OPTIMAL MASS TRANSPORT IN THIN DOMAINS

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ABSTRACT. We find the behavior of the solution of the optimal transport problem for the Euclidean distance (and its approximation by $p-$Laplacian problems) when the involved measures are supported in a domain that is contracted in one direction.

1. INTRODUCTION.

In this paper we study the behaviour of the solutions (Kantorovich potentials and mass transport plans) for the Monge-Kantorovich mass transport problem when the involved masses (that we assume to be absolutely continuous with respect to the usual Lebesgue measure) are contained in a domain that is contracted (and therefore thin) in one direction.

Thin domains occur in applications as they can be found in problems in mechanics. For example, in ocean dynamics, one is dealing with fluid regions which are thin compared to the horizontal length scales. Other examples include lubrication, meteorology, blood circulation, etc.; they are a part of a broader study of the behaviour of various PDEs on thin $n-$dimensional domains, where $n \geq 2$ (for a review see [24]).

In order to formulate precise statements as well as to put this work in context, we first need to introduce some notations, concepts and results from the Monge-Kantorovich Mass Transport Theory (we refer to [1], [13], [25] and [26] for details) that will be used in the rest of the paper.

1.1. **Monge-Kantorovich Mass Transport Theory.** We denote by $\mathcal{M}(\Omega)$ the set of Radon measures on Ω and by $\mathcal{M}^+(\Omega)$ the non-negative elements of $\mathcal{M}(\Omega)$. Given $\mu, \nu \in \mathcal{M}^+(\Omega)$ satisfying the mass balance condition $\mu(\Omega) = \nu(\Omega)$ we denote by $\mathcal{A}(\mu, \nu)$ the set of transport maps pushing μ to ν, that is, the set of Borel maps $T : \Omega \to \Omega$ such that $T\#\mu = \nu$, that is, $\mu(T^{-1}(E)) = \nu(E)$ for all $E \subset \Omega$ Borel.

The Monge problem. *The Monge problem, associated with the measures μ and ν, is to find a map $T^* \in \mathcal{A}(\mu, \nu)$ which minimizes the cost functional*

$$\tilde{\mathcal{F}}(T) := \int_\Omega |x - T(x)| \, d\mu(x) \tag{1.1}$$

in the set $\mathcal{A}(\mu, \nu)$. When μ and ν are absolutely continuous with respect to the Lebesgue measure, $\mu = f\mathcal{L}^N \llcorner \Omega$ and $\nu = g\mathcal{L}^N \llcorner \Omega$, there exists such an optimal map T. A map $T^* \in \mathcal{A}(\mu, \nu)$ satisfying $\tilde{\mathcal{F}}(T^*) = \min\{\tilde{\mathcal{F}}(T) : T \in \mathcal{A}(\mu, \nu)\}$, is called an optimal transport map of μ to ν.

In general, the Monge problem is ill-posed. To overcome the difficulties of the Monge problem, in 1942, L. V. Kantorovich in [17] proposed a relaxed version of the problem and introduced a dual variational principle. Let $\pi_t(x,y) := (1-t)x + ty$. Given a Radon measure γ in $\Omega \times \Omega$, its marginals are defined by $proj_x(\gamma) := \pi_0\#\gamma$, $proj_y(\gamma) := \pi_1\#\gamma$.

The Monge-Kantorovich problem. *The Monge-Kantorovich problem,* [17]*, is the minimization problem*

$$\min \left\{ \int_{\Omega\times\Omega} |x-y| \, d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\},$$

where $\Pi(\mu,\nu) := \{$*Radon measures* γ *in* $\Omega\times\Omega : \pi_0\#\gamma = \mu, \pi_1\#\gamma = \nu\}$. *The elements* $\gamma \in \Pi(\mu,\nu)$ *are called transport plans between* μ *and* ν*, and a minimizer* γ^* *an optimal transport plan. A minimizer always exists.*

The Monge-Kantorovich problem has a dual formulation that can be stated in this case as follows (see for instance [25, Theorem 1.14]).

Kantorovich-Rubinstein Theorem. *It holds the following duality result,*

$$\min \left\{ \int_{\Omega\times\Omega} |x-y| \, d\gamma(x,y) \ : \ \gamma \in \Pi(\mu,\nu) \right\} = \max \left\{ \int_{\Omega} u \, d(\mu-\nu) \ : \ u \in K_1(\overline{\Omega}) \right\}, \tag{1.2}$$

where $K_1(\overline{\Omega}) := \{u : \overline{\Omega} \to \mathbb{R} \ : \ |u(x)-u(y)| \le |x-y| \ \ \forall\, x,y \in \overline{\Omega}\}$ *is the set of 1-Lipschitz functions in* $\overline{\Omega}$. *The maximizers* u^* *of the right hand side of* (1.2) *are called Kantorovich potentials.*

Kantorovich potentials can be obtained taking the limit as $p \to \infty$ in a $p-$Laplacian problem. Assume that $\mu = f\mathcal{L}^N \llcorner \Omega$ and $\nu = g\mathcal{L}^N \llcorner \Omega$ and consider

$$\begin{cases} -\Delta_p u_p = f - g & \text{in } \Omega, \\ |\nabla u_p|^{p-2} \frac{\partial u_p}{\partial \eta} = 0 & \text{on } \partial\Omega, \\ u_p(0) = 0. \end{cases} \tag{1.3}$$

The condition $u_p(0) = 0$ is just a normalization (we assume here that $0 \in \Omega$). We have the following result, see [14] and Section 5 in this paper.

Evans-Gangbo Theorem. *The solutions to* (1.3) *converge, along subsequences, uniformly in* $\overline{\Omega}$*,*

$$\lim_{p\to\infty} u_p = u^*,$$

where u^* *is a Kantorovich potential, that is, a maximizer for the right hand side of* (1.2)*. In fact, this limit procedure gives much more since it allows to construct an optimal transport map.*

For later reference, we will call $TC(f,g)_\Omega$ the total cost of the transport of $f\mathcal{L}^N \llcorner \Omega$ to $g\mathcal{L}^N \llcorner \Omega$, that is given by the minimum or the maximum in (1.2).

1.2. **The Monge-Kantorovich problem in a thin domain.** We consider a product domain $\Omega_1 \times \Omega_2 = \Omega \subset \mathbb{R}^n$, with $\Omega_1 \subset \mathbb{R}^k$, $\Omega_2 \subset \mathbb{R}^l$ and, for simplicity, we assume that

$|\Omega_1| = |\Omega_2| = 1$ and that $(0,0) \in \Omega_1 \times \Omega_2$. We are given two nonnegative L^1 functions $f_+(x,y)$ and $f_-(x,y)$, with $x \in \mathbb{R}^k$, $y \in \mathbb{R}^l$, supported in Ω, with the same total mass,

$$\int_\Omega f_+(x,y)\,dxdy = \int_\Omega f_-(x,y)\,dxdy := M. \tag{1.4}$$

Now we take $\varepsilon > 0$ small and contract the second variable, y, that is, we consider

$$\Omega_\varepsilon = \Omega_1 \times \varepsilon\Omega_2 = \{(x, \varepsilon y) \,:\, x \in \Omega_1, y \in \Omega_2\}.$$

In this set Ω_ε we define

$$f_+^\varepsilon(\bar{x},\bar{y}) = f_+\left(\bar{x}, \frac{\bar{y}}{\varepsilon}\right)\frac{1}{\varepsilon^l}, \qquad \text{and} \qquad f_-^\varepsilon(\bar{x},\bar{y}) = f_-\left(\bar{x}, \frac{\bar{y}}{\varepsilon}\right)\frac{1}{\varepsilon^l}, \qquad \text{for } (\bar{x},\bar{y}) \in \Omega_\varepsilon.$$

These functions still satisfy the mass balance condition in Ω_ε, indeed, it holds that,

$$\int_{\Omega_\varepsilon} f_+^\varepsilon(\bar{x},\bar{y})\,d\bar{x}d\bar{y} = \int_{\Omega_\varepsilon} f_-^\varepsilon(\bar{x},\bar{y})\,d\bar{x}d\bar{y} = M.$$

We will keep the notation (x,y) for the variables in the reference domain, $\Omega_1 \times \Omega_2$, and $(\bar{x},\bar{y})$ for the variables in the contracted domain, $\Omega_1 \times \varepsilon\Omega_2$, along the whole paper.

Now we consider the Monge-Kantorovich problem for the measures f_+^ε and f_-^ε in the thin domain Ω_ε.

From previous results (see [1], [13], [25] and [26]) we know that there exist $\bar{\mu}^\varepsilon$ an optimal transport plan and $\bar{u}^\varepsilon$ a Kantorovich potential for this problem defined in Ω_ε. In addition if we consider the $p-$Laplacian approximation given by (1.3) with $f = f_+^\varepsilon$ and $g = f_-^\varepsilon$ in the thin domain Ω_ε we know that the solutions $\bar{u}_p^\varepsilon$ to the $p-$Laplacian type problems (1.3) in Ω_ε provide an approximation to a Kantorovich potential.

Main goal. Our main concern in this paper is to study the behaviour as $\varepsilon \to 0$ of all the relevant variables for this problem; the total costs $TC(f_+^\varepsilon, f_-^\varepsilon)_{\Omega_\varepsilon}$, the optimal transport plans, $\bar{\mu}^\varepsilon$, the Kantorovich potentials, $\bar{u}^\varepsilon$, and the $p-$Laplacian approximations, $\bar{u}_p^\varepsilon$.

We find that when $\varepsilon \to 0$ the limit problem that appears is the mass transport problem in Ω_1 where the involved masses are given by the projections of f_+ and f_- in the x variable, that is,

$$g_+(x) = \int_{\Omega_2} f_+(x,y)dy \qquad \text{and} \qquad g_-(x) = \int_{\Omega_2} f_-(x,y)dy. \tag{1.5}$$

Associated with the mass transport problem for the projections we have optimal transport plans (denoted by η in the sequel) and Kantorovich potentials (denoted by u) and approximating sequences of solutions to $p-$Laplacians (denoted by u_p).

Our main results can be summarized as follows:

Theorem 1.1. *With the above notations we have the following commutative diagram (for all the involved functions rescaled to the fixed reference domain Ω)*

$$\begin{array}{ccccc} u_p^\varepsilon & \longrightarrow & u^\varepsilon & \\ \downarrow & \circlearrowright & \downarrow & (\varepsilon \to 0). \\ u_p & \longrightarrow & u & \end{array}$$

$$(p \to \infty)$$

This means that Kantorovich potentials (and their $p-$Laplacian approximations) for the problem in the thin domain converge to a Kantorovich potential (and to the $p-$Laplacian approximation) for the problem for the projections of the involved measures.

Concerning optimal plans, it holds that the optimal plans in the thin domain $\bar{\mu}^\varepsilon$ rescaled back to $\Omega \times \Omega$ converge weakly- in the sense of measures to a measure, ν, that allows us to construct an optimal plan for the projections, η.*

In addition, we find that the error is of order ε, in the sense that the difference of the total cost of transporting f_+^ε to f_-^ε and the total cost of transporting the projections g_+ to g_- is less or equal to $2M diam(\Omega_2)\varepsilon$.

Remark 1.1. With the same methods and ideas we can handle the case of Ω being a general domain in $\mathbb{R}^{k+l}$ (not necessarily a product domain). In this case we just consider

$$\Omega_\varepsilon = \{(x, \varepsilon y) \,:\, (x,y) \in \Omega\},$$

$f_\pm^\varepsilon$ are defined as above and the projections are given by $g_\pm(x) = \int_{\mathbb{R}^l} f_\pm(x,y)\,dy$. All our results (and their proofs) can be obtained for this more general case. The only place at which there is a difference is when we take the limit as $\varepsilon \to 0$ of the approximations sequence $\bar{u}_p^\varepsilon$ (with fixed p). In this case there appears a weight in the limit PDE (that is the constant $|\Omega_2|$ for a product domain, but that depends on x in the general case). We include a remark on this point when appropriate (in Section 5). We prefer to present our results for a product domain to clarify the arguments involved.

Remark 1.2. The same ideas can be used to handle the situation in which the measures are contained in a domain that lies between two parallel hyperplanes that are close one to each other. We don't include the details for simplicity. Also, the methods used here could be extended with domains that concentrate along a surface, that is, domains of the form $\Omega_\varepsilon = S + B(0,\varepsilon)$ where S is a k-dimensional surface in $\mathbb{R}^n$.

Remark 1.3. In general, the transport problem for the projections is simpler than the original one (since it involves measures in a smaller dimension). This fact together with the bound for the error allows us to build approximate transport maps when the projections are one-dimensional, that is, $\Omega_1 = (a,b) \subset \mathbb{R}$. We provide examples in Section 6.

To finish the introduction we briefly comment on the previous bibliography and the methods and ideas involved in the proofs. Optimal transport problems is by now a classical subject that still deserves attention. We refer to [2], [3], [4], [6], [21], [22], [23] and the surveys and books [1], [13], [25] and [26]. It has many applications, for example in economics (matching problems), [5], [7], [8], [9], [10], [11], [20]. Closely related to this article is the case in which the involved measures are concentrated in a small strip around the boundary of a fixed domain. This has been considered in [15] (see also [16] for singular measures supported on the boundary). In [19] the role of boundary conditions (Dirichlet and/or Newmann) in the $p-$Laplacian approximation was clarified (note that in our case we use Newmann boundary conditions since no mass is to be taken/bringed to/from outside of the domain). The first paper that uses the approximation by $p-$Laplacian type problems is [14] where the authors use Dirichlet boundary conditions in a sufficiently large ball, we can not use Dirichlet boundary conditions here since, as we want to contract the domain in one direction, is it likely that some mass will be taken to/from the boundary of the domain if we impose Dirichlet boundary conditions (we will elaborate more on this issue in Section 7).

Concerning the methods used in the proofs we have: to pass to the limit in the Kantorovich potentials, we first rescale back to Ω and then, using that Kantorovich potentials are Lipschitz functions to gain compactness and that they are solutions to a variational formulation we find that any possible uniform limit is a solution to a maximization limit problem. Then we find that the limit function is independent of the y variable and just observe that integration in y gives the projections of $f_\pm$. The proof of the convergence of the optimal transport plans is similar but we have to work in the space of Borel measures. To obtain convergence of the $p-$Laplacian approximations we use mainly the variational characterization of the solutions to the $p-$Laplacian as minimizers of an adequate functional in the Sobolev space $W^{1,p}$. We include here the details of the approximation of a Kantorovich potential with solutions to the $p-$Laplacian problems as $p \to \infty$ for completeness.

The paper is organized as follows: In Section 2 we prove the existence of Kantorovich potentials $\bar{u}_\varepsilon$ and study their limit as $\varepsilon \to 0$; in Section 3 we study the behaviour of the optimal transport plans; in Section 4 we show estimates for the difference of the total costs of the $\varepsilon-$problem and the limit problem; in Section 5 we deal with the $p-$Laplacian approximations and their behaviour as $\varepsilon \to 0$; in Section 6 we collect some examples that show that we can construct approximate transport maps when the limit problem is one-dimensional; finally in Section 7 we comment on the possibility of considering other boundary conditions than homogeneous Neumann ones in the $p-$Laplacian approximations.

2. Behavior of the Kantorovich potentials.

Lemma 2.1. *Given f_+, f_- and Ω, for each ε there exists a Kantorovich potential, $\bar{u}^\varepsilon$, that is, a solution to*

$$\max_{\substack{|\nabla \bar{v}(\bar{x},\bar{y})| \leq 1 \\ \bar{v}(0,0)=0}} \int_{\Omega_\varepsilon} \bar{v}(\bar{x},\bar{y})(f_+^\varepsilon(\bar{x},\bar{y}) - f_-^\varepsilon(\bar{x},\bar{y}))\, d\bar{x}d\bar{y}. \tag{2.1}$$

Proof. Let $K = \left\{ \bar{v} : \overline{\Omega}_\varepsilon \to \mathbb{R} \,:\, |\nabla \bar{v}| \leq 1, v(0,0) = 0 \right\}$, and, for $\bar{v} \in K$, consider

$$L(\bar{v}) = \int_{\Omega_\varepsilon} \bar{v}(\bar{x},\bar{y})(f_+^\varepsilon(\bar{x},\bar{y}) - f_-^\varepsilon(\bar{x},\bar{y}))d\bar{x}d\bar{y}.$$

If we take $(\bar{x},\bar{y})\,,\,(\bar{z},\bar{w}) \in \overline{\Omega}_\varepsilon$ we have,

$$|\bar{v}(\bar{x},\bar{y}) - \bar{v}(\bar{z},\bar{w})| \leq |\nabla \bar{v}(\bar{\xi})|\,|(\bar{x},\bar{y}) - (\bar{z},\bar{w})| \leq |(\bar{x},\bar{y}) - (\bar{z},\bar{w})| \leq diam(\Omega_\varepsilon), \tag{2.2}$$

where $\bar{\xi}$ lies on the segment between $(\bar{x},\bar{y})$ and $(\bar{z},\bar{w})$. Now, (1.4) implies

$$\begin{aligned} L(\bar{v}) &= \int_{\Omega_\varepsilon} \bar{v}(\bar{x},\bar{y}) f_+^\varepsilon(\bar{x},\bar{y}) d\bar{x}d\bar{y} - \int_{\Omega_\varepsilon} \bar{v}(\bar{x},\bar{y}) f_-^\varepsilon(\bar{x},\bar{y})\, d\bar{x}d\bar{y} \\ &\leq 2diam(\Omega_\varepsilon) \int_{\Omega_\varepsilon} f_+^\varepsilon(\bar{x},\bar{y}) d\bar{x}d\bar{y} = 2M diam(\Omega_\varepsilon), \end{aligned}$$

for all $\bar{v} \in K$. Hence L is bounded above in K. Let $(\bar{v}_j)_{j\in\mathbb{N}}$ be a sequence in K such that

$$L(\bar{v}_j) \nearrow \sup_{\bar{v}\in K} L(\bar{v}).$$

This sequence is equicontinuos and equibounded by (2.2), using the condition $\bar{v}(0,0) = 0$. So we can extract a subsequence $(\bar{v}_{j_k})_{k\in\mathbb{N}}$ such that $\bar{v}_{j_k} \rightrightarrows \bar{u}^\varepsilon$ in $\bar{\Omega}_\varepsilon$, uniformly. We have,

$$\lim_{k\to\infty} \int_{\Omega_\varepsilon} \bar{v}_{j_k}(\bar{x},\bar{y})(f_+^\varepsilon(\bar{x},\bar{y}) - f_-^\varepsilon(\bar{x},\bar{y}))d\bar{x}d\bar{y} = L(\bar{u}^\varepsilon) = \sup_{\bar{v}\in K} L(\bar{v}).$$

To conclude we need to check that $\bar{u}^\varepsilon \in K$. This follows from the fact that $\bar{v}_{j_k}(0,0) = 0$ and that, from (2.2) we get, $|\bar{v}_{j_k}(\bar{x},\bar{y}) - \bar{v}_{j_k}(\bar{z},\bar{w})| \leq |(\bar{x},\bar{y}) - (\bar{z},\bar{w})|$. When we take the limit as $k \to \infty$, we obtain, $\bar{u}^\varepsilon(0,0) = 0$ and $|\bar{u}^\varepsilon(\bar{x},\bar{y}) - \bar{u}^\varepsilon(\bar{z},\bar{w})| \leq |(\bar{x},\bar{y}) - (\bar{z},\bar{w})|$. So $\bar{u}^\varepsilon \in K$ and then it is the desired maximizer. □

Now we can state the following theorem concerning the behaviour as $\varepsilon \to 0$ of the Kantorovich potentials.

Theorem 2.1. *Let $\bar{u}^\varepsilon$ be a maximizer of* (2.1) *defined in Ω_ε and rescale it to Ω as*

$$u^\varepsilon(x,y) = \bar{u}^\varepsilon(x,\varepsilon y).$$

Then

$$u^\varepsilon(x,y) \rightrightarrows u(x), \qquad \text{when } \varepsilon \to 0, \tag{2.3}$$

uniformly in $\overline{\Omega}$ along subsequences. The limit u only depends on x and is a Kantorovich potential for the projections of f_+ and f_-, that is, u is a maximizer for

$$\max_{\substack{|\nabla_x v(x)| \leq 1 \\ v(0)=0}} \int_{\Omega_1} v(x)(g_+(x) - g_-(x))\, dx, \tag{2.4}$$

with g_+ and g_- given by (1.5).

Proof. We have that $\bar{u}^\varepsilon$ is defined in Ω_ε and we want to rescale it to Ω, we let $\bar{x} = x$, $\bar{y} = \varepsilon y$, and we obtain, using that $\bar{u}^\varepsilon$ is a Kantorovich potential that

$$\begin{aligned}\int_{\Omega_\varepsilon} \bar{u}^\varepsilon(\bar{x},\bar{y})(f_+^\varepsilon(\bar{x},\bar{y}) - f_-^\varepsilon(\bar{x},\bar{y}))\, d\bar{x}d\bar{y} &= \varepsilon^l \int_\Omega \bar{u}^\varepsilon(x,\varepsilon y)(f_+^\varepsilon(x,\varepsilon y) - f_-^\varepsilon(x,\varepsilon y))\, dxdy \\ &\geq \varepsilon^l \int_\Omega v(x)(f_+^\varepsilon(x,\varepsilon y) - f_-^\varepsilon(x,\varepsilon y))\, dxdy,\end{aligned} \tag{2.5}$$

for any v such that $|\nabla_x v(x)| \leq 1$ and $v(0) = 0$. The function u^ε verifies $u^\varepsilon(0,0) = \bar{u}^\varepsilon(0,0) = 0$ and

$$\begin{aligned} |\nabla_x u^\varepsilon(x,y)| &= |\nabla_x \bar{u}^\varepsilon\,(x,\varepsilon y)| \quad &\Rightarrow \quad |\nabla_x u^\varepsilon(x,y)| \leq 1, \\ |\nabla_y u^\varepsilon(x,y)| &= |\nabla_y \bar{u}^\varepsilon\,(x,\varepsilon y)|\,\varepsilon \quad &\Rightarrow \quad |\nabla_y u^\varepsilon(x,y)| \leq \varepsilon. \end{aligned}$$

Hence u^ε ia a equicontinuos and equibounded family and therefore we can extract a uniformly convergent subsequence, that is, there is $(\varepsilon_j)_{j\in\mathbb{N}}$, with $\varepsilon_j \to 0$ such as $u^{\varepsilon_j} \rightrightarrows u$, uniformly in $\overline{\Omega}$. Now we check that u only depends on x. First we have,

$$|u^\varepsilon(x,y_1) - u^\varepsilon(x,y_2)| \leq |\nabla_y u^\varepsilon(x,\xi)||y_1 - y_2| \leq \varepsilon diam(\Omega_2)$$

where ξ lies on the segment between y_1 and y_2. Now if $\varepsilon_j \to 0$ we conclude

$$|u(x,y_1) - u(x,y_2)| \leq 0.$$

Hence, $u(x,y)$ only depends on x. So we write $u(x)$ and next we show that u is a Kantorovich potential for the projections of f_+ and f_-. We need to check that $u(x)$ satisfy $|\nabla_x u(x)| \leq 1$. We have

$$|u^\varepsilon(x_1,y) - u^\varepsilon(x_2,y)| \leq |\nabla_x u^\varepsilon(\xi,y)||x_1 - x_2| \leq |x_1 - x_2|$$

where ξ lies on the segment between x_1 and x_2. Now taking $\varepsilon_j \to 0$ we conclude that

$$|u(x_1) - u(x_2)| \leq |x_1 - x_2|.$$

So $|\nabla_x u(x)| \leq 1$ and, therefore the limit u is $1-$Lipschitz. To see that u is a Kantorovich potential for the projections of f_+ and f_- we argue as follows:

$$\varepsilon^l \int_\Omega \bar{u}^{\varepsilon_j}(x,\varepsilon y)(f_+^\varepsilon(x,\varepsilon y) - f_-^\varepsilon(x,\varepsilon y))\, dxdy = \int_\Omega u^{\varepsilon_j}(x,y)(\varepsilon^l f_+^\varepsilon(x,\varepsilon y) - \varepsilon^l f_-^\varepsilon(x,\varepsilon y))\, dxdy.$$

Using (2.5) we obtain

$$\begin{aligned} &\varepsilon^l \int_\Omega \bar{u}^{\varepsilon_j}(x,\varepsilon y)(f_+^\varepsilon(x,\varepsilon y) - f_-^\varepsilon(x,\varepsilon y))\, dxdy \\ &\quad = \int_\Omega u^{\varepsilon_j}(x,y)(f_+(x,y) - f_-(x,y))\, dxdy \quad \geq \varepsilon^l \int_\Omega v(x)(f_+^\varepsilon(x,\varepsilon y) - f_-^\varepsilon(x,\varepsilon y))\, dxdy \\ &\qquad = \int_\Omega v(x)(f_+(x,y) - f_-(x,y))\, dxdy \quad = \int_{\Omega_1} v(x) \int_{\Omega_2} (f_+(x,y) - f_-(x,y))\, dydx, \end{aligned}$$

for all v such that $|\nabla_x v(x)| \leq 1$ and $v(0) = 0$. Now we take limits as $\varepsilon_j \to 0$, using that $u^{\varepsilon_j} \rightrightarrows u$, and (1.5), we get,

$$\int_{\Omega_1} u(x)(g_+(x) - g_-(x))\, dxdy \geq \int_{\Omega_1} v(x)(g_+(x) - g_-(x))\, dx,$$

for all v such that $|\nabla_x v(x)| \leq 1$ and $v(0) = 0$. □

Also from the previous proof we obtain the following result:

Corollary 2.1. *Under the same hypothesis of Theorem 2.1 we have,*

$$\lim_{\varepsilon\to 0} \int_{\Omega_\varepsilon} \bar{u}^\varepsilon(\bar{x},\bar{y})(f_+^\varepsilon(\bar{x},\bar{y}) - f_-^\varepsilon(\bar{x},\bar{y}))\, d\bar{x}d\bar{y} = \int_{\Omega_1} u(x)(g_+(x) - g_-(x))\, dx.$$

That is, we have that

$$\lim_{\varepsilon\to 0} TC(f_+^\varepsilon, f_-^\varepsilon)_{\Omega_\varepsilon} = TC(g_+, g_-)_{\Omega_1}.$$

3. Behaviour of the transport plans.

We consider measures $\bar{\mu}^\varepsilon$ in $\Omega_\varepsilon \times \Omega_\varepsilon$ that are solutions to the minimization problem

$$\min_{\substack{proj_{(\bar{x},\bar{y})}(\bar{\mu}) = f_+^\varepsilon \\ proj_{(\bar{\theta},\bar{\xi})}(\bar{\mu}) = f_-^\varepsilon}} \int_{\Omega_\varepsilon}\int_{\Omega_\varepsilon} |(\bar{x},\bar{y}) - (\bar{\theta},\bar{\xi})| d\bar{\mu}((\bar{x},\bar{y}),(\bar{\theta},\bar{\xi})). \tag{3.1}$$

Now, for $F \subset \Omega$ we let $S_\varepsilon(F) = \{(\theta, \varepsilon\xi) : (\theta,\xi) \in F\}$ and we define the rescaled measure as

$$\mu^\varepsilon(E \times F) = \bar{\mu}^\varepsilon(S_\varepsilon(E) \times S_\varepsilon(F)). \tag{3.2}$$

Concerning the limit as $\varepsilon \to 0$ of optimal transport plans we have the following result:

Theorem 3.1. *Let μ^ε be the measure in $\Omega \times \Omega$ given by* (3.2) *where $\bar{\mu}^\varepsilon$ is a minimizer of* (3.1)*. Then*

$$\mu^\varepsilon \to \nu$$

weakly- as $\varepsilon \to 0$ along a subsequence. If we let*

$$\eta(x,\theta) = \int_{\Omega_2}\int_{\Omega_2} d\nu((x,y),(\theta,\xi)), \tag{3.3}$$

it holds that η depends only on the first coordinates (x,θ) and is an optimal transport plan for the projections of f_+ and f_-, that is, η is a minimizer of

$$\min_{\substack{proj_x(\eta) = g_+ \\ proj_\theta(\eta) = g_-}} \int_{\Omega_1}\int_{\Omega_1} |x - \theta| d\eta(x,\theta).$$

Proof. First, let us compute the projections of μ^ε. We have

$$\mu^\varepsilon(\Omega \times F) = \bar{\mu}^\varepsilon(\Omega_1 \times \varepsilon\Omega_2 \times S_\varepsilon(F)) = \int_{S_\varepsilon(F)} f_-^\varepsilon(\bar{\theta},\bar{\xi})\, d\bar{\theta}d\bar{\xi} = \int_F f_-(\theta,\xi)\, d\theta d\xi.$$

Therefore, we have that $proj_{\theta,\xi}(\mu^\varepsilon) = f_-$. Analogously, we obtain $proj_{x,y}(\mu^\varepsilon) = f_+$. Hence, μ^ε are nonnegative measures with bounded total mass,

$$\mu^\varepsilon(\Omega \times \Omega) = \int_\Omega f_+ = M,$$

and therefore there exists a sequence $\varepsilon_j \to 0$ such that

$$\mu^{\varepsilon_j} \rightharpoonup \nu$$

weakly-* in the sense of measures. It follows that $proj_{\theta,\xi}(\nu) = f_-$, and $proj_{x,y}(\nu) = f_+$. Now we observe that, taking into account (3.2),

$$\int_{\Omega_{\varepsilon_j}} \int_{\Omega_{\varepsilon_j}} |(\bar{x},\bar{y}) - (\bar{\theta},\bar{\xi})| d\bar{\mu}^{\varepsilon_j}((\bar{x},\bar{y}),(\bar{\theta},\bar{\xi})) = \int_\Omega \int_\Omega |(x,\varepsilon_j y) - (\theta, \varepsilon_j \xi)| d\mu^{\varepsilon_j}((x,y),(\theta,\xi)).$$

Hence, the limit as $\varepsilon_j \to 0$ is given by

$$\int_\Omega \int_\Omega |x-\theta| d\nu((x,y),(\theta,\xi)).$$

Finally, we easily obtain that the measure η given by (3.3) is a minimizer for

$$\min_{\substack{proj_x(\eta) = g_+ \\ proj_\theta(\eta) = g_-}} \int_{\Omega_1} \int_{\Omega_1} |x-\theta| d\eta(x,\theta).$$

□

4. A bound for the error.

In this section our main goal is to estimate the error committed in the total cost when we replace the optimal transport problem in Ω_ε with the transport problem of the projections, that is, we want to obtain a bound for

$$\left| TC(f_+^\varepsilon, f_-^\varepsilon)_{\Omega_\varepsilon} - TC(g_+,g_-)_{\Omega_1} \right| = \left| \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) - \int_{\Omega_1} u(g_+ - g_-) \right|$$

in terms of ε. Our main result in this direction is the following:

Theorem 4.1. *There exists a constant $C := 2M diam(\Omega_2)$ independent of ε such that*

$$\left| \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) - \int_{\Omega_1} u(g_+ - g_-) \right| \le C\varepsilon.$$

Proof. Changing variables as before $\bar{x} = x$, $\bar{y} = \varepsilon y$ and $\bar{u}^\varepsilon(\bar{x},\bar{y}) = u^\varepsilon(x,y)$ we get

$$\int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon)(\bar{x},\bar{y})\, d\bar{x} d\bar{y} = \int_\Omega u^\varepsilon (f_+ - f_-)(x,y)\, dxdy$$

with u^ε verifying $|\nabla_x u^\varepsilon|^2 + \varepsilon^{-2} |\nabla_y u^\varepsilon|^2 \le 1$. As u depends only on x and verifies $|\nabla_x u| \le 1$ it competes with u^ε in the maximization problem, hence we have

$$\int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \ge \int_{\Omega_1} u(g_+ - g_-).$$

Let

$$h^\varepsilon(x) = \int_{\Omega_2} u^\varepsilon(x,y) dy.$$

Now, we observe that, from the fact that $|\nabla_x u^\varepsilon| \leq 1$ we get that this function h^ε competes with u in its maximization problem, then,

$$\int_{\Omega_1} h^\varepsilon (g_+ - g_-) \leq \int_{\Omega_1} u(g_+ - g_-).$$

In addition, we have

$$|u^\varepsilon(x,y) - h^\varepsilon(x)| \leq diam(\Omega_2)\varepsilon.$$

It follows that (recall that we assumed $|\Omega_2| = 1$)

$$\left| \int_{\Omega_1} u(g_+ - g_-) - \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \right| \leq \int_{\Omega_1} \int_{\Omega_2} |h^\varepsilon - u^\varepsilon| (f_+ + f_-) \leq 2M diam(\Omega_2)\varepsilon.$$

This ends the proof. ☐

Remark 4.1. The bound depends in a sharp way of the relevant quantities as it can be seen taken two masses concentrated near points (x_1, y_1) and (x_1, y_2) with $|y_1 - y_2| \sim diam(\Omega_2)$. Note that since both concentration points have the same first coordinate, we have $TC(g_+, g_-)_{\Omega_1} \sim 0$ and for the total cost $TC(f_+^\varepsilon, f_-^\varepsilon)_{\Omega_\varepsilon} \sim \varepsilon |y_1 - y_2| M \sim diam(\Omega_2) M\varepsilon$.

We can also characterize when we have equality of the total cost for the original functions and the projections.

Theorem 4.2. *There is a Kantorovich potential for the transport of f_+^ε to f_-^ε that depends only in the x variable, that is, of the form $\bar{u}(\bar{x}, \bar{y}) = \widehat{u}(\bar{x})$, if and only if the total cost of sending f_+^ε to f_-^ε is the same as the total cost for the projections g_+ to g_-.*

Proof. Using that $\widehat{u}(\bar{x})$ is a Kantorovich potential for the transport of f_+^ε to f_-^ε and the previous proof we obtain that

$$\begin{aligned}
&\max_{\substack{|\nabla \bar{v}(\bar{x}, \bar{y})| \leq 1 \\ \bar{v}(0) = 0}} \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y}))\, d\bar{x}\, d\bar{y} \\
&= \int_{\Omega_\varepsilon} \widehat{u}(\bar{x})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y}))\, d\bar{x}\, d\bar{y} = \int_{\Omega_1} \widehat{u}(x)(g_+(x) - g_-(x))\, dx \\
&\leq \max_{\substack{|\nabla_x v(x)| \leq 1 \\ v(0) = 0}} \int_{\Omega_1} v(x)(g_+(x) - g_-(x))\, dx \\
&\leq \max_{\substack{|\nabla \bar{v}(\bar{x}, \bar{y})| \leq 1 \\ \bar{v}(0) = 0}} \int_{\Omega_\varepsilon} \bar{v}(\bar{x}, \bar{y})(f_+^\varepsilon(\bar{x}, \bar{y}) - f_-^\varepsilon(\bar{x}, \bar{y}))\, d\bar{x}\, d\bar{y},
\end{aligned}$$

and hence we conclude that the total costs for f_+^ε to f_-^ε and for g_+ to g_- coincide.

Conversely, if the costs coincide, then take $\widehat{u}(x)$ a Kantorovich potential for the projections and observe that

$$\int_{\Omega_1} \widehat{u}(x)(g_+(x) - g_-(x))\,dx = \max_{\substack{|\nabla_x v(x)| \leq 1 \\ v(0)=0}} \int_{\Omega_1} v(x)(g_+(x) - g_-(x))\,dx$$

$$= \max_{\substack{|\nabla \bar{v}(\bar{x},\bar{y})| \leq 1 \\ \bar{v}(0)=0}} \int_{\Omega_\varepsilon} \bar{v}(\bar{x},\bar{y})(f_+^\varepsilon(\bar{x},\bar{y}) - f_-^\varepsilon(\bar{x},\bar{y}))\,d\bar{x}\,d\bar{y},$$

and we conclude that $\widehat{u}$ is a Kantorovich potential for f_+^ε to f_-^ε that depends only on x. □

5. A $p-$Laplacian approximation and its behaviour as $\varepsilon \to 0$.

We consider

$$\min_{\substack{\bar{v} \in W^{1,p}(\Omega_\varepsilon) \\ \bar{v}(0)=0}} \frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \int_{\Omega_\varepsilon} \bar{v}(f_+^\varepsilon - f_-^\varepsilon). \tag{5.1}$$

Note that we have normalized the gradient term in the functional with $\frac{1}{\varepsilon^l}$. This is the right scale to compensate the fact that $|\Omega_\varepsilon| \sim \varepsilon^l$. This scaling factor is not needed in the second term since we have normalized $f_\pm^\varepsilon$ in such a way that they have constant total mass M.

Lemma 5.1. *There exists a unique minimizer of* (5.1)*, that we will call* $\bar{u}_p^\varepsilon$.

Proof. We just observe that the functional

$$L_p(\bar{v}) = \frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \int_{\Omega_\varepsilon} \bar{v}(f_+^\varepsilon - f_-^\varepsilon)$$

is bounded below in $W^{1,p}(\Omega_\varepsilon)$. Indeed, for $\bar{v} \in W^{1,p}(\Omega_\varepsilon)$ with $\bar{v}(0,0) = 0$, calling $f^\varepsilon = f_+^\varepsilon - f_-^\varepsilon$ we have,

$$\int_{\Omega_\varepsilon} (\bar{v} f^\varepsilon) \leq \|\bar{v}\|_{L^p(\Omega_\varepsilon)} \|f^\varepsilon\|_{L^{p'}(\Omega_\varepsilon)} \leq C_1 \|\nabla \bar{v}\|_{L^p(\Omega_\varepsilon)},$$

where C_1 is a constant that depends on f^ε. So

$$L_p(\bar{v}) = \frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \int_{\Omega_\varepsilon} \bar{v}(f_+^\varepsilon - f_-^\varepsilon) \geq \frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - C_1 \|\nabla \bar{v}\|_{L^p(\Omega_\varepsilon)}. \tag{5.2}$$

Using Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^{p'}}{p'}$ with $a = \varepsilon^{l/p} C_1$, $b = \|\nabla \bar{v}\|_{L^p(\Omega_\varepsilon)}$, we get

$$L_p(\bar{v}) \geq \frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla \bar{v}|^p - \frac{\varepsilon^{l/(p-1)}(C_1)^{p'}}{p'} - \frac{(\|\nabla \bar{v}\|_{L^p(\Omega_\varepsilon)})^p}{\varepsilon^l p} = -\frac{\varepsilon^{l/(p-1)}(C_1)^{p'}}{p'}.$$

So $L_p(\bar{v}) \geq C$ for all $\bar{v} \in W^{1,p}(\Omega_\varepsilon)$ with $\bar{v}(0,0) = 0$. Take $\bar{v}_n$ a minimizing sequence. From (5.2) and the fact that $\bar{v}_n(0,0) = 0$ we get that $\bar{v}_n$ is bounded in $W^{1,p}(\Omega_\varepsilon)$ and extracting a subsequence if necessary we can assume that $\bar{v}_n \to \bar{u}_p^\varepsilon$ weakly in $W^{1,p}(\Omega_\varepsilon)$. From the lower semicontinuity of L_p we conclude that $\bar{u}_p^\varepsilon$ is a minimizer of L_p.

Uniqueness follows from the strict convexity of L_p. □

From the fact that $\bar{u}_p^\varepsilon$ is a minimizer of (5.1) we have that $\bar{u}_p^\varepsilon$ is a weak solution to the following PDE problem

$$\begin{cases} -\frac{1}{\varepsilon^l}\Delta_p \bar{u}_p^\varepsilon = f_+^\varepsilon - f_-^\varepsilon & \text{in } \Omega_\varepsilon, \\ \frac{1}{\varepsilon^l}|\nabla \bar{u}_p^\varepsilon|^{p-2}\frac{\partial \bar{u}_p^\varepsilon}{\partial \eta} = 0 & \text{on } \partial\Omega_\varepsilon, \\ \bar{u}_p^\varepsilon(0) = 0. \end{cases} \tag{5.3}$$

Theorem 5.1. *Let $\bar{u}_p^\varepsilon$ be a minimizer of* (5.1). *Then, extracting a subsequence if necessary,*

$$\bar{u}_p^\varepsilon \to \bar{u}^\varepsilon$$

as $p \to \infty$ uniformly in Ω_ε where $\bar{u}^\varepsilon$ is a Kantorovich potential for the transport problem of the mass f_+^ε to the mass f_-^ε.

Proof. Along this proof ε is fixed and C denotes a constant that is independent of p but may depend on ε and change from one line to another. Let $\bar{u}^\varepsilon$ be a Kantorovich potential for the transport of f_+^ε to f_-^ε (its existence is guaranteed by Lemma 2.1). We have $|\nabla \bar{u}^\varepsilon| \le 1$ and $\bar{u}^\varepsilon(0) = 0$ and hence $\bar{u}^\varepsilon$ is bounded in Ω_ε and $\bar{u}^\varepsilon \in W^{1,p}(\Omega_\varepsilon)$. Using that $\bar{u}_p^\varepsilon$ is a minimizer of L_p we get

$$\begin{aligned} &\frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla \bar{u}_p^\varepsilon|^p - \int_{\Omega_\varepsilon} \bar{u}_p^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \le \frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla \bar{u}^\varepsilon|^p - \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \\ &\le \frac{1}{\varepsilon^l}\frac{|\Omega_\varepsilon|}{p} - \int_{\Omega_\varepsilon} \bar{u}^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \le C. \end{aligned} \tag{5.4}$$

It follows that

$$\frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon} |\nabla \bar{u}_p^\varepsilon|^p \le C + \int_{\Omega_\varepsilon} \bar{u}_p^\varepsilon (f_+^\varepsilon - f_-^\varepsilon) \le C + C\|\bar{u}_p^\varepsilon\|_{L^p(\Omega_\varepsilon)} \le C + CS_p\|\nabla \bar{u}_p^\varepsilon\|_{L^p(\Omega_\varepsilon)},$$

here S_p is the best Sobolev constant that can be bounded by Cp (see [12]). Therefore, we get

$$\|\nabla \bar{u}_p^\varepsilon\|_{L^p(\Omega_\varepsilon)} \le (Cp)^{1/p}.$$

Now, fix q with $n < q < p$ and observe that

$$\|\nabla \bar{u}_p^\varepsilon\|_{L^q(\Omega_\varepsilon)} \le |\Omega_\varepsilon|^{\frac{p-q}{pq}} \|\nabla \bar{u}_p^\varepsilon\|_{L^p(\Omega_\varepsilon)} \le |\Omega_\varepsilon|^{\frac{p-q}{pq}} (Cp)^{1/p}.$$

Hence, we have that $(\bar{u}_p^\varepsilon)_{p>q}$ is bounded in $W^{1,q}(\Omega_\varepsilon)$. Therefore, by a diagonal procedure, we can extract a subsequence (that we call $\bar{u}_{p_n}^\varepsilon$) such that

$$\bar{u}_{p_n}^\varepsilon \to \bar{v} \qquad \text{as } p_n \to \infty$$

weakly in every $W^{1,q}(\Omega_\varepsilon)$ and, therefore, uniformly in $\overline{\Omega}_\varepsilon$ (we are using here the compact embedding $W^{1,q}(\Omega_\varepsilon) \hookrightarrow C^\alpha(\Omega_\varepsilon)$ when $q > n$). Since $\bar{u}_{p_n}^\varepsilon(0) = 0$ we get $\bar{v}(0) = 0$. From the semicontinuity of the norm we get $\|\nabla \bar{v}\|_{L^q(\Omega_\varepsilon)} \leq |\Omega_\varepsilon|^{1/q}$, and hence, taking $q \to \infty$, we obtain

$$\|\nabla \bar{v}\|_{L^\infty(\Omega_\varepsilon)} \leq 1.$$

From (5.4) we have

$$-\int_{\Omega_\varepsilon} \bar{u}^\varepsilon_{p_n}(f^\varepsilon_+ - f^\varepsilon_-) \leq \frac{1}{\varepsilon^l}\frac{|\Omega_\varepsilon|}{p_n} - \int_{\Omega_\varepsilon} \bar{u}^\varepsilon(f^\varepsilon_+ - f^\varepsilon_-).$$

Now, taking $p \to \infty$ we obtain

$$-\int_{\Omega_\varepsilon} \bar{v}(f^\varepsilon_+ - f^\varepsilon_-) \leq -\int_{\Omega_\varepsilon} \bar{u}^\varepsilon(f^\varepsilon_+ - f^\varepsilon_-),$$

from where we conclude that $\bar{v}$, the limit of $\bar{u}^\varepsilon_{p_n}$, as $p_n \to \infty$ is a Kantorovich potential. □

Remark 5.1. With the arguments used in the previous proof we can obtain an alternative proof of the existence of a Kantorovich potential for the transport of f^ε_+ to f^ε_-.

Now we study the limit as $\varepsilon \to 0$ of $\bar{u}^\varepsilon_p$.

Theorem 5.2. *Let*

$$u^\varepsilon_p(x,y) = \bar{u}^\varepsilon_p(\bar{x},\bar{y}), \qquad x = \bar{x},\ \varepsilon y = \bar{y},$$

where $\bar{u}^\varepsilon_p$ is a minimizer of (5.1)*. Then*

$$u^\varepsilon_p \to u_p$$

as $\varepsilon \to 0$ uniformly in $\overline{\Omega}$ and weakly in $W^{1,p}(\Omega)$ where u_p depends only on x and is a solution to the minimization problem

$$\min_{\substack{v \in W^{1,p}(\Omega_1)\\ v(0)=0.}} \frac{1}{p}\int_{\Omega_1} |\nabla_x v|^p - \int_{\Omega_1} v(g_+ - g_-) \tag{5.5}$$

Proof. We have $\nabla_{\bar{x}}\bar{u}^\varepsilon_p(\bar{x},\bar{y}) = \nabla_x u^\varepsilon_p(x,y)$ and $\varepsilon\nabla_{\bar{y}}\bar{u}^\varepsilon_p(\bar{x},\bar{y}) = \nabla_y u^\varepsilon_p(x,y)$. Hence, u^ε_p is a minimizer of

$$\frac{1}{p}\int_{\Omega_1}\int_{\Omega_2}\left(\sqrt{|\nabla_x v|^2 + \varepsilon^{-2}|\nabla_y v|^2}\right)^p dxdy - \int_{\Omega_1}\int_{\Omega_2} v(f_+ - f_-)\,dxdy$$

in $W^{1,p}(\Omega)$ with $v(0) = 0$.

By the same arguments used in Lemma 5.1 we obtain the existence of a unique minimizer of (5.5) that we call u_p. As $u_p \in W^{1,p}(\Omega_1)$ we can consider it as a function of $W^{1,p}(\Omega_1 \times \Omega_2)$ and then it competes with u^ε_p. We get

$$\begin{aligned} &\frac{1}{p}\int_{\Omega_1}\int_{\Omega_2}\left(\sqrt{|\nabla_x u^\varepsilon_p|^2 + \varepsilon^{-2}|\nabla_y u^\varepsilon_p|^2}\right)^p dxdy - \int_{\Omega_1}\int_{\Omega_2} u^\varepsilon_p(f_+ - f_-)\,dxdy \\ &\qquad \leq \frac{1}{p}\int_{\Omega_1}|\nabla_x u_p|^p dx - \int_{\Omega_1} u_p(g_+ - g_-)\,dx \end{aligned} \tag{5.6}$$

(we recall that, for simplicity, we have assumed that $|\Omega_2| = 1$). Therefore there exists a constant C independent of ε such that

$$\frac{1}{p}\int_{\Omega_1}\int_{\Omega_2}\left(\sqrt{|\nabla_x u_p^\varepsilon|^2+\varepsilon^{-2}|\nabla_y u_p^\varepsilon|^2}\right)^p dxdy \le C + \int_{\Omega_1}\int_{\Omega_2} u_p^\varepsilon(f_+-f_-)\,dxdy. \tag{5.7}$$

Taking $\varepsilon < 1$ and arguing as in the proof of Theorem 5.1 we get that u_p^ε is bounded in $W^{1,p}(\Omega)$ uniformly in ε. Therefore, we can extract a subsequence such that

$$u_p^{\varepsilon_n} \to v, \qquad \text{as } \varepsilon_n \to 0,$$

weakly in $W^{1,p}(\Omega)$ and (using that $p > n$) uniformly in Ω. In addition we have

$$\nabla_x u_p^{\varepsilon_n} \to \nabla_x v \text{ and } \nabla_y u_p^{\varepsilon_n} \to \nabla_y v \qquad \text{weackly in } L^p(\Omega).$$

Now we observe that from (5.7) we obtain that there exists a constant C independent of ε such that

$$\left(\int_{\Omega_1}\int_{\Omega_2}|\nabla_y u_p^{\varepsilon_n}|^p dxdy\right)^{1/p} \le C\varepsilon_n.$$

Therefore,

$$\nabla_y u_p^{\varepsilon_n} \to 0 \qquad \text{strongly in } L^p(\Omega)$$

and we obtain that the limit v is independent of y.

Now, from (5.6) we get

$$\begin{aligned}\frac{1}{p}\int_{\Omega_1}\int_{\Omega_2}|\nabla_x u_p^\varepsilon|^p dxdy - \int_{\Omega_1}\int_{\Omega_2} u_p^\varepsilon(f_+-f_-)\,dxdy\\ \le \frac{1}{p}\int_{\Omega_1}|\nabla_x u_p|^p dx - \int_{\Omega_1} u_p(g_+-g_-)\,dx.\end{aligned}$$

Taking $\varepsilon_n \to 0$ and using that v is independent of y we conclude that

$$\frac{1}{p}\int_{\Omega_1}|\nabla_x v|^p dydx - \int_{\Omega_1} v(g_+-g_-)\,dx \le \frac{1}{p}\int_{\Omega_1}|\nabla_x u_p|^p dx - \int_{\Omega_1} u_p(g_+-g_-)\,dx.$$

Hence the limit v is a minimizer. By uniqueness we must have $v = u_p$ and then it holds that $\lim_{\varepsilon\to 0} u_p^\varepsilon = u_p$. $\square$

Corollary 5.1. *Under the same assumptions of Theorem 5.2 we have that*

$$\begin{aligned}\lim_{\varepsilon\to 0}\left\{\min_{\substack{\bar v \in W^{1,p}(\Omega_\varepsilon)\\ \bar v(0)=0}} \frac{1}{\varepsilon^l}\frac{1}{p}\int_{\Omega_\varepsilon}|\nabla \bar v|^p - \int_{\Omega_\varepsilon}\bar v(f_+^\varepsilon - f_-^\varepsilon)\right\}\\ = \min_{\substack{v \in W^{1,p}(\Omega_1)\\ v(0)=0.}} \frac{1}{p}\int_{\Omega_1}|\nabla_x v|^p - \int_{\Omega_1} v(g_+-g_-).\end{aligned}$$

Remark 5.2. The unique minimizer u_p of (5.5) is a weak solution to

$$\begin{cases} -\Delta_p u_p = g_+ - g_- & \text{in } \Omega_1, \\ |\nabla u_p|^{p-2} \frac{\partial u_p}{\partial \eta} = 0 & \text{on } \partial\Omega_1, \\ u_p(0) = 0. \end{cases}$$

Remark 5.3. When we deal with a general domain Ω (instead of a product domain) and we take the limit as $\varepsilon \to 0$ the limit problem that appears involve the weight

$$\omega(x) = |\{y \,:\, (x,y) \in \Omega\}|.$$

In fact, with the same arguments used before, we get that the uniform limit of u_p^ε as $\varepsilon \to 0$ is a weak solution to

$$\begin{cases} -\mathrm{div}(\omega |\nabla u_p|^{p-2} \nabla u_p) = g_+ - g_- & \text{in } \Omega_1, \\ \omega |\nabla u_p|^{p-2} \frac{\partial u_p}{\partial \eta} = 0 & \text{on } \partial\Omega_1, \\ u_p(0) = 0. \end{cases}$$

Theorem 5.3. *Let u_p be the unique minimizer of* (5.5). *Then*

$$u_p \to u$$

uniformly in $\overline{\Omega}_1$ where u is Kantorovich potential for the transport of the projections, g_+ to g_-.

Proof. The proof is analogous to the one of Theorem 5.1 and hence we omit the details. ☐

6. Examples.

In this section we look for a method to define, using an optimal transport map from the projections, an approximation for the original problem. The construction of such a transport map is known in the literature as the Knothe map, [18].

To simplify let us suppose that we are in $\mathbb{R}^2$, and we have $\Omega_1 = (a,b)$ and $\Omega_2 = (c,d)$. Hence the projections are defined as $g_+ : \Omega_1 = (a,b) \to \mathbb{R}$ and $g_- : \Omega_1 = (a,b) \to \mathbb{R}$. Let us assume that the support of the projections are also intervals, that is, $supp(g_+(x)) = [\alpha, \beta]$ and $supp(g_-(y)) = [\gamma, \delta]$.

Now in one dimension we are going to see two ways to define an optimal transport map for the projections $T : [\alpha, \beta] \to [\gamma, \delta]$. This optimal transport map must satisfy for all $E \in (\gamma, \delta)$,

$$\int_{T^{-1}(E)} g_+(x)dx = \int_E g_-(y)dy.$$

Therefore, assuming that T is differentiable, we get

$$\int_{T^{-1}(E)} g_+(x)dx = \int_E g_-(y)dy = \int_{T^{-1}(E)} g_-(T(x))|T'(x)|dx.$$

Now we have two options, to consider $T'(x) \geq 0$ or $T'(x) \leq 0$. We will call this possibilities applications as T_D and T_I. First we will take $T'(x) \geq 0$ and look for T_D a solution to the ODE problem,

$$\begin{cases} g_+(x) = g_-(T_D(x))T_D'(x), \\ T_D(\alpha) = \gamma. \end{cases}$$

Observe that we move the mass "directly", it means T_D preserves orientation. An alternative way to define T_D for all $x \in [\alpha, \beta]$ is the following:

$$T(x) = inf\left\{ y \in [\gamma, \delta] : \int_\alpha^x g_+ = \int_\gamma^y g_- \right\}.$$

The other choice to define T is to consider $T'(x) \leq 0$. We call it T_I and have the ODE,

$$\begin{cases} g_+(x) = -g_-(T_I(x))T_I'(x), \\ T_I(\alpha) = \delta. \end{cases}$$

Observe that this time we move the mass reversing the orientation of the interval. An alternative way to define T_I for all $x \in [\alpha, \beta]$ is given by,

$$T(x) = sup\left\{ y \in [\gamma, \delta] : \int_\alpha^x g_+ = \int_y^\delta g_- \right\}.$$

The two options are optimal.

Now we go back to the original problem and show how we can use this optimal maps in $\mathbb{R}^2$ to obtain a transport map $S : supp(f_+) \to supp(f_-)$. Let us suppose further that exist g_{11}, g_{12}, g_{21} and g_{22} functions which allow us to write: $supp(f_+) = \{(x, y) \in \mathbb{R}^2 : g_{11}(x) \leq y \leq g_{12}(x)\}$ and $supp(f_-) = \{(x, y) \in \mathbb{R}^2 : g_{21}(x) \leq y \leq g_{22}(x)\}$. We will propose S to be of the form $S(x, y) = (T_1(x), T_2(x, y))$ (with T_1 equal to T_D or T_I). Hence we want for all $E \in \Omega_1 \times \Omega_2$,

$$\int_E f_+(x, y)dxdy = \int_{S^{-1}(E)} f_-(x, y)dxdy = \int_E f_-\left(S(x, y)\right)|det(DS(x, y))|dxdy.$$

Since $S(x, y) = (T_1(x), T_2(x, y))$ with T_1 independent of y, we have,

$$DS = \begin{pmatrix} T_1'(x) & \dfrac{\partial T_2}{\partial x}(x, y) \\ 0 & \dfrac{\partial T_2}{\partial y}(x, y) \end{pmatrix}.$$

Therefore,

$$|det(DS(x, y))| = \left| T_1'(x)\frac{\partial T_2}{\partial y}(x, y) \right|.$$

And we obtain,

$$f_+(x,y) = f_-(T_1(x), T_2(x,y)) \left| T_1'(x) \frac{\partial T_2}{\partial y}(x,y) \right|. \tag{6.1}$$

This equation can be seen as an ODE for T_2 as a function of y (here x plays the role of a parameter). Now, again, we have two options for T_2 given by consider T_2 increasing or decreasing as a function of y. In each case we choose as initial conditions to complement (6.1),

$$\begin{cases} T_2(x, g_{11}(x)) = g_{21}(x), & \text{if } \frac{\partial T_2}{\partial y} \geq 0, \\ T_2(x, g_{12})(x) = g_{22}(x), & \text{if } \frac{\partial T_2}{\partial y} \leq 0. \end{cases}$$

In this way we can construct a transport map S (that is in general not optimal) moving f_+ to f_-.

Example 1. To start with, let us consider the simplest situation. In $\mathbb{R}^2$ consider f_+ and f_- two measures supported on two points with mass 1/2, that is

$$f_+ = \frac{1}{2}\delta_{(0,0)} + \frac{1}{2}\delta_{(1,0)} \qquad \text{and} \qquad f_- = \frac{1}{2}\delta_{(1,1)} + \frac{1}{2}\delta_{(2,1)}.$$

So, for the projections we have the optimal transport maps $T_D = x+1$ and $T_I = 2-x$, and then all possible transport maps S are given by all possible assignments of $\{(0,0),(1,0)\} \to \{(1,1),(2,1)\}$. We obtain,

$$S_1(x,y) = (x+1, y+1), \qquad \text{and} \qquad S_2(x,y) = (2-x, y+1).$$

Let us compute the total costs corresponding to these maps. We have,

$$\tilde{\mathcal{F}}(S_1) = \sqrt{2} = 1,4142 < \tilde{\mathcal{F}}(S_2) = \frac{1}{2}(1+\sqrt{5}) = 1,6180.$$

In the contracted domain $\Omega_1 \times \varepsilon\Omega_2$ we get

$$S_1(x,y) = (x+1, \varepsilon(y+1)), \qquad \text{and} \qquad S_2(x,y) = (2-x, \varepsilon(y+1)),$$

with approximate costs (up to the first nontrivial order in ε),

$$\tilde{\mathcal{F}}(S_1) \sim 1 + \frac{\varepsilon^2}{2} + o(\varepsilon^2) < \tilde{\mathcal{F}}(S_2) \sim 1 + \frac{\varepsilon}{2} + o(\varepsilon).$$

Example 2. As a second example we consider as $f_\pm$ the characteristic functions of the triangles, $C_1 = conv\{(0,0),(1,0),(1,1)\}$ and $C_2 = conv\{(3,0),(3,1),(2,1)\}$. So, for the projections we have the optimal transport maps,

$$T_D = \sqrt{-(x^2-2x)} + 2, \qquad \text{and} \qquad T_I = 3 - x.$$

Then we can obtain four different $S(x,y)$ transport maps given by the construction that we explained before, these are given by,

$$S_1(x,y) = (\sqrt{-(x^2-2x)}+2, y\frac{\sqrt{-(x^2-2x)}}{-x+1}+3-x),$$
$$S_2(x,y) = (\sqrt{-(x^2-2x)}+2, y\frac{\sqrt{-(x^2-2x)}}{-x+1}+1),$$
$$S_3(x,y) = (3-x, y+x),$$
$$S_4(x,y) = (3-x, 1-y).$$

Now, we approximate the total cost in the thin triangles $E_1 = conv\{(0,0),(1,0),(1,\varepsilon)\}$ and $E_2 = conv\{(3,0),(3,\varepsilon),(2,\varepsilon)\}$ with the transport maps

$$R_1(x,y) = (3-x, \varepsilon(1-\frac{y}{\varepsilon})) = (3-x, \varepsilon - y),$$
$$R_2(x,y) = (3-x, y+\varepsilon x).$$

We estimate the cost as follows:

$$\begin{aligned}
\tilde{\mathcal{F}}(R_1) &= \int_0^1 \int_0^{\varepsilon x} \|(x,y)-(3-x,\varepsilon-y)\| f_+^{\varepsilon}(x,y)dydx,\\
&= \int_0^1 \int_0^{\varepsilon x} \|(2x-3, 2y-\varepsilon)\| \frac{1}{\varepsilon}dydx,\\
&= \int_0^1 \int_0^{\varepsilon x} \sqrt{(2x-3)^2+(2y-\varepsilon)^2}\,\frac{1}{\varepsilon}dydx.
\end{aligned}$$

We take $z = \frac{y}{\varepsilon}$ and we obtain

$$\tilde{\mathcal{F}}(R_1) = \int_0^1 \int_0^{x} \sqrt{(2x-3)^2+\varepsilon^2(2y-1)^2}\,\frac{1}{\varepsilon}\,\varepsilon dydx = A(\varepsilon^2),$$

and hence

$$\tilde{\mathcal{F}}(R_1) = A(0) + A'(0)\,\varepsilon^2 + O(\varepsilon^4) = \frac{5}{6} + \frac{1}{78}\left(27\ln(3) - 26\right)\varepsilon^2 + O(\varepsilon^4).$$

We perform the same computations for $R_2(x,y) = (3-x, y+\varepsilon x)$ and we obtain,

$$\tilde{\mathcal{F}}(R_2) = \frac{5}{6} + \frac{27}{32}\left(\ln(3) - 1\right)\varepsilon^2 + O(\varepsilon^4).$$

Since $\frac{1}{78}(27\ln(3)-26) < \frac{27}{32}(\ln(3)-1)$, we see that $\tilde{\mathcal{F}}(R_1) < \tilde{\mathcal{F}}(R_2)$ for ε small.

We just note that in this example we obtain that the two possible transport maps, constructed as explained before, considering T_1 increasing or decreasing, may have different costs.

7. Boundary conditions

In this last section we comment briefly on the possibility of using Dirichlet boundary conditions instead of Neumann. Along this paper we have used Neumann boundary conditions for the $p-$Laplacian approximations. This choice is due to the fact that we want to transport the whole mass of f_+^ε to cover the whole mass of f_-^ε inside Ω_ε. If we impose Dirichlet boundary conditions we allow for some mass to be imported (created) at some point on the boundary or exported (eliminated) at other points on the boundary, paying in this case an extra import/export tax per unit of mass given by the value of the Dirichlet datum in addition to the usual transport cost given by the Euclidean distance. This problem was analyzed in detail in [19]. Here we contract the domain in one direction. Therefore, if we impose Dirichlet boundary conditions on the boundary $\Omega_1 \times \partial\varepsilon\Omega_2$ it will be more convenient to import/export some part of the mass trough the boundary than to transport it inside Ω_ε (since the distance of our masses to that part of the boundary is of order ε and hence negligible as $\varepsilon \to 0$ while the distance between masses remains of order one as $\varepsilon \to 0$). Hence, the choice of homogeneous Neumann boundary conditions on $\Omega_1 \times \partial\varepsilon\Omega_2$ seems natural. However, we can impose Dirichlet boundary conditions on $\partial\Omega_1 \times \varepsilon\Omega_2$, but to pass to the limit as $\varepsilon \to 0$ we need to take a constant as Dirichlet datum. If we do this we arrive to a limit problem that corresponds to an optimal mass transport problem between the projections in Ω_1 with import/export taxes at the boundary of Ω_1 equal to the constant Dirichlet datum.

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