0$ is a solution of (1.1) with $(w * u_0)(x, 0) = u_0(x)$. $\square$

3. Asymptotic behavior. Proof of Theorem 1.2

Next we prove the first part of Theorem 1.2.

Theorem 3.1. *Let u be a solution of (1.1) with $u_0, \hat{u}_0 \in L^1(\mathbb{R}^N)$. Then, the asymptotic behavior of $u(x, t)$ is given by*

$$\lim_{t \rightarrow +\infty} t^{\frac{N}{\alpha n}} \max_x |u(x, t) - v(x, t)| = 0,$$

where v is the solution of $v_t(x, t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x, t)$, with initial condition $v(x, 0) = u_0(x)$.

Proof. As in the previous section, we have in Fourier variables,

$$\hat{u}_t(\xi, t) = (-1)^{n-1} (\hat{J}(\xi) - 1)^n \hat{u}(\xi, t).$$

Hence

$$\hat{u}(\xi, t) = e^{(-1)^{n-1} (\hat{J}(\xi) - 1)^n t} \hat{u}_0(\xi).$$

On the other hand, let $v(x, t)$ be a solution of $v_t(x, t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x, t)$, with the same initial datum $v(x, 0) = u_0(x)$. Solutions of this equation are understood in the sense that

$$\hat{v}(\xi, t) = e^{-A^n |\xi|^{\alpha n} t} \hat{u}_0(\xi).$$

Hence in Fourier variables

$$\begin{aligned}
\int_{\mathbb{R}^N} |\hat{u} - \hat{v}|(\xi, t) d\xi &= \int_{\mathbb{R}^N} \left| (e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} - e^{-A^n |\xi|^{\alpha n} t}) \hat{u}_0(\xi) \right| d\xi \\
&\leq \int_{|\xi| \geq r(t)} \left| (e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} - e^{-A^n |\xi|^{\alpha n} t}) \hat{u}_0(\xi) \right| d\xi \\
&\quad + \int_{|\xi| < r(t)} \left| (e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} - e^{-A^n |\xi|^{\alpha n} t}) \hat{u}_0(\xi) \right| d\xi \\
&= I + II,
\end{aligned}$$

where I and II denote the first and the second integral respectively. To get a bound for I we decompose it in two parts,

$$\begin{aligned}
I &\leq \int_{|\xi| \geq r(t)} \left| e^{-A^n |\xi|^{\alpha n} t} \hat{u}_0(\xi) \right| d\xi + \int_{|\xi| \geq r(t)} \left| e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} \hat{u}_0(\xi) \right| d\xi \\
&= I_1 + I_2.
\end{aligned}$$

First we consider I_1 . Setting $\eta = \xi t^{1/(\alpha n)}$ and writing I_1 in the new variable η we get,

$$I_1 \leq \|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta| \geq r(t)t^{\frac{1}{\alpha n}}} e^{-A^n |\eta|^{\alpha n} t^{-\frac{N}{\alpha n}}} d\eta,$$

and hence

$$t^{\frac{N}{\alpha n}} I_1 \leq \|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta| \geq r(t)t^{\frac{1}{\alpha n}}} e^{-A^n |\eta|^{\alpha n}} d\eta \xrightarrow{t \rightarrow \infty} 0$$

if we impose that

$$r(t)t^{\frac{1}{\alpha n}} \xrightarrow{t \rightarrow \infty} \infty. \quad (3.1)$$

To deal with I_2 we have to use different arguments for n even and n odd. Let us begin with the easier case of an even n .

- *n even* - Using our hypotheses on J we get

$$I_2 \leq C e^{-t},$$

with $r(t) \xrightarrow{t \rightarrow \infty} 0$ and therefore

$$t^{\frac{N}{\alpha n}} I_2 \leq C e^{-t} t^{\frac{N}{\alpha n}} \xrightarrow{t \rightarrow \infty} 0.$$

Now consider the case when n is odd.

- *n odd* - From our hypotheses on J we have that $\hat{J}$ verifies

$$\hat{J}(\xi) \leq 1 - A |\xi|^\alpha + |\xi|^\alpha h(\xi),$$

where h is bounded and $h(\xi) \rightarrow 0$ as $\xi \rightarrow 0$. Hence there exists $D > 0$ and a constant a such that

$$\hat{J}(\xi) \leq 1 - D |\xi|^\alpha, \quad \text{for } |\xi| \leq a.$$

Moreover, because $|\hat{J}(\xi)| \leq 1$ and J is a radial function, there exists a $\delta > 0$ such that

$$\hat{J}(\xi) \leq 1 - \delta, \quad \text{for } |\xi| \geq a.$$

Therefore I_2 can be bounded by

$$\begin{aligned} I_2 &\leq \int_{a \geq |\xi| \geq r(t)} \left| e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} \hat{u}_0(\xi) \right| d\xi \\ &\quad + \int_{|\xi| \geq a} \left| e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} \hat{u}_0(\xi) \right| d\xi \\ &\leq \|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \int_{a \geq |\xi| \geq r(t)} e^{-D^n |\xi|^{\alpha n} t} d\xi + C e^{-\delta^n t}. \end{aligned}$$

Changing variables as before, $\eta = \xi t^{1/(\alpha n)}$, we get

$$\begin{aligned} t^{\frac{N}{\alpha n}} I_2 &\leq \|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \int_{at^{\frac{1}{\alpha n}} \geq |\eta| \geq r(t)t^{\frac{1}{\alpha n}}} e^{-D^n |\eta|^{\alpha n}} d\eta + C t^{\frac{N}{\alpha n}} e^{-\delta^n t} \\ &\leq \|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta| \geq r(t)t^{\frac{1}{\alpha n}}} e^{-D^n |\eta|^{\alpha n}} d\eta + C t^{\frac{N}{\alpha n}} e^{-\delta^n t} \rightarrow 0, \end{aligned}$$

as $t \rightarrow \infty$ if (3.1) holds.

It remains only to estimate II. We proceed as follows

$$II = \int_{|\xi| < r(t)} e^{-A^n |\xi|^{\alpha n} t} \left| e^{t[(-1)^{n-1}(\hat{J}(\xi)-1)^n + A^n |\xi|^{\alpha n}]} - 1 \right| |\hat{u}_0(\xi)| d\xi.$$

Applying the binomial formula and taking into account the two different cases when n is even and odd we can conclude

$$t^{\frac{N}{\alpha n}} II \leq C t^{\frac{N}{\alpha n}} \int_{|\xi| < r(t)} e^{-A^n |\xi|^{\alpha n} t} t (|\xi|^{\alpha n} h(\xi) + K(|\xi|^{\alpha k} h(\xi)^k)) d\xi,$$

where $K(|\xi|^{\alpha k} h(\xi)^k)$ is a polynomial in $|\xi|^\alpha$ and $h(\xi)$ with $0 < k \leq n$ and provided we impose

$$t(r(t))^{\alpha n} h(r(t)) \rightarrow 0 \quad \text{as } t \rightarrow \infty. \quad (3.2)$$

In this case we have

$$\begin{aligned} t^{\frac{N}{\alpha n}} II &\leq C \int_{|\eta| < r(t)t^{\frac{1}{\alpha n}}} e^{-A^n |\eta|^{\alpha n}} (|\eta|^{\alpha n} h(\eta/t^{1/(\alpha n)}) \\ &\quad + K(|\eta|^{\alpha k} h(\eta/t^{1/(\alpha n)})^k) \frac{1}{t^{(\alpha k)/(\alpha n)}}) d\eta. \end{aligned}$$

To show the convergence of II to zero we use dominated convergence. Because of our assumption on h we know $h(\eta/t^{1/(\alpha n)}) \rightarrow 0$ as $t \rightarrow \infty$ (note that clearly also $h(\eta/t^{1/(\alpha n)})^k$ converges to zero for every $k > 0$). Further the integrand is dominated by $\|h\|_{L^\infty(\mathbb{R}^N)} e^{-A^n |\eta|^{\alpha n}} |\eta|^{\alpha n}$, which belongs to $L^1(\mathbb{R}^N)$.

Combining this with our previous results we have that

$$t^{\frac{N}{\alpha n}} \int_{\mathbb{R}^N} |\hat{u} - \hat{v}|(\xi, t) d\xi \leq t^{\frac{N}{\alpha n}} (I + II) \rightarrow 0 \quad \text{as } t \rightarrow \infty, \quad (3.3)$$

provided we can find a $r(t) \rightarrow 0$ as $t \rightarrow \infty$ which fulfills both conditions (3.1) and (3.2). This is done in Lemma 3.1, which is postponed just after the present proof. To conclude we only have to observe that from the convergence of the Fourier transforms $\hat{u}(\cdot, t) - \hat{v}(\cdot, t) \rightarrow 0$ in L^1 the convergence of $u - v$ in L^∞ follows. Indeed, from (3.3) we obtain

$$t^{\frac{N}{\alpha n}} \max_x |u(x, t) - v(x, t)| \leq t^{\frac{N}{\alpha n}} \int_{\mathbb{R}^N} |\hat{u} - \hat{v}|(\xi, t) d\xi \rightarrow 0, \quad t \rightarrow \infty,$$

which ends the proof of the theorem. $\square$

The following Lemma shows that there exists a function $r(t)$ satisfying (3.1) and (3.2), as required in the proof of the previous theorem.

Lemma 3.1. *Given a function $h \in C(\mathbb{R}, \mathbb{R})$ such that $h(\rho) \rightarrow 0$ as $\rho \rightarrow 0$ with $h(\rho) > 0$ for small ρ , there exists a function r with $r(t) \rightarrow 0$ as $t \rightarrow \infty$ which satisfies*

$$\lim_{t \rightarrow \infty} r(t) t^{\frac{1}{\alpha n}} = \infty$$

and

$$\lim_{t \rightarrow \infty} t(r(t))^{\alpha n} h(r(t)) = 0.$$

Proof. For fixed t large enough, we choose $r(t)$ as a solution of

$$r(h(r))^{\frac{1}{2\alpha n}} = t^{-\frac{1}{\alpha n}}. \quad (3.4)$$

This equation defines a function $r = r(t)$ which, by continuity arguments goes to zero as t tends to infinity, satisfying also the additional asymptotic conditions in the lemma. Indeed, if there exists $t_n \rightarrow \infty$ with no solution of (3.4) for $r \in (0, \delta)$ then $h(r) \equiv 0$ in $(0, \delta)$, which is a contradiction to our assumption that $h(r) > 0$ for r small. $\square$

As a consequence of Theorem 3.1, we obtain the following corollary which completes the results gathered in Theorem 1.2 in the Introduction.

Corollary 3.1. *The asymptotic behavior of solutions of (1.1) is given by*

$$\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} \leq \frac{C}{t^{\frac{N}{\alpha n}}}.$$

Moreover, the asymptotic profile is given by

$$\lim_{t \rightarrow +\infty} \max_y \left| t^{\frac{N}{\alpha n}} u(y t^{\frac{1}{\alpha n}}, t) - \|u_0\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G}_A(\xi) = e^{-A^n |\xi|^{\alpha n}}$.

Proof. From Theorem 3.1 we obtain that the asymptotic behavior is the same as the one for solutions of the evolution given by a power n of the fractional Laplacian. It is easy to check that the asymptotic behavior is in fact the one described in the statement of the corollary. In Fourier variables we have

$$\begin{aligned} \lim_{t \rightarrow \infty} \hat{v}(\eta t^{-\frac{1}{\alpha n}}, t) &= \lim_{t \rightarrow \infty} e^{-A^n |\eta|^{\alpha n}} \hat{u}_0(\eta t^{-\frac{1}{\alpha n}}) \\ &= e^{-A^n |\eta|^{\alpha n}} \hat{u}_0(0) \\ &= e^{-A^n |\eta|^{\alpha n}} \|u_0\|_{L^1(\mathbb{R}^N)}. \end{aligned}$$

Therefore

$$\lim_{t \rightarrow +\infty} \max_y \left| t^{\frac{N}{\alpha n}} v(y t^{\frac{1}{\alpha n}}, t) - \|u_0\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G}_A(\xi) = e^{-A^n |\xi|^{\alpha n}}$. $\square$

With similar arguments as in the proof of Theorem 3.1 one can prove that also the asymptotic behavior of the derivatives of solutions u of (1.1) is the same as the one for derivatives of solutions v of the evolution of a power n of the fractional Laplacian, assuming sufficient regularity of the solutions u of (1.1).

Theorem 3.2. *Let u be a solution of (1.1) with $u_0 \in W^{k,1}(\mathbb{R}^N)$, $k \leq \alpha n$ and $\hat{u}_0 \in L^1(\mathbb{R}^N)$. Then, the asymptotic behavior of $D^k u(x, t)$ is given by*

$$\lim_{t \rightarrow +\infty} t^{\frac{N+k}{\alpha n}} \max_x |D^k u(x, t) - D^k v(x, t)| = 0,$$

where v is the solution of $v_t(x, t) = -A^n (-\Delta)^{\frac{\alpha n}{2}} v(x, t)$ with initial condition $v(x, 0) = u_0(x)$.

Proof. We begin again by transforming our problem for u and v in a problem for the corresponding Fourier transforms $\hat{u}$ and $\hat{v}$. For this we consider

$$\begin{aligned} \max_x |D^k u(x, t) - D^k v(x, t)| &= \max_{\xi} \left| (\widehat{D^k u(\xi, t)})^\vee - (\widehat{D^k v(\xi, t)})^\vee \right| \\ &\leq \int_{\mathbb{R}^N} \left| \widehat{D^k u(\xi, t)} - \widehat{D^k v(\xi, t)} \right| d\xi = \int_{\mathbb{R}^N} |\xi|^k |\hat{u}(\xi, t) - \hat{v}(\xi, t)| d\xi. \end{aligned}$$

Showing $\int_{\mathbb{R}^N} |\xi|^k |\hat{u}(\xi, t) - \hat{v}(\xi, t)| d\xi \rightarrow 0$ as $t \rightarrow \infty$ works analogue to the proof of Theorem 3.1. The additional term $|\xi|^k$ is always dominated by the exponential terms. $\square$

4. Scaling the kernel. Proof of Theorem 1.3

In this section we show that the problem $v_t(x, t) = -A^n (-\Delta)^{\frac{\alpha n}{2}} v(x, t)$ can be approximated by nonlocal problems like (1.1) when rescaled in an appropriate way.

Proof of Theorem 1.3. The proof uses once more the explicit formula for the solutions in Fourier variables. We have, arguing exactly as before,

$$\hat{u}_\varepsilon(\xi, t) = e^{(-1)^{n-1} \frac{(\hat{J}_\varepsilon(\xi)-1)^n}{\varepsilon^{\alpha n}} t} \hat{u}_0(\xi).$$

and

$$\hat{v}(\xi, t) = e^{-A^n |\xi|^{\alpha n} t} \hat{u}_0(\xi).$$

Now, we just observe that $\hat{J}_\varepsilon(\xi) = \hat{J}(\varepsilon\xi)$ and therefore we obtain

$$\begin{aligned} \int_{\mathbb{R}^N} |\hat{u}_\varepsilon - \hat{v}|(\xi, t) d\xi &= \int_{\mathbb{R}^N} \left| e^{(-1)^{n-1} \frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{\alpha n}} t} - e^{-A^n |\xi|^{\alpha n} t} \right| \hat{u}_0(\xi) d\xi \\ &\leq \|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \left(\int_{|\xi| \geq r(\varepsilon)} \left| e^{(-1)^{n-1} \frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{\alpha n}} t} - e^{-A^n |\xi|^{\alpha n} t} \right| d\xi \right. \\ &\quad \left. + \int_{|\xi| < r(\varepsilon)} \left| e^{(-1)^{n-1} \frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{\alpha n}} t} - e^{-A^n |\xi|^{\alpha n} t} \right| d\xi \right). \end{aligned}$$

For $t \in [0, T]$ we can proceed as in the proof of Theorem 1.2 (Section 3) to obtain that

$$\max_x |u_\varepsilon(x, t) - v(x, t)| \leq \int_{\mathbb{R}^N} |\hat{u}_\varepsilon - \hat{v}|(\xi, t) d\xi \rightarrow 0, \quad \varepsilon \rightarrow 0.$$

We leave the details to the reader. $\square$

Acknowledgments

CBS partially supported by the WWTF project nr.CI06 003, the PhD program taking place at the University of Vienna, the FWF Wittgenstein award of Peter Markowich, project nr. Z-50 MAT, and the EPSRC fund at the University of Cambridge. JDR is partially supported by UBA X066, CONICET (Argentina) and SIMUMAT (Spain).

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