A LOGISTIC EQUATION WITH REFUGE AND NONLOCAL DIFFUSION

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Abstract. In this work we consider the nonlocal stationary nonlinear problem $(J*u)(x) - u(x) = -\lambda u(x) + a(x)u^p(x)$ in a domain Ω, with the Dirichlet boundary condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ and $p > 1$. The kernel J involved in the convolution $(J*u)(x) = \int_{\mathbb{R}^N} J(x-y)u(y)\,dy$ is a smooth, compactly supported nonnegative function with unit integral, while the weight $a(x)$ is assumed to be nonnegative and is allowed to vanish in a smooth subdomain Ω_0 of Ω. Both when $a(x)$ is positive and when it vanishes in a subdomain, we completely discuss the issues of existence and uniqueness of positive solutions, as well as their behavior with respect to the parameter λ.

1. Introduction

In this paper we deal with the following stationary nonlocal diffusion problem:

$$
\left\{
\begin{array}{ll}
(J*u)(x) - u(x) = -\lambda u(x) + a(x)u^p(x) & x \in \Omega, \\
\qquad u = 0 & x \in \mathbb{R}^N \setminus \Omega.
\end{array}
\right. \tag{1.1}
$$

Here Ω is a bounded domain of $\mathbb{R}^N$, $p > 1$ and λ is a real parameter. The operator $J*u-u$ is a nonlocal diffusion operator that has been used in biological models, see [20]. The kernel J is assumed to be a smooth, compactly supported nonnegative function (see precise assumptions below), and $J*u$ is the usual convolution, that is,

$$
(J*u)(x) = \int_{\mathbb{R}^N} J(x-y)u(y)\,dy.
$$

In (1.1) the condition $u = 0$ in $\mathbb{R}^N \setminus \Omega$ is the nonlocal analogue to the usual Dirichlet boundary condition $u|_{\partial\Omega} = 0$ imposed when one considers the usual Laplacian as the diffusion operator.

Problems related to (1.1) have been widely treated in the literature. The general problem

$$
\left\{
\begin{array}{ll}
(J*u)(x) - u(x) = f(x,u(x)) & x \in \Omega, \\
\qquad u = 0 & x \in \mathbb{R}^N \setminus \Omega,
\end{array}
\right. \tag{1.2}
$$

and its parabolic counterpart have been the subject of several works. In most of them, $\Omega = \mathbb{R}^N$, so that the Dirichlet condition is not present. We quote for instance [3], [5], [6], [8], [15], [16], [18], [19], [32] and [33], devoted to travelling front type solutions to the parabolic problem when $\Omega = \mathbb{R}$, and [4], [9], [10], [17], [31], which dealt with the study of problem (1.2) with a logistic type, bistable or power-like nonlinearity. The particular instance of the parabolic problem in $\mathbb{R}^N$ when $f = 0$ is considered in [7], [27], while the "Neumann" boundary condition for the same problem is treated in [1], [13] and [14]. See also [28] for the appearance of convective terms and [11], [12] for interesting features in other related nonlocal problems. We finally mention the paper [26], where some logistic equations and systems of Lotka-Volterra type are studied.

It is easy to see that solutions to (1.2) are critical points of the functional

$$\begin{aligned} H(u) = &\frac{1}{4}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))^2\,dx\,dy \\ &- \int_\Omega F(x,u(x))\,dx, \end{aligned} \tag{1.3}$$

where all functions are assumed to vanish in $\mathbb{R}^N\setminus\Omega$ and $F(\cdot,u)=\int_0^u f(\cdot,s)$. When the first integral in (1.3) is expanded in a Taylor series and only the first term is taken into account, we obtain the "approximate energy"

$$\widetilde{H}(u) = c\int_\Omega\int_\Omega |\nabla u(x)|^2 dx - \int_\Omega F(x,u(x))dx,$$

where $c = 1/(4N)\int_{\mathbb{R}^N} J(y)|y|^2 dy$ (see for instance [4]). Assuming with no loss of generality that $c = 1$, the critical points of this energy are weak solutions to the local problem

$$\begin{cases} -\Delta u(x) = f(x,u(x)) & x\in\Omega, \\ \quad u(x) = 0 & x\in\partial\Omega. \end{cases} \tag{1.4}$$

Thus it seems reasonable to expect that solutions to (1.2) share many properties with those of (1.4).

Hence before proceeding to the study of problem (1.1) is seems convenient to briefly discuss its local analogue

$$\begin{cases} -\Delta u(x) = \lambda u(x) - a(x)u^p(x) & x\in\Omega, \\ \quad u(x) = 0 & x\in\partial\Omega. \end{cases} \tag{1.5}$$

The structure of solutions to problem (1.5) is nowadays well understood. When $a(x)$ is strictly positive in $\overline{\Omega}$, there exists a unique positive solution if and only if $\lambda > \lambda_1(\Omega)$, where $\lambda_1(\Omega)$ is the first Dirichlet eigenvalue of the Laplacian in Ω. The situation changes drastically when the weight $a(x)$ is allowed to vanish in a smooth subdomain Ω_0 of Ω. It is shown that positive solutions can only exist when $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$ (see [21], [22], [30]). In this range, the positive solution is unique. Moreover, the exact behavior of this solution as $\lambda \to \lambda_1(\Omega)$ is determined: it diverges to infinity in Ω_0 while it remains bounded in $D = \Omega\setminus\overline{\Omega}_0$, converging to the minimal positive solution to

$$\begin{cases} -\Delta u(x) = \lambda_1(\Omega_0)u(x) - a(x)u^p(x) & x\in\Omega, \\ \quad u(x) = +\infty & x\in\partial\Omega, \end{cases}$$

(see also [23] for the study of this problem).

We will prove in the present paper that problem (1.1) behaves like (1.5) with respect to the existence and uniqueness issues, but it is completely different with regard to the asymptotic behavior of solutions.

We also mention that problem (1.5) has been frequently proposed to model the diffusion of a single species in Ω whose density is given by the function u. In that case, the coefficient λ represents the birth rate of the species, while $a(x)$ measures the intraspecific competition. In particular, the case $a = 0$ in Ω_0 means that Ω_0 can be considered as a "refuge" for the species, since it is free from competition there. In this regard, the main difference between problems (1.5) and (1.1) is that in the former the effect of diffusion is only local, while in the latter also long range effects are being taken into account.

We now state the precise assumptions that we impose on J.

Hypotheses: The kernel J will be assumed to be a C^1 function with a compact support, which, with no loss o generality, we always take to be the unit ball of $\mathbb{R}^N$. We moreover suppose that $J > 0$ in B_1, that J is even, i.e. $J(-x) = J(x)$ for every x, and that $\int_{\mathbb{R}^N} J(x)\, dx = 1$.

Without further mention, we are always assuming that J verifies these hypotheses.

Before coming to the statements of our theorems, let us make some comments on solutions of (1.1). By a solution to (1.1) we mean a function $u \in L^1(\mathbb{R}^N)$ which verifies (1.1) almost everywhere. However, we remark that with $u \in L^1(\mathbb{R}^N)$ it always follows $J * u \in C(\overline{\Omega})$ thanks to the regularity of J, so that $(1-\lambda)u + a(x)u^p \in C(\overline{\Omega})$. We will show that it is possible to invert u and then obtain that $u \in C(\overline{\Omega})$ for all possible positive solutions. This is not always the situation for nonlocal problems, since – at least for some bistable-like nonlinearities – discontinuous front-wave solutions for the associated evolution problem can be constructed (see [5]). We mention in passing that the solutions are strictly positive in $\overline{\Omega}$, and hence there is a jump discontinuity across $\partial\Omega$.

We can now state our results for problem (1.1). In all our subsequent results, the eigenvalue problem

$$\begin{cases} (J * u)(x) - u(x) = -\lambda u(x) & x \in \Omega, \\ \qquad u = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

whose properties will be analyzed in Section 2 (see Theorem 9) will play an important role. We only mention for the moment that there exists a unique principal eigenvalue, which will be denoted by $\lambda_1(\Omega)$. Moreover, we have that $0 < \lambda_1(\Omega) < 1$.

We begin with the case $a(x) > 0$ in $\overline{\Omega}$, which is somehow easier.

Theorem 1. *Assume $a \in C(\overline{\Omega})$ is strictly positive, and let $p > 1$. Then problem* (1.1) *admits a positive solution u_λ if and only if $\lambda > \lambda_1(\Omega)$. In that case, $u_\lambda \in C(\overline{\Omega})$, it is unique, increasing with respect to λ and verifies $u_\lambda \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \downarrow \lambda_1(\Omega)$. In addition, we have for $\lambda \geq 1$:*

$$\left(\frac{\lambda - 1}{a(x)}\right)^{\frac{1}{p-1}} \leq u_\lambda \leq \left(\frac{\lambda}{\inf_\Omega a}\right)^{\frac{1}{p-1}}, \tag{1.6}$$

and hence $u_\lambda \to \infty$ uniformly in $\overline{\Omega}$ as $\lambda \to \infty$.

As the estimates (1.6) show, the solution behaves essentially as $\lambda^{\frac{1}{p-1}}$ when $\lambda \to +\infty$. It would be of course desirable to know the exact asymptotic behavior as $\lambda \to +\infty$. This exact behavior is the content of the next theorem.

Theorem 2. *Assume $a \in C(\overline{\Omega})$ is strictly positive, and let $p > 1$. Let u_λ be the unique positive solution to* (1.1) *with $\lambda > \lambda_1(\Omega)$ given by Theorem 1. Then*

$$u_\lambda(x) \sim \left(\frac{\lambda}{a(x)}\right)^{\frac{1}{p-1}} \qquad \text{as } \lambda \to \infty,$$

uniformly in $\overline{\Omega}$.

Next, we consider problem (1.1) when the weight $a(x)$ is allowed to vanish in a subdomain of Ω. We assume

$$\text{(A)} \qquad a(x) \text{ vanishes in a smooth subdomain } \Omega_0,$$

and we always denote $D = \Omega \setminus \overline{\Omega}_0$. As in the local problem, the range of existence of solutions is now bounded above by a critical value, which can be characterized as a principal eigenvalue for the operator $J * u - u$.

Theorem 3. *Assume $a \in C(\overline{\Omega})$ verifies* (A)*, and let $p > 1$. Then* (1.1) *has a positive solution u_λ if and only if $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$. In that case, $u_\lambda \in C(\overline{\Omega})$, it is unique, increasing with respect to λ and verifies $u_\lambda \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \downarrow \lambda_1(\Omega)$ and $u_\lambda \to \infty$ uniformly in $\overline{\Omega}$ as $\lambda \uparrow \lambda_1(\Omega_0)$.*

Notice that the behavior of u_λ as $\lambda \uparrow \lambda_1(\Omega_0)$ differs from the one that holds for the local equation (1.5) where solutions are bounded above in a subdomain of Ω.

As in the situation where a does not vanish, we could ask for the asymptotic behavior of u_λ as $\lambda \uparrow \lambda_1(\Omega_0)$. As was to be expected, the behavior is more subtle than in Theorem 2. We first compare the solution u_λ with its maximum M_λ. It turns out that the behavior of u_λ depends on the "domain of influence" of the convolution with J. Since the support of J is assumed to be the unit ball, the sets

$$B_0 := \Omega_0, \qquad B_n = \{x \in \Omega \setminus B_{n-1} : \ \mathrm{dist}(x, B_{n-1}) < 1\},$$

and

$$\Gamma_0 = \partial\Omega_0 \cap \Omega, \qquad \Gamma_n = \{x \in \Omega \setminus B_{n-1} : \ \mathrm{dist}(x, B_{n-1}) = 1\},$$

will be important. Notice that, since Ω is bounded, there are only finitely many – say k – nonempty such sets. The solution u_λ diverges to infinity with a different rate in each set B_n.

Theorem 4. *Let $M_\lambda = \max_{\overline{\Omega}} u_\lambda$, so that $M_\lambda \to +\infty$ as $\lambda \uparrow \lambda_1(\Omega_0)$. Then*

$$\frac{u_\lambda(x)}{M_\lambda} \sim \phi(x) \qquad \text{uniformly in } \overline{\Omega}_0 \tag{1.7}$$

where ϕ is the positive eigenfunction associated to $\lambda_1(\Omega_0)$ normalized so that $\|\phi\|_\infty = 1$. Moreover, there exists functions $\Psi_1(x), \Psi_2(x), \ldots, \Psi_k(x)$ such that $\Psi_i > 0$ in B_n for $n = 1, \ldots, k$ and for $n = 1, 2, \ldots, k$,

$$\frac{u_\lambda(x)}{M_\lambda^{1/p^n}} \sim \Psi_n(x) \qquad \text{uniformly in compacts of } B_n \cup \Gamma_n. \tag{1.8}$$

Remark 1. It is clear from the change of rate of divergence of solutions in each B_n that $\Psi_n = 0$ on $\partial B_n \cap \partial B_{n+1}$ while $\Psi_n = \infty$ on $\partial B_n \cap \partial B_{n-1}$.

It seems desirable to obtain the exact behavior of the solution u_λ in terms of λ rather than M_λ. However, we content ourselves with a lower bound, assuming that the function $a(x)$ has a convenient decay near ∂D. We believe that the information provided by a is not enough to get a similar upper bound. We denote $D_\delta = \{x \in D : \operatorname{dist}(x, \partial D) < \delta\}$. Then we have

Theorem 5. *Assume $a \in C(\overline{\Omega})$ verifies* (A) *and $\sup_{D_\delta} a(x) = O(\delta^{1+\gamma})$ as $\delta \to 0$ for some $\gamma > 0$. Then there exists a positive constant C such that the unique positive solution u_λ to* (1.1) *verifies*

$$u_\lambda(x) \geq C(\lambda_1(\Omega_0) - \lambda)^{-\frac{\gamma}{p-1}} \quad \text{in } \Omega_0. \tag{1.9}$$

Moreover, there exist positive functions $c_1(x), \ldots, c_k(x)$ such that

$$u_\lambda(x) \geq c_n(x)(\lambda_1(\Omega) - \lambda)^{-\frac{\gamma}{p^n(p-1)}} \quad \text{in } B_n \tag{1.10}$$

for $n = 1, 2, \ldots, k$.

For the proof of this result we need in particular an important property of the principal eigenvalue $\lambda_1(\Omega)$ of the operator $J * u - u$ in the domain Ω, which will be considered in Section 2: $\lambda_1(\Omega)$ is differentiable with respect to differentiable perturbations of the domain. We refer to [24] for complete proofs.

Most of our results are also valid when the kernel J is assumed to be strictly positive in $\mathbb{R}^N$ instead of having compact support. One of the main differences being the asymptotic behavior of solutions as $\lambda \uparrow \lambda_1(\Omega_0)$ in Theorems 4 and 5.

Finally, we remark that with the same methods it is possible to deal with more general "logistic type" problems like

$$\begin{cases} (J * u)(x) - u(x) = -\lambda u(x) + a(x) f(u(x)) & x \in \Omega, \\ \qquad u = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

where $a(x)$ is as before (strictly positive or vanishing in a whole subdomain Ω_0) and $f(u)$ is a C^1 function which verifies:

(i) $\dfrac{f(t)}{t}$ is increasing.

(ii) $\displaystyle\lim_{t\to 0+} \frac{f(t)}{t} = 0$, $\displaystyle\lim_{t\to+\infty} \frac{f(t)}{t} = +\infty$.

The paper is organized as follows: in Section 2 we state and prove some auxiliary results which deal with the maximum principle, the sweeping principle and the eigenvalue problem for the nonlocal operator. Section 3 is dedicated to prove Theorems 1 and 2, while the proofs of Theorems 3 and 4 will be carried out in Section 4. We finally include and Appendix with the method of sub and supersolutions for nonlocal problems like (1.2).

2. Preliminaries

We devote this section to collect some results which will be needed in the proofs of our theorems. We begin by briefly considering the maximum principle for the operator

$$L_M u(x) = (J * u)(x) - (1+M)u(x),$$

where $M \geq 0$ (for simplicity we denote $L_M := L$ when $M = 0$). An important consequence of the maximum principle will be the *sweeping principle*, which we also consider. We include proofs for completeness (see [15], [16]; and also [26] for a parabolic version).

Theorem 6. *Let $u \in C(\overline{\Omega}) \cap L^1(\mathbb{R}^N)$ verify $L_M u \leq 0$ in Ω with $u \geq 0$ in $\mathbb{R}^N \setminus \Omega$. Then either $u > 0$ or $u \equiv 0$ in $\overline{\Omega}$.*

Proof. Let $m = \inf_\Omega u$, and assume $m < 0$. If u attains the infimum in $x_0 \in \overline{\Omega}$, then

$$\begin{aligned} 0 = u(x_0) - m &\geq \frac{1}{1+M} \int_\Omega J(x_0 - y)u(y)\,dy - m \\ &\geq m \left(\frac{1}{1+M} \int_\Omega J(x_0 - y)\,dy - 1 \right). \end{aligned} \tag{2.1}$$

This implies that $\int_\Omega J(x_0 - y)\,dy \geq 1 + M$, which is a clear contradiction if $M > 0$. When $M = 0$ we have $\int_\Omega J(x_0 - y)\,dy = 1$, that is, the ball $B_1(x_0)$ is contained in Ω. Thus the infimum of u cannot be attained on $\partial\Omega$, and in particular u is not constant. Hence we can choose x_0 so that $u(x_0) = m$ but $u(x_0) \not\equiv m$ in $B_1(x_0)$. Going back to (2.1) we obtain

$$\int_\Omega J(x_0 - y)(u(y) - m)\,dy = 0$$

and hence $u \equiv m$ in $B_1(x_0)$, a contradiction. Thus $m \geq 0$, that is, $u \geq 0$. Furthermore, if $u(x_1) = 0$ for some $x_1 \in \overline{\Omega}$, then

$$\int_{\mathbb{R}^N} J(x_1 - y)u(y)\,dy = 0$$

which implies that $u \equiv 0$ in a neighborhood of x_1. A standard connectedness argument gives $u \equiv 0$ in $\overline{\Omega}$. □

Remark 2. As an immediate corollary to the maximum principle, if $u \in L^2(\mathbb{R}^N)$ verifies $L_M u = 0$ in Ω with $u = 0$ in $\mathbb{R}^N \setminus \Omega$ then $u \equiv 0$. Thus it is standard to conclude that the problem

$$\begin{cases} (J * u)(x) - (1+M)u(x) = f(x) & x \in \Omega, \\ u = h(x) & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

admits a unique solution $u \in L^2(\mathbb{R}^N)$ for every $f \in L^2(\Omega)$ and $h \in L^2(\mathbb{R}^N)$.

We consider next the sweeping principle for problem (1.1). It will be applied to prove uniqueness of positive solutions to (1.1). We state it in a slightly more general form, for supersolutions to the problem

$$\begin{cases} (J * u)(x) - u(x) = f(x, u(x)) & x \in \Omega, \\ u = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.2}$$

where f is locally Lipschitz with respect to u uniformly in $x \in \overline{\Omega}$. Let us introduce the definition of a supersolution.

Definition 7. *We say that a function $\underline{u} \in L^1(\mathbb{R}^N)$ is a supersolution if*

$$\begin{cases} (J * \underline{u})(x) - \underline{u}(x) \leq f(x, \underline{u}(x)) & a.e.\ x \in \Omega, \\ \quad u \geq h(x) & a.e.\ x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Subsolutions are defined by reversing the above inequalities. We are assuming throughout that the subsolutions and supersolutions are bounded.

We say that u is a strict supersolution if it is not a solution to (1.1). Then we have the standard sweeping principle for supersolutions (of course, a similar statement holds for subsolutions).

Theorem 8. *Let $\{u_t\}_{t \geq t_0}$ be a family of continuous strict supersolutions to* (1.1) *such that the function $t \mapsto u_t$ is continuous considered with values in $C(\overline{\Omega})$. Assume that u is a continuous solution to* (1.1) *with $u < u_{t_1}$ in $\overline{\Omega}$ for some $t_1 > t_0$, and that $f(x,u)$ is locally Lipschitz in u uniformly in $x \in \overline{\Omega}$. Then $u \leq u_{t_0}$ in $\overline{\Omega}$.*

Proof. Notice that the continuity of $u_t(x)$ with respect to x in $\overline{\Omega}$ and t in $[t_0, t_1]$ implies that u_t is uniformly bounded in $\overline{\Omega}$. Hence thanks to the regularity of f we can take $M > 0$ such that the function $g(x,u) = f(x,u) - Mu$ is decreasing in the interval $[\inf_\Omega u, \sup_{t_0 \leq t \leq t_1} (\sup_\Omega u_t)]$.

Now set $\bar{t} = \inf\{t > t_0 :\ u < u_t \text{ in } \overline{\Omega}\}$. We have $t_0 \leq \bar{t} \leq t_1$, and we prove next that $\bar{t} = t_0$. Thus assume $\bar{t} > t_0$. By continuity we have $u \leq u_{\bar{t}}$. Then

$$L_M u = g(x,u) \geq g(x, u_{\bar{t}}) \geq L_M u_{\bar{t}}.$$

Theorem 6 implies then $u \equiv u_{\bar{t}}$ or $u < u_{\bar{t}}$ in $\overline{\Omega}$. Since $u_{\bar{t}}$ is a strict supersolution, the first possibility is ruled out. But thanks to the continuity of u_t in t, we can assert that $u < u_t$ for $t < \bar{t}$, $t \sim \bar{t}$, which contradicts the definition of $\bar{t}$ as an infimum. Hence $\bar{t} = t_0$, and $u \leq u_{t_0}$, as was to be proved. $\square$

We close this section by considering an eigenvalue problem which will be a fundamental tool when constructing solutions to (1.1), and also for the study of their behavior with varying λ. The eigenvalue problem is

$$\begin{cases} (J * u)(x) - u(x) = -\lambda u(x) & x \in \Omega, \\ \quad u = 0 & x \in \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.3}$$

where the minus sign in the right-hand side is used for convenience. We refer to [26] or [17] for existence of the principal eigenvalue in related situations and to [24] for details on differentiability with respect to perturbations of the domain.

Theorem 9. *Problem* (2.3) *admits an eigenvalue $\lambda_1(\Omega)$ associated to an eigenfunction $\phi \in C(\overline{\Omega})$ which is positive in $\overline{\Omega}$. Moreover, it is simple and unique, and it verifies $0 < \lambda_1(\Omega) < 1$. In addition it can be variationally characterized as*

$$\lambda_1(\Omega) = 1 - \sup_{\substack{u \in L^2(\Omega) \\ u \neq 0}} \frac{\displaystyle\int_\Omega \int_\Omega J(x-y) u(x) u(y)\, dy\, dx}{\displaystyle\int_\Omega u(x)^2\, dx}. \tag{2.4}$$

This eigenvalue $\lambda_1(\Omega)$ *is decreasing with respect to the domain, that is, if* $\Omega \subsetneq \Omega'$, *then* $\lambda_1(\Omega) > \lambda_1(\Omega')$. *On the other hand, if we let* $\Omega_\delta = \Omega \cup \{x \in \mathbb{R}^N : \operatorname{dist}(x, \partial\Omega) < \delta\}$, *then* $\lambda_1(\Omega_\delta)$ *is differentiable with respect to* δ *at* $\delta = 0$ *and*

$$\frac{\partial \lambda_1(\Omega_\delta)}{\partial \delta}\Big|_{\delta=0} = -(1-\lambda_1(\Omega)) \int_{\partial\Omega} u_0(x)^2 dS(x) < 0. \tag{2.5}$$

Sketch of the proof. It is clear that λ is an eigenvalue of (2.3) if and only if $\mu = 1 - \lambda$ is an eigenvalue of $Lu = J * u$ in $L^2(\Omega)$. Since L is self-adjoint, it follows by standard spectral theory (see for example [2]) that $\mu_1 = \|L\|$ is an eigenvalue of L (moreover notice that L is positive). On the other hand, the eigenvalue problem can also be considered in $C(\overline{\Omega})$, and an application of the Krein-Rutman theorem (see Theorem 6.2 in [29]) gives that μ_1 possesses the properties stated in the theorem. Note that the operator L is not strongly positive, but it has the property that, for a given $u \in C(\overline{\Omega})$, nonnegative and nontrivial, there exists a positive integer n so that $L^n u > 0$ in $\overline{\Omega}$.

The monotonicity property is a consequence of the variational characterization (2.4), since for $\Omega \subsetneq \Omega'$, every function in $u \in L^2(\Omega)$ can be considered in $L^2(\Omega')$ when extended to be zero in $\mathbb{R}^N \setminus \Omega$. Thus $\lambda_1(\Omega) \geq \lambda_1(\Omega')$. The strict inequality follows from the maximum principle.

We now briefly sketch the proof of the differentiability with respect to δ (we refer to [24] for a detailed proof). In fact, we consider Ω_δ as a perturbation of Ω in the sense that $\Omega_\delta = T_\delta(\Omega)$, where $T_\delta = I + \delta\Phi$ and Φ is a suitable smooth function. By means of a change of variables in (2.4), we obtain

$$1 - \lambda_1(\Omega_\delta) = \int_\Omega \int_\Omega J_\delta(z,w) u_\delta(z + \delta\Phi(z)) u_\delta(w + \delta\Phi(w)) \, \Delta(z)\Delta(w) \, dz \, dw$$

where $J_\delta(z,w) = J(z - w + \delta(\Phi(z) - \Phi(w)))$, $\Delta(z) = \det(I + \delta D\Phi(z))$, and u_δ is the associated positive eigenfunction, normalized by

$$\int_{\Omega_\delta} u_\delta(x)^2 dx = 1.$$

Performing an expansion in δ in the previous identity, we obtain

$$\begin{aligned}
1 - \lambda_1(\Omega_\delta) = 2 \Bigg(& \int_\Omega \int_\Omega DJ(x-y)\Phi(x) u_0(x) u_0(y) \, dx \, dy \\
& + \int_\Omega \int_\Omega J(x-y)\Phi(x)\nabla u_0(x) u_0(y) \, dx \, dy \\
& + \int_\Omega \int_\Omega J(x-y) u_0(x) u_0(y) \operatorname{div} \Phi(x) \, dx \, dy \Bigg) \delta \\
& -(1 - \lambda_1(\Omega)) \left(2 \int_\Omega u_0(x)\Phi(x)\nabla u_0(x) \, dx + \int_\Omega u_0(x)^2 \operatorname{div} \Phi(x) \, dx \right) \delta \\
& + o(\delta).
\end{aligned}$$

By means of an integration by parts in the integrals containing the divergence we arrive at

$$
\begin{aligned}
1-\lambda_1(\Omega_\delta) &= \Big(2\int_\Omega\int_{\partial\Omega} J(x-y)u_0(x)u_0(y)\Phi(x)\nu(x)\,dS(x)\,dy \\
&\qquad -(1-\lambda_1(\Omega))\int_{\partial\Omega} u_0(x)^2\Phi(x)\nu(x)\,dS(x)\Big)\,\delta+o(\delta) \\
&= (1-\lambda_1(\Omega))\Big(\int_{\partial\Omega} u_0(x)^2\Phi(x)\nu(x)\,dS(x)\Big)\,\delta+o(\delta),
\end{aligned}
$$

where we have used Fubini's theorem and the equation satisfied by u_0. Here we have denoted by ν the outward pointing unit normal vector field.

It follows that

$$
\frac{\partial\lambda_1(\Omega_\delta)}{\partial\delta}\Big|_{\delta=0} = -(1-\lambda_1(\Omega))\int_{\partial\Omega} u_0(x)^2\Phi(x)\nu(x)\,dS(x).
$$

It is now possible to select the perturbation term Φ such that $\Phi=\nu$ on $\partial\Omega$ (see [25]) and this gives (2.5). This concludes the proof. $\square$

3. Strictly positive weights

We dedicate this section to problem (1.1) when the weight $a(x)$ is assumed to be strictly positive in $\overline{\Omega}$.

Proof of Theorem 1. We first prove that if a nontrivial solution to (1.1) exists, then $\lambda>\lambda_1(\Omega)$. To this aim, assume $u\in L^1(\Omega)$ is a positive solution to (1.1) and let ϕ be a positive eigenfunction associated to $\lambda_1(\Omega)$. If we multiply (1.1) by ϕ and integrate in Ω, we arrive at

$$
\begin{aligned}
&\int_\Omega \phi(x)\int_\Omega J(x-y)u(y)\,dy\,dx-\int_\Omega u(x)\phi(x)\,dx \\
&\qquad = -\lambda\int_\Omega u(x)\phi(x)\,dx+\int_\Omega a(x)u^p(x)\phi(x)\,dx.
\end{aligned} \tag{3.1}
$$

If we apply Fubini's theorem in the first integral, and use that ϕ is an eigenfunction associated to $\lambda_1(\Omega)$ (the symmetry of J is needed here), then (3.1) becomes

$$
-\lambda_1(\Omega)\int_\Omega u(y)\phi(y)\,dy = -\lambda\int_\Omega u(x)\phi(x)\,dx+\int_\Omega a(x)u^p(x)\phi(x)\,dx.
$$

It follows from this equation that necessarily $\lambda>\lambda_1(\Omega)$.

Now assume $\lambda>\lambda_1(\Omega)$, and let us show that there exists a positive solution to (1.1) by means of the method of sub and supersolutions (see the Appendix). We claim that $\underline{u}=\varepsilon\phi$, $\overline{u}=M$ are a pair of ordered sub and supersolutions if ε is small enough and M large enough. Indeed, $\underline{u}$ will be a subsolution provided

$$
J*(\varepsilon\phi)-\varepsilon\phi\ge -\lambda(\varepsilon\phi)+a(x)(\varepsilon\phi)^p,
$$

that is $\lambda_1(\Omega)\le\lambda-a(x)(\varepsilon\phi)^{p-1}$, which can be achieved by taking ε sufficiently small since $p>1$. On the other hand, $\overline{u}$ is a supersolution if

$$
J*M-M=0\le -\lambda M+a(x)M^p
$$

and this can also be clearly fulfilled by selecting a large M, since $a > 0$ in $\overline{\Omega}$. Hence thanks to the method of sub and supersolutios we obtain that (1.1) admits a positive (bounded) weak solution.

Let us prove next that all possible positive weak solutions $u \in L^1(\Omega)$ are indeed continuous in $\overline{\Omega}$. To this aim we first prove that the leftmost inequality in (1.6) holds for $\lambda > 1$, that is $(\lambda - 1) \leq a(x)u^{p-1}$ almost everywhere in Ω. However, this is immediate since the integral term in (1.1) is positive.

On the other hand, we have

$$(1 - \lambda)u + au^p = \int_\Omega J(\cdot - y)u(y)\,dy \in C(\overline{\Omega}). \tag{3.2}$$

We now observe that $(1 - \lambda) + pau^{p-1} > (1 - \lambda) + au^{p-1}$, and this last quantity is positive in $\overline{\Omega}$, both for $\lambda_1(\Omega) < \lambda \leq 1$ and for $\lambda > 1$ thanks to the recently proved lower bound. This implies that the function $(1-\lambda)t+at^p$ is invertible (with respect to t) in the range of u and thus $u \in C(\overline{\Omega})$ by (3.2).

Having shown that all solutions are continuous in $\overline{\Omega}$, we now prove the upper bound in (1.6). Let $x_0 \in \overline{\Omega}$ be a point where u achieves its maximum. If $a(x_0)u^{p-1}(x_0) > \lambda$, then we would have the contradiction

$$u(x_0) < \int_\Omega J(x_0 - y)u(y)\,dy \leq u(x_0) \int_\Omega J(x_0 - y)\,dy \leq u(x_0).$$

Hence $a(x_0)u^{p-1}(x_0) \leq \lambda$, and this shows in particular the rightmost inequality in (1.6).

We now prove uniqueness. Let u, v be (continuous) solutions to (1.1). It is not hard to see that $v_t = tv$ is a strict supersolution to (1.1) for $t > 1$. Since v_t is continuous in t and $u < v_t$ in $\overline{\Omega}$ for large t, it follows from the sweeping principle (Theorem 8) that $u \leq v$. A symmetric argument gives $u = v$, and hence uniqueness follows.

We prove next that u_λ is increasing in λ. For this notice that if $\lambda < \mu$, then u_μ is a supersolution to (1.1). As there are arbitrarily small subsolutions, it follows that $u_\lambda \leq u_\mu$, and by the strong maximum principle $u_\lambda < u_\mu$ in $\overline{\Omega}$, as was to be shown.

To conclude the proof of the theorem, it only remains to show that $u_\lambda \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \downarrow \lambda_1(\Omega)$. Indeed, by the monotonicity of u_λ, the limit $u_0 = \lim_{\lambda \downarrow \lambda_1(\Omega)} u_\lambda$ exists, and it is a bounded function. Passing to the limit in (1.1) with the aid of dominated convergence theorem, we obtain that u_0 is a nonnegative solution to (1.1) with $\lambda = \lambda_1(\Omega)$. Hence $u_0 = 0$, and since the convergence is monotonic we obtain that it is uniform thanks to Dini's theorem. This finishes the proof. $\square$

Remark 3. As can be seen from the proof, the upper bound in (1.6) continues to hold for $\lambda_1(\Omega) < \lambda < 1$ (the lower bound being trivial in this case).

Proof of Theorem 2. Thanks to the lower bound in (1.6), it suffices to prove that

$$\limsup_{\lambda \to \infty} \frac{u_\lambda(x)}{\lambda^{\frac{1}{p-1}}} \leq a(x)^{-\frac{1}{p-1}}$$

uniformly in Ω. To see this we construct a supersolution for large λ. We claim that $\overline{u} = \lambda^{\frac{1}{p-1}}(a(x) - \varepsilon)^{-\frac{1}{p-1}}$ is a supersolution provided λ is large enough and $\varepsilon < \inf_\Omega a$. Indeed, since

$$-\lambda\overline{u} + a(x)\overline{u}^p = \lambda^{\frac{p}{p-1}}\varepsilon(a(x)-\varepsilon)^{-\frac{p}{p-1}} \tag{3.3}$$

and $J * \overline{u} - \overline{u} = \lambda^{\frac{1}{p-1}}c(x)$ for a certain fixed function $c(x)$, it follows that $J * \overline{u} - \overline{u} < -\lambda\overline{u} + a(x)\overline{u}^p$ in Ω for large λ. By uniqueness, $u_\lambda \le \overline{u}$ and this immediately implies (3.3). $\square$

4. Vanishing weights

We now consider problem (1.1) when the weight $a(x)$ is assumed to vanish in a whole smooth subdomain Ω_0 of Ω. The proof of Theorem 3 is similar in some respects to that of Theorem 1, and we are only stressing the main differences.

Proof of Theorem 3. It follows as in Theorem 1 that positive solutions can only exist if $\lambda > \lambda_1(\Omega)$. Let us show next that $\lambda < \lambda_1(\Omega_0)$ is also necessary for existence. Let ψ be an eigenfunction associated to $\lambda_1(\Omega_0)$. Multiplying (1.1) by ψ and integrating in Ω_0, we arrive at

$$\begin{aligned}&\int_{\Omega_0}\int_\Omega J(x-y)u(y)\psi(x)\,dy\,dx - \int_{\Omega_0} u(x)\psi(x)\,dx\\ &\qquad = -\lambda\int_{\Omega_0} u(x)\psi(x)\,dx.\end{aligned} \tag{4.1}$$

We apply Fubini's theorem in the first integral and split the integration with respect to y in two parts, corresponding to Ω_0 and $\Omega\setminus\Omega_0$:

$$\begin{aligned}\int_{\Omega_0}\int_\Omega J(x-y)u(y)\psi(x)\,dy\,dx &= \int_{\Omega_0}\int_{\Omega_0} J(x-y)u(y)\psi(x)\,dx\,dy\\ &\quad + \int_{\Omega\setminus\Omega_0}\int_{\Omega_0} J(x-y)u(y)\psi(x)\,dx\,dy\\ &> \int_{\Omega_0}\int_{\Omega_0} J(x-y)u(y)\psi(x)\,dx\,dy\\ &= (1-\lambda_1(\Omega_0))\int_{\Omega_0} u(y)\psi(y)\,dy.\end{aligned}$$

Going back to (4.1) we obtain that $\lambda < \lambda_1(\Omega_0)$, as we wanted to see.

We now assume that $\lambda_1(\Omega) < \lambda < \lambda_1(\Omega_0)$. To prove the existence of a positive solution we employ again the method of sub and supersolutions. The subsolution can be taken as $\underline{u} = \varepsilon\phi$, where ϕ is the eigenfunction associated to $\lambda_1(\Omega)$ and ε is small enough, exactly as in Theorem 1. The construction of the supersolution, however, is slightly different, and is inspired in [21].

Let $\delta > 0$ be small, and define $\Omega_\delta = \Omega_0 \cup \{x \in \Omega :\ \operatorname{dist}(x, \partial\Omega_0) < \delta\}$. Thanks to the continuity and monotonicity of λ_1 with respect to the domain, Theorem 9, we have $\lambda < \lambda_1(\Omega_\delta) < \lambda_1(\Omega_0)$ for sufficiently small δ. Let ϕ_δ be a positive eigenfunction associated to $\lambda_1(\Omega_\delta)$, and extend ϕ_δ as a positive continuous function to the whole Ω (we recall that $\phi_\delta > 0$ on $\partial\Omega_\delta$, so this extension is possible). We claim that $\overline{u} = M\phi_\delta$ is a supersolution provided M is large enough. Indeed, in Ω_δ

$$J * (M\phi_\delta) - M\phi_\delta = -\lambda_1(\Omega_\delta)M\phi_\delta \le -\lambda M\phi_\delta \le -\lambda M\phi_\delta + a(x)(M\phi_\delta)^p$$

since $a \geq 0$ and $\lambda < \lambda_1(\Omega_\delta)$. On the other hand, since $a \geq a_0 > 0$ in $\Omega \setminus \Omega_\delta$, $\overline{u}$ will be a supersolution provided that

$$J * \phi_\delta - \phi_\delta \leq -\lambda \phi_\delta + a_0 M^{p-1} \phi_\delta^p,$$

in $\Omega \setminus \Omega_\delta$, which holds for large enough M. Thus the method of sub and supersolutions yields the existence of a positive bounded weak solution u to (1.1).

All positive weak solutions $u \in L^1(\Omega)$ are indeed continuous in $\overline{\Omega}$. To see this, it suffices to show as before that $(1-\lambda) + pa(x)u^{p-1} > 0$ in $\overline{\Omega}$. Notice however that in the present situation $pa(x) > a(x)$ is no longer true in Ω_0. Nevertheless, we still have $(1-\lambda) + pa(x)u^{p-1} > 0$ for all solutions, since $1 - \lambda > 1 - \lambda_1(\Omega_0) > 0$, according to Theorem 9.

Hence all weak solutions are continuous in $\overline{\Omega}$, and uniqueness is proved exactly as in Theorem 1. The assertions about the monotonicity of u_λ and the uniform convergence to zero as $\lambda \downarrow \lambda_1(\Omega)$ are also shown in the same way.

We finally show that $u_\lambda \to +\infty$ uniformly in $\overline{\Omega}$ as $\lambda \uparrow \lambda_1(\Omega_0)$. We first prove that

$$\int_\Omega u_\lambda(x)\, dx \to +\infty \tag{4.2}$$

as $\lambda \uparrow \lambda_1(\Omega_0)$. Indeed, assume that (4.2) does not hold. Since u_λ is increasing in λ, thanks to monotone convergence theorem, it follows that $w = \sup u_\lambda \in L^1(\Omega)$, and hence $w < \infty$ almost everywhere in Ω. Passing to the limit in (1.1) it follows that w is a weak solution to (1.1) with $\lambda = \lambda_1(\Omega_0)$, which is impossible, as has been already shown. Hence (4.2) holds.

Now let Q be a partition of Ω into cubes of diameter $h < 1/4$ (recall that we are assuming the support of J to be the unit ball). Thanks to the monotonicity of u_λ, there exists a cube Q_0 such that

$$\int_{Q_0} u_\lambda(x)\, dx \to +\infty.$$

Now if $x \in \Omega$ is such that $\operatorname{dist}(x, Q_0) \leq 1/4$ it follows that

$$\int_\Omega J(x-y) u_\lambda(y)\, dy \geq \inf_{B_{2/3}} J \int_{Q_0} u_\lambda(y)\, dy$$

and thus the first integral tends to infinity as $\lambda \uparrow \lambda_1(\Omega_0)$. Since $a \geq 0$ and $\lambda_1(\Omega_0) < 1$, passing to the limit in (1.1) it follows that $u_\lambda(x) \to +\infty$ for x such that $\operatorname{dist}(x, Q_0) \leq 1/4$.

This argument shows that $u_\lambda \to \infty$ not only in Q_0, but in all neighboring cubes. After finitely many steps we arrive at $u_\lambda \to +\infty$, and it is clear that the limit is uniform in $\overline{\Omega}$. $\square$

We now prove Theorem 4. The proof is completely different from that of Theorem 2.

Proof of Theorem 4. Let $v_\lambda = u_\lambda / M_\lambda$. Then v_λ verifies

$$J * v_\lambda - v_\lambda = -\lambda v_\lambda + a(x) M_\lambda^{p-1} v_\lambda^p \qquad \text{in } \Omega. \tag{4.3}$$

Take an arbitrary sequence $\lambda_n \to \lambda_1(\Omega_0)$. Since v_{λ_n} is bounded in $L^2(\Omega)$, we may assume (passing to a subsequence) that $v_{\lambda_n} \to v_0$ weakly in $L^2(\Omega)$ for some $v_0 \in L^2(\Omega)$. Hence $J * v_{\lambda_n} \to J * v_0$ strongly in $L^2(\Omega)$ and then $(1-\lambda_n)v_{\lambda_n} + a(x)M_{\lambda_n}^{p-1}v_{\lambda_n}^p$ converges to $J*v_0$ in $L^2(\Omega)$. In particular, since $M_{\lambda_n} \to \infty$, it follows that $v_{\lambda_n} \to 0$ almost everywhere in D and hence $v_0 = 0$ almost everywhere in D.

On the other hand, $J * v_{\lambda_n} = (1-\lambda_n)v_{\lambda_n}$ in Ω_0, and thus v_{λ_n} converges in $L^2(\Omega_0)$ to v_0 which then verifies $J * v_0 = (1-\lambda_1(\Omega_0))v_0$ in Ω_0. Notice that this implies, together with the previous discussion, that $v_{\lambda_n} \to v_0$ in $L^2(\Omega)$. Hence $(1-\lambda_n)v_{\lambda_n} + a(x)v_n^p$ converges uniformly in $\overline{\Omega}$, and this implies that v_{λ_n} converges to v_0 uniformly in $\overline{\Omega}_0$ and $v_{\lambda_n} \to 0$ uniformly in compact subsets of D.

The proof of (1.7) will be concluded if we show that $\|v_0\|_\infty = 1$, since in that case v_0 will be a normalized eigenfunction associated to $\lambda_1(\Omega_0)$.

It is clear that $\|v_0\|_\infty \le 1$. Now take points $x_n \in \overline{\Omega}$ such that $v_{\lambda_n}(x_n) = 1$, and assume with no loss of generality that $x_n \to x_0$. Notice that from (4.3) we have

$$1 - \lambda_n + a(x_n)M_{\lambda_n}^{p-1} \to (J * v_0)(x_0), \tag{4.4}$$

and this immediately leads to $a(x_0) = 0$, i. e. $x_0 \in \overline{\Omega}_0$, for otherwise the right-hand side of (4.4) is unbounded. Thus $(J*v_0)(x_0) = (1-\lambda_1(\Omega_0))v_0(x_0)$. We then have from (4.4) that

$$a(x_n)M_n^{p-1} \to (1-\lambda_1(\Omega_0))(v_0(x_0) - 1),$$

and since the left-hand side is nonnegative while the right-hand side is nonpositive, it follows that $v_0(x_0) = 1$, i. e. $\|v_0\|_\infty = 1$, as we wanted to prove.

To verify (1.8), observe that thanks to the convergence in $L^2(\Omega)$ of any subsequence v_{λ_n} to v_0, we get

$$J * v_{\lambda_n} \to J * v_0 =: \Phi_1$$

uniformly in $\overline{\Omega}$, and since $v_{\lambda_n} \to 0$ uniformly in compacts of $B_1 \cup \Gamma_1$, (4.3) implies that $a(x)M_\lambda^{p-1}v_\lambda^p \to \Phi_1(x)$ uniformly in compacts of $B_1 \cup \Gamma_1$, and $a(x)M_\lambda^{p-1}v_\lambda^p \to 0$ uniformly in $\overline{B}_2 \cup \cdots \cup \overline{B}_k$. Hence

$$\frac{u_\lambda(x)}{M_\lambda^{1/p}} \sim \left(\frac{\Phi_1(x)}{a(x)}\right)^{1/p} =: \Psi_1(x) \quad \text{uniformly in compacts of } B_1 \cup \Gamma_1,$$

which proves (1.8) for $n = 1$, and

$$\frac{u_\lambda(x)}{M_\lambda^{1/p}} \to 0 \quad \text{uniformly in } \overline{B}_2 \cup \ldots \cup \overline{B}_k.$$

We now introduce $w_\lambda = u_\lambda / M^{1/p}$. It follows that $w_\lambda \to \Psi_1$ uniformly in compacts of B_1 and $w_\lambda \to 0$ uniformly in $\overline{B}_2 \cup \ldots \cup \overline{B}_k$, while w_λ solves the equation

$$J * w_\lambda - w_\lambda = -\lambda w_\lambda + a(x)M^{1-\frac{1}{p}}w_\lambda^p \qquad \text{in } \Omega. \tag{4.5}$$

We then obtain that

$$J * w_\lambda \to J * \Psi_1 =: \Phi_2$$

uniformly in compacts of $B_2 \cup \Gamma_2$, so that (4.5) gives that $a(x)M^{1-\frac{1}{p}} w_\lambda(x)^p \to \Phi_2(x)$ uniformly in compacts of $B_2 \cup \Gamma_2$, that is

$$\frac{u_\lambda(x)}{M_\lambda^{1/p^2}} \sim \left(\frac{\Phi_2(x)}{a(x)}\right)^{1/p} =: \Psi_2(x) \quad \text{uniformly in compacts of } B_2 \cup \Gamma_2,$$

which proves (1.8) for $n = 2$. The rest of the proof follows in the same way. $\square$

We finally prove Theorem 5.

Proof of Theorem 5. Since the inequalities (1.10) follow from (1.9) as (1.8) in Theorem 4, we only show (1.9).

First we construct a subsolution to (1.1) for λ close to $\lambda_1(\Omega_0)$. As in the proof of Theorem 3, let $\Omega_\delta = \Omega_0 \cup \{x \in \Omega : \ \mathrm{dist}(x, \partial\Omega_0) < \delta\}$ for small positive δ, and choose the positive eigenfunction ϕ_δ associated to $\lambda_1(\Omega_\delta)$ with the normalization $\|\phi_\delta\|_\infty = 1$.

Now take λ such that $\lambda_1(\Omega_\delta) < \lambda_1(\Omega_{\delta/2}) < \lambda < \lambda_1(\Omega_{\delta/4}) < \lambda_1(\Omega_0)$ for small δ (cf. Theorem 9). It is not hard to see that $\varepsilon\phi_\delta$ is a subsolution in Ω with the "optimal" choice

$$\varepsilon = \left(\frac{\lambda - \lambda_1(\Omega_\delta)}{\sup_{D_\delta} a}\right)^{\frac{1}{p-1}}, \tag{4.6}$$

where $D_\delta = \Omega_\delta \setminus \Omega_0$. By uniqueness of solutions we have $u_\lambda \geq \varepsilon\phi_\delta$ in Ω.

Now thanks to Theorem 9, $\lambda_1(\Omega_\delta)$ is differentiable with respect to δ, so that there exists a constant $C > 0$ such that

$$\lambda - \lambda_1(\Omega_\delta) > \lambda_1(\Omega_{\delta/2}) - \lambda_1(\Omega_\delta) \geq C\delta$$

for small enough δ. Since, by the hypothesis on $a(x)$, $\sup_{D_\delta} a \leq C\delta^{1+\gamma}$, we arrive using (4.6) at $u_\lambda(x) \geq C\delta^{-\frac{\gamma}{p-1}}$ in Ω_0. Moreover, $\lambda_1(\Omega_0) - \lambda > \lambda_1(\Omega_0) - \lambda_1(\Omega_{\delta/4}) \geq C\delta$, so we arrive at

$$u_\lambda(x) \geq C(\lambda_1(\Omega_0) - \lambda)^{-\frac{\gamma}{p-1}} \quad \text{in } \Omega_0,$$

which establishes (1.9). As we have already pointed out, the proof of (1.10) follows from (1.9) as in Theorem 4. $\square$

Appendix: the method of sub and supersolutions

The purpose of this appendix is to develop the well-known method of sub and supersolutions for stationary nonlocal problems

$$\begin{cases} (J * u)(x) - u(x) = f(x, u(x)) & x \in \Omega, \\ u = h(x) & x \in \mathbb{R}^N \setminus \Omega. \end{cases} \tag{A.1}$$

We provide complete proofs for completeness, although the method has been used before in the context of nonlocal problems for instance in [17].

Recall that in Definition 7 we have defined a supersolution as follows: a function $\overline{u} \in L^1(\mathbb{R}^N)$ is a supersolution to (A.1) if

$$\begin{cases} (J * \overline{u})(x) - \overline{u}(x) \leq f(x, \overline{u}(x)) & x \in \Omega, \\ \overline{u} \geq h(x) & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Subsolutions are defined by reversing the above inequalities. We are assuming throughout that the subsolutions and supersolutions are *bounded.*

Theorem 10. *Assume $h \in L^\infty(\Omega)$, f is locally Lipschitz with respect to u, uniformly for $x \in \overline{\Omega}$, and that there exists a bounded subsolution $\underline{u}$ and a bounded supersolution $\overline{u}$ of problem* (A.1) *such that $\underline{u} \le \overline{u}$ in Ω. Then problem* (A.1) *admits a minimal solution u_- and a maximal solution u_+ in the interval $[\underline{u}, \overline{u}]$.*

Proof. Since f is locally Lipschitz with respect to u, uniformly for $x \in \overline{\Omega}$, we can choose $M > 0$ so that the function $g(x,u) = f(x,u) - Mu$ is decreasing in $[\inf \underline{u}, \sup \overline{u}]$. Let u_1 be the unique solution of the problem

$$\text{(A.2)} \qquad \begin{cases} (J * u)(x) - (1+M)u(x) = g(x, \underline{u}(x)) & x \in \Omega, \\ \qquad u = h(x) & x \in \mathbb{R}^N \setminus \Omega, \end{cases}$$

(see Remark 2). We claim that $\underline{u} \le u_1 \le \overline{u}$. Indeed, we have $L_M u_1 = g(x, \underline{u}) \le L_M \underline{u}$ in Ω with $u_1 \ge \underline{u}$ in $\mathbb{R}^N \setminus \Omega$. Thus by the maximum principle $u_1 \ge \underline{u}$. Similarly, $L_M u_1 = g(x, \underline{u}) \ge g(x, \overline{u}) \ge L_M \overline{u}$ in Ω, with $u_1 \le \overline{u}$ in $\mathbb{R}^N \setminus \Omega$ implies $u_1 \le \overline{u}$.

We now define u_2 to be the solution to (A.2) with $\underline{u}$ replaced by u_1. It follows in the same way that $\underline{u} \le u_1 \le u_2 \le \overline{u}$. We continue in this way and define an increasing sequence of functions $\{u_n\}$ which verify $\underline{u} \le u_n \le \overline{u}$ and

$$\text{(A.3)} \qquad \begin{cases} (J * u_n)(x) - (1+M)u_n(x) = g(x, u_{n-1}(x)) & x \in \Omega, \\ \qquad u_n = h(x) & x \in \mathbb{R}^N \setminus \Omega. \end{cases}$$

Let $u_-(x) = \sup u_n(x)$. It is clear that $\underline{u} \le u_- \le \overline{u}$. Moreover, we can pass to the limit in (A.3) using the monotone convergence theorem and the continuity of g, and obtain that u_- is a solution to (A.1).

To show that u_- is the minimal solution in $[\underline{u}, \overline{u}]$, let u be another solution verifying $\underline{u} \le u \le \overline{u}$. It is easily checked as before that $u_n \le u$, and hence $u_- \le u$.

In a similar fashion, we construct a decreasing sequence of functions $\{v_n\}$ such that $\underline{u} \le v_n \le \overline{u}$ and $v_n \to u_+ = \inf v_n$, which is the maximal solution to (A.1) in $[\underline{u}, \overline{u}]$. □

Remark 4. (a) As the above proof shows, if the function f is decreasing with respect to u, or if f is globally Lipschitz, the subsolution and the supersolution need not be bounded.

(b) If the function u_- (resp. u_+) is continuous, then it follows thanks to Dini's theorem that the convergence of u_n (resp. v_n) is uniform in $\overline{\Omega}$.

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