Short time behavior near the boundary for the heat equation with a nonlinear boundary condition

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Abstract

We study the short time behavior near the boundary of solutions of the heat equation, $u_t = \Delta u$, in a bounded smooth domain with a nonlinear boundary condition, $\frac{\partial u}{\partial \eta} = \varphi(u)$ and constant initial data. We give an asymptotic expansion for $u(x,t)$ in powers of the distance of x to the boundary. As a consequence we obtain that u grows faster, for short times, near points of the boundary where the mean curvature is bigger.

1 Introduction and description of the results.

Let Ω be a bounded smooth domain in $\mathbf{R}^N$ and let φ be a smooth positive function. In this article we are concerned with the behavior near the

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boundary of the solution of the following problem :

$$\begin{array}{ll} u_t = \Delta u & \text{in } \Omega \times (0,T), \\ \frac{\partial u}{\partial \eta} = \varphi(u) & \text{on } \partial\Omega \times (0,T), \\ u(x,0) \equiv M & \text{in } \Omega. \end{array} \tag{1.1}$$

Existence and regularity of solutions to a wide class of parabolic problems, that include the case of (1.1), can be found in [A].

In order to state our result more precisely we need some preliminaries. Let $\bar{x}$ be a point in $\partial\Omega$. We denote by $\hat{n}(\bar{x})$ the inner unit normal to $\partial\Omega$ at the point $\bar{x}$. Since the boundary is smooth it is well known that there exists $\delta > 0$ such that the mapping $\Phi : \partial\Omega \times [0,\delta) \to \mathbf{R}^N$, given by $\Phi(\bar{x}, s) = \bar{x} + s\hat{n}(\bar{x})$, defines new coordinates $(\bar{x}, s)$ in a neighborhood U of $\partial\Omega$ in $\overline{\Omega}$. With this notation we can regard u as a function of $(\bar{x}, s, t)$ in a neighborhood of $\partial\Omega \times [0,T]$.

We state now our result.

Theorem 1.1 *Let u be the solution of (1.1) then, with the previous notation, there exist universal functions $F_i(s,t)$, $i = 1,2,3$, such that*

$$\begin{aligned} u(\bar{x}, \varepsilon s, \varepsilon^2 t) = M + \varphi(M)F_1(s,t)\varepsilon + \\ \{\varphi(M)\varphi'(M)F_2(s,t) + \varphi(M)H(\bar{x})F_3(s,t)\}\varepsilon^2 + o(\varepsilon^2). \end{aligned} \tag{1.2}$$

Where $H(\bar{x}) = \sum_{j=1}^{N-1} H_j(\bar{x})$ and $H_j(\bar{x})$ denote the principal curvatures of $\partial\Omega$ at $\bar{x}$. Moreover $F_i(s,t) > 0$ for $i = 1,2,3$.

Expansion (1.2) gives a precise description of the growth of u near the boundary at short times. In particular, for a given φ it shows that u grows fastest, for small times, near points of the boundary where the mean curvature is maximized. This is a quite natural fact because at points of the boundary where the mean curvature is bigger there is less room for diffusion. Moreover it also shows that the dependence on the curvature is *'linear'*.

The idea of the proof of expansion (1.2) is to nest, by means of a comparison argument, the solution of (1.1) between a sub and a supersolution that have the same behavior near the boundary for small times. This idea has been also used in [MS1] and [MS2] to characterize the small time behavior near the boundary in a problem with a noncompatibility between a zero Dirichlet boundary condition and the initial data. We note that our method for constructing the sub and supersolutions are completely different from the ones used in [MS1] and [MS2]. We also note that the results in later cited articles were used in [FSW] to develop a theory of heat equations in weighted Lebesgue spaces.

We give now a formal idea of the proof of Theorem 1.1. Fix $\bar{x} \in \partial\Omega$. After a rotation and a translation we can assume that $\bar{x}$ is the origin and that the tangent space to $\partial\Omega$ at $\bar{x}$ is given by the equation $x_N = 0$. Moreover we can assume that the coordinates $(x_1, ..., x_{N-1})$ in $x_N = 0$ point in the principal directions at $\bar{x}$. We can also assume that, in a neighborhood of $\bar{x}$, $\partial\Omega$ is given by the equation $x_N = h(x_1, ..., x_{N-1})$ where $h \in C^4$. For $j = 1, ..., N-1$ we set $p_j = \frac{\partial h}{\partial x_j}$.

Since we are interested in the behavior of u near the boundary at short times, we re-scale u by defining

$$u_\varepsilon(\bar{x}, s, t) = u(\bar{x}, \varepsilon s, \varepsilon^2 t).$$

For ε small, u_ε satisfies

$$(u_\varepsilon)_t = \Delta_\varepsilon(u_\varepsilon) \tag{1.3}$$

in $D = \partial\Omega \times (0, s_0/\varepsilon) \times (0, 1)$ where, as a long but straightforward calculation shows that, $\Delta_\varepsilon(u_\varepsilon)$, when computed in these new variables and evaluated at a point of the form $(\bar{x}, s, t)$, is given by

$$\Delta_\varepsilon(u_\varepsilon)(\bar{x}, s, t) = \frac{\partial^2 u_\varepsilon}{\partial s^2}(\bar{x}, s, t) - \varepsilon \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1 - H_j(\bar{x})\varepsilon s)} \frac{\partial u_\varepsilon}{\partial s}(\bar{x}, s, t) +$$

$$\varepsilon^2 \sum_{j=1}^{N-1} \frac{1}{(1 - \varepsilon s H_j(\bar{x}))^2} \frac{\partial^2 u_\varepsilon}{\partial x_j^2}(\bar{x}, s, t) +$$

$$\varepsilon^3 \; s \sum_{i,j=1}^{N-1} \frac{1}{(1 - \varepsilon s H_j(\bar{x}))(1 - \varepsilon s H_i(\bar{x}))^2} \frac{\partial^2 p_j}{\partial x_i^2} \frac{\partial u_\varepsilon}{\partial x_j}(\bar{x}, s, t).$$

We stress the fact that in this calculation we are evaluating at a point $(\bar{x}, s, t)$ and, the coordinates $x_1, ..., x_{N-1}$ point in the tangent directions to

$\partial\Omega$ at $\bar{x}$. This implies that $p_j(\bar{x}) = 0$ for $j = 1, ..., N-1$, a fact that simplifies a little the resulting formula.

We note that u_ε also satisfies

$$-\frac{\partial u_\varepsilon}{\partial s}(\bar{x}, 0, t) = \varepsilon\varphi(u_\varepsilon(\bar{x}, 0, t)) \tag{1.4}$$

and

$$u_\varepsilon(\bar{x}, s, 0) = M. \tag{1.5}$$

Let now $w_\varepsilon(\bar{x}, s, t)$ be the solution of (1.3), (1.4) and (1.5) in D with the following boundary condition at $s = s_0/\varepsilon$,

$$w_\varepsilon(\bar{x}, s_0/\varepsilon, t) = M. \tag{1.6}$$

Formally expanding w_ε in powers of ε in the form

$$w_\varepsilon(\bar{x}, s, t) = M + b(s, t)\varepsilon + c(\bar{x}, s, t)\varepsilon^2 + o(\varepsilon^2) \tag{1.7}$$

one finds that b and c must satisfy

$$\begin{aligned} &b_t = b_{ss}, \\ &-b_s(0, t) = \varphi(M), \\ &b(s, 0) = 0, \end{aligned} \tag{1.8}$$

and

$$\begin{aligned} &c_t = c_{ss} - H(\bar{x})b_s, \\ &-c_s(0, t) = \varphi'(M)b(0, t), \\ &c(s, 0) = 0. \end{aligned} \tag{1.9}$$

The rest of this article is organized as follows. Section 2 of is devoted to find explicit solutions of problems (1.8) and (1.9). In Section 3 we first follow ideas of [CDE] to construct, based in the functions b and c, sub and super solutions of problem (1.3)-(1.6). Then we use these sub and super solutions to give an actual proof of the expansion (1.7) for w_ε. Finally we show that the expansions for w_ε and u_ε coincide up to second order, proving in this fashion our main result.

2 The functions b and c.

We look for self similar solutions of (1.8) and (1.9), so we define $\xi = s/\sqrt{t}$. In order to find a solution of (1.8) we set

$$b(s,t) = \sqrt{t}\; f(\xi)$$

and note that b satisfies (1.8) if and only if f satisfies

$$\begin{aligned} &\frac{1}{2}f(\xi) - \frac{1}{2}f'(\xi)\xi = f''(\xi), \\ &-f'(0) = \varphi(M). \end{aligned} \tag{2.1}$$

Let

$$f_1(\xi) = \frac{1}{\sqrt{\pi}} \int_\xi^{+\infty} \int_l^{+\infty} e^{-\frac{w^2}{4}}\; dw\; dl.$$

It can be checked directly, or otherwise one can mimic the argument for finding the function g below, that $f(\xi) = \varphi(M) f_1(\xi)$ solves (2.1).

We claim that the function

$$c(\bar{x}, s, t) = t\frac{\varphi'(M)\varphi(M)}{\sqrt{\pi}} g_1(\xi) + tH(\bar{x})\frac{\varphi(M)}{\sqrt{\pi}}[g_2(\xi) - g_1(\xi)]$$

where

$$g_1(\xi) = \int_\xi^{+\infty} \int_l^{+\infty} \int_s^{+\infty} e^{-\frac{w^2}{4}}\; dw\; ds\; dl$$

and

$$g_2(\xi) = \int_\xi^{+\infty} \int_l^{+\infty} \int_s^{+\infty} \frac{w^2}{4} e^{-\frac{w^2}{4}}\; dw\; ds\; dl$$

is a solution of (1.9).

To prove the claim we fix $\bar{x}$ and we look for solutions of the form $t\; g(\xi)$. We note that such a function is a solution of (1.9) if g satisfies

$$\begin{aligned} &g(\xi) - \frac{1}{2}g'(\xi)\xi = g''(\xi) - H(\bar{x})f'(\xi), \\ &-g'(0) = \varphi'(M)f(0). \end{aligned} \tag{2.2}$$

Since $f'''(\xi) = -\frac{f(0)}{2}e^{-\frac{\xi^2}{4}}\frac{\xi}{2}$, differentiating twice the equation in (2.2), we get

$$-\frac{1}{2}g'''(\xi)\xi = g^{(iv)}(\xi) + H(\bar{x})\frac{f(0)}{2}e^{-\frac{\xi^2}{4}}\frac{\xi}{2}.$$

Integrating the last equation we obtain

$$g'''(\xi) = -H(\bar{x})\frac{f(0)}{2}e^{-\frac{\xi^2}{4}}\frac{\xi^2}{4} + Ce^{-\frac{\xi^2}{4}}.$$

Hence

$$g(\xi) = H(\bar{x})\frac{f(0)}{2}\int_{\xi}^{+\infty}\int_{l}^{+\infty}\int_{s}^{+\infty} e^{-\frac{w^2}{4}}\frac{w^2}{4}\,dw\,ds\,dl -$$
$$C\int_{\xi}^{+\infty}\int_{l}^{+\infty}\int_{s}^{+\infty} e^{-\frac{w^2}{4}}\,dw\,ds\,dl.$$

Choosing C such that

$$g'(0) = -H(\bar{x})\frac{f(0)}{2}\int_{0}^{+\infty}\int_{l}^{+\infty} e^{-\frac{w^2}{4}}\frac{w^2}{4}\,dw\,dl +$$
$$C\int_{0}^{+\infty}\int_{l}^{+\infty} e^{-\frac{w^2}{4}}\,dw\,dl = -\varphi'(M)f(0)$$

the claim follows after disentangling the notation and observing that $g_1'(0) = g_2'(0) = -2$ and $f(0) = \frac{2\varphi(M)}{\sqrt{\pi}}$.

3 Proof of the main result.

We will construct a super-solution and a sub-solution of (1.3)-(1.6) using the functions b and c already found in Section 2.

By a super-solution, of (1.3)–(1.6) in $D = \partial\Omega \times (0, s_0/\varepsilon) \times (0,1)$, we understand a function $\overline{w}_\varepsilon$ such that

$$
\begin{aligned}
(\overline{w}_\varepsilon)_t &\geq \Delta_\varepsilon(\overline{w}_\varepsilon) \text{ in } D, && (3.1)\\
-\frac{\partial \overline{w}_\varepsilon}{\partial s}(\bar{x},0,t) &\geq \varepsilon\varphi(\overline{w}_\varepsilon(\bar{x},0,t))\ \forall \bar{x}\in\partial\Omega,\ \forall t\in(0,1) && (3.2)\\
\overline{w}_\varepsilon(\bar{x}, s_0/\varepsilon, t) &> M\ \forall\bar{x}\in\partial\Omega\ \forall t\in(0,1) && (3.3)\\
\text{and}\ \ \overline{w}_\varepsilon(\bar{x}, s, 0) &> M\ \forall\bar{x}\in\partial\Omega\ \forall s\in(0, s_0/\varepsilon). && (3.4)
\end{aligned}
$$

A sub-solution is a function $\underline{w}_\varepsilon$ that satisfies (3.1) to (3.4) with the reversed inequalities.

Now we can state a comparison lemma,

Lemma 3.1 *Let $\overline{w}_\varepsilon$ be a super-solution of (1.3)-(1.6), $\underline{w}_\varepsilon$ a sub-solution and w a solution then*

$$\underline{w}_\varepsilon < w < \overline{w}_\varepsilon$$

in D.

Proof: Assume, for a contradiction, that $\overline{w}_\varepsilon \leq w$ at some point. Set $z = \overline{w}_\varepsilon - w$. Then there exists a point $(\bar{x}_0, s_0, t_0)$ in D such that $z(\bar{x}_0, s_0, t_0) = 0$ and $z(\bar{x}, s, t) > 0$ for all $t < t_0$. As z satisfies (3.1), $(\bar{x}_0, s_0, t_0)$ must belong to the parabolic boundary of D. Moreover, by (3.3) and (3.4) we must have $s_0 = 0$ and $t_0 > 0$, but at such a point we have

$$-\frac{\partial z}{\partial s}(\bar{x}_0, 0, t_0) \geq 0$$

a contradiction with Hopf's lemma. This proves that $w \leq \overline{w}_\varepsilon$. The other inequality is proved similarly. $\square$

The following lemma provides the desired super-solution.

Lemma 3.2 *Let $1 < \beta < \alpha < 2 < \gamma$, then there exists ε_0 such that the function*

$$\overline{w}_\varepsilon(\bar{x}, s, t) = M + \varepsilon b((1+\varepsilon^\alpha)s, (1+\varepsilon^\beta)t) + \varepsilon^2 c(\bar{x}, (1+\varepsilon^\alpha)s, (1+\varepsilon^\beta)t) + \varepsilon^\gamma$$

is a super-solution of (1.3)-(1.6) for every $\varepsilon < \varepsilon_0$.

Proof: To simplify the notation we set $\tilde{s} = (1+\varepsilon^\alpha)s$. A straightforward calculation shows that

$$(\overline{w}_\varepsilon)_t - \Delta_\varepsilon(\overline{w}_\varepsilon) = \varepsilon^{\beta+1} b_{ss} + \varepsilon^{\beta+2} c_t + \varepsilon^2 \sum_{j=1}^{N-1} \left(\frac{H_j(\bar{x})}{(1 - H_j(\bar{x})\varepsilon\tilde{s})} - H_j(\bar{x})\right) b_s$$
$$+\varepsilon^3 \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1 - H_j(\bar{x})\varepsilon\tilde{s})} c_s - \varepsilon^4 \sum_{j=1}^{N-1} \frac{1}{(1 - \varepsilon\tilde{s}H_j(\bar{x}))^2} \frac{\partial^2 c}{\partial x_j^2} -$$

$$\varepsilon^5\ \tilde{s} \sum_{i,j=1}^{N-1} \frac{1}{(1-\varepsilon\tilde{s}H_j(\bar{x}))(1-\varepsilon\tilde{s}H_i(\bar{x}))^2} \frac{\partial^2 p_j}{\partial x_i^2} \frac{\partial c}{\partial x_j} =$$

$$\varepsilon^{\beta+1} b_{ss} + \varepsilon^{\beta+2} c_t + \varepsilon^3 \sum_{j=1}^{N-1} \frac{H_j^2(\bar{x})\tilde{s}}{1-H_j(\bar{x})\varepsilon\tilde{s}} b_s +$$

$$\varepsilon^3 \sum_{j=1} \sqrt{\pi}^{N-1} \frac{H_j(\bar{x})}{(1-H_j(\bar{x})\varepsilon\tilde{s})} c_s - \varepsilon^4 \sum_{j=1}^{N-1} \frac{1}{(1-\varepsilon\tilde{s}H_j(\bar{x}))^2} \frac{\partial^2 c}{\partial x_j^2} -$$

$$\varepsilon^5\ \tilde{s} \sum_{i,j=1}^{N-1} \frac{1}{(1-\varepsilon\tilde{s}H_j(\bar{x}))(1-\varepsilon\tilde{s}H_i(\bar{x}))^2} \frac{\partial^2 p_j}{\partial x_i^2} \frac{\partial c}{\partial x_j}.$$

Now using that $1 < \beta < \alpha < 2$, that $0 < t < 1$, the explicit formulae for b and c and, the fact that the functions $f'(\xi)/f''(\xi)$, $g'(\xi)/f''(\xi)$, $\frac{\partial^2 c}{\partial {x_j}^2}(\bar{x},\xi)/f''(\xi)$ and $\frac{\partial c}{\partial x_j}(\bar{x},\xi)/f''(\xi)$ are bounded, one gets that

$$(\overline{w}_\varepsilon)_t - \Delta_\varepsilon(\overline{w}_\varepsilon) = \varepsilon^{1+\beta} \frac{f''(\xi)}{\sqrt{(1+\varepsilon^\beta)t}} (1+o(1)) \geq 0$$

for ε small enough.

Next we check the boundary conditions. One has

$$-(\overline{w}_\varepsilon)_s(\bar{x},0,t) = \varepsilon\varphi(\overline{w}_\varepsilon)(\bar{x},0,t) - \varepsilon^3(\frac{\varphi''(\theta)}{2}(b+\varepsilon c)^2 - \varphi'(M)c)(\bar{x},0,t) +$$
$$+\varepsilon^{\alpha+1}(\varphi(M) + \varepsilon\varphi'(M)b)(\bar{x},0,t) + \varepsilon^{\gamma+1}\varphi'(\xi),$$

for some θ, ξ such that $M < \theta < M + \varepsilon b + \varepsilon^2 c + \varepsilon^\gamma$ and $M + \varepsilon b + \varepsilon^2 c < \xi < M + \varepsilon b + \varepsilon^2 c + \varepsilon^\gamma$.

Since $1 < \alpha < 2$ and $\gamma > 2$ we get that

$$-\varepsilon^3(\varphi''(\theta)(b+\varepsilon c)^2 - \varphi'(M)c)(\bar{x},0,t) + \varepsilon^{\alpha+1}(\varphi(M)+\varepsilon\varphi'(M)b)(\bar{x},0,t) +$$
$$\varepsilon^{\gamma+1}\varphi'(\xi) \geq 0,$$

and hence

$$-(\overline{w}_\varepsilon)_s(\bar{x},0,t) \geq \varepsilon\varphi(\overline{w}_\varepsilon)(\bar{x},0,t)$$

for ε small enough.

Moreover, it is clear that $\overline{w}_\varepsilon(\bar{x}, s_0/\varepsilon, t) > M$ and $\overline{w}_\varepsilon(\bar{x}, s, 0) > M$. The lemma is proved. □

The same arguments of the proof of the previous lemma allow us to prove the following result whose proof we omit.

Lemma 3.3 *Let $1 < \beta < \alpha < 2 < \gamma$, then there exists ε_0 such that the function*

$$\underline{w}_\varepsilon(l,s,t) = M + \varepsilon b((1-\varepsilon^\alpha)s,(1-\varepsilon^\beta)t) + \varepsilon^2 c(\bar{x},(1-\varepsilon^\alpha)s,(1-\varepsilon^\beta)t) - \varepsilon^\gamma$$

is a sub-solution of (1.3)-(1.6) for every $\varepsilon < \varepsilon_0$.

With these two lemmas we are ready to prove Theorem 1.1.

Proof of Theorem 1.1 :

We note that $\overline{w}_\varepsilon$ and $\underline{w}_\varepsilon$ have the same expansion in powers of ε up to second order, namely

$$M + b(s,t)\varepsilon + c(\bar{x},s,t)\varepsilon^2 + o(\varepsilon^2).$$

Moreover, by a comparison argument in D (see Lemma 3.1), we have

$$\underline{w}_\varepsilon(\bar{x},s,t) \le w_\varepsilon(\bar{x},s,t) \le \overline{w}_\varepsilon(\bar{x},s,t)$$

if ε is small enough. Consequently

$$w_\varepsilon(\bar{x},s,t) = M + b(s,t)\varepsilon + c(\bar{x},s,t)\varepsilon^2 + o(\varepsilon^2).$$

So, in order to finish the proof we have to check that the expansions in powers of ε of w_ε and u_ε coincide up to order two. To see this let us define

$$v = \frac{u_\varepsilon - w_\varepsilon}{\varepsilon^2}.$$

We note that $v \ge 0$ and satisfies

$$v_t = \Delta_\varepsilon v \quad \text{in} \quad \partial\Omega \times [0, s_0/\varepsilon] \times (0,1), \tag{3.5}$$

$$-\frac{\partial v}{\partial s}(\bar{x},0,t) \le K v(\bar{x},0,t), \qquad v(\bar{x},s,0) = 0, \tag{3.6}$$

where K is a constant that depends on the values of φ' near M. Moreover we claim that there exists a function $A(\varepsilon)$ with $A(\varepsilon) \to 0$ as $\varepsilon \to 0$ such that

$$0 \le v(\bar{x}, s_0/\varepsilon, t) \le A(\varepsilon) \text{ for all } \bar{x} \in \partial\Omega \text{ and all } t \in (0,1). \tag{3.7}$$

To prove the claim let z be the solution of

$$z_t = \Delta z \quad \text{in} \quad \Omega \times (0,1),$$
$$\frac{\partial z}{\partial \eta} = \varphi(z) \quad \text{on} \quad \partial\Omega \times (0,1),$$

with an initial condition, z_0, constructed as follows. Let $G : \mathbf{R} \to \mathbf{R}$ be a C^3 function such that $G \geq M$, $G(s) = M$ if $s \geq s_0/2$ and $-G'(0) = \varphi(G(0))$. Now we define

$$z_0(\bar{x}, s) = G(s) \qquad \text{for } s < s_0$$

and $z_0 \equiv M$ elsewhere in Ω. This z_0 verifies

$$\frac{\partial z_0}{\partial \eta} = \varphi(z_0).$$

Under this compatibility condition it was proved in [A] that z has continuous derivatives up to $t = 0$. See also [ChFQ] and [LGMW].

Now by a comparison argument,

$$0 \leq \frac{u(\bar{x}, s_0, \varepsilon^2 t) - M}{\varepsilon^2} \leq \frac{z(\bar{x}, s_0, \varepsilon^2 t) - M}{\varepsilon^2}$$

and the claim follows by the continuity of z_t in $\overline{\Omega} \times [0,1]$.

It can be directly checked that the function $\overline{v}(s,t) = e^{4K^2 t}(1 + e^{-2Ks})A(\varepsilon)$ is a super solution of (3.5)-(3.7), therefore, for ε sufficiently small

$$0 \leq v(\bar{x}, s, t) \leq 2e^{4K^2} A(\varepsilon) \text{ in } \partial\Omega \times [0, s_0/\varepsilon] \times [0,1].$$

Therefore w_ε and u_ε have the same expansion in powers of ε up to second order and, consequently, we have proved (1.2) with

$$F_1(s,t) = \sqrt{t} f_1(s/\sqrt{t}),$$
$$F_2(s,t) = \frac{t}{\sqrt{\pi}} g_1(s/\sqrt{t}),$$

and

$$F_3(s,t) = \frac{t}{\sqrt{\pi}}[g_2(s/\sqrt{t}) - g_1(s/\sqrt{t})].$$

Finally it is clear that $F_1 > 0$ and $F_2 > 0$. Moreover F_3 satisfies $(F_3)_t = (F_3)_{ss} - f_1'$, $(F_3)_s(0,t) \equiv 0$ and $F_3(s,0) \equiv 0$. Hence, as $f_1' < 0$, it follows that $F_3 > 0$. This ends the proof of the theorem. $\square$

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