A NONLOCAL MONGE-KANTOROVICH PROBLEM

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Abstract. This paper is concerned with a nonlocal version of the classical Monge-Kantorovich mass transport problem in which we replace the Euclidean distance with a discrete distance to measure the transport cost. We fix the length of a step and the distance that measures the cost of the transport depends on the number of steps that is needed to transport the involved mass from its origin to its destination. For this problem we deal with the issues of existence of Monge maps, Kantorovich potentials and optimal transport plans. We find that, even in one dimension, and for absolutely continuous measures respect to the Lebesgue measure, the existence of an optimal transport map depends on the involved masses. We also prove that when the length of the step tends to zero these nonlocal problems give an approximation to the classical Monge-Kantorovich mass transport problem. Finally, we study how the approach developed by Evans and Gangbo for the classical case works in this context. We find an equation for the potentials, obtained as a limit of nonlocal $p$–Laplacian problems, and we use it to construct optimal transport plans. We also obtain the transport density for the classical Monge-Kantorovich problem by rescaling.

To the memory of Fuensanta Andreu, our friend and colleague.

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1. Introduction and Main Results

1.1. Introduction. The Monge mass transport problem, as proposed by Monge in 1781, consists in finding the optimal way of moving points from one mass distribution to another so that the work done is minimized. In general, the total work is proportional to some cost function. In the classical Monge problem the cost function is the Euclidean distance, and this problem has been intensively studied and generalized in different directions that correspond to different classes of cost functions. We refer to the surveys and books [1], [3], [10], [17], [19], [20] for further discussion of Monge’s problem, its history, and applications.

However, even being the case of discontinuous cost functions very interesting for concrete situations and applications, it seems not to be well covered in the literature. For instance, assume that you want to transport an amount of bricks to a hole, then you count the number of steps that you have to move each brick to its final destination in the hole and try to move the total amount of bricks making as less as possible steps. This amounts to the classical Monge-Kantorovich problem for the discrete distance (that count the number of steps)

\[ d_1(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \lfloor |x - y| \rfloor + 1 & \text{if } x \neq y, \end{cases} \]

where \( |\cdot| \) is the Euclidean norm and \( \lfloor r \rfloor \) is defined for \( r > 0 \) by \( \lfloor r \rfloor := n, \) if \( n < r \leq n + 1, \) \( n = 0, 1, 2, \ldots \). That is,

\[ d_1(x, y) = \begin{cases} 0 & \text{if } x = y, \\ 1 & \text{if } 0 < |x - y| \leq 1, \\ 2 & \text{if } 1 < |x - y| \leq 2, \\ \vdots \end{cases} \]

This transport problem with this discrete distance also appears naturally when one considers a transport problem between cities in which the cost is measured by the toll in the road (that in a simplified version is a discrete function of the number of kilometers). We want to mention another motivation for the study of this mass transport problem that comes from a nonlocal model for sandpiles studied in [5] (see also [14]), and from which we have decided to add the word nonlocal to this transport problem.

In [10] Evans gives an interpretation of the sandpile model introduced by Aronsson, Evans and Wu in [6], in terms of Monge-Kantorovich mass transport theory for the particular case of the Euclidean distance \( d_{1,1} \) in \( \mathbb{R}^N \) as cost. The model for the evolution of a sandpile with source \( f \geq 0 \) is the evolution equation

\[ \begin{cases} f - u_t \in \partial \Pi_{K_{d_{1,1}}(\mathbb{R}^N)}(u), & t > 0, \\ u(0, x) = 0, \end{cases} \]

where \( u(t, x) \) is the height of the sandpile at time \( t \) in \( x \) and

\[ K_{d_{1,1}}(\mathbb{R}^N) := L^2(\mathbb{R}^N) \cap \text{Lip}_1(\mathbb{R}^N, d_{1,1}) = \{ u \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) : |\nabla u| \leq 1 \text{ a.e.} \}. \]

The interpretation reads as follows: at each moment of time, the mass \( \mu^+ = f(t, \cdot) \, dx \) is instantly and optimally transported downhill by the potential \( u(t, \cdot) \) into the mass \( \mu^- = u_t(t, \cdot) \, dy \). In other words, the height function \( u(t, \cdot) \) of the sandpile is deemed also to be the potential generating the Monge-Kantorovich reallocation of \( \mu^+ \) to \( \mu^- \).
In [5] it is studied the following nonlocal version of the sandpile model given in [6]. Let

\[
J : \mathbb{R}^N \to \mathbb{R}
\]

be a nonnegative continuous radial function with

\[
\text{supp}(J) = B(0,1), \quad J(0) > 0 \quad \text{and} \quad \int_{\mathbb{R}^N} J(x) \, dx = 1,
\]

and set

\[
K^1_J := \{ u \in L^2(\mathbb{R}^N) : |u(x) - u(y)| \leq 1 \text{ for } x - y \in \text{supp}(J) \text{ a.e.} \}.
\]

The nonlocal problem is written as

\[
\begin{aligned}
\{ f(t,\cdot) - u_t(t,\cdot) \in \partial I_{K^1_J}(u(t,\cdot)) \quad &\text{a.e. } t \in (0,T), \\
u(x,0) = u_0(x).
\end{aligned}
\]

To give a mass transport interpretation of this model, we consider the previously mentioned distance \(d_1\). Observe that \(K^1_J = L^2(\mathbb{R}^N) \cap \text{Lip}_1(\mathbb{R}^N,d_1)\). The following result is proved in [5]: the solution \(u(t,\cdot)\) of the limit problem (1.3) is a solution of the Kantorovich dual problem

\[
\max_{u \in K^1_J} \int_{\mathbb{R}^N} u(f^+ - f^-)
\]

when the involved measures are the source term \(f^+ = f(t,x)\) and the time derivative of the solution \(f^- = u_t(t,x)\). Hence, as for the local model, at each moment of time, the mass \(\mu^+ = f(t,\cdot) \, dx\) is instantly and optimally transported downhill by the potential \(u(t,\cdot)\) into the mass \(\mu^- = u_t(t,\cdot) \, dy\), but in this case the discrete function \(d_1\) is what gives the transport cost.

The aim of this paper is, on the one hand, the study of the Monge and Kantorovich problems for the discrete cost function \(d_1(x,y)\). And, on the other hand, the study of rescaled problems with costs \(d_{\varepsilon}\) in which the size of the step is \(\varepsilon\) (see (1.21) for a precise definition) in order to approach the Monge and Kantorovich problems for the Euclidean distance as the parameter \(\varepsilon\) goes to 0. We perform a detailed study of the one dimensional case with several concrete examples which illustrate the general obstructions to existence and the difficulties that may appear with this particular cost. We also find an equation for the potentials, obtained as a limit of nonlocal \(p\)-Laplacian problems, and we use it to construct optimal transport plans. Moreover, we obtain the transport density for the classical Monge-Kantorovich problem by rescaling.

It is clear that our problem falls into the scope of lower semi-continuous cost functions, so that standard results, like the existence of a solution for the relaxed problem, the so called Monge-Kantorovich problem, or the Kantorovich duality remain true for \(d_1\). Moreover, since \(d_1\) is a metric, it is well known that one of the interesting features of the theory is the existence of a potential, the so called Kantorovich potential. However, since \(d_1\) is discrete, the connection between Monge and Monge-Kantorovich problems, the characterization of the potential transport, the Evans-Gangbo approach, as well as the computation of the optimal plan and/or map transport are not covered in the literature for this concrete distance. In particular, the potential can not be characterized in a standard way, i.e., by using standard differentiation. In this work, we characterize the Kantorovich potential by using a nonlocal equation and we develop a complete theory concerning the Monge problem with the discrete distance \(d_1\). Moreover, we give the connection between the Monge-Kantorovich problem with the discrete distance and the classical Monge-Kantorovich problem with the Euclidean distance, proving that when the length of the step tends to zero these nonlocal problems give an approximation to the classical Monge-Kantorovich mass transport problem.
1.2. Preliminaries. To fix notation and for convenience of the reader, let us start recalling with some detail what are the problems in which we are interested and also state some known results and useful properties of their solutions. Although most of them hold true in a more general framework we will state them for $\Omega$ a bounded convex domain in $\mathbb{R}^N$ which will be the framework of our results. Whenever $T$ is a map from a measure space $(X, \mu)$ to an arbitrary space $Y$, we denote by $T\# \mu$ the pushforward measure of $\mu$ by $T$. Explicitly, $(T\# \mu)[B] = \mu[T^{-1}(B)]$. When we write $T\# f = g$, where $f$ and $g$ are nonnegative functions, this means that the measure having density $f$ is pushed-forward to the measure having density $g$.

The Monge problem. Fix $c : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty]$ a lower semi-continuous cost function, and two non-negative Borel function $f^+, f^- \in L^1(\Omega)$ satisfying the mass balance condition

\begin{equation}
\int_{\Omega} f^+(x) \, dx = \int_{\Omega} f^-(y) \, dy.
\end{equation}

Let $\mathcal{A}(f^+, f^-)$ be the set of Borel maps $T : \Omega \to \Omega$ which push the measure $\mu^+ := f^+(x) \, dx$ forward to $\mu^- := f^-(y) \, dy$, that is

$T\# f^+ = f^- \iff \int_{T^{-1}(B)} f^+(x) \, dx = \int_B f^-(y) \, dy, \quad \forall B \subset \Omega$ Borel.

The Monge problem is to find a map $T^* \in \mathcal{A}(f^+, f^-)$ which minimizes the cost functional

\begin{equation}
\mathcal{F}_c(T) := \int_{\Omega} c(x, T(x)) f^+(x) \, dx
\end{equation}

in the set $\mathcal{A}(f^+, f^-)$. A map $T^* \in \mathcal{A}(f^+, f^-)$ satisfying

$\mathcal{F}_c(T^*) = \min\{ \mathcal{F}_c(T) : T \in \mathcal{A}(f^+, f^-) \}$,

is called an optimal transport map pushing $f^+$ to $f^-$. The original problem studied by Monge corresponds to the cost function

$c(x, y) = d_{\text{eu}}(x, y) := |x - y|$, the Euclidean distance. In general, the Monge problem is ill-posed. To overcome the difficulties of the Monge problem, in 1942 L. V. Kantorovich, [15], proposed to study a relaxed version of the Monge problem and, what is more relevant here, introduced a dual variational principle.

We will use the usual convention of denoting by $\pi : \mathbb{R}^N \times \mathbb{R}^N$ the projections, $\pi_1(x, y) := x$, $\pi_2(x, y) := y$. Given a Radon measure $\mu$ in $\Omega \times \Omega$, its marginals are defined by $\text{proj}_x(\mu) := \pi_1 \# \mu$, $\text{proj}_y(\mu) := \pi_2 \# \mu$. The relaxed problem is the following.

The Monge-Kantorovich problem. Fix $c$, $f^+$ and $f^-$ as above. Let

$\pi(f^+, f^-) := \{ \text{Nonnegative Radon measures } \mu \text{ in } \Omega \times \Omega : \text{proj}_x(\mu) = f^+(x) \, dx, \text{proj}_y(\mu) = f^-(y) \, dy \}$,

and consider the functional

\begin{equation}
\mathcal{K}_c(\mu) := \int_{\Omega \times \Omega} c(x, y) \, d\mu(x, y).
\end{equation}
The Monge-Kantorovich problem is to find a measure \( \mu^* \in \pi(f^+, f^-) \) which minimizes the cost functional \( K_c(\mu) \), that is, to find a solution to the minimization problem

\[
K_c(\mu^*) = \min \{ K_c(\mu) : \mu \in \pi(f^+, f^-) \}.
\]

The elements \( \mu \in \pi(f^+, f^-) \) are called transport plans between \( f^+ \) and \( f^- \), and \( \mu^* \) satisfying (1.7) is called an optimal transport plan between \( f^+ \) and \( f^- \). Remark that we say plans between \( f^+ \) and \( f^- \) since this problem is reversible, which is not true in general for the Monge problem. Given two non-negative Borel function \( f^+, f^- \in L^1(\Omega) \) satisfying the mass balance condition (1.4), there exists always the trivial transport plan, \( f^+ \boxtimes f^- \), defined as

\[
(f^+ \boxtimes f^-)(A \times B) := \frac{\int_{A \times B} f^+(x)f^-(y) \, dx \, dy}{\int_{\Omega} f^+(x) \, dx} = \frac{\int_{A \times B} f^+(x)f^-(y) \, dx \, dy}{\int_{\Omega} f^-(y) \, dy}.
\]

The relation between transport maps and transport plans is given in the following result (see, for instance, [1, Proposition 2.1]).

**Proposition 1.1.** Any Borel transport map \( T \in \mathcal{A}(f^+, f^-) \) induces a transport plan \( \mu_T \in \pi(f^+, f^-) \) defined by \( \mu_T := (\text{Id} \times T)\#f^+ = f^+(x)\delta_{y=T(x)} \), that is, \( \int_{\Omega \times \Omega} c(x,y) \, d\mu_T(x,y) = \int_{\Omega} c(x,T(x))f^+(x) \, dx \). Conversely, a transport plan \( \mu \in \pi(f^+, f^-) \) is induced by a transport map if \( \mu \) is supported on a \( \mu \)-measurable graph.

An immediate consequence of the above result is that

\[
\inf \{ K_c(\mu) : \mu \in \pi(f^+, f^-) \} \leq \inf \{ F_c(T) : T \in \mathcal{A}(f^+, f^-) \}.
\]

Under the above assumptions it is not difficult to show the existence of an optimal transport plan (see [1, 16] and the references therein).

**Proposition 1.2.** Let \( c : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty] \) a lower semi-continuous cost function, and let \( f^+, f^- \in L^1(\Omega) \) two non-negative Borel function satisfying the mass balance condition (1.4). Then, there exists an optimal transport plan \( \mu^* \in \pi(f^+, f^-) \) solving the Monge-Kantorovich problem

\[
K_c(\mu^*) = \min \{ K_c(\mu) : \mu \in \pi(f^+, f^-) \}.
\]

Moreover, if \( c \) is continuous we have that (1.8) is, in fact, an equality:

\[
\min \{ K_c(\mu) : \mu \in \pi(f^+, f^-) \} = \inf \{ F_c(T) : T \in \mathcal{A}(f^+, f^-) \}.
\]

The dual formulation by Kantorovich of the Monge problem is the following.

**The Kantorovich dual problem.** Fix \( c : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty] \) a lower semi-continuous cost function, and two non-negative Borel function \( f^+, f^- \in L^1(\Omega) \) satisfying the mass balance condition (1.4). The dual problem is to find \( (u^*, v^*) \) that maximize

\[
\mathcal{D}(u, v) := \int_{\Omega} u(x)f^+(x) \, dx + \int_{\Omega} v(y)f^-(y) \, dy
\]

in the set

\[
\Phi_c(f^+, f^-) := \{(u, v) \in L^1(\Omega, f^+(x) \, dx) \times L^1(\Omega, f^-(y) \, dy) : u(x) + v(y) \leq c(x,y) \}.
\]
We have the following result (see for instance [3, Theorem 3.1] or [19, Theorem 1.3]).

**Theorem 1.3. (Kantorovich duality).** Let \( c : \mathbb{R}^N \times \mathbb{R}^N \to [0, +\infty] \) a lower semi-continuous cost function, and let \( f^+, f^- \in L^1(\Omega) \) two non-negative Borel function satisfying the mass balance condition (1.4). Then,

\[
(1.11) \quad \min \{ K_c(\mu) : \mu \in \pi(f^+, f^-) \} = \sup \{ D(u, v) : (u, v) \in \Phi_c(f^+, f^-) \}.
\]

Furthermore, it does not change the value of the supremum in the right-hand side of (1.11) if one restricts the definition of \( \Phi_c(f^+, f^-) \) to those functions \( (u, v) \) which are bounded and continuous.

When the cost function is a metric, \( c(x, y) = d(x, y) \) on \( \mathbb{R}^N \), then there is more structure in the Kantorovich duality principle. We denote by \( K_d(\Omega) \) the set of functions in \( \Omega \) which are Lipschitz continuous with Lipschitz constant no greater than one,

\[
K_d(\Omega) := \text{Lip}_1(\Omega, d) = \{ u \in L^1(\Omega) : |u(x) - u(y)| \leq d(x, y) \text{ a.e.} \}.
\]

We have the following result (see for instance [19, Theorem 1.14]).

**Theorem 1.4. (Kantorovich-Rubinstein Theorem).** Let \( d \) be a lower semi-continuous metric on \( \mathbb{R}^N \), and let \( f^+, f^- \in L^1(\Omega) \) two non-negative Borel function satisfying the mass balance condition (1.4). Then,

\[
(1.12) \quad \min \{ K_d(\mu) : \mu \in \pi(f^+, f^-) \} = \sup \{ P_{f^+, f^-}(u) : u \in K_d(\Omega) \},
\]

where

\[
P_{f^+, f^-}(u) := \int_{\Omega} u(x)(f^+(x) - f^-(x)) \, dx.
\]

The maximizers \( u^* \) of (1.12) are called Kantorovich (transport) potentials.

Now, given a convex set \( K \subset L^2(\Omega) \), if we denote by \( \mathbb{I}_K \) its indicator function, that is, the function defined as

\[
\mathbb{I}_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K, \end{cases}
\]

we have that the Euler-Lagrange equation associated with the variational problem

\[
\max \left\{ \int_{\Omega} u(x)(f^+(x) - f^-(x)) \, dx : u \in \text{Lip}_1(\Omega, d) \right\}
\]

is the equation

\[
(1.13) \quad f^+ - f^- \in \partial \mathbb{I}_K(\Omega)(u).
\]

That is, the Kantorovich potentials are solutions of (1.13).

In the particular case of the Euclidean distance \( d(x, y) = d_{|1}(x, y) = |x - y| \) and for adequate masses \( f^+ \) and \( f^- \), Evans and Gangbo in [11] find a solution of (1.13), that is, a Kantorovich potential for the mass transport problem of \( f^+ \) to \( f^- \), as a limit as \( p \to \infty \) of solutions to the local \( p \)-Laplace equation with Dirichlet boundary conditions in a sufficiently large ball \( B(0, R) \),

\[
\begin{align*}
-\Delta_p u_p &= f^+ - f^- & & B(0, R), \\
u_p &= 0, & & \partial B(0, R),
\end{align*}
\]

Moreover, they characterize the solutions to the limit equation (1.13) by means of a PDE.
Theorem 1.5. (Evans-Gangbo Theorem). Let \( d_{1} \) be the Euclidean distance on \( \mathbb{R}^N \), and let \( f^{+}, f^{-} \in L^1(\Omega) \) two non-negative Borel function satisfying the mass balance condition (1.4). Assume additionally that \( f^{+} \) and \( f^{-} \) are Lipschitz continuous functions with compact support such that \( \text{supp}(f^{+}) \cap \text{supp}(f^{-}) = \emptyset \). Then, there exists \( u^{\ast} \in K_{d_{1}} \) such that

\[
\int_{\Omega} u^{\ast}(x)(f^{+}(x) - f^{-}(x)) \, dx = \max \left\{ \int_{\Omega} u(x)(f^{+}(x) - f^{-}(x)) \, dx \mid u \in K_{d_{1}} \right\} ;
\]

and there exists \( 0 \leq a \in L^{\infty}(\Omega) \) such that

\[
(f^{+} - f^{-}) = -\text{div}(a \nabla u^{\ast}) \quad \text{in} \quad \mathcal{D}'(\Omega).
\]

Furthermore

\[
|\nabla u^{\ast}| = 1 \quad \text{a.e. on the set} \quad \{a > 0\}.
\]

The function \( a \) that appear in the previous result is the Lagrange multiplier corresponding to the constraint \( |\nabla u^{\ast}| \leq 1 \), and it is called the transport density.

Moreover, what is very important from the point of view of mass transport, Evans and Gangbo use this PDE to find a proof of the existence of an optimal transport map for the classical Monge problem, different to the first one given by Sudakov in 1979 by means of probability methods ([18], see also [1] and [3]).

1.3. Statement of the main results. Let us state the main results of this paper. We use the same notation previously introduced.

Since \( d_{1} \) is lower semi-continuous, then, for any \( f^{+}, f^{-} \in L^1(\Omega) \) satisfying the mass balance condition (1.4), we deduce that

- there exists an optimal plan transport \( \mu^{\ast} \) minimizing the Monge-Kantorovich problem
  \[ K_{d_{1}}(\mu^{\ast}) = \min \{ K_{d_{1}}(\mu) \mid \mu \in \pi(f^{+}, f^{-}) \} . \]
- The Kantorovich duality remains true, i.e.,
  \[ \min \{ K_{d_{1}}(\mu) \mid \mu \in \pi(f^{+}, f^{-}) \} = \sup \{ D_{d_{1}}(u, v) \mid (u, v) \in \Phi_{c}(f^{+}, f^{-}) \} . \]

Moreover, since \( d_{1} \) is a metric, the interesting way to study the problem is by studying the Kantorovich potential, indeed the Kantorovich-Rubinstein Theorem states that

\[ \min \{ K_{d}(\mu) \mid \mu \in \pi(f^{+}, f^{-}) \} = \sup \{ P_{f^{+}, f^{-}}(u) \mid u \in K_{d}(\Omega) \} . \]

The metric \( d_{1} \) makes the problem independent from processes related to differentiation. Here, we characterize the potential \( u^{\ast} \) by means of a nonlocal equation. Following the ideas from [11], we first construct Kantorovich potentials for the cost function \( d_{1}(x, y) \) taking limit as \( p \to \infty \) in some nonlocal \( p \)-Laplacian type problems. This limit procedure is nontrivial due to the lack of regularity that these nonlocal problems enjoy. Moreover this result is a remarkable fact of the parallelism between local and nonlocal \( p \)-Laplacian problems and their approach to Monge-Kantorovich problems.

Next, we prove existence of Kantorovich potentials with a finite number of jumps of size one. This result is of special interest for searching/constructing optimal transport maps and plans.

Theorem 1.6. Given \( f^{+}, f^{-} \in L^{\infty}(\Omega) \) two non-negative Borel functions satisfying the mass balance condition (1.4) and such that

\[
|\text{supp}(f^{+}) \cap \text{supp}(f^{-})| = 0,
\]

This theorem provides the existence of optimal transport maps with a finite number of jumps, which is a significant result in the study of mass transport problems.
there exists a Kantorovich potential $u^*$ such that $u^*(\Omega) \subset \mathbb{Z}$ and takes a finite number of values. More precisely, there exists a partition $\{A_j\}_{j=0}^k$ of $\Omega$ such that

$$u^* = \sum_{j=0}^k j \chi_{A_j}.$$ 

In addition, we have

$$f^+ \chi_{A_0} = f^- \chi_{A_k} = 0.$$ 

Next, we deal with the equality between Monge’s infimum and Kantorovich’s minimum. It is well known (see [16]) that equality between Monge’s infimum and Kantorovich’s minimum is not true in general if the cost function is not continuous. The example given by Pratelli in [16] can be adapted to get a counterexample also for the case of the cost function given by the metric $d_1$, see Section 2. We prove that the cause of the above counterexample is not the discontinuity of $d_1$ being the cause that the measures $f^+$ and $f^-$ are not absolutely continuous respect the Lebesgue measure. Indeed, we get the following result.

**Theorem 1.7.** Let $f^+, f^- \in L^1(\Omega)$ two non-negative Borel function satisfying the mass balance condition (1.4). Then,

$$(1.16) \quad \min \{ K_{d_1}(\mu) : \mu \in \pi(f^+, f^-) \} = \inf \{ \mathcal{F}_{d_1}(T) : T \in \mathcal{A}(f^+, f^-) \}.$$ 

We study the existence and nonexistence for optimal transport maps, there are several examples that show that the existence of a transport map is a delicate issue that depends strongly on the configuration of the data $f^+$ and $f^-$ and the distance $d_1$. For example, we obtain the following result.

**Theorem 1.8.** In one space dimension, let $f^+ = L \chi_{[0,1]}$ and $f^- = \chi_{[-L,0]}$, then there exists an optimal transport map $T$ if and only if $L \notin \mathbb{N}$ or $L = 1$.

Another main task is to show how the Evans-Gangbo approach, studied in [11] for the Euclidean distance, works in the setting of the discrete distance $d_1$. Recall that for the Euclidean distance, the Monge-Kantorovich equation (1.14) contains all the information concerning the optimal transportation of $f^+$ into $f^-$. In our case, we use an analogous analysis by providing a new nonlocal equation and proving how this enables to construct the optimal plan transportation of $f^+$ into $f^-$. Actually, we characterize the solutions of the Euler-Lagrange equation

$$(1.17) \quad f^+ - f^- \in \partial \Pi_{K_{d_1}}(u),$$ 

as the functions $u^* \in K_1$, for which there exists an antisymmetric bounded measure $\sigma^*$ in $\Omega$, such that

$$[\sigma^*]^+ L \{(x, y) \in \Omega \times \Omega : u^*(x) - u^*(y) = 1, |x - y| \leq 1\},$$

$$(1.18) \quad [\sigma^*]^- L \{(x, y) \in \Omega \times \Omega : u^*(y) - u^*(x) = 1, |x - y| \leq 1\},$$

satisfying

$$(1.19) \quad \int_{\Omega} d\sigma^*(x, y) = f^+(x) - f^-(x), \quad x \in \Omega.$$
and

\begin{equation}
|\sigma^*(\Omega \times \Omega) = 2 \int_\Omega (f^+(x) - f^-(x))u^*(x) \, dx.
\end{equation}

We see that the measure \( \sigma^* \) jointly with the potential \( u^* \) contain all the information needed to construct an optimal transport plan and we show how to find it. Using Theorem 1.6 and (1.18), we have that the positive part of \( \sigma^* \) may be written as

\[ [\sigma^*]^+ = \sum_{j=0}^{k}[\sigma^*]^+ \mathbb{L}(A_j \times A_{j-1}) \]

and, we use the sequence of measures \([\sigma^*]^+ \mathbb{L}(A_j \times A_{j-1})\), jointly with the Gluing Lemma 5.4, that follows from [19, Lemma 7.6], to give an exact description of an optimal transport of \( f^+ \) into \( f^- \) (see Section 5.2 for the details).

Note that, as we will show, there is no optimal transport map in general, therefore, we can not expect to obtain such a map from the limit equation, however, we find a general construction of optimal transport plans.

The last task is the connection between our optimal mass transport problem and the classical Monge transport problem. Actually, we approximate the usual Monge-Kantorovich problem with the Euclidean distance with nonlocal Monge-Kantorovich problems when we reduce the step size. To this end let us set the definition of the discrete distance for step of size \( \varepsilon \),

\begin{equation}
d_\varepsilon(x, y) = \begin{cases}
0 & \text{if } x = y, \\
\varepsilon \left( \left| \frac{x-y}{\varepsilon} \right| + 1 \right) & \text{if } x \neq y.
\end{cases}
\end{equation}

Recall that for the Euclidean distance, we have

\[ \inf \left\{ F_{d_\varepsilon}(T) : T \in \mathcal{A}(f^+, f^-) \right\} = \min \left\{ K_{d_\varepsilon}(\mu) : \mu \in \mathbb{P}(f^+, f^-) \right\} \]

\[ = \sup \left\{ \mathcal{P}_{f^+, f^-}(u) : u \in K_{d_\varepsilon} \right\} =: \mathcal{W}_{d_\varepsilon}. \]

For the discrete distance \( d_\varepsilon \), thanks to Theorem 1.16, we deduce that

\[ \inf \left\{ F_{d_\varepsilon}(T) : T \in \mathcal{A}(f^+, f^-) \right\} = \min \left\{ K_\varepsilon(\mu) : \mu \in \mathbb{P}(f^+, f^-) \right\} \]

\[ = \sup \left\{ \mathcal{P}_{f^+, f^-}(u) : u \in K_{\varepsilon} \right\} =: \mathcal{W}_\varepsilon. \]

And we prove the following result.

**Theorem 1.9.** Let \( f^+, f^- \in L^2(\Omega) \) two non-negative Borel function satisfying the mass balance condition (1.4).

1. The sequence \( \mathcal{W}_\varepsilon \) is nonincreasing as \( \varepsilon \to 0 \) and

\[ 0 \leq \mathcal{W}_\varepsilon - \mathcal{W}_{d_\varepsilon} \leq \varepsilon \int f^+(x) \, dx \quad \text{for any} \ \varepsilon > 0. \]

2. The optimal transport plans \( \mu_\varepsilon \) converge as \( \varepsilon \to 0 \) to an optimal transport plan for the Monge-Kantorovich problem with respect to the Euclidean distance: there exists \( \mu^* \in \mathbb{P}(f^+, f^-) \), such that, after a subsequence,

\[ \mu_\varepsilon \rightharpoonup \mu^* \quad \text{as measures} \]
and
\[ K_{d_{\varepsilon}}(\mu^*) = \min \{ K_{d_{\varepsilon}}(\mu) : \mu \in \pi(f^+, f^-) \} . \]

(3) Convergence for the Kantorovich potentials: let \( u^*_\varepsilon \) be a Kantorovich potential associated with the metric \( d_\varepsilon \), then, after a subsequence,

\[ u^*_\varepsilon \rightharpoonup u^* \quad \text{in} \quad L^2 \quad \text{as} \quad \varepsilon \to 0 , \]

where \( u^* \) is a Kantorovich potential associated with the Euclidean distance.

Finally, we obtain the transport density for the classical Monge-Kantorovich problem by rescaling. We prove that there exists a subsequence \( \varepsilon_n \to 0 , \) such that

\[ \mu_n := (\pi_1, T_{\varepsilon_n})\# [\varepsilon_n \sigma^*_{\varepsilon_n}]^+ \rightharpoonup \sigma^+ \quad \text{weakly as measures} , \]

where

\[ T_{\varepsilon}(x, y) = \frac{y - x}{\varepsilon} , \]

and

\[ a = \pi_1 \# \sigma^+ \]

is the transport density for the classical Monge-Kantorovich problem given in Theorem 1.5.

2. A nonlocal Monge-Kantorovich problem

Given \( \varepsilon > 0 \), we consider the convex set

\[ K_\varepsilon := \{ u \in L^2(\Omega) : |u(x) - u(y)| \leq \varepsilon \quad \text{if} \quad |x - y| \leq \varepsilon \quad \text{a.e.} \} . \]

For the distance \( d_\varepsilon \) defined in (1.21) it is easy to see that

\[ K_\varepsilon = K_{d_\varepsilon}(\Omega) = \{ u \in L^2(\Omega) : |u(x) - u(y)| \leq d_\varepsilon(x, y) \quad \text{a.e.} \} , \]

that is,

\[ K_\varepsilon = \text{Lip}_1(\Omega, d_\varepsilon) . \]

Given \( f^+, f^- \in L^1(\Omega) \) two non-negative Borel functions satisfying the mass balance condition (1.4), we have that the Monge problem associated with the distance \( d_\varepsilon \) is to find an optimal transport map \( T^* \in \mathcal{A}(f^+, f^-) \) satisfying

\[ \mathcal{F}_\varepsilon(T^*) = \min \{ \mathcal{F}_\varepsilon(T) : T \in \mathcal{A}(f^+, f^-) \} , \]

where \( \mathcal{F}_\varepsilon(T) := \mathcal{F}_{d_\varepsilon}(T) . \) We will see in Example 3.2 that, in general and even for masses for which the classical Monge problem has solution, the Monge problem associated with the distance \( d_\varepsilon \) does not have a solution.

The Monge-Kantorovich problem associated with the distance \( d_\varepsilon \) is to find and optimal plan \( \mu^* \in \pi(f^+, f^-) \) solving the minimization problem

\[ K_\varepsilon(\mu^*) = \min \{ K_\varepsilon(\mu) : \mu \in \pi(f^+, f^-) \} , \]

where \( K_\varepsilon(\mu) := K_{d_\varepsilon}(\mu) . \) By Proposition 1.2, we have that there is an optimal plan \( \mu^* \in \pi(f^+, f^-) \) solving the Monge-Kantorovich problem (2.3). Now, since \( d_\varepsilon \) is not continuous, we do not know, in principle, if

\[ \min \{ K_\varepsilon(\mu) : \mu \in \pi(f^+, f^-) \} = \inf \{ \mathcal{F}_\varepsilon(T) : T \in \mathcal{A}(f^+, f^-) \} . \]
On the other hand, from Theorems 1.3 and 1.4, it follows that, for $f^+, f^- \in L^2(\Omega)$,
\begin{equation}
(2.4) \quad \min\{K_\varepsilon(\mu) : \mu \in \pi(f^+, f^-)\} = \sup \left\{ \int_{\Omega} u(x)(f^+(x) - f^-(x)) \, dx : u \in K_\varepsilon \right\}.
\end{equation}

The aim of this section is the study of (2.2), (2.3) and the Kantorovich potentials in (2.4). Let us first prove, working as in the proof of [9, Lemma 6], the following Dual Criteria for Optimality.

**Lemma 2.1.** Fix $u^* \in K_\varepsilon$ and let $T^* \in A(f^+, f^-)$. If
\begin{equation}
(2.5) \quad u^*(x) - u^*(T^*(x)) = d_\varepsilon(x, T^*(x)) \quad \text{for almost all } x \in \text{supp}(f^+),
\end{equation}
then
(i) $u^*$ is a Kantorovich potential for the metric $d_\varepsilon$.
(ii) $T^*$ is an optimal map for the Monge problem associated to the metric $d_\varepsilon$.
(iii) \[ \inf \{ \mathcal{F}_\varepsilon(T) : T \in A(f^+, f^-)\} = \sup \{ \mathcal{P}_{f^+, f^-}(u) : u \in K_\varepsilon \}. \]
(iv) Every optimal map $\hat{T}$ for the Monge problem associated to the metric $d_\varepsilon$ and Kantorovich potential $\hat{u}$ for the metric $d_\varepsilon$ satisfy (2.5).

**Proof.** We will write $\mathcal{P} = \mathcal{P}_{f^+, f^-}$ for simplicity. For every $T \in A(f^+, f^-)$ and $u \in K_\varepsilon$, we have
\begin{align*}
\mathcal{F}_\varepsilon(T) &= \int_{\Omega} d_\varepsilon(x, T(x)) f^+(x) \, dx \geq \int_{\Omega} (u(x) - u(T(x))) f^+(x) \, dx \\
&= \int_{\Omega} u(x) f^+(x) \, dx - \int_{\Omega} u(y) f^-(y) \, dy = \mathcal{P}(u).
\end{align*}
Hence,
\begin{equation}
(2.6) \quad \inf \{ \mathcal{F}_\varepsilon(T) : T \in A(f^+, f^-)\} \geq \sup \{ \mathcal{P}(u) : u \in K_\varepsilon \}.
\end{equation}

On the other hand, the assumption (2.5) imply $\mathcal{F}_\varepsilon(T^*) = \mathcal{P}(u^*)$. Therefore,
\[ \mathcal{P}(u^*) = \mathcal{F}_\varepsilon(T^*) \geq \inf \{ \mathcal{F}_\varepsilon(T) : T \in A(f^+, f^-)\} \geq \sup \{ \mathcal{P}(u) : u \in K_\varepsilon \} \geq \mathcal{P}(u^*), \]
and consequently (iii) holds. Moreover, we also get
\[ \mathcal{P}(u^*) = \max \{ \mathcal{P}(u) : u \in K_\varepsilon \}, \]
from where it follows (i), and
\[ \mathcal{F}_\varepsilon(T^*) = \min \{ \mathcal{F}_\varepsilon(T) : T \in A(f^+, f^-)\}, \]
from where (ii) follows.

Finally, let $\hat{T}$ be an optimal map for the Monge problem associated to the metric $d_\varepsilon$ and $\hat{u}$ a Kantorovich potential for the metric $d_\varepsilon$. Then, $\mathcal{F}_\varepsilon(\hat{T}) = \mathcal{F}_\varepsilon(T^*)$ and $\mathcal{P}(\hat{u}) = \mathcal{P}(u^*)$. Thus, by (iii), we obtain that $\mathcal{F}_\varepsilon(\hat{T}) = \mathcal{P}(\hat{u})$, and consequently,
\[ \int_{\Omega} d_\varepsilon(x, \hat{T}(x)) f^+(x) \, dx = \int_{\Omega} (u(x) - u(\hat{T}(x))) f^+(x) \, dx. \]
From where it follows that
\[ \hat{u}(x) - \hat{u}(\hat{T}(x)) = d_\varepsilon(x, \hat{T}(x)) \quad \text{for almost all } x \in \text{supp}(f^+), \]
since $d_\varepsilon(x, T(x)) \geq u(x) - u(\hat{T}(x))$ and $f^+ \geq 0$. \qed
Remark 2.2. Observe also that if $u^*$ is a Kantorovich potential for the metric $d$, from (1.12) and the inequality $u^*(x) - u^*(y) \leq d(x, y)$ it follows that, if $\mu^* \in \Pi(f^+, f^-)$,

(2.7) \[ \mu^* \text{ is optimal } \iff u^*(x) - u^*(y) = d(x, y) \text{ } \mu^* - \text{a.e. in } \Omega \times \Omega. \]

2.1. Kantorovich potentials. We will show, like in [11], that it is possible to obtain Kantorovich potentials associated with the metric $d_1$ taking limit as $p \to \infty$ in some related nonlocal $p$–Laplacian type problems. Let $J$ be as in (1.2), $f \in L^\infty(\Omega)$, and consider the functional

(2.8) \[ F_p(u) = \frac{1}{2p} \int_\Omega \int_\Omega J(x - y)|u(y) - u(x)|^p \, dy \, dx - \int_\Omega f(x)u(x) \, dx \]

in the space

(2.9) \[ S_p = \left\{ u \in L^p(\Omega) : \int_\Omega u(x) \, dx = 0 \right\}. \]

We will use the following Poincaré type inequality from [4].

Proposition 2.3. Given $p \geq 1$, $J$ and $\Omega$, there exists $\beta_{p-1} = \beta_{p-1}(J, \Omega, p) > 0$ such that

(2.10) \[ \beta_{p-1} \int_\Omega \left| u - \frac{1}{|\Omega|} \int_\Omega u \right|^p \leq \frac{1}{2} \int_\Omega \int_\Omega J(x - y)|u(y) - u(x)|^p \, dy \, dx \quad \forall u \in L^p(\Omega). \]

Using this proposition we can show that there is a unique minimum of $F_p$ in $S_p$.

Theorem 2.4. There exists a unique $u_p \in S_p$ such that

(2.11) \[ F_p(u_p) = \min_{v \in S_p} F_p(v). \]

Proof. Let $u_n$ be a minimizing sequence. Hence, $F_p(u_n) \leq C$, that is

\[ \frac{1}{2p} \int_\Omega \int_\Omega J(x - y)|u_n(y) - u_n(x)|^p \, dy \, dx - \int_\Omega f(x)u_n(x) \, dx \leq C. \]

Then,

\[ \frac{1}{2p} \int_\Omega \int_\Omega J(x - y)|u_n(y) - u_n(x)|^p \, dy \, dx \leq \int_\Omega f(x)u_n(x) \, dx + C. \]

From the Poincaré inequality (2.10) and Hölder’s inequality, we get

\[ \frac{1}{2p} \int_\Omega \int_\Omega J(x - y)|u_n(y) - u_n(x)|^p \, dy \, dx \leq \|f\|_{L^{p'}(\Omega)} \|u_n\|_{L^p} + C \]

\[ \leq C(f) \left( \frac{1}{2p} \int_\Omega \int_\Omega J(x - y)|u_n(y) - u_n(x)|^p \, dy \, dx \right)^{1/p} + C. \]

Therefore, we have that

\[ \frac{1}{2p} \int_\Omega \int_\Omega J(x - y)|u_n(y) - u_n(x)|^p \, dy \, dx \leq C. \]

Then, applying again Poincaré’s inequality (2.10), we have \( \{u_n : n \in \mathbb{N}\} \) is bounded in $L^p(\Omega)$. Hence, we can extract a subsequence that converges weakly in $L^p(\Omega)$ to some $u$ (that clearly has to verify $\int_\Omega u = 0$) and we obtain

\[ \liminf_{n \to +\infty} \frac{1}{2p} \int_\Omega \int_\Omega J(x - y)|u_n(y) - u_n(x)|^p \, dy \, dx \geq \frac{1}{2p} \int_\Omega \int_\Omega J(x - y)|u(y) - u(x)|^p \, dy \, dx \]
and
\[ \lim_{n \to +\infty} \int_{\Omega} f(x)u_n(x) \, dx = \int_{\Omega} f(x)u(x) \, dx \]
Therefore, \( u \) is a minimizer of \( F_p \).

Uniqueness is a direct consequence of the fact that \( F_p \) is strictly convex. \( \square \)

Now we want to study the limit as \( p \to \infty \) of the sequence \( u_p \) of minimizers of \( F_p \) in \( S_p \).

**Theorem 2.5.** Let \( f^+, f^- \in L^\infty(\Omega) \) two non-negative Borel function satisfying the mass balance condition (1.4). Let \( u_p \) the solution of (2.11) for \( f = f^+ - f^- \). Then, there exists a subsequence \( \{u_{p_n}\}_{n \in \mathbb{N}} \) having a weak limit \( u \) which is a Kantorovich potential associated with the metric \( d_1 \), that is,
\[ \int_{\Omega} u(x)(f^+(x) - f^-(x)) \, dx = \max_{v \in K_1} \int_{\Omega} v(x)(f^+(x) - f^-(x)) \, dx. \]

**Proof.** For \( 1 \leq q \leq p \), by Hölder’s and Poincare’s inequalities, we have
\[ |||u|||_q := \left( \int_{\Omega} \int_{\Omega} J(x - y)|u(y) - u(x)|^q \, dy \, dx \right)^{\frac{1}{q}}. \]
Let us write \( f := f^+ - f^- \). Since \( F_p(u_p) \leq F_p(0) = 0 \),
\[ |||u_p|||_p^p \leq 2p \int_{\Omega} f(x)u_p(x) \, dx \leq 2p |||f|||_\infty |||u_p|||_1. \]

For \( 1 \leq q < p \), by Hölder’s and Poincare’s inequalities, we have
\[ |||u_p|||_q \leq \left( \int_{\Omega} \int_{\Omega} J(x - y)|u_p(y) - u_p(x)|^p \, dy \, dx \right)^{\frac{1}{p}} \left( \int_{\Omega} \int_{\Omega} J(x - y) \, dy \, dx \right)^{\frac{p-q}{pq}} \]
\[ \leq (2p)^{\frac{1}{q}} |||f|||_\infty \frac{1}{q} \left( \int_{\Omega} \int_{\Omega} J(x - y) \, dy \, dx \right)^{\frac{p-1}{pq}}. \]

Consequently,
\[ (2.12) \quad |||u_p|||_q \leq \left( \frac{2p |||f|||_\infty}{\beta_0} \right)^{\frac{1}{q}} \left( \int_{\Omega} \int_{\Omega} J(x - y) \, dy \, dx \right)^{\frac{1}{q}}. \]
Then, \( \{|||u_p|||_q : p > q\} \) is bounded. Hence, by Poincaré’s inequality (2.10), we have that \( \{u_p : p > q\} \) is bounded in \( L^q(\Omega) \). Therefore, we can assume that \( u_p \rightharpoonup u \) weakly in \( L^q(\Omega) \). By a diagonal process, we have there is a sequence \( p_n \to \infty \), such that \( u_{p_n} \rightharpoonup u \) weakly in \( L^m(\Omega) \), as \( n \to +\infty \), for all \( m \in \mathbb{N} \). Thus, \( u \in L^\infty(\Omega) \). Since the functional \( v \mapsto |||v|||_q \) is weakly lower semi-continuous, having in mind (2.12), we have
\[ |||u|||_q \leq \left( \int_{\Omega} \int_{\Omega} J(x - y) \, dy \, dx \right)^{\frac{1}{q}}. \]
Therefore,
\[ \lim_{q \to +\infty} |||u|||_q \leq 1, \]
from where it follows that 
\[ |u(x) - u(y)| \leq d_1(x, y) \text{ a.e.} \]

Let us see that \( u \) is a Kantorovich potential associated with the metric \( d_1 \). Fix \( v \in K_1 \). Then,
\[
- \int_{\Omega} f u_p \leq \frac{1}{2p} \int_{\Omega} \int_{\Omega} J(x - y) \left| u_p(y) - u_p(x) \right|^p dy dx - \int_{\Omega} f(x) u_p(x) dx 
= F_p(u_p) \leq F_p \left( v - \frac{1}{|\Omega|} \int_{\Omega} v \right) 
= \frac{1}{2p} \int_{\Omega} \int_{\Omega} J(x - y) v(y) - v(x)^p dy dx - \int_{\Omega} f(x) v(x) dx 
\leq \frac{1}{2p} \int_{\Omega} \int_{\Omega} J(x - y) dy dx - \int_{\Omega} f(x) v(x) dx,
\]

where we have used \( \int_{\Omega} f = 0 \) for the second equality and the fact that \( v \in K_1 \) for the last inequality.

Hence, taking limit as \( p \to \infty \), we obtain that \( \int_{\Omega} u(x)(f^+(x) - f^-(x)) dx \geq \int_{\Omega} v(x)(f^+(x) - f^-(x)) dx. \)

Therefore, \( u \) is a Kantorovich potential associated with the metric \( d_1 \). \( \square \)

Our next step is to study the existence of some special Kantorovich potentials which will play an important role in the rest of the paper. We will see that, under quite general assumptions, there is a Kantorovich potential \( u^* \) associated with \( d_1 \) such that its image has a finite number of values in \( \mathbb{Z} \), that is, \( u^* : \Omega \to \mathbb{Z} \) with \( u^*(\Omega) \) finite. To this end, let us first prove the following result.

**Lemma 2.6.** Assume that \( v \in K_1 \) takes a finite number of values. Then there exists \( u \in K_1 \) that also takes a finite number of values but with jumps of length 1, the number of points in its image is less or equal than the number of points in the image of \( v \) and improves in the maximization problem, that is,
\[
\int_{\Omega} u(x)(f^+(x) - f^-(x)) dx \geq \int_{\Omega} v(x)(f^+(x) - f^-(x)) dx.
\]

**Proof.** The proof runs by induction in the number of levels of \( v \). Take \( f := f^+ - f^- \) and suppose that \( v \in K_1 \) is given by, without loss of generality,
\[ v(x) = a_0 \chi_{A_0} + a_1 \chi_{A_1} + \cdots + a_k \chi_{A_k}, \]
with \( a_0 = 0, |A_i| > 0, A_i \cap A_j = \emptyset \) for any \( i \neq j \).

Set \( s := \text{Sign} \left( \int_{A_0} f \right) \), where \( \text{Sign}(r) = \begin{cases} 1 & \text{if } r \geq 0, \\ -1 & \text{if } r < 0, \end{cases} \)
and consider
\[ t_0 = \max \{ t \geq 0 : u_t := (a_0 + st) \chi_{A_0} + a_1 \chi_{A_1} + \cdots + a_k \chi_{A_k} \in K_1 \}. \]
So, $t_0$ is such that $\exists \ i \neq 0$, dist($A_i, A_0$) $\leq 1$ and $|a_0 + st_0 - a_i| = 1$ and
\[
\int_{\Omega} f(x)v(x) \, dx \leq \int_{\Omega} f(x)u_t(x) \, dx.
\]
Hence, replacing $v$ by $u_{t_0}$, we can assume that $A_i$ are disjoint sets, dist($A_0, A_1$) $\leq 1$ and $|u_0 - u_1| = 1$.

Now, we set $s := \text{Sign} \left( \int_{A_0 \cup A_1} f \right)$ and we consider
\[
t_0 = \max \{ t \geq 0 \mid u_t := (a_0 + st)\chi_{A_0} + (a_1 + st)\chi_{A_1} + a_2\chi_{A_2} + \cdots + a_k\chi_{A_k} \in K_1 \}.
\]
So, $t_0$ is such that $\exists \ i \in \{0, 1\}$ and $\exists \ j_i \notin \{0, 1\}$ such that dist($A_i, A_{j_i}$) $\leq 1$, $|a_i + st_0 - a_{j_i}| = 1$ and
\[
\int_{\Omega} f(x)v(x) \, dx \leq \int_{\Omega} f(x)u_t(x) \, dx.
\]
Hence, replacing $v$ by $u_{t_0}$, we can assume that $A_i$ are disjoint sets and $|u_i - u_j| \in \{0, 1, 2\}$, for any $i, j \in \{0, 1, 2\}$.

Now, by induction assume that we have $u = a_0\chi_{A_0} + \cdots + a_l\chi_{A_l} + \cdots + a_k\chi_{A_k}$, where $A_i$ are disjoint sets, and $|a_i - a_j| \in \mathbb{N}$, for any $i, j = 0, 1, \ldots, l$, and let us prove that we can assume that $A_i$ are disjoint compact sets, and $|a_i - a_j| \in \mathbb{N}$, for any $i, j \in \{0, 1, \ldots, l+1\}$. We set
\[
s := \text{Sign} \left( \int_{A_0 \cup \cdots \cup A_l} f \right),
\]
and we consider
\[
t_0 = \max \{ t \geq 0 \mid u_t := (a_0 + st)\chi_{A_0} + \cdots + (a_l + st)\chi_{A_l} + a_{l+1}\chi_{A_{l+1}} + \cdots + a_k\chi_{A_k} \in K_1 \}.
\]
So, $t_0$ is such that $\exists \ i \in \{0, 1, \ldots, l\}$ and $\exists \ j_i \notin \{0, 1, \ldots, l\}$ such that dist($A_i, A_{j_i}$) $\leq 1$, $|u_i + st_0 - u_{j_i}| = 1$ and
\[
\int_{\Omega} f(x)v(x) \, dx \leq \int_{\Omega} f(x)u_t(x) \, dx.
\]
Hence, replacing $u$ by $u_{t_0}$, we can assume that $A_i$ are disjoint sets, and $|a_i - a_j| \in \mathbb{N}$, for any $i, j \in \{0, 1, \ldots, l+1\}$.

At last, by induction we deduce that we can assume that $A_i$ are disjoint compact sets, and $|a_i - a_j| \in \mathbb{N}$, for any $i, j \in \{0, 1, \ldots, k\}$. \hfill $\square$

**Theorem 2.7.** Let $f^+, f^- \in L^\infty(\Omega)$ two non-negative Borel functions satisfying the mass balance condition (1.4) and such that $|\text{supp}(f^+) \cap \text{supp}(f^-)| = 0$. Then there exists a Kantorovich potential $u^*$, associated with the metric $d_1$, such that $u^*(\Omega) \subset \mathbb{Z}$ and takes a finite number of values.

**Proof.** Take $f := f^+ - f^-$. By density, we have that there exists a maximizing sequence $v_n \in K_1$ such that $v_n$ takes a finite number of values and
\[
\int_{\Omega} v_n f \to \max_{w \in K_1} \int_{\Omega} w f.
\]
Thanks to the previous lemma, there exists $u_n \in K_1$ such that all the jumps of the levels of $u_n$ are of length one and it is a new maximizing sequence,

$$\int_{\Omega} u_n f \to \max_{w \in K_1} \int_{\Omega} w f.$$  

(2.13)

Notice that the number of levels of $u_n \in K_1$ is bounded by a constant that only depends on $\Omega$. Indeed, if $u \in K_1$ is like

$$u(x) = j \chi_{C_1} + (j + 1) \chi_{C_2} + \cdots + (j + k) \chi_{C_k},$$

with $|C_i| > 0$, $C_i \cap C_j = \emptyset$ for $i \neq j$, then, there exists a null set $N \subset \Omega \times \Omega$ such that $|x - y| > 1 \quad \forall (x, y) \in (C_{i-1} \times C_{i+1}) \setminus N, \quad \forall i,$

otherwise $u \not\in K_1$. Therefore, since $\Omega$ has finite diameter, this provides a bound for the number of possible sets $k$ that is the same as the number of possible levels.

Let us take $k$ an uniform bound for the number of levels of functions in $K_1$ and let us suppose also that the lower level for the minimizing functions is 0. By Fatou’s Lemma and having in mind (2.13) we get

$$\max_{w \in K_1} \int_{\Omega} w f \leq \int_{\Omega} \limsup_{n \to \infty} (u_n f).$$

Now, since $\text{supp}(f^+) \cap \text{supp}(f^-)$ is null,

$$\limsup_{n \to \infty} (f u_n) \leq f^+ \limsup_{n \to \infty} u_n - f^- \liminf_{n \to \infty} u_n$$

$$= f^+ \sum_{i=0}^{k} i \chi_{A_i} - f^- \sum_{i=0}^{k} i \chi_{B_i} = f \sum_{i=0}^{k} i \chi_{C_i},$$

where

$$C_i = (A_i \cap \{f^+(x) > 0\}) \cup (B_i \cap \{f^-(x) > 0\}), \quad i > 0,$$

and

$$C_0 = \Omega \setminus \bigcup_{i=0}^{k} C_i.$$

So, setting

$$w^* = \sum_{i=0}^{k} i \chi_{C_i},$$

we have

$$\max_{w \in K_1} \int_{\Omega} w f \leq \int_{\Omega} f w^*.$$

To finish the proof let us see that $u^* \in K_1$. Indeed, let $N_n \subset \Omega \times \Omega$ the null set where

$$|u_n(x) - u_n(y)| > d_1(x, y),$$

and set $N := \bigcup_{n \in \mathbb{N}} N_n$. Take $(x, y) \in (\Omega \times \Omega) \setminus N$ such h that $|x - y| \leq 1$. Let us suppose that $x \in A_i \cap \{f^+ > 0\}$ and $y \in B_j \cap \{f^- > 0\}$ (the other cases being similar), then since $|u_n(x) - u_n(y)| \leq 1$ for all $n \in \mathbb{N}$, by the definition of $\limsup$ and $\liminf$, we have

$$|u^*(x) - u^*(y)| = |i - j| \leq 1.$$
If not, that is, if $|i - j| > 1$, assuming for instance that $i < j$, we have that there exists $0 < \epsilon < 1$ such that

$$i < i + \epsilon < j - \epsilon < j$$

and there exist $n \in \mathbb{N}$ such that $u_n(x) \in [i, i + \epsilon]$, and $u_n(y) \in [j - \epsilon, j]$, that is, $u_n(x) = i$ and $u_n(y) = j$, which contradicts that $|u_n(x) - u_n(y)| \leq d_1(x, y)$ (we have taken $(x, y) \in (\Omega \times \Omega) \setminus N_n$).

**Remark 2.8.** Note that with a similar proof, for the distance $d_\epsilon$ we obtain potentials that take finite values in $\{j\epsilon : j \in \mathbb{Z}\}$.

**Remark 2.9.** By Lemma 2.1, for an optimal transport map $T^*$ pushing $f^+$ to $f^-$, we have

$$u^*(x) - u^*(T^*(x)) = d_1(x, T^*(x)) \quad \text{for almost all } x \in \text{supp}(f^+),$$

being $u^*$ a Kantorovich potential. This relation says that a point $x \in \text{supp}(f^+)$ has to be carried to a point $T^*(x)$ in such a way that the number of steps is given by the jumps of $u^*$, that is the number of steps is equal to $u^*(x) - u^*(T^*(x))$. To search or to construct an optimal transport map $T^*$ we must have this observation in mind.

Let us make another observation. If we assume that $u^*$ takes only the values $\{j, j + 1, j + 2, ..., j + k\}$, $j \in \mathbb{Z}$, that is, $u^* = j\chi_{A_0} + (j + 1)\chi_{A_1} + (j + 2)\chi_{A_2} + ... + (j + k)\chi_{A_k}$, then,

$$|A_k \cap \text{supp}(f^-)| = 0 \quad \text{and} \quad |A_0 \cap \text{supp}(f^+)| = 0.$$

In fact, if not, just redefine $u^*$ to be

$$\tilde{u}^*(x) = \begin{cases} j + k - 1 & \text{in } A_k \cap \text{supp}(f^-), \\ u^*(x) & \text{otherwise}, \end{cases}$$

and we get that $\tilde{u}^* \in K_1$ with

$$\int_{\Omega} u^*f < \int_{\Omega} \tilde{u}^*f,$$

a contradiction.

We also observe that

$$\int_{A_k} f^+ \geq \int_{A_{k-1}} f^-.$$

In fact, if not, we define

$$\tilde{u}^*(x) = \begin{cases} j + k - 1 & \text{in } A_k, \\ j + k - 2 & \text{in } A_{k-1} \cap \text{supp}(f^-), \\ u^*(x) & \text{otherwise}, \end{cases}$$

and we get that $\tilde{u}^* \in K_1$ with

$$\int_{\Omega} u^*f < \int_{\Omega} \tilde{u}^*f,$$

a contradiction. The inequality (2.16) will be of special interest.
2.2. Equality between Monge’s infimum and Kantorovich’s minimum. It is well known (see, [16]) that equality (1.9) between Monge’s infimum and Kantorovich’s minimum is not true in general if the cost function is not continuous. The example given by Pratelli in [16], can be adapted to get a counterexample also for the case of the cost function given by the metric \( d_1 \).

Example 2.10. Consider \( R, S \) and \( T \) are the parallel segments in \( \mathbb{R}^2 \), given by \( R := \{(-1, y) : y \in [-1, 1]\} \), \( S := \{(0, y) : y \in [-1, 1]\} \), \( Q := \{(1, y) : y \in [-1, 1]\} \). Let \( f^+ := 2H^1 \mathbb{L} S \) and \( f^- := H^1 \mathbb{L} R + H^1 \mathbb{L} Q \). It is not difficult to see that
\[
\min \{K_{d_1}(\mu) : \mu \in \pi(f^+, f^-)\} = 2
\]
and the the minimum is achieved by the transport plan splitting the central segment \( S \) in two parts and translating them on the left and on the right. On the other hand, we claim that \( (2.17) \)
\[
\inf \{F_1(T) : T \in A(f^+, f^-)\} \geq 4.
\]
To prove (2.17), fix \( T \in A(f^+, f^-) \) and consider \( I(T) := \{x \in S : d_1(x, T(x)) = 1\} \). If we see that
\[
(2.18) \quad f^+(I(T)) = 0,
\]
then
\[
F_1(T) = \int_S d_1(x, T(x)) \, df^+(x) = 2 \int_{S \setminus I(T)} d_1(x, T(x)) \, dH^1(x) \geq 4,
\]
and (2.17) follows. Finally, let us see that (2.18) holds. If we define
\[
I(T)_R := \{x \in I(T) : T(x) \in R\} \quad \text{and} \quad I(T)_Q := \{x \in I(T) : T(x) \in Q\},
\]
we have \( I(T) = I(T)_R \cup I(T)_Q \) and \( I(T)_R \cap I(T)_Q = \emptyset \), and by the definition of \( I(T) \), if \( E = T(I(T)) \), it is easy to see that
\[
H^1(E) = H^1(E \cap R) + H^1(E \cap Q) = H^1(I(T)_R) + H^1(I(T)_R) = H^1(I(T)).
\]
Therefore,
\[
f^+(I(T)) = 2f^-(E).
\]
But since \( T \in A(f^+, f^-) \) one has
\[
f^-(E) = f^+(T^{-1}(E)) \geq f^+(I(T)) = 2f^-(E),
\]
that implies \( f^+(I(T)) = 0 \) and (2.18) is proved.

Let us see that the cause of the above counterexample is not the discontinuity of \( d_1 \) being the cause the fact that the measures \( f^+ \) and \( f^- \) are not absolutely continuous respect the Lebesgue measure.

Theorem 2.11. Let \( f^+, f^- \in L^1(\Omega) \) two non-negative Borel function satisfying the mass balance condition (1.4). Then,
\[
(2.19) \quad \min \{K_{d_1}(\mu) : \mu \in \pi(f^+, f^-)\} = \inf \{F_1(T) : T \in A(f^+, f^-)\}.
\]

Proof. For \( n \in \mathbb{N}, n \geq 2 \), let \( \varphi_n \) be the function defined by
\[
\varphi_n(r) := \begin{cases} 
 m & \text{if } m \leq r \leq m + 1 - \frac{1}{n}, \\
 n(r - (m + 1)) + m + 1 & \text{if } m + 1 - \frac{1}{n} < r < m + 1,
\end{cases}
\]
with \( m = 0, 1, 2, \ldots \). Consider the continuous cost function
\[
c_n(x, y) := \begin{cases} 
0 & \text{if } x = y, \\
\varphi_n(|x - y|) + 1 & \text{if } x \neq y.
\end{cases}
\]
By construction, \( c_1 \geq c_2 \geq \ldots c_n \geq d_1 \) and \( \lim_{n \to +\infty} c_n(x, y) = d_1(x, y) \). Let \( \mu^* \in \pi(f^+, f^-) \), such that
\[
K_{d_1}(\mu^*) = \min \{ K_{d_1}(\mu) : \mu \in \pi(f^+, f^-) \}.
\]
Since \( c_n \) is continuous, by Proposition 1.2, we have there exists \( \mu_n \in \pi(f^+, f^-) \), such that
\[
K_{c_n}(\mu_n) = \min \{ K_{c_n}(\mu) : \mu \in \pi(f^+, f^-) \} = \inf \left\{ \int_{\Omega} c_n(x, T(x)) f^+(x) \, dx : T \in \mathcal{A}(f^+, f^-) \right\}.
\]
Therefore, for every \( n \geq 2 \), there exists \( T_n \in \mathcal{A}(f^+, f^-) \) such that
\[
\int_{\Omega} \int_{\Omega} c_n(x, y) \, d\mu^*(x, y) = \int_{\Omega} \int_{\Omega} c_n(x, y) \, d\mu_n(x, y) \geq \int_{\Omega} c_n(x, T_n(x)) \, f^+(x) \, dx - \frac{1}{n}.
\]
Then, by the Monotone Convergence Theorem, we get
\[
\int_{\Omega} \int_{\Omega} d_1(x, y) \, d\mu^*(x, y) = \lim_{n \to +\infty} \int_{\Omega} \int_{\Omega} c_n(x, y) \, d\mu^*(x, y) \geq \limsup_{n \to +\infty} \int_{\Omega} c_n(x, T_n(x)) \, f^+(x) \, dx
\]
and since \( c_n \geq d_1 \), we have
\[
\limsup_{n \to +\infty} \int_{\Omega} d_1(x, T_n(x)) \, f^+(x) \, dx \leq \limsup_{n \to +\infty} \int_{\Omega} c_n(x, T_n(x)) \, f^+(x) \, dx \leq \int_{\Omega} \int_{\Omega} d_1(x, y) \, d\mu^*(x, y) = K_{d_1}(\mu^*).
\]
On the other hand,
\[
\inf \left\{ \mathcal{F}_1(T) : T \in \mathcal{A}(f^+, f^-) \right\} \leq \int_{\Omega} d_1(x, T_n(x)) \, f^+(x) \, dx,
\]
hence
\[
K_{d_1}(\mu^*) \leq \inf \left\{ \mathcal{F}_1(T) : T \in \mathcal{A}(f^+, f^-) \right\} \leq \liminf_{n \to +\infty} \int_{\Omega} d_1(x, T_n(x)) \, f^+(x) \, dx.
\]
Consequently,
\[
K_{d_1}(\mu^*) = \inf \left\{ \mathcal{F}_1(T) : T \in \mathcal{A}(f^+, f^-) \right\} = \lim_{n \to +\infty} \int_{\Omega} d_1(x, T_n(x)) \, f^+(x) \, dx.
\]
This ends the proof. \( \square \)

**Remark 2.12.** Let us remark that the results we have obtained are also true if in the metric \( d_\varepsilon \) we change the Euclidean norm by any norm \( \| \| \) of \( \mathbb{R}^N \), that is if we consider the distance
\[
d_\varepsilon(x, y) = \begin{cases} 
0 & \text{if } x = y, \\
\varepsilon \left( \frac{\| x - y \|}{\varepsilon} \right) + 1 & \text{if } x \neq y.
\end{cases}
\]
Especially interesting is the case in which we consider the \( \| \cdot \|_\infty \) norm since in this case it counts the maximum of steps moving parallel to the coordinate axes. That is, in this case we measure the distance cost as the number of blocks that the taxi has to cover going from \( x \) to \( y \) in a city.
3. The one-dimensional case

In this section we will see that in the one dimensional case we can be more precise and provide a better description of the abstract situation together with some examples that illustrate the difficulties of this problem.

3.1. Kantorovich potentials. We assume first that the functions \( f^+ \) and \( f^- \) are \( L^\infty \)-functions satisfying

\[
(3.1) \quad f^- = f^- \chi_{[a,0]}, \quad f^+ = f^+ \chi_{[c,d]}, \quad c \geq 0, \quad \text{supp}(f^\pm) \subset [-L,L], \quad \text{for some } L \in \mathbb{N}.
\]

Set \( \Omega \) any interval containing \([-L,L]\).

By Theorem 2.7, there exists a Kantorovich potential \( u^* \) associated with the metric \( d_1 \), such that

\[
(3.2) \quad u^*(x) = \theta_\alpha(x) := \begin{cases} 
-1 & \text{if } \alpha - 2 < x \leq \alpha - 1, \\
0 & \text{if } \alpha - 1 < x \leq \alpha, \\
1 & \text{if } \alpha < x \leq \alpha + 1, \\
\vdots & 
\end{cases}
\]

for some \( 0 < \alpha \leq 1 \). In order to find which \( \alpha \)'s give the Kantorovich potential, we need to maximize

\[
\int_{\Omega} u^*(x)(f^+(x) - f^-(x)) \, dx = - \int_{-L}^{0} u^*(x)f^-(x) \, dx + \int_{0}^{L} u^*(x)f^+(x) \, dx
\]

\[
= -\sum_{j=-L}^{-1} \int_{0}^{1} (\theta_\alpha(x) + j)f^-(x+j) \, dx + \sum_{j=0}^{L-1} \int_{0}^{1} (\theta_\alpha(x) + j)f^+(x+j) \, dx
\]

\[
= -\sum_{j=-L}^{-1} \int_{0}^{1} \theta_\alpha(x)f^-(x+j) \, dx + \sum_{j=0}^{L-1} \int_{0}^{1} \theta_\alpha(x)f^+(x+j) \, dx
\]

\[
- \sum_{j=-L}^{-1} \int_{0}^{1} jf^-(x+j) \, dx + \sum_{j=0}^{L-1} \int_{0}^{1} jf^+(x+j) \, dx.
\]

Since the last two integrals are independent of \( \theta_\alpha \), we only need to maximize

\[
-\sum_{j=-L}^{-1} \int_{0}^{1} (\theta_\alpha(x))f^-(x+j) \, dx + \sum_{j=0}^{L-1} \int_{0}^{1} (\theta_\alpha(x))f^+(x+j) \, dx
\]

\[
= \int_{0}^{1} \theta_\alpha(x)M(x) \, dx = \int_{\alpha}^{1} M(x) \, dx,
\]

for \( 0 < \alpha \leq 1 \), where

\[
(3.3) \quad M(x) = -\sum_{j=-L}^{-1} f^-(x+j) + \sum_{j=0}^{L-1} f^+(x+j), \quad 0 < x \leq 1.
\]
Observe that

$$\int_0^1 M(x) \, dx = \int (f^+ - f^-) = 0.$$  

If $M(x)$ is non decreasing, it is clear that, for $0 < x \leq 1$,

$$\theta_\alpha(x) = \begin{cases} 0 & \text{if } M(x) < 0, \\ 1 & \text{if } M(x) > 0, \end{cases}$$

is the best choice (unique for points where $M(x) \neq 0$). If $M(x)$ is non increasing, it is also clear that $\alpha = 1$ is the best choice.

**Remark 3.1.** Let us suppose now that the supports of the masses are not ordered. For example, let us search for a Kantorovich potential associated with the metric $d_1$ for $f^- = f_1 + f_2$, $f_1 = f_1^+ \chi_{(a_1,a_2)}$, $f_2 = f_2^+ \chi_{(c_1,c_2)}$, and $f^+ = f_1^+ \chi_{(b_1,b_2)}$, with $a_1 < a_2 < b_1 < b_2 < c_1 < c_2$. Let $b \in (b_1, b_2)$ be such that $\int f_1 = \int f \chi_{(b_1,b)}$ and $\int f_2 = \int f \chi_{(b_2,b)}$. Let us call $f_1^+ := f \chi_{(b_1,b)}$ and $f_2^+ := f \chi_{(b_2)}$. By the previous example we construct a non decreasing stair function, $\theta_1$, as Kantorovich potential for $f_1^+$ and $f_1^-$ with value at $b$ equals to some $\lambda$ fixed, and a non increasing stair function, $\theta_2$, as Kantorovich potential for $f_2^+$ and $f_2^-$ with the same value $\lambda$ at $b$. Then, $\theta = \theta_1 \chi_{(a_1,b)} + \theta_2 \chi_{(b,c_2)}$ gives a Kantorovich potential for $f^+$ and $f^-$. This construction can be done for any configuration $f^+ = \sum_{i=1}^m \chi_{(b_{i-1},b_i)}$ and $f^- = \sum_{i=1}^m \chi_{(c_{i-1},c_i)}$.

### 3.2. Nonexistence of optimal transport maps

Let us see with a simple example that, in general, an optimal transport map does not exist for $d_1$ as cost function. Let us point out that for the Euclidean distance it is well known (see for instance [1] or [19]) the existence of an optimal transport map in the case $f^\pm \in L^1(a,b)$, even more, the existence of an unique optimal transport map in the class of non-decreasing functions which is given by

$$T_0(x) := \sup \left\{ y \in \mathbb{R} : \int_a^y f^-(t) \, dt \leq \int_a^x f^+(t) \, dt \right\} \quad \text{if } x \in (a,b).$$

**Example 3.2.** Let $f^+ = L \chi_{[0,1]}$ and $f^- = \chi_{[-L,0]}$ with $L \in \mathbb{R}$. Set $\Omega$ an interval containing $[-L, L]$. Let us see that if $L \in \mathbb{N}$, $L \geq 2$, then there is no optimal transport map $T$ with distance $d_1$ pushing $f^+$ to $f^-$, nevertheless we will see later in Example 3.5 that if $L \notin \mathbb{N}$ then there is an optimal transport map pushing $f^+$ to $f^-$. With a similar proof it can be proved that there is no transport map $T$ between $f^+ = L \chi_{[0,1]}$ and $f^- = \chi_{[-L,0]}$ with $L \in \mathbb{N}$ if one considers the distance $d_{1/k}$ with $k \in \mathbb{N}$.

Arguing by contradiction assume there is an optimal transport map $T$ pushing $f^+$ to $f^-$. Let $A_i = \{ x \in [0,1] : d_1(x, T(x)) = i \}, \quad i = 1, ..., L$.

A Kantorovich potential for this configuration of masses $f^+$ and $f^-$ is given by

$$u^*(x) = \begin{cases} 0, & x \in (0,1) \\ -1, & x \in (-1,0] \\ \vdots \\ -L, & x \in (-L,-L+1]. \end{cases}$$
From Lemma 2.1 we have the equality 
\[ u^*(x) - u^*(T(x)) = d_1(x, T(x)). \]
Therefore, if \( x \in A_i \) then \( T(x) \in (-i, -i + 1] \) and hence we can conclude that 
\[ |A_i| = 1/L, \]
in fact, \( A_i = T^{-1}((-i, -i + 1]) \). Moreover, 
\[ T(x) \geq x - i \quad \text{for all} \ x \in A_i. \]

Now, we claim that 
\[ T(x) = x - i \quad \text{for all} \ x \in A_i, \]
for every \( i = 1, \ldots, L \).

Therefore \( |T(A_i)| = 1/L \) which gives a contradiction with the fact that \( |T([0, 1])| = L \).

To prove that claim we argue as follows: assume, without lose of generality, that there is a set of positive measure \( K \subset A_1 \) such that \( T(x) > x - 1 \) in \( K \). Then, it is easy to see that there exists \( \theta \in (0, 1) \) such that 
\[ |T^{-1}((-1, \theta - 1))| < |A_1 \cap (0, \theta)|. \]

Therefore, since 
\[ T^{-1}((-i, \theta - i)) \subset A_i \cap (0, \theta) \quad \text{for all} \ i, \]
we have 
\[ \theta = \frac{1}{L} \left| \bigcup_{i=1}^{L} (-i, \theta - i) \right| = \left| T^{-1} \left( \bigcup_{i=1}^{L} (-i, \theta - i) \right) \right| = \left| \bigcup_{i=1}^{L} T^{-1}((-i, \theta - i)) \right| < \bigcup_{i=1}^{L} |A_i \cap (0, \theta)| = \theta, \]
and we arrive to a contradiction.

Nevertheless, since \( d_1 \) is lower semi-continuous, by Proposition 1.2, there exists an optimal plan \( \mu^* \in \pi(f^+, f^-) \) solving the Monge-Kantorovich problem 
\[ K_1(\mu^*) = \min \{ K_1(\mu) : \mu \in \pi(f^+, f^-) \} = \sup \{ P(u) : u \in K_1 \} \]
\[ = \int_{\Omega} u^*(x)(f^+(x) - f^-(x)) \, dx = 1 + 2 + 3 + \cdots + L = \frac{L(L + 1)}{2}. \]

Now, if define the measure \( \mu^* \) in \( \Omega \times \Omega \) by 
\[ \mu^*(x, y) := L\chi_{[0,1]}(x) \left( \frac{1}{L} \delta_{y=-1+x} + \frac{1}{L} \delta_{y=-2+x} + \cdots + \frac{1}{L} \delta_{y=-L+x} \right), \]
then \( \mu^* \in \pi(f^+, f^-) \) and, moreover, since 
\[ K_1(\mu^*) = \int_{\Omega \times \Omega} d_1(x, y) \, d\mu^*(x, y) \]
\[ = L \int_0^1 \left( \frac{1}{L} d_1(x, -1 + x) + \frac{1}{L} d_1(x, -2 + x) + \cdots + \frac{1}{L} d_1(x, -L + x) \right) \, dx = \frac{L(L + 1)}{2}, \]
we have that \( \mu^* \) is an optimal plan.

Let us finish this example with the following construction. We consider \( L = 2 \) for simplicity. By Theorem 2.11 we know that there exists \( t_n \in A(f^+, f^-) \) such that 
\[ F_{d_1}(t_n) = \int_{\Omega} d_1(x, t_n(x)) f^+(x) \, dx \xrightarrow{n \to 0^+} 3. \]
Nevertheless we will find an explicit approximation \( t_n \) from which we can infer a different approach for the proof of Theorem 2.11. This \( t_n \) can be constructed following the subsequent ideas. Push \( f^+\chi_{\left[1-1/2^n,1\right]} \) to \( f^-\chi_{\left[-2,-2+1/2^n\right]} \) with a plan induced by a map (see the picture below), paying \( \frac{3}{2^n} \), and \( f^+\chi_{\left[0,1-1/2^n\right]} \) to \( f^-\chi_{\left[-2+1/2^n,0\right]} \) with a plan induced also by a map (see the picture below), paying \( 3 - \frac{2}{2^n} \). Therefore,

\[
F_{d_1}(t_n) = \int_{\Omega} d_1(x, t_n(x)) f^+(x) \, dx = 3 + \frac{1}{2^n} \rightarrow^{n \to 0^+} 3,
\]

and the Monge and Kantorovich problems have the same cost.

Support of \( 2\chi_{[0,1]}(x) \delta_{y=t_n(x)} \) (\( n = 2 \))

Observe that all the segments have slope 2.

3.3. **Optimal transport plans.** Let us construct an optimal transport plan under the assumptions (3.1). This will show, on account of Remark 3.1, how to work in a more general situation.
Let \( u^* = \theta_\alpha \) the Kantorovich potential given by (3.2) and construct a new configuration of equal masses as follows,

\[
\int_{-L}^{L} u^*(x)(f^+(x) - f^-(x)) \, dx = \int_{-1}^{1} u^*(x)(f_0^+(x) - f_0^-(x)) \, dx
\]

(3.5)

For these masses, the same \( u^* \) is a Kantorovich potential. Moreover,

\[
\left\{ \begin{array}{l}
\sup \left\{ y \in \mathbb{R} : \int_{-1+\alpha}^{y} f_0^- = \int_{\alpha}^{x} f_0^+ \right\} & \text{if } x \in (\alpha, \beta), \\
\sup \left\{ y \in \mathbb{R} : \int_{-1}^{y} f_0^- = \int_{\beta}^{x} f_0^+ \right\} & \text{if } x \in (\beta, 1), \\
\sup \left\{ y \in \mathbb{R} : \int_{-1}^{y} f_0^- = \int_{0}^{x} f_0^+ \right\} & \text{if } x \in (0, \alpha).
\end{array} \right.
\]

(3.7)

Consider the smallest of such \( \beta \). Take also the smallest \( \gamma \in [-1, -1+\alpha] \) such that

\[
\int_{\gamma}^{\beta} f_0^+ = \int_{-1}^{0} f_0^-.
\]

(3.6)

And (3.5) follows. By (2.16) there exists \( \beta \in [\alpha, 1] \) such that

\[
\int_{\alpha}^{\beta} f_0^+ = \int_{-1+\alpha}^{0} f_0^-.
\]

For \( x \in (0, 1) \), we define \( T_0 \) by
Observe that $T_0(0) \geq T_0(1) = \gamma$, $T_0(\alpha) = -1 + \alpha$ and

$$-1 \leq \lim_{x \to \beta^+} T_0(x) \leq \lim_{x \to \beta^-} T_0(x) \leq 0.$$ 

Moreover,

$$d_1(x, T_0(x)) = 1 \quad \text{for all} \; x \in [0, \beta],$$

and

$$d_1(x, T_0(x)) = 2 \quad \text{for all} \; x \in (\beta, 1].$$

In fact, as $M(x) = f_0^+(x) - f_0^-(x - 1)$ (see (3.3)), we have, for any $s \in (\alpha, 1)$,

$$\int_\alpha^1 (f_0^+(x) - f_0^-(x - 1)) \, dx \geq \int_s^1 (f_0^+(x) - f_0^-(x - 1)) \, dx;$$

therefore,

$$\int_\alpha^s f_0^+(x) \, dx \geq \int_\alpha^s f_0^-(x - 1) \, dx = \int_{-1+\alpha}^{-1+s} f_0^-(x) \, dx,$$

and consequently,

$$\int_{-1+\alpha}^{T_0(s)} f_0^-(x) \, dx \geq \int_{-1+\alpha}^{-1+s} f_0^-(x) \, dx,$$

which implies $-1 + s \leq T_0(s) < 0$. Similarly, one can obtain the other cases. Then,

$$\mu_{00}(x, y) = f_0^+(x) \delta_{y=T_0(x)}$$

is an optimal transport plan between $f_0^+$ and $f_0^-$ for the cost function $d_1$.

The straight lines are only illustrative.

In fact,

$$\int_{-1}^1 u^\ast(f_0^+ - f_0^-) = \int_{-1}^\gamma 1 \cdot f_0^- + \int_\gamma^{-1+\alpha} 1 \cdot f_0^- + \int_{-1+\alpha}^0 0 \cdot f_0^- + \int_0^\alpha 0 \cdot f_0^+ + \int_\alpha^\beta 1 \cdot f_0^+ + \int_\beta^1 1 \cdot f_0^+.$$
Now, since
\[
\int_{-1}^{1} f_0^+ = \int_{0}^{1} f_0^+ \quad \text{and} \quad \int_{-1}^{1+\alpha} f_0^- = \int_{0}^{\alpha} f_0^+, \]
we have
\[
\int_{-1}^{1} u^*(f_0^+ - f_0^-) = \int_{0}^{\beta} 1 \cdot f_0^+ + \int_{\beta}^{1} 2 \cdot f_0^- = \int_{0}^{1} d_1(x, T_0(x)) f_0^+(x) \, dx.
\]

Also,
\[
\mu_0(x, y) = \sum_{j=0}^{L-1} f^+(x) \chi_{(j, j+1)}(x) \delta_{[y=T_0(x-j)]}
\]
is an optimal transport plan between \( f^+ \) and \( f_0^- \) for the cost function \( d_1 \). In fact,
\[
\int_{-1}^{L} u^*(f^+ - f_0^-) = \int_{0}^{1} u^*(f_0^+ - f_0^-) + \sum_{j=0}^{L-1} \int_{0}^{1} j f^+(x+j) \, dx
\]
\[
= \int_{0}^{1} d_1(x, T_0(x)) f_0^+(x) \, dx + \sum_{j=0}^{L-1} \int_{0}^{1} j f^+(x+j) \, dx
\]
\[
= \sum_{j=0}^{L-1} \int_{0}^{1} d_1(x, T_0(x)) f^+(x+j) \, dx + \sum_{j=0}^{L-1} \int_{0}^{1} j f^+(x+j) \, dx
\]
\[
= \sum_{j=0}^{L-1} \int_{0}^{1} (d_1(x, T_0(x)) + j) f^+(x+j) \, dx
\]
\[
= \sum_{j=0}^{L-1} \int_{j}^{j+1} (d_1(x-j, T_0(x-j)) + j) f^+(x) \, dx
\]
\[
= \sum_{j=0}^{L-1} \int_{j}^{j+1} d_1(x, T_0(x-j)) f^+(x) \, dx = \int_{\Omega \times \Omega} d_1(x, y) \mu_0(x, y).
\]

A remarkable observation is that these \( \mu_{00} \) and \( \mu_0 \) are induced by optimal transport maps.

Now, by splitting the mass
\[
f^+(x) \chi_{(j, j+1)}(x) = \sum_{i=0}^{L-1} g_{i,j}(x), \quad j = 0, 1, \ldots, L-1,
\]
is such a way that, for \( i = 0, 1, \ldots, L-1, \)
\[
\sum_{j=0}^{L-1} \int_{j}^{x+j} g_{i,j} = \int_{\gamma_{-i}}^{T_0(x)-i} f^- \quad \text{if} \ x \in (0, \beta),
\]
and
\[
\sum_{j=0}^{L-1} \int_{\beta+j}^{x+j} g_{i,j} = \int_{-1-i}^{T_0(x)-i} f^- \quad \text{if} \ x \in (\beta, 1),
\]
we have that
\[
\mu(x, y) = \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} g_{i,j} \chi_{(j,j+1)}(x) \delta_{y=-i+T_{0}(x-j)}
\]
is a transport plan between \(f^+\) and \(f^-\) for the cost function \(d_{1}\). Let us see that \(\mu\) is an optimal transport plan. In fact, taking \(x = \beta\) in (3.9), and \(x = 1\) in (3.10), respectively, we get
\[
\sum_{j=0}^{L-1} \int_{\beta+j}^{\gamma+i} g_{i,j} = \int_{\gamma-i}^{\beta-i} f^- \quad \text{and} \quad \sum_{j=0}^{L-1} \int_{\beta+j}^{\gamma+i} g_{i,j} = \int_{\gamma-i}^{\beta-i} f^-.
\]
Adding the last two equalities, we obtain
\[
\sum_{j=0}^{L-1} \int_{j}^{1+j} g_{i,j}(x) \, dx = \int_{-1-i}^{-i} f^- (x) \, dx = \int_{-1}^{0} f^- (x-i) \, dx.
\]
Hence,
\[
\int_{-L}^{L} u^*(f^+ - f^-) = \int \int d_{1}(x, y) \mu_{0}(x, y) + \sum_{j=0}^{L-1} \int_{-1}^{0} j f^- (x-j) \, dx
\]
\[
= \sum_{j=0}^{L-1} \int_{j}^{j+1} d_{1}(x, T_{0}(x-j)) f^+ (x) + \sum_{j=0}^{L-1} i \int_{-1}^{0} f^- (x-i) \, dx
\]
\[
= \sum_{j=0}^{L-1} \int_{j}^{j+1} d_{1}(x, T_{0}(x-j)) \left( \sum_{i=0}^{L-1} g_{i,j}(x) \right) \, dx + \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \int_{j}^{j+1} g_{i,j}(x) \, dx
\]
\[
= \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \int_{j}^{j+1} (d_{1}(x, T_{0}(x-j)) - i) g_{i,j}(x) \, dx
\]
\[
= \sum_{i=0}^{L-1} \sum_{j=0}^{L-1} \int_{-1}^{L-1} d_{1}(x, -i + T_{0}(x-j)) g_{i,j}(x) \, dx = \int_{\Omega \times \Omega} d_{1}(x, y) \mu(x, y).
\]
In the following example, \(\mu(x, y) = f^+(x) \delta_{[y=T_{1}(x)]}\) draws the above construction.

**Example 3.3.** Set \(f^- = \chi_{[-1/4, 0]}\) and \(f^+ = \chi_{[0, 1]}\). Then \(M = -\frac{1}{4} \chi_{[0, 1]} + \frac{3}{4} \chi_{[\frac{1}{2}, 1]}\) and therefore
\[
u^*(x) = \begin{cases} 
0 & \text{if } -\frac{5}{4} < x < -\frac{1}{4}, \\
1 & \text{if } -\frac{1}{4} < x < \frac{3}{4}, \\
2 & \text{if } \frac{3}{4} < x < \frac{7}{4}, \\
3 & \text{if } \frac{7}{4} < x < \frac{11}{4}, \\
\vdots & \text{otherwise}.
\end{cases}
\]
is (up to adding a constant) the unique Kantorovich potential associated with the metric $d_1$ for $f^-$ and $f^+$, and

$$
\int u^*(f^+ - f^-) = \frac{11}{16}
$$

Nevertheless, there exist infinitely many optimal transport maps. For example, the following two are optimal transport maps,

$$
T_1^*(x) = \begin{cases} 
4x - \frac{29}{4} & \text{if } \frac{28}{16} < x < \frac{29}{16}, \\
4x - \frac{33}{4} & \text{if } \frac{29}{16} < x < 2, \\
x & \text{otherwise},
\end{cases}
$$

$$
T_2^*(x) = \begin{cases} 
4x - \frac{29}{4} & \text{if } \frac{28}{16} < x < \frac{57}{32}, \\
-4x + \frac{57}{8} & \text{if } \frac{57}{32} < x < \frac{29}{16}, \\
-4x + 7 & \text{if } \frac{29}{16} < x < 2, \\
x & \text{otherwise}.
\end{cases}
$$

Observe that both push the mass $f^+\chi_{\left]-\frac{1}{4},\frac{1}{4}\right]}$ toward $f^-\chi_{\left]-\frac{1}{4},\frac{1}{4}\right]}$ paying, after 2 steps, $2 \times \frac{1}{16}$, and push the rest from $f^+\chi_{\left]\frac{29}{16},\frac{57}{32}\right]}$ toward $f^-\chi_{\left]-\frac{1}{4},0\right]}$ paying, after 3 steps, $3 \times \frac{3}{16}$. Therefore the total cost is, as known,

$$
2 \times \frac{1}{16} + 3 \times \frac{3}{16} = \frac{11}{16}.
$$
Support of \( f^+(x)\delta_{y=T_0^*(x)} \)

\[
\begin{align*}
\text{Remark 3.4.} & \quad \text{Observe that the unique non-decreasing optimal transport map, } T_0, \text{ for the Euclidean distance as cost function that pushes } f^+ \text{ forward to } f^-, \text{ given by (3.4), in this particular case is } T_0(x) = 4x - 8. \text{ Now, } T_0 \text{ is not an optimal transport map for } d_1, \text{ the transport cost with this map is, in fact, } \frac{12}{16}. \text{ However, it is well known (see [3]) that if the cost function } c(x, y) \text{ is equal to } \phi(|x - y|) \text{ with } \phi \text{ non-decreasing and convex then } T_0 \text{ is an optimal transport, but in our situation } \phi \text{ fails to be convex.}

\text{On the other hand, the following transport plan, not induced by a map, between } f^+ \text{ and } f^- \text{ is very simple and it is optimal: } \\
\mu = \chi_{(\frac{3}{4}, 2)}(x) \left( \frac{1}{4} \delta_{|y-x-2|} + \frac{1}{4} \delta_{|y-x+2|} + \frac{1}{4} \delta_{|y-x-2|} + \frac{1}{4} \delta_{|y-x-11|} \right).
\]

\text{Example 3.5.} \quad \text{Let } f^+ = L\chi_{[0,1]} \text{ and } f^- = \chi_{[-L,0]} \text{ with } L \notin \mathbb{N}. \text{ Let us see that there is optimal transport map } T \text{ with distance } d_1 \text{ pushing } f^+ \text{ to } f^- \text{ (compare this example with Example 3.2).}

\text{In order to simplify the exposition we take } 2 < L < 3. \text{ This particular case shows clearly how to handle the general case with any other } L.

\text{Using the procedure introduced in Subsection 3.3 we have that } (\alpha = 1 = \beta \text{ and } \gamma = -1)

\[
T_0(x) = \begin{cases} 
\frac{L}{2}x - 1 & \text{if } 0 < x < \frac{2(3-L)}{L}, \\
\frac{L}{3}(x - 1) & \text{if } \frac{2(3-L)}{L} < x < 1,
\end{cases}
\]
is an optimal transport map pushing $f_0^+$ to $f_0^-$. 

Now, in order to obtain an optimal transport map $T$ pushing $f^+$ to $f^-$, we do the splitting procedure (3.8) (there are many different ways) in the following adequate way. For $x < \frac{2(3-L)}{L}$ we have to distribute the mass $f^+$ in two *equiweighted* parts, so, set the rectangles with corner coordinates,

upper-left, $ul_i = (x_{i+1}, y_i)$,
upper-right, $ur_i = (x_i, y_i)$,
lower-left, $ll_i = (x_{i+1}, y_{i+1})$,
lower-right, $lr_i = (x_i, y_{i+1})$,
i = 1, 2, \ldots ,

where

\[
x_1 = \frac{2(3-L)}{L}, \quad y_1 = 2 - L,
\]
\[
y_{i+1} = x_i - 1, \quad x_{i+1} = x_i - \frac{2}{L}(y_i - y_{i+1}) = \frac{2}{L}(y_{i+1} + 1).
\]

Observe that $lr_i \in [y = x - 1]$ and $ll_i, ur_i \in [y = \frac{L}{2}x - 1]$. In each rectangle we can trace 2 parallel segments of slope $L$ defined by the lines

\[
y = L(x - x_i) + y_i,
\]
and

\[
y = L(x - \hat{x}_i) + y_i, \quad \hat{x}_i = x_i - \frac{x_i - x_{i+1}}{2}.
\]

Then

\[
T_i(x) = f^+(x)\chi_{[x_{i+1}, x_i]}(x)\delta_{y = L(x - x_i) + y_i} + f^+(x)\chi_{[x_i, x_{i+1}]}(x)\delta_{y = L(x - \hat{x}_i) + y_i - 1}
\]
push in an optimal way $f^+\chi_{[x_{i+1}, x_i]}$ to $f^-\chi_{[y_i, y_{i+1}]}$, for $i = 1, 2, \ldots$
For $x > \frac{2(3-L)}{L}$ we have to distribute the mass $f^+$ in three equiweighted parts, so, set the rectangles with corner coordinates,

lower-left, $ll_i = (x_i, y_i), \quad$ lower-right, $lr_i = (x_{i+1}, y_i), \quad$ upper-left, $ul_i = (x_i, y_{i+1}), \quad$ upper-right, $ur_i = (x_{i+1}, y_{i+1}), i = 1, 2, \ldots$, where now

$$x_1 = \frac{2(3-L)}{L}, \quad y_1 = 2 - L,$$

$$x_{i+1} = y_i + 1, \quad y_{i+1} = y_i + \frac{L}{3}(x_{i+1} - x_i) = \frac{L}{3}(x_{i+1} - 1).$$

Observe that $lr_i \in [y = x - 1]$ and $ll_i, ur_i \in [y = \frac{L}{3}(x - 1)]$. In each rectangle we can trace 3 parallel segments of slope $L$ defined by the lines

$$y = L(x - x_i) + y_i, \quad y = L(x - \hat{x}_i) + y_i, \quad \hat{x}_i = x_i + \frac{x_{i+1} - x_i}{3},$$

and

$$y = L(x - \hat{x}_i) + y_i, \quad \hat{x}_i = x_i + 2\frac{x_{i+1} - x_i}{3}.$$ Then

$$T_i(x) = f^+(x)\chi_{(x_i, \hat{x}_i)}(x)\delta_{y=L(x-x_i)+y_i} + f^+(x)\chi_{(\hat{x}_i, x_{i+1})}(x)\delta_{y=L(x-\hat{x}_i)+y_i-1} + f^+(x)\chi_{(x_i, x_{i+1})}(x)\delta_{y=L(x-x_i)+y_i-2}$$

push in an optimal way $f^+\chi_{(x_i, x_{i+1})}$ to $f^-\chi_{(y_i, y_{i+1})}\cup(y_{i-1}, y_{i+1}-1)\cup(y_{i-2}, y_{i+1}-2)$, for $i = 1, 2, \ldots$

4. APPROXIMATION OF THE CLASSICAL MONGE-KANTOROVICH PROBLEM BY THE NONLOCAL PROBLEMS

Consider the classical Monge problem, with the Euclidean distance $d_{|\cdot|}$ as cost, for two non-negative Borel function $f^+, f^- \in L^2(\Omega)$ satisfying the mass balance condition

$$\int_{\Omega} f^+(x) \, dx = \int_{\Omega} f^-(y) \, dy.$$

To approach an optimal transport map $T^* \in A(f^+, f^-)$ which minimizes the cost functional

$$\mathcal{F}_{d_{|\cdot|}}(T) = \int_{\Omega} |x - T(x)| f^+(x) \, dx.$$

by means of nonlocal optimal transport maps is going to be quite difficult since even in 1—dimension we have proved that there is not optimal transport map for some concrete configurations of the data with respect to the metric $d_\varepsilon$, even if one takes $\varepsilon$ small. However, as we will see below, the Monge infimum for the nonlocal problems converges to the Monge infimum for the Euclidean distance.

On the other hand, we will approach $\mu^*$, an optimal plan, and $u^*$, a Kantorovich potential, for which

$$\mathcal{F}_{d_{|\cdot|}}(T^*) = K_{d_{|\cdot|}}(\mu^*) = \mathcal{P}(u^*),$$
by nonlocal optimal transport plans and nonlocal Kantorovich potentials, respectively. And, moreover, we will see in the one dimensional case that the local optimal transport plan induced by the unique non decreasing local optimal transport map can be approximated by nonlocal optimal transport plans.

First of all, in the following result we state the convergence to the Monge/Kantorovich problems.

**Proposition 4.1.** We have, for the dual problems,

\[
\lim_{\varepsilon \to 0^+} \sup \{ P_{f+} f^- (u) : u \in K_{\varepsilon} \} = \sup \{ P_{f+} f^- (u) : u \in K_{d_{| \cdot |}} \},
\]

or equivalently, for the relaxed problems,

\[
\lim_{\varepsilon \to 0^+} \min \{ K_{\varepsilon} (\mu) : \mu \in \pi (f^+, f^-) \} = \min \{ K_{d_{| \cdot |}} (\mu) : \mu \in \pi (f^+, f^-) \},
\]

and also, for the primal problems,

\[
\lim_{\varepsilon \to 0^+} \inf \{ F_{\varepsilon} (\mu) : \mu \in \pi (f^+, f^-) \} = \inf \{ F_{d_{| \cdot |}} (T) : T \in A(f^+, f^-) \}.
\]

Moreover,

\[
0 \leq \sup \{ P_{f+} f^- (u) : u \in K_{\varepsilon} \} - \sup \{ P_{f+} f^- (u) : u \in K_{d_{| \cdot |}} \} \leq \varepsilon \int_{\Omega} f^+ (x) \, dx
\]

and

\[
0 \leq \min \{ K_{\varepsilon} (\mu) : \mu \in \pi (f^+, f^-) \} - \min \{ K_{d_{| \cdot |}} (\mu) : \mu \in \pi (f^+, f^-) \} \leq \varepsilon \int_{\Omega} f^+ (x) \, dx.
\]

**Proof.** We have (see [5]) that \( K_{d_{| \cdot |}} \subset K_{\varepsilon_1} \subset K_{\varepsilon_2} \), if \( \varepsilon_1 < \varepsilon_2 \). Therefore, for any \( \varepsilon > 0 \),

\[
\sup \{ P_{f+} f^- (u) : u \in K_{\varepsilon} \} \geq \sup \{ P_{f+} f^- (u) : u \in K_{d_{| \cdot |}} \}.
\]

Hence

\[
\lim_{\varepsilon \to 0} \sup \{ P_{f+} f^- (u) : u \in K_{\varepsilon} \} \geq \sup \{ P_{f+} f^- (u) : u \in K_{d_{| \cdot |}} \}.
\]

For a given \( \mu \in \pi (f^+, f^-) \) we have

\[
\lim_{\varepsilon \to 0} \int_{\Omega \times \Omega} d_{\varepsilon} (x, y) \, d\mu (x, y) = \int_{\Omega \times \Omega} d_{| \cdot |} (x, y) \, d\mu (x, y),
\]

that is

\[
\lim_{\varepsilon \to 0} K_{\varepsilon} (\mu) = K_{d_{| \cdot |}} (\mu).
\]

This holds since for any two points \( x, y \) in \( \Omega \) we have

\[
|d_{\varepsilon} (x, y) - |x - y|| \leq \varepsilon, \quad d_{\varepsilon} (x, y) \leq 2|x - y| \quad \text{for } |x - y| \geq \varepsilon
\]

and we have \( \mu \) has compact support.

Hence, if \( \mu^* \) is an optimal plan for \( K_{d_{| \cdot |}} \), we get

\[
\lim_{\varepsilon \to 0} \sup \{ P_{f+} f^- (u) : u \in K_{\varepsilon} \} = \lim_{\varepsilon \to 0} \sup \min \{ K_{\varepsilon} (\mu) : \mu \in \pi (f^+, f^-) \}
\]

\[
\leq \lim_{\varepsilon \to 0} \sup K_{\varepsilon} (\mu^*) = K_{d_{| \cdot |}} (\mu^*) = \sup \{ P_{f+} f^- (u) : u \in K_{d_{| \cdot |}} \}.
\]
Then, by (4.6), we obtain (4.1) and (4.2). On the other hand, we also have

\[
\lim_{\varepsilon \to 0^+} \min \{ K_{\varepsilon}(\mu) : \mu \in \Pi(f^+, f^-) \} = \inf \left\{ F_{d_{|}}(T) : T \in \mathcal{A}(f^+, f^-) \right\}.
\]

Take now \( T' \) a transport map. Thanks to (4.7),

\[
\limsup_{\varepsilon \to 0} \inf \left\{ F_{\varepsilon}(T) : T \in \mathcal{A}(f^+, f^-) \right\} = \limsup_{\varepsilon \to 0} \inf \left\{ \int_{\Omega} d_{\varepsilon}(x, T'(x)) f^+(x) \, dx : T \in \mathcal{A}(f^+, f^-) \right\} 
\]

Therefore,

\[
\limsup_{\varepsilon \to 0} \inf \left\{ F_{\varepsilon}(T) : T \in \mathcal{A}(f^+, f^-) \right\} \leq \inf \left\{ F_{d_{|}}(T) : T \in \mathcal{A}(f^+, f^-) \right\}.
\]

On the other hand, by (1.8),

\[
\min \{ K_{\varepsilon}(\mu) : \mu \in \Pi(f^+, f^-) \} \leq \inf \left\{ F_{\varepsilon}(T) : T \in \mathcal{A}(f^+, f^-) \right\},
\]

taking \( \liminf_{\varepsilon \to 0} \) and taking into account (4.8) and (4.9) we obtain (4.3).

By Kantorovich-Rubinstein's Theorem (Theorem 1.4), to finished the proof is enough to prove (4.5). Since

\[
d_{\varepsilon}(x, y) - \varepsilon \leq d_{|}(x, y) \leq d_{\varepsilon}(x, y),
\]

given \( \mu \in \Pi(f^+, f^-) \), we have

\[
\int_{\Omega \times \Omega} (d_{\varepsilon}(x, y) - \varepsilon) \, d\mu(x, y) \leq \int_{\Omega \times \Omega} d_{|}(x, y) \, d\mu(x, y) \leq \int_{\Omega \times \Omega} d_{\varepsilon}(x, y) \, d\mu(x, y).
\]

Then, taking the minimum over all \( \mu \in \Pi(f^+, f^-) \), and having in mind that

\[
\int_{\Omega \times \Omega} d\mu(x, y) = \int_{\Omega} f^+(x) \, dx,
\]

we obtain (4.5). \( \square \)

**Remark 4.2.** Observe that in the above proof we have not made use of Theorem 2.11. Also note that since \( d_{\varepsilon} \leq d_{\varepsilon}' \), the sequence of costs is nonincreasing as \( \varepsilon \) decreases to zero.

Let us now proceed to approach optimal transport plans. Since \( d_{\varepsilon} \) is lower semi-continuous, by Proposition 1.2 we have there exists an optimal transport plans \( \mu_{\varepsilon} \), that is, \( \mu_{\varepsilon} \in \Pi(f^+, f^-) \) and

\[
K_{\varepsilon}(\mu_{\varepsilon}) = \min \{ K_{\varepsilon}(\mu) : \mu \in \Pi(f^+, f^-) \}.
\]

Let us see that \( \mu_{\varepsilon} \) converge as \( \varepsilon \to 0 \) to an optimal transport plan for the Monge-Kantorovich problem with respect to the euclidean distance.

**Proposition 4.3.** There exists \( \mu^* \in \Pi(f^+, f^-) \), such that, after a subsequence,

\[
\mu_{\varepsilon} \rightharpoonup \mu^* \quad \text{as measures}
\]

and

\[
K_{d_{|}}(\mu^*) = \min \{ K_{d_{|}}(\mu) : \mu \in \Pi(f^+, f^-) \}.
\]
Proof. To prove this we just observe that
\[ d_1(x, y) = |x - y| \leq d_\varepsilon(x, y) \leq |x - y| + \varepsilon. \]

Therefore
\[ d_\varepsilon(x, y) \to |x - y| \quad \text{uniformly as } \varepsilon \to 0. \]

Hence,
\[ \int_{\Omega \times \Omega} |x - y| d\mu_\varepsilon(x, y) \leq \int_{\Omega \times \Omega} d_1(x, y) d\mu_\varepsilon(x, y) \leq \int_{\Omega \times \Omega} (|x - y| + \varepsilon) d\mu_\varepsilon(x, y). \]

On the other hand, by Prokhorov’s Theorem, we can assume that \( \mu_\varepsilon \) converges weakly* in the sense of measures to a limit \( \mu^* \). Therefore, we conclude that
\[ \int_{\Omega \times \Omega} |x - y| d\mu_\varepsilon(x, y) = \lim_{\varepsilon \to 0} \int_{\Omega \times \Omega} d_\varepsilon(x, y) d\mu_\varepsilon(x, y). \]

Finally, by Proposition 4.1 we obtain that \( \mu^* \) is a minimizer for the usual euclidean distance. \( \square \)

Example 4.4. Let \( f^+, f^- \) the masses considered in Example 3.2 for \( L = 2 \). Let us see that for each \( n \in \mathbb{N} \), there is \( \tilde{\mu}_n \in \Pi(f^+, f^-) \), optimal transport plan for the distance \( d_{\frac{1}{2^n}} \), that is,
\[ K_{\frac{1}{2^n}}(\mu_n) = \min \{ K_{\frac{1}{2^n}}(\mu) : \mu \in \Pi(f^+, f^-) \}, \]

such that
\[ \tilde{\mu}_n \rightharpoonup f^+ \boxplus f^- \quad \text{weakly*.} \]

For the above masses the trivial plan \( f^+ \boxplus f^- \) is an optimal transport plan between \( f^+ \) and \( f^- \) for the euclidean distance.

In fact, for \( n \in \mathbb{N} \) and \( m = 0, \ldots, 3 \cdot 2^n - 1 \), we define the functions
\[ s_m(x) = \left( x - \frac{m}{2^n} \right) \chi_{(0, \frac{m}{2^n})}(x), \quad m = 1, \ldots, 2^n, \]
\[ s_m(x) = \left( x - \frac{m}{2^n} \right) \chi_{(0,1)}(x), \quad m = 2^n + 1, \ldots, 2 \cdot 2^n, \]
\[ s_m(x) = \left( x - \frac{m}{2^n} \right) \chi_{(\frac{m-2^n}{2^n},1)}(x), \quad m = 2 \cdot 2^n + 1, \ldots, 3 \cdot 2^n - 1. \]

Using these functions we construct the measure \( \tilde{\mu}_n \in \Pi(f^+, f^-) \), as
\[ \tilde{\mu}_n(x, y) = \frac{1}{2^n} \sum_{m=0}^{3 \cdot 2^n - 1} \delta_{[y = s_m(x)]}. \]
Here is a picture where we schematize the support of $\tilde{\mu}_n$.

Let us see that $\tilde{\mu}_n \rightharpoonup f^+ \boxtimes f^-$, weakly*. If $\varphi \in C_c(\Omega \times \Omega)$, we have

$$\lim_{n \to \infty} \int_{\Omega \times \Omega} \varphi(x, y) \, d\tilde{\mu}_n(x, y) = \lim_{n \to \infty} \frac{1}{2^n} \sum_{m=1}^{3 \cdot 2^n - 1} \int_0^1 \varphi(x, s_m(x)) \, dx$$

$$= \lim_{n \to \infty} \frac{1}{2^n} \sum_{m=1}^{2^n} \int_0^{\frac{m}{2^n}} \varphi \left( x, x - \frac{m}{2^n} \right) \, dx + \lim_{n \to \infty} \frac{1}{2^n} \sum_{m=2^{n+1}}^{2 \cdot 2^n} \int_0^1 \varphi \left( x, x - \frac{m}{2^n} \right) \, dx$$

$$+ \lim_{n \to \infty} \frac{1}{2^n} \sum_{m=2 \cdot 2^n + 1}^{3 \cdot 2^n} \int_{\frac{m-2 \cdot 2^n}{2^n}}^{\frac{m}{2^n}} \varphi \left( x, x - \frac{m}{2^n} \right) \, dx.$$ 

Now, if

$$\phi_1(y) := \int_0^y \varphi(x, x - y) \, dx,$$

using Fubini’s Theorem, we have

$$\lim_{n \to \infty} \frac{1}{2^n} \sum_{m=1}^{2^n} \int_0^{\frac{m}{2^n}} \varphi \left( x, x - \frac{m}{2^n} \right) \, dx = \lim_{n \to \infty} \frac{1}{2^n} \sum_{m=1}^{2^n} \phi_1 \left( \frac{m}{2^n} \right) = \int_0^1 \phi_1(y) \, dy$$

$$= \int_0^1 \left( \int_0^y \varphi(x, x - y) \, dx \right) \, dy = \int_0^1 \left( \int_x^1 \varphi(x, x - y) \, dy \right) \, dx$$

$$= \int_0^1 \left( \int_{x-1}^0 \varphi(x, z) \, dz \right) \, dx.$$ 

Similarly, if

$$\phi_2(y) := \int_0^1 \varphi(x, x - 1 - y) \, dx.$$
we have

\[
\lim_{n \to \infty} \frac{1}{2^n} \sum_{m=2^{n+1}}^{2^{2n}} \int_{0}^{1} \varphi \left( x, x - \frac{m}{2^n} \right) \, dx = \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{2^n} \int_{0}^{1} \varphi \left( x, x - \frac{k}{2^n} \right) \, dx
\]

\[
= \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{2^n} \phi_2 \left( \frac{k}{2^n} \right) = \int_{0}^{1} \phi_2(y) \, dy = \int_{0}^{1} \left( \int_{0}^{1} \varphi(x, x - y) \, dx \right) \, dy
\]

\[
= \int_{0}^{1} \left( \int_{0}^{x} \varphi(x, x - y) \, dy \right) \, dx = \int_{0}^{1} \left( \int_{x}^{x-1} \varphi(x, z) \, dz \right) \, dx.
\]

Finally, when

\[
\phi_3(y) := \int_{y}^{1} \varphi(x, x - y) \, dx,
\]

we obtain

\[
\lim_{n \to \infty} \frac{1}{2^n} \sum_{m=2^{2n+1}}^{2^{3n}} \int_{0}^{1} \varphi \left( x, x - \frac{m}{2^n} \right) \, dx = \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{2^n-1} \int_{0}^{1} \varphi \left( x, x - 2 - \frac{k}{2^n} \right) \, dx
\]

\[
= \lim_{n \to \infty} \frac{1}{2^n} \sum_{k=1}^{2^n-1} \phi_3 \left( \frac{k}{2^n} \right) = \int_{0}^{1} \phi_3(y) \, dy = \int_{0}^{1} \left( \int_{0}^{1} \varphi(x, x - 2 - y) \, dx \right) \, dy
\]

\[
= \int_{0}^{1} \left( \int_{0}^{x} \varphi(x, x - 2 - y) \, dy \right) \, dx = \int_{0}^{1} \left( \int_{-2}^{x-2} \varphi(x, z) \, dz \right) \, dx.
\]

Consequently, we get that

\[
\lim_n \int_{\Omega \times \Omega} \varphi(x, y) \, d\tilde{\mu}_n(x, y)
\]

\[
= \int_{0}^{1} \left( \int_{x}^{x-1} \varphi(x, z) \, dz \right) \, dx + \int_{0}^{1} \left( \int_{-2}^{x-2} \varphi(x, z) \, dz \right) \, dx + \int_{0}^{1} \left( \int_{-2}^{x-2} \varphi(x, z) \, dz \right) \, dx
\]

\[
= \int_{0}^{1} \left( \int_{-2}^{x-2} \varphi(x, z) \, dz \right) \, dx = \int_{\Omega \times \Omega} \varphi(x, y) \, d\tilde{\mu}(x, y),
\]

where \(\tilde{\mu} = f^+ \otimes f^-\).

Let us see now that for each \(n \in \mathbb{N}\), there is \(\mu_n \in \Pi(f^+, f^-)\), optimal transport plan for the distance \(d_{2^n}^\star\), such that \(\mu_n \rightharpoonup \mu = f^+(x)\delta_{[|y=T(x)|]}\) weakly*, where \(T(x) = 2x - 2\) is the unique nondecreasing optimal transport map for the euclidean distance between \(f^+\) and \(f^-\). Set, for \(n \in \mathbb{N}\),

\[
\mu_n(x, y) = \chi_{\left[\frac{2^n-1}{2^{2n}}, 1\right]}(x)\delta_{[y=x-1]} + \sum_{m=1}^{2^n-1} \chi_{\left[\frac{2^n-m}{2^{2n}}, \frac{2^n-m+1}{2^{2n}}\right]}(x)\delta_{[y=x-1-\frac{m}{2^n}]} + \chi_{(0, \frac{1}{2^{2n}}]}(x)\delta_{[y=x-2]}.
\]
Then, for $\varphi \in C_c(\Omega \times \Omega)$, we have

$$
\lim_{n \to \infty} \int_{\Omega \times \Omega} \varphi(x, y) \, d\mu_n(x, y) = \lim_{n \to \infty} \sum_{m=1}^{2^n-1} \int_{\frac{2^n-m-1}{2^n}}^{\frac{2^n-m+1}{2^n}} \varphi \left( x, x - \frac{m}{2^n} \right) \, dx
$$

and, by the continuity of $\varphi$,

$$
\lim_{n \to \infty} \frac{1}{2^n} \sum_{m=1}^{2^n-1} \int_{\frac{2^n-m}{2^n}}^{\frac{2^n}{2^n}} 2\varphi \left( x + 1 - \frac{m}{2^n}, x - \frac{2m}{2^n} \right) \, dx
$$

$$
= \int_0^1 2\varphi(1-z, -2z) \, dz = \int_0^1 2\varphi(x, 2x - 2) \, dx = \int_{\Omega \times \Omega} \varphi(x, y) \, d\mu(x, y).
$$

**Remark 4.5.** The above example illustrates how to design an splitting process (3.8) which allows to obtain $d_\epsilon$-optimal transport plans between $f^+$ and $f^-$ that converge to the optimal transport plan for the euclidean distance induced by the unique optimal transport map pushing $f^+$ to $f^-$, $T_0$, which is nondecreasing. This splitting process should charge the maximum of mass from the right to the left and from the top to the bottom. Let us see how it works for $\epsilon = 1$.

Since $T_0$ is nondecreasing we have that

$$
\int_{-j+T_0(x)}^{-j+T_0(x)} f^- \quad \text{and} \quad \int_{-j}^{-j+T_0(x)} f^-
$$

are functions of bounded variation with positive locally integrable derivatives, and we can chose in the splitting process (3.8) the following functions,
Let us finish this section with a convergence result for Kantorovich potentials.

**Theorem 4.6.** Let \( u^*_\varepsilon \) be a Kantorovich potential associated with the metric \( d_\varepsilon \). Then, after a subsequence, 

\[ u^*_\varepsilon \to u^* \quad \text{in } L^2 \quad \text{as } \varepsilon \to 0, \]

where \( u^* \) is a Kantorovich potential associated with the metric \( d_{\vert_1} \).

**Proof.** It is an obvious fact that \( \{u_\varepsilon\} \) is \( L^\infty \)-bounded, then, there exists a sequence, 

\[ u^*_\varepsilon \to v \quad \text{in } L^2. \]

Therefore, 

\[ \lim_{n \to +\infty} \int_{\Omega} u^*_\varepsilon(x)(f^+(x) - f^-(x)) \, dx = \int_{\Omega} v(x)(f^+(x) - f^-(x)) \, dx. \]

Now, since 

\[ \int_{\Omega} u^*_\varepsilon(x)(f^+(x) - f^-(x)) \, dx = \sup \{ \mathcal{P}_{f^+, f^-}(u) : u \in K_{\varepsilon_1} \}, \]

by Proposition 4.1, we conclude that 

\[ \int_{\Omega} v(x)(f^+(x) - f^-(x)) \, dx = \sup \{ \mathcal{P}_{f^+, f^-}(u) : u \in K_{d_{\vert_1}} \}. \]
In order to have that the limit $v$ is a maximizer $u^*$ we need to show that $v \in K_{d_{|1}|}$, and this follows by the Mosco-convergence of $l\!l_{K_{\nu}}$ to $l\!l_{K_{d_{|1}|}}$ (see [5]).

**Remark 4.7.** For $0 < \varepsilon \leq 1$, let $u^*_\varepsilon$ be a Kantorovich potential for $f = f^+ - f^-$ respect to the distance $d_\varepsilon$, that is, 

$$\mathcal{P}_{f^+,f^-}(u^*_\varepsilon) = \max\{\mathcal{P}_{f^+,f^-}(u) : u \in K_{\varepsilon}\}.$$ 

Let us see what are the relation between $u^*_\varepsilon$ and $u^*_1$ for a fixed $0 < \varepsilon < 1$. Given $u \in K_{\varepsilon}$, we have

$$\int u(x)f\left(\frac{x}{\varepsilon}\right)dx = \varepsilon^N \int u(\varepsilon y)f(y)dy = \varepsilon^{N+1} \int \frac{1}{\varepsilon}u(\varepsilon y)f(y)dy$$

$$\leq \varepsilon^{N+1} \int u_1^*(y)f(y)dy = \int \varepsilon u_1^*\left(\frac{x}{\varepsilon}\right)f\left(\frac{x}{\varepsilon}\right)dx$$

since the map $y \mapsto \frac{1}{\varepsilon}u(\varepsilon y)$ is an element of $K_1$. Therefore, we have shown that if $u^*_1$ is a Kantorovich potential for $f$ respect to the distance $d_1$, then $\varepsilon u_1^*\left(\frac{\cdot}{\varepsilon}\right)$ is a Kantorovich potential for $f\left(\frac{\cdot}{\varepsilon}\right)$ respect to the distance $d_\varepsilon$.

5. The nonlocal version of the Evans-Gangbo approach

Let us consider again the Euler-Lagrange equation

$$(5.1) \quad f^+ - f^- \in \partial l\!l_{K_d(\Omega)}(u),$$

$K_d(\Omega) = \text{Lip}_1(\Omega,d)$, associated with the variational problem

$$(5.2) \quad \max\left\{\int_{\Omega} u(x)(f^+(x) - f^-(x))dx : u \in K_d(\Omega)\right\}.$$ 

Recall that we assume that $\Omega$ is convex. In the particular case of the Euclidean distance $d(x,y) = d_{||1||}(x,y)$ in $\Omega$, and as we have mentioned in the Introduction, Evans and Gangbo, in [11], characterize solutions to this equation by means of a PDE and use this PDE to find a proof of the existence of an optimal transport map for the classical Monge problem. The aim of this section is to perform a similar approach with the step distance $d_\varepsilon$.

5.1. Characterizing the Euler-Lagrange equation. The Euler-Lagrange equation to be considered is

$$(5.3) \quad f^+ - f^- \in \partial l\!l_{K_\varepsilon(\Omega)}(u),$$

where $K_\varepsilon(\Omega) = K_{d_{|\varepsilon|}}(\Omega)$. We characterize $\partial l\!l_{K_\varepsilon}$ by introducing the following operator $B_\varepsilon$ in $L^2(\Omega)$. Let

$$\mathcal{M}_0^\varepsilon(\Omega \times \Omega) := \{\text{bounded antisymmetric Radon measures in } \Omega \times \Omega\}.$$ 

We define the operator $B_\varepsilon$ in $L^2(\Omega)$ as follows, $(u,v) \in B_\varepsilon$ if and only if $u \in K_\varepsilon$, $v \in L^2(\Omega)$, and there exists $\sigma \in \mathcal{M}_0^\varepsilon(\mathbb{R}^N \times \Omega)$ such that

$$\sigma \mathcal{L}\{(x,y) \in \Omega \times \Omega : |x-y| \leq \varepsilon\},$$

$$\int_{\Omega} \int_{\Omega} \xi(x)d\sigma(x,y) = \int_{\Omega} \xi(x)v(x)dx, \quad \forall \xi \in C_c(\Omega),$$

and

$$|\sigma|(\Omega \times \Omega) \leq \frac{2}{\varepsilon} \int_{\Omega} v(x)u(x)dx.$$
Theorem 5.1. We have
\[ \partial I_{K_\varepsilon} = B_\varepsilon. \]

Proof. Let us first see that \( B_\varepsilon \subset \partial I_{K_\varepsilon} \). Let \((u, v) \in B_\varepsilon \), to see that \((u, v) \in \partial I_{K_\varepsilon} \) we need to prove that
\[ 0 \leq \int_\Omega v(x)(u(x) - \xi(x)) \, dx, \quad \forall \xi \in K_\varepsilon. \]
Using an approximation procedure taking a convolution, we can assume that \( \xi \in K_\varepsilon \) is continuous. Then,
\[
\begin{align*}
\int_\Omega v(x)(u(x) - \xi(x)) \, dx &\geq \frac{\varepsilon}{2} |\xi(\Omega \times \Omega) - \int_\Omega v(x)\xi(x) \, dx \\
&= \frac{\varepsilon}{2} |\xi(\Omega \times \Omega) - \int_\Omega \int_\Omega \xi(x) \, d\sigma(x, y) \\
&= \frac{\varepsilon}{2} |\xi(\Omega \times \Omega) - \frac{1}{2} \int_\Omega \int_\Omega (\xi(x) - \xi(y)) \, d\sigma(x, y) \geq 0,
\end{align*}
\]
where in the last equality we have used the antisymmetry of \( \sigma \). Therefore, we have \( B_\varepsilon \subset \partial I_{K_\varepsilon} \).

Since \( \partial I_{K_\varepsilon} \) is a maximal monotone operator, to see that the operators are equal we only need to show that for every \( f \in L^2(\Omega) \) there exists \( u \in K_\varepsilon \) such that
\[ u + B_\varepsilon(u) = f. \]
Let \( J : \mathbb{R}^N \to \mathbb{R} \) as in (1.2) and
\[ J_\varepsilon(z) = \frac{1}{\varepsilon^N} J\left(\frac{z}{\varepsilon}\right). \]
By results in [5], given \( p > N \) and \( f \in L^2(\Omega) \) there exists a unique solution \( u_p \in L^\infty(\Omega) \) of the nonlocal \( p \)-Laplacian problem
\[ u_p(x) + \int_\Omega J_\varepsilon(x - y) \left| \frac{u_p(y) - u_p(x)}{\varepsilon} \right|^{p-2} (u_p(y) - u_p(x)) \, dy = T_p(f)(x) \quad \forall x \in \Omega, \]
where \( T_k(r) := \max\{\min\{k, r\}, -r\} \). And we also know that there exists \( u \in K_\varepsilon \) such that
\[ u_p \to u \quad \text{in} \quad L^2(\Omega) \quad \text{as} \quad p \to +\infty, \]
with
\[ u + \partial I_{K_\varepsilon}(u) \ni f, \]
from where it follows that
\[ \int_\Omega (f(x) - u(x))(w(x) - u(x)) \, dx \leq 0, \quad \forall w \in K_\varepsilon, \]
and consequently, \( u = P_{K_\varepsilon}(f) \).

Multiplying (5.6) by \( u_p \) and integrating, we get
\[ \int_\Omega (T_p(f)(x) - u_p(x)) u_p(x) \, dx = \frac{1}{2\varepsilon^{p-2}} \int_{\Omega \times \Omega} J_\varepsilon(x - y) \left| u_p(y) - u_p(x) \right|^p \, dx dy, \]
from where it follows that
\[ \frac{1}{\varepsilon^{p-2}} \int_{\Omega \times \Omega} J_\varepsilon(x - y) \left| u_p(y) - u_p(x) \right|^p \, dx dy + \int_\Omega |u_p(x)|^2 \, dx \leq \|f\|_{L^2(\Omega)}^2. \]
If we set
\[ \sigma_p(x, y, \varepsilon) := \frac{1}{\varepsilon^{p-2}} J_\varepsilon(x - y) \left| u_p(y) - u_p(x) \right|^{p-2} \left( u_p(y) - u_p(x) \right), \]
by Hölder’s inequality,
\[ \int_{\Omega \times \Omega} |\sigma_p(x, y, \varepsilon)| \, dxdy = \frac{1}{\varepsilon^{p-2}} \int_{\Omega \times \Omega} J_\varepsilon(x - y) \left| u_p(y) - u_p(x) \right|^{p-1} \, dxdy \leq \frac{1}{\varepsilon^{p-2}} \left( \int_{\Omega \times \Omega} J_\varepsilon(x - y) \, dxdy \right)^{\frac{p-1}{p}} \left( \int_{\Omega \times \Omega} J_\varepsilon(x - y) \, dxdy \right)^{\frac{1}{p}}. \]
Now, by (5.9), we have
\[ \int_{\Omega \times \Omega} |\sigma_p(x, y, \varepsilon)| \, dxdy \leq \left( \| f \|_{L^2(\Omega)}^2 \right)^{\frac{p-1}{p}} \varepsilon^{\frac{2-p}{p}}. \]
Hence, \( \{\sigma_p : p \geq 2\} \) is bounded in \( L^1(\Omega \times \Omega) \), and consequently we can assume that
\[ \sigma_p(\cdot, \cdot, \varepsilon) \rightharpoonup \pi_\varepsilon =: \sigma \quad \text{weakly}^* \quad \text{in} \quad M_b(\Omega \times \Omega). \]
Obviously, since each \( \sigma_p \) is antisymmetric, \( \sigma \in M_b^a(\Omega \times \Omega) \). Moreover, since \( \text{supp}(J_\varepsilon) = B_\varepsilon(0) \), we have
\[ \sigma| \{ (x, y) \in \Omega \times \Omega : |x - y| \leq \varepsilon \}. \]
On the other hand, given \( \xi \in C_c(\Omega) \), by (5.6), (5.7) and (5.10), we get
\[ \int_{\Omega} \int_{\Omega} \xi(x) \, d\sigma(x, y) = \lim_{p \to +\infty} \int_{\Omega} \int_{\Omega} \xi(x) \sigma_p(x, y, \varepsilon) \, dx \, dy = \lim_{p \to +\infty} \frac{1}{\varepsilon^{p-2}} \int_{\Omega} \int_{\Omega} J_\varepsilon(x - y) \left| u_p(y) - u_p(x) \right|^{p-2} \left( u_p(y) - u_p(x) \right) \xi(x) \, dx \, dy = \lim_{p \to +\infty} \int_{\Omega} (T_p(f)(x) - u_p(x)) \xi(x) \, dx = \int_{\Omega} (f(x) - u(x)) \xi(x) \, dx. \]
Then, to prove (5.5), we only need to show that
\[ |\sigma|(\Omega \times \Omega) \leq \frac{2}{\varepsilon} \int_{\Omega} (f(x) - u(x))u(x) \, dx. \]
In fact, by (5.10), we have
\[ |\sigma|(\Omega \times \Omega) \leq \liminf_{p \to +\infty} \int_{\Omega} \int_{\Omega} |\sigma_p(x, y)| \, dx \, dy. \]
Now, by (5.8),
\[
\int_{\Omega \times \Omega} |\sigma_p(x,y)| \, dx \, dy \leq \frac{1}{\varepsilon^{p-2}} \left( \int_{\Omega \times \Omega} J_p(x - y) |u_p(y) - u_p(x)|^p \, dxdy \right)^{\frac{p-1}{p}}
\]
\[
= \frac{1}{\varepsilon^{p-2}} \left( 2\varepsilon^{p-2} \int_{\Omega} (T_p(f)(x) - u_p(x))u_p(x) \, dx \right)^{\frac{p-1}{p}}
\]
\[
= 2^{\frac{p-1}{r}} \varepsilon^{\frac{2}{r}} \left( \int_{\Omega} (T_p(f)(x) - u_p(x))u_p(x) \, dx \right)^{\frac{p-1}{p}}.
\]
Therefore,
\[
|\sigma|_{\Omega \times \Omega} \leq \frac{2}{\varepsilon^2} \int_{\Omega} (f(x) - u(x))u(x) \, dx.
\]
This ends the proof. \(\square\)

We can rewrite the operator \(B_\varepsilon\) as follows.

**Corollary 5.2.** \((u,v) \in B_\varepsilon\) if and only if \(u \in K_\varepsilon\), \(v \in L^2(\Omega)\), and there exists \(\sigma \in \mathcal{M}_b^\varepsilon(\Omega \times \Omega)\) such that

\[
\sigma^+ \mathbb{L}\{(x,y) \in \Omega \times \Omega : |x - y| \leq \varepsilon, u(x) - u(y) = \varepsilon\},
\]
\[
\sigma^- \mathbb{L}\{(x,y) \in \Omega \times \Omega : |x - y| \leq \varepsilon, u(y) - u(x) = \varepsilon\},
\]

\[
\int_{\Omega} \int_{\Omega} \xi(x) d\sigma(x,y) = \int_{\Omega} \xi(x)v(x) \, dx, \quad \forall \xi \in C_c(\Omega),
\]

and

\[
|\sigma|_{\Omega \times \Omega} = \frac{2}{\varepsilon^2} \int_{\Omega} v(x)u(x) \, dx.
\]

**Proof.** Let \((u,v) \in B_\varepsilon\), then

\[(5.11) \quad \int_{\Omega} \int_{\Omega} \xi(x) d\sigma(x,y) = \int_{\Omega} \xi(x)v(x) \, dx, \quad \forall \xi \in C_c(\Omega).
\]

Hence,

\[
\int_{\Omega} d\sigma(x,y) = v(x).
\]

Therefore, by approximation, we can take \(\xi \in L^2(\Omega)\) in (5.11) and \(\int_{\Omega} \xi(x) d\sigma(x,y)\) has this sense. Taking \(\xi = u\) in (5.11) and using the antisymmetric of \(\sigma\) and the previous result we get

\[
|\sigma|_{\Omega \times \Omega} \geq \int_{\Omega} \int_{\Omega} \frac{(u(x) - u(y))}{\varepsilon} d\sigma(x,y)
\]
\[
= \frac{2}{\varepsilon} \int_{\Omega} \int_{\Omega} u(x) d\sigma(x,y) = \frac{2}{\varepsilon} \int_{\Omega} u(x) v(x) \, dx \geq |\sigma|_{\Omega \times \Omega},
\]
from where the result follows. \(\square\)
As a consequence of the above result, we have that the Kantorovich potentials \( u^*_\varepsilon \) associated with the metric \( d_\varepsilon \), that is, the functions \( u^*_\varepsilon \in K_\varepsilon \) such that
\[
\int_\Omega u^*_\varepsilon(x)(f^+(x) - f^-(x)) \, dx = \max \left\{ \int_\Omega u(x)(f^+(x) - f^-(x)) \, dx : u \in K_\varepsilon \right\}
\]
are the solutions of the nonlocal equation
\[
f^+ - f^- \in B_\varepsilon(u).
\]
In other words, the functions \( u^*_\varepsilon \in K_\varepsilon \), for which there exists \( \sigma^*_\varepsilon \in M^*_b(\Omega \times \Omega) \), such that
\[
[\sigma^*_\varepsilon]^+_\varepsilon \mathcal{L}_\varepsilon \{ (x, y) \in \Omega \times \Omega : u^*_\varepsilon(x) - u^*_\varepsilon(y) = \varepsilon, \, |x - y| \leq \varepsilon \},
\]
\[
[\sigma^*_\varepsilon]^-_\varepsilon \mathcal{L}_\varepsilon \{ (x, y) \in \Omega \times \Omega : u^*_\varepsilon(y) - u^*_\varepsilon(x) = \varepsilon, \, |x - y| \leq \varepsilon \},
\]
satisfying
\[
\int_\Omega d\sigma^*_\varepsilon(x, y) = f^+(x) - f^-(x), \quad x \in \Omega.
\]
and
\[
|\sigma^*_\varepsilon|(\Omega \times \Omega) = \frac{2}{\varepsilon} \int_\Omega (f^+(x) - f^-(x))u^*_\varepsilon(x) \, dx,
\]
that is,
\[
|\sigma^*_\varepsilon|(\Omega \times \Omega) = \frac{2}{\varepsilon} \mathcal{P}(u^*_\varepsilon).
\]

We want to highlight that (5.13) plays the role of (1.14). Moreover we will see in the next subsection that we can construct optimal transport plans from it, more precisely, we shall see that the potential \( u^*_\varepsilon \) and the measure \( \sigma^*_\varepsilon \) encode all the information that we need to construct an optimal transport plan associated with the problem.

5.2. Constructing optimal transport plans. Let us begin by constructing \( \sigma^* \) for some concrete masses \( f^+ \) and \( f^- \) and trying to obtain from it and the potential, information to construct an optimal transport plan between them.

Example 5.3. In the Example 3.3, for the masses \( f^+ = \chi_{[7/4,2]} \) and \( f^- = \frac{1}{4} \chi_{[-1,0]} \), we have seen that
\[
\inf \{ \mathcal{F}_1(T) : T \in \mathcal{A}(f^+, f^-) \} = \min \{ K_1(\mu) : \, \mu \in \pi(f^+, f^-) \}
\]
\[
= \sup \{ \mathcal{P}_{f^+, f^-}(u) : \, u \in K_1 \} = \frac{11}{16},
\]
is attained at
\[
u^*(x) = \begin{cases} 
0 & \text{if } -\frac{5}{4} < x < -\frac{1}{4}, \\
1 & \text{if } -\frac{1}{4} < x < \frac{3}{4}, \\
2 & \text{if } \frac{3}{4} < x < \frac{7}{4}, \\
\vdots & \text{elsewhere}.
\end{cases}
\]
For these masses, we have that the following antisymmetric measure satisfies the above conditions (5.14), (5.15) and (5.16),

\[
\sigma^*(x, y) = \chi_{(7/4, 2)}(x)\delta_{[y=x-1]} - \chi_{(3/4, 1)}(x)\delta_{[y=x+1]}
\]

\[+ \frac{1}{4}\chi_{(\frac{3}{4}, 1)}(x) \left( \delta_{[y=x-1]} + \delta_{[y=x-\frac{1}{2}]} + \delta_{[y=x-\frac{1}{4}]} \right)
\]

\[ - \frac{1}{4}\chi_{(-\frac{1}{4}, -\frac{1}{4})}(x)\delta_{[y=x-1]} - \frac{1}{4}\chi_{(0, \frac{1}{4})}(x)\delta_{[y=x+\frac{1}{4}]} - \frac{1}{4}\chi_{(\frac{1}{4}, \frac{1}{4})}(x)\delta_{[y=x+\frac{3}{4}]} - \frac{1}{4}\chi_{(\frac{1}{4}, \frac{1}{4})}(x)\delta_{[y=x+\frac{3}{4}]}
\]

\[+ \frac{1}{4}\chi_{(0, \frac{1}{4})}(x)\delta_{[y=x-1]} - \frac{1}{4}\chi_{(-1, -\frac{1}{4})}(x)\delta_{[y=x+1]}.
\]

Let us point out that also the antisymmetric measure

\[
\nu^*(x, y) = \chi_{(7/4, 2)}(x)\delta_{[y=x-1]} - \chi_{(3/4, 1)}(x)\delta_{[y=x+1]}
\]

\[ - \frac{1}{4}\chi_{(-1, -\frac{1}{4})}(x)\delta_{[y=x+1]} + \frac{1}{4}\chi_{(0, 1)}(x)\delta_{[y=x-1]}
\]

\[+ \chi_{(3/4, 1)}(x)\chi_{(0, 3/4)}(y) - \chi_{(0, 3/4)}(x)\chi_{(3/4, 1)}(y)
\]

satisfies the above conditions.

We try now to recover the optimal transport plan \(\mu^*\) given by

\[
\mu^*(x, y) = \chi_{(7/4, 2)}(x) \left( \frac{1}{4}\delta_{[y=x-2]} + \frac{1}{4}\delta_{[y=x-\frac{3}{2}]} + \frac{1}{4}\delta_{[y=x-\frac{3}{4}]} + \frac{1}{4}\delta_{[y=x-\frac{1}{4}]} \right)
\]

from the measure \(\sigma^*(x, y)\). To do that we proceed as follows. Let \(\mu_3\) be the restriction of \([\sigma^*]^+\) to the set where \(u^*(x) = 3\), that is, \(\mu_3(x, y) = \chi_{(7/4, 2)}(x)\delta_{[y=x-1]}\). Then, given \(\varphi, \psi \in C_c(\mathbb{R})\), we have

\[
\int_{\mathbb{R} \times \mathbb{R}} (\varphi(x) + \psi(y)) \, d\mu_3(x, y) = \int_{7/4}^2 (\varphi(x) + \psi(x - 1)) \, dx = \int_{7/4}^2 \varphi(x) \, dx + \int_{3/4}^1 \psi(y) \, dy.
\]

Therefore

\[\mu_3 \in \pi \left( f^+, \chi_{(3/4, 1)} \right)\]

Moreover, since

\[\chi_{(7/4, 2)}(x) - \chi_{(3/4, 1)}(y) = 1 = d_1(x, y) \quad \mu_3 - a.e.,\]

by (2.7), we have \(\mu_3\) is an optimal transport plan.

Consider now \(\mu_2\) the restriction of \([\sigma^*]^+\) to the set where \(u^*(x) = 2\), that is,

\[
\mu_2(x, y) = \chi_{(\frac{3}{4}, 1)}(x) \left( \frac{1}{4}\delta_{[y=x-1]} + \frac{1}{4}\delta_{[y=x-\frac{1}{2}]} + \frac{1}{4}\delta_{[y=x-\frac{1}{4}]} + \frac{1}{4}\delta_{[y=x-\frac{1}{4}]} \right).
\]

Then, for \(\varphi, \psi \in C_c(\mathbb{R})\), we have

\[
\int_{\mathbb{R} \times \mathbb{R}} (\varphi(x) + \psi(y)) \, d\mu_2(x, y) = \int_{3/4}^1 \varphi(x) \, dx + \frac{1}{4}\int_{-1/4}^{3/4} \psi(y) \, dy.
\]

Therefore

\[\mu_2 \in \pi \left( \chi_{(3/4, 1)}, \frac{1}{4}\chi_{(-1/4, 3/4)} \right),\]

and again by (2.7), we have \(\mu_2\) is an optimal transport plan.
Now, we split the mass $\frac{1}{4} \chi_{(-1/4,3/4)}$ as follows
\[
\frac{1}{4} \chi_{(-1/4,3/4)} = \frac{1}{4} \chi_{(-1/4,0)} + \frac{1}{4} \chi_{(0,3/4)} = f^- \chi_{(-1/4,0)} + \frac{1}{4} \chi_{(0,3/4)}.
\]
Let $\mu_1$ be the restriction of $[\sigma^+]^*$ to the set where $u^*(x) = 1$, that is, $\mu_1 = \frac{1}{4} \chi_{(0,3/4)}(x) \delta_{y=x-1}$. It is easy to see that
\[
\mu_1 \in \pi \left( \frac{1}{4} \chi_{(0,3/4)}, f^- \chi_{(-1,-1/4)} \right),
\]
and that it is an optimal transport plan.

Let us see how to glue together these transport plans to find an optimal transport plan between $f^+$ and $f^-$. We will first perform the above construction in general and then we will glue in the general situation.

**A general construction of optimal transport plans.**

Given $f^+, f^- \in L^\infty(\Omega)$ two non-negative Borel functions satisfying the mass balance condition (1.4) and such that $\text{supp}(f^+) \cap \text{supp}(f^-) = 0$, by Theorems 1.4 and 1.6, there exists a Kantorovich potential $u^*$ such that $u^*(\Omega) \subset \mathbb{Z}$, taking a finite number of values, and
\[
\min\{\mathcal{K}_{d_1}(\mu) : \mu \in \pi(f^+, f^-)\} = \mathcal{K}_{d_1}(u^*) = \int_\Omega u^*(x)(f^+(x) - f^-(x)) \, dx.
\]
Now, $u^*$ verifies the Euler-Lagrange equation
\[
f^+ - f^- \in \partial \mathcal{K}_{d_1}(u^*).
\]
Then, by Theorem 5.1, there exists $\sigma \in \mathcal{M}_b^+(\Omega \times \Omega)$ such that
\[
\begin{cases}
\sigma^+ \mathbb{L}\{ (x, y) \in \Omega \times \Omega : |x - y| \leq 1, \ u^*(x) - u^*(y) = 1 \}, \\
\sigma^- \mathbb{L}\{ (x, y) \in \Omega \times \Omega : |x - y| \leq 1, \ u^*(y) - u^*(x) = 1 \}, \\
\int_\Omega \int_\Omega \xi(x) d\sigma(x, y) = \int_\Omega \int_\Omega \xi(x)(f^+(x) - f^-(x)) \, dx \quad \forall \xi \in C_c(\Omega),
\end{cases}
\]
and
\[
|\sigma|(\Omega \times \Omega) = 2 \int_\Omega u^*(x)(f^+(x) - f^-(x)) \, dx = 2 \min\{\mathcal{K}_{d_1}(\mu) : \mu \in \pi(f^+, f^-)\}.
\]
We are going to give a method to obtain an optimal transport plan $\mu^*$ from the measure $\sigma$. To do that we shall use the Gluing Lemma (see Lemma 7.6 in [19]), which permits to glue together two transport plans in an adequate way. As it is remarked in [19], it is possible to state the following result, that we present for the distance $d_1$.

**Lemma 5.4.** Let $f_1, f_2, g$ be three positive measures in $\Omega$. If $\mu_1 \in \pi(f_1, g)$ and $\mu_2 \in \pi(g, f_2)$, there exists a measure $\mathcal{G}(\mu_1, \mu_2) \in \pi(f_1, f_2)$ such that
\[
\mathcal{K}_{d_1}(\mathcal{G}(\mu_1, \mu_2)) \leq \mathcal{K}_{d_1}(\mu_1) + \mathcal{K}_{d_1}(\mu_2).
\]
We divide the construction in two steps. We assume without loss of generality that

\[ u^* = 0\chi_{A_0} + 1\chi_{A_1} + \cdots + k\chi_{A_k}, \]

with \( A_i = \{ x \in \Omega : u^*(x) = i \} \).

**Step 1. How the measures \( \sigma^+ \mathcal{L}(A_j \times A_{j-1}) \) work.**

Taking into account the antisymmetry of \( \sigma \) and (5.18), we have that

\[ \text{proj}_x(\sigma^+) - \text{proj}_y(\sigma^+) = f^+ - f^-, \]

which implies

\[ g := \text{proj}_x(\sigma^+) - f^+ = \text{proj}_y(\sigma^+) - f^- . \]

By (2.15), \( \text{proj}_x(\sigma^+) \mathcal{L} A_k = f^+ \chi_{A_k} \) and \( \text{proj}_x(\sigma^+) \mathcal{L} A_0 = f^+ \chi_{A_0} = 0 \), then

\[ g \mathcal{L} A_k = g \mathcal{L} A_0 = 0 . \]

Moreover we have

\[ \text{proj}_x(\sigma^+) \mathcal{L} (A_j \times A_{j-1})) = \text{proj}_x(\sigma^+) \mathcal{L} A_j \quad \text{and} \quad \text{proj}_y(\sigma^+) \mathcal{L} (A_j \times A_{j-1})) = \text{proj}_x(\sigma^+) \mathcal{L} A_{j-1} , \]

then

\[ \text{proj}_x(\sigma^+) \mathcal{L} (A_j \times A_{j-1})) = f^+ \chi_{A_j} + g \mathcal{L} A_j \]

and

\[ \text{proj}_y(\sigma^+) \mathcal{L} (A_j \times A_{j-1})) = f^- \chi_{A_{j-1}} + g \mathcal{L} A_{j-1} . \]

Let us call

\[ \mu_j := \sigma^+ \mathcal{L} (A_j \times A_{j-1}) . \]

Let us briefly comment what these measures do. The first one, \( \mu_k \), transports \( f^+ \chi_{A_k} \) into \( f^- \chi_{A_{k-1}} \) plus something else, that is \( g \mathcal{L} A_{k-1} \). Afterwards, \( \mu_j \) transports \( f^+ \chi_{A_j} + g \mathcal{L} A_j \) into \( f^- \chi_{A_{j-1}} \) again plus something else, that is \( g \mathcal{L} A_{j-1} \). The last one, \( \mu_1 \), transports \( f^+ \chi_{A_1} + g \mathcal{L} A_1 \) to \( f^- \chi_{A_0} \).

**Step 2. The Gluing.**

Now we would like to glue this transportations, and, in order to apply the Gluing Lemma, we consider the measures

\[ \mu_k^l(x, y) := \mu_k(x, y) + f^+(x)\chi_{A_{k-1}}(x)\delta_{[x=\sigma]} , \]

and

\[ \mu_k^{r-1}(x, y) := \mu_{k-1}(x, y) + f^-(x)\chi_{A_{k-1}}(x)\delta_{[x=\sigma]} . \]

It is easy to see that

\[ \mu_k^l \in \pi(f^+ \chi_{A_k} + f^+ \chi_{A_{k-1}}; f^- \chi_{A_{k-1}} + \text{proj}_x(\sigma^+) \mathcal{L} A_{k-1}) \]

and

\[ \mu_{k-1}^l \in \pi(f^- \chi_{A_{k-1}} + \text{proj}_x(\sigma^+) \mathcal{L} A_{k-1}, f^- \chi_{A_{k-1}} + f^- \chi_{A_{k-2}} + g \mathcal{L} A_{k-2}) . \]

Therefore, by the Gluing Lemma,

\[ G(\mu_k^l, \mu_k^{r-1}) \in \pi(f^+ \chi_{A_k} + f^+ \chi_{A_{k-1}}; f^- \chi_{A_{k-1}} + f^- \chi_{A_{k-2}} + g \mathcal{L} A_{k-2}) . \]

Let us now consider the measures

\[ \mu_k^{l-1}(x, y) := G(\mu_k^l, \mu_k^{r-1})(x, y) + f^+(x)\chi_{A_{k-2}}(x)\delta_{[x=\sigma]} \]

and

\[ \mu_k^{r-2}(x, y) := \mu_{k-2}(x, y) + (f^-(x)\chi_{A_{k-2}}(x) + f^-(x)\chi_{A_{k-2}}(x))\delta_{[x=\sigma]} . \]
Then we have
\[ \mu^l_{k-1} \in \pi(f^+X^k_1 + f^+X^k_{-1} + f^+X^k_{-2} + f^+X^k_{-3} + \mu X^k_{-3}) \]
and
\[ \mu^r_{k-2} \in \pi(f^-X^k_1 + f^-X^k_{-1} + f^-X^k_{-2} + f^-X^k_{-3} + \mu X^k_{-3}) \]
Consequently,
\[ \mathcal{G}(\mu^l_{k-1}, \mu^r_{k-2}) \in \pi(f^+X^k_1 + f^+X^k_{-1} + f^+X^k_{-2} + f^+X^k_{-3} + g \mathbb{L} A_k) \]
Proceeding in this way we arrive to the construction of
\[ \mu_2(x, y) = \mathcal{G}(\mu_2^l, \mu_2^r)(x, y) + f^+(x)X^0_1(x)\delta_{y=x}, \]
\[ \mu_1(x, y) = \mu_1(x, y) + \sum_{i=1}^{k-1} f^-(x)X^0_i(x)\delta_{y=x} \]
and
\[ \mu^* = \mathcal{G}(\mu_2^l, \mu_1^r) \in \pi(f^+, f^-), \]
which is, in fact, an optimal transport plan since, by (5.19),
\[ K_{d_1}(\mu^*) \leq K_{d_1}(\mu_2^l) + K_{d_1}(\mu_1^r) \leq K_{d_1}(\mathcal{G}(\mu_3^l, \mu_2^r)) + K_{d_1}(\mu_1) \]
\[ \leq K_{d_1}(\mu_3^l) + K_{d_1}(\mu_2^r) + K_{d_1}(\mu_1^r) \]
\[ \leq K_{d_1}(\mu_3^l) + \sum_{j=1}^{k-1} K_{d_1}(\mu_j) = \sum_{j=1}^{k} K_{d_1}(\mu_j). \]
Hence,
\[ K_{d_1}(\mu^*) \leq \sum_{j=1}^{k} K_{d_1}(\mu_k) = \sum_{j=1}^{k} \int_{\Omega} \int_{\Omega} d\sigma^+(A_j \times A_{j-1}) = \int_{\Omega} \int_{\Omega} d\sigma^+ \]
\[ = \frac{1}{2} |\sigma|((\Omega \times \Omega) = \min\{K_{d_1}(\mu) : \mu \in \pi(f^+, f^-)\}. \]
We want to remark that the a similar construction works for any Kantorovich potential \( u^* \),
without assuming that \( u^*(\Omega) \subset \mathbb{Z} \). We have presented the above one for simplicity.

With this general gluing construction it is easy to find an optimal measure in Example 5.3.
Let us present another example.

**Example 5.5.** Let us see how the above method works in the particular case of Example 3.2, for
which we know that, for the masses \( f^+ = 2\chi_{[0,1]} \) and \( f^- = \chi_{[-2,0]} \), there is no optimal transport
map pushing \( f^+ \) to \( f^- \) for the cost function \( d_1 \). A Kantorovich potential for this configuration
of masses is given by
\[ u^*(x) = \begin{cases} 
2, & x \in (0,1), \\
1, & x \in (-1,0), \\
0, & x \in (-2, -1).
\end{cases} \]
For these masses, we have that the following antisymmetric measure satisfies the above conditions (5.14), (5.15) and (5.16),

\[ \sigma^*(x, y) = 2\chi_{(0,1)}(x)\delta_{[y=x-1]} - 2\chi_{(-1,0)}(x)\delta_{[y=x+1]} - \chi_{(-2,-1)}(x)\delta_{[y=x+1]} + \chi_{(-1,0)}(x)\delta_{[y=x-1]}. \]

In this case, \( \mu_2 \), the restriction of \( [\sigma^*]^+ \) to the set \((0, 1) \times (-1, 0)\), is

\[ \mu_2(x, y) = 2\chi_{(0,1)}(x)\delta_{[y=x-1]} \in \pi \left( 2\chi_{(0,1)}, f^-\chi_{(-1,0)} + \chi_{(-1,0)} \right), \]

and \( \mu_1 \), the restriction of \( [\sigma^*]^+ \) to the set \((-1, 0) \times (-2, -1)\), is

\[ \mu_1(x, y) = \chi_{(-1,0)}(x)\delta_{[y=x-1]} \in \pi \left( \chi_{(-1,0)}, -\chi_{(-2,-1)} \right). \]

Therefore, for \( \mu_1^2 = \mu_2 \) and \( \mu_1^1 = \mu_1 + f^-\chi_{(-1,0)} \), we have that

\[ \mu = \mathcal{G}(\mu_1^2, \mu_1^1) \]

is an optimal transport plan between \( f^+ \) and \( f^- \).

5.3. Construction of the transport density for the classical MK-problem by rescaling.

Under the hypothesis of Theorem 1.5 we know that there exists a transport density \( 0 \leq a \in L^\infty(\Omega) \) such that

\[
\begin{align*}
-\text{div}(a\nabla u) &= f^+ - f^- \quad \text{in } D'(\Omega) \\
|\nabla u| &= 1 \quad \text{a.e. on the set } \{a > 0\},
\end{align*}
\]

(5.20)

where \( u \) is a Kantorovich potential associated with the Euclidean distance \( d_{1\cdot} \).

By Theorem 4.6, if \( u^*_\varepsilon \) is a Kantorovich potential associated with the metric \( d_\varepsilon \), then, after a subsequence,

\[ u^*_\varepsilon \rightharpoonup u^* \quad \text{in } L^2(\Omega) \text{ as } \varepsilon \to 0, \]

(5.21)

where \( u^* \) is a Kantorovich potential associated with the metric \( d_{1\cdot} \).

In this subsection we shall see that also it is possible to get the transport density \( a \) by rescaling, to be more precise from the measures \( \sigma^*_\varepsilon \) associated with the Euler-Lagrange equation (5.13).

Let us fix

\[ \Omega' \subset \subset \Omega'' \subset \subset \Omega \]

(5.22)

be such that \( |x - y| > r = \text{diam}(\text{supp}(f^+ - f^-)) \) for any \( x \in \text{supp}(f^+ - f^-) \) and any \( y \in \Omega \setminus \Omega' \).

By Theorem 5.1 we have that there exists \( \sigma^*_\varepsilon \in \mathcal{M}^1_b(\Omega \times \Omega) \) satisfying (5.14), (5.15) and (5.16).

Hence,

\[ \int_{\Omega} \xi(x)(f^+(x) - f^-(x)) \, dx = \int_{\Omega} \int_{\Omega} \xi(x) \, d\sigma^*_\varepsilon(x, y), \quad \forall \, \xi \in C_c(\Omega). \]

(5.23)

If we take \( \xi \in C_c^1(\Omega) \), from (5.23) and taking into account the antisymmetry of \( \sigma^*_\varepsilon \), we have that

\[ \int_{\Omega} \xi(x)(f^+(x) - f^-(x)) \, dx = \int_{\Omega} \int_{\Omega} \xi(x) \, d\sigma^*_\varepsilon(x, y) = \int_{\Omega} \int_{\Omega} \frac{\xi(x) - \xi(y)}{\varepsilon} \, d\left( \frac{\varepsilon}{2} \sigma^*_\varepsilon(x, y) \right), \]

and

\[ \int_{\Omega} \xi(x)(f^+(x) - f^-(x)) \, dx = \int_{\Omega} \int_{\Omega} \xi(x) \, d\sigma^*_\varepsilon(x, y) = \int_{\Omega} \int_{\Omega} \frac{\xi(x) - \xi(y)}{\varepsilon} \, d(\varepsilon[\sigma^*_\varepsilon]^+(x, y)). \]

(5.24)

(5.25)
Now observe that for \( \varphi \in C_c(\Omega \times \Omega) \), if \( \phi(x, z) = \varphi(x, x + \varepsilon z) \) and \( T_\varepsilon(x, y) = \frac{y-x}{\varepsilon} \), then
\[
\int \int \varphi(x, y) \, d[\sigma_\varepsilon^*]^+(x, y) = \int \int \phi((\pi_1, T_\varepsilon)(x, y)) \, d[\sigma_\varepsilon^*]^+(x, y)
\]
\[
= \int \int \phi(x, z) \, d((\pi_1, T_\varepsilon)\#[\sigma_\varepsilon^*]^+)(x, z) = \int \int \varphi(x, x + \varepsilon z) \, d((\pi_1, T_\varepsilon)\#[\sigma_\varepsilon^*]^+)(x, z).
\]
Therefore, we can rewrite (5.25) as
\[
\int \Omega \xi(x)(f^+(x) - f^-(x)) \, dx = \int \int \frac{\xi(x) - \xi(x + \varepsilon z)}{\varepsilon} \, \xi((\pi_1, T_\varepsilon)\#[\varepsilon \sigma_\varepsilon^*]^+)(x, z).
\]
On the other hand, by (5.16),
\[
|((\pi_1, T_\varepsilon)\#[\varepsilon \sigma_\varepsilon^*]^+)| \text{ is bounded by a constant independent of } \varepsilon.
\]
Therefore there exists a subsequence \( \varepsilon_n \to 0 \), such that
\[
\mu_n := ((\pi_1, T_{\varepsilon_n})\#[\varepsilon_n \sigma_{\varepsilon_n}^*]^+) \to \sigma_+ \text{ weakly as measures.}
\]
Then, taking limit in (5.26), for \( \varepsilon = \varepsilon_n \), as \( n \) goes to infinity, we obtain
\[
\int \Omega \xi(x)(f^+(x) - f^-(x)) \, dx = \int \int \nabla \xi(x) \cdot (-z) \, ds_+(x, z).
\]
Since
\[
[\varepsilon_n \sigma_{\varepsilon_n}^*]^+ \cup \{(x, y) \in \Omega \times \Omega : u_{\varepsilon_n}^*(x) - u_{\varepsilon_n}^*(y) = \varepsilon_n, |x - y| \leq \varepsilon_n\},
\]
and \( (\pi_1, T_\varepsilon) \) is one to one and continuous, we have
\[
\mu_n \cup (\pi_1, T_{\varepsilon_n}) \left( \{ (x, y) \in \Omega \times \Omega : u_{\varepsilon_n}^*(x) - u_{\varepsilon_n}^*(y) = \varepsilon_n, |x - y| \leq \varepsilon_n \} \right)
\]
that is,
\[
\mu_n \cup \{(x, z) : x \in \Omega, x + \varepsilon_n z \in \Omega, |z| \leq 1, u_{\varepsilon_n}^*(x) - u_{\varepsilon_n}^*(x + \varepsilon_n z) = \varepsilon_n\},
\]
and consequently
\[
\sigma_+ \cup \{(x, z) : x \in \Omega, |z| \leq 1\}.
\]
Now, by disintegration of the measure \( \sigma_+ \) (see [2]),
\[
\sigma_+ = (\sigma_+) \otimes \mu,
\]
with
\[
\mu = \pi_1 \# \sigma_+,
\]
that is a nonnegative measure. Moreover, if we define
\[
\nu(x) := \int_{\mathbb{R}^N} (-z) \, d(\sigma_+) \otimes \mu(z), \quad x \in \Omega,
\]
then, \( \nu \in L^1_\mu(\Omega, \mathbb{R}^N) \) and we can rewrite (5.28) as
\[
\int \Omega \xi(x)(f^+(x) - f^-(x)) \, dx = \int \Omega \nabla \xi(x) \cdot \nu(x) \, d\mu(x), \quad \forall \xi \in C^1_c(\Omega).
\]
Let us see that
\[
(5.30) \quad \text{supp}(\mu) \subset \subset \Omega.
\]
The proof of (5.30) follows the argument of [1, Lemma 5.1] (we include this argument here for the sake of completeness). In fact, let \( x_0 \in \text{supp}(f^+ - f^-) \) be a minimum point for the restriction of \( u^* \) to \( \text{supp}(f^+ - f^-) \) and define

\[
w(x) := \min \{(u^*(x) - u^*(x_0))^+, \text{dist}(x, \Omega \setminus \Omega')\},
\]

where \( \Omega' \) verifies (5.22). Then, \( w(x) = u^*(x) - u^*(x_0) \) on \( \text{supp}(f^+ - f^-) \) and \( w \equiv 0 \) on \( \Omega \setminus \Omega' \). On the other hand,

\[
\mu(\Omega) = \sigma_+ (\Omega \times \mathbb{R}^N) \leq \liminf_{\varepsilon \to 0} \mu_\varepsilon (\Omega \times \mathbb{R}^N) \leq \liminf_{\varepsilon \to 0} \varepsilon [\sigma^*_\varepsilon]_+ (\Omega \times \mathbb{R}^N)
\]

(5.31)

\[
= \liminf_{\varepsilon \to 0} \int_{\Omega} u^*_\varepsilon(x) (f^+(x) - f^-(x)) \, dx = \int_{\Omega} u^*(x) (f^+(x) - f^-(x)) \, dx.
\]

and, for a regularizing sequence \( \rho_{\frac{1}{n}} \), on account of (5.29) and using that \( |\nu(x)| \leq 1 \), we have

\[
\int_{\Omega} u^*(x) (f^+(x) - f^-(x)) \, dx = \lim_{n} \int_{\Omega} (u^*(x) - u^*(x_0)) (f^+(x) - f^-(x)) \, dx
\]

\[
= \lim_{n} \int_{\Omega} (u^*(x) - u^*(x_0)) (f^+(x) - f^-(x)) \, dx = \lim_{n} \int_{\Omega} \nabla (w * \rho_{\frac{1}{n}})(x) \cdot \nu(x) \, d\mu(x) \leq \mu(\Omega''),
\]

where \( \Omega'' \) verifies (5.22). So, \( \mu(\Omega \setminus \Omega'') = 0 \), and (5.30) is satisfied.

Let us now recall some tangential calculus for measures (see [7], [8]). We introduce the tangent space \( T_\mu \) to the measure \( \mu \) which is defined \( \mu \)-a.e. by setting \( T_\mu(x) := N^1_\mu(x) \) where:

\[
N_\mu(x) = \{ \xi \in L^\infty(\Omega, \mathbb{R}^N) : \exists u_n \text{ smooth, } u_n \to 0 \text{ uniformly, } \nabla u_n \rightharpoonup \xi \text{ weakly}^* \text{ in } L^\infty \}.
\]

In [7], given \( u \in D(\Omega) \), for \( \mu \)-a.e. \( x \in \Omega \), the tangential derivative \( \nabla_\mu u(x) \) is defined as the projection of \( \nabla u(x) \) on \( T_\mu(x) \). Now, by [8, Proposition 3.2], there is an extension of the linear operator \( \nabla_\mu \) to \( \text{Lip}_1(\Omega, d_{1,i}) \) the set of Lipschitz continuous functions.

Let us see that

\[
(5.32) \quad \nu(x) \in T_\mu(x) \quad \mu \text{-a.e. } x \in \Omega.
\]

For that we need to show that

\[
(5.33) \quad \int_{\Omega} \nu(x) \cdot \xi(x) \, d\mu(x) = 0, \quad \forall \xi \in N_\mu.
\]

In fact, given \( \xi \in N_\mu \), there exists \( u_n \) smooth, \( u_n \to 0 \) uniformly, \( \nabla u_n \rightharpoonup \xi \text{ weakly}^* \text{ in } L^\infty \). Then, taking \( \xi = u_n \) in (5.29), which is possible on account of (5.30), we obtain

\[
\int_{\Omega} u_n(x)(f^+(x) - f^-(x)) \, dx = \int_{\Omega} \nabla u_n(x) \cdot \nu(x) \, d\mu(x),
\]

from here, taking limit as \( n \to +\infty \), we get

\[
\int_{\Omega} \nu(x)\xi(x) \cdot \nu(x) \, d\mu(x) = 0, \quad \forall v \in D(\Omega),
\]

from where (5.33) follows.

Now, if we set \( \Phi := \nu \mu \), by (5.29) we have

\[
(5.34) \quad -\text{div}(\Phi) = f^+ - f^- \text{ in } D'(\Omega).
\]
Then, having in mind (5.32), by [8, Proposition 3.5], we get

\[(5.35) \quad \int_{\Omega} u^*(x)(f^+(x) - f^-(x)) \, dx = \int_{\Omega} \nu(x) \nabla u^*(x) \, d\mu(x),\]

where \(\nabla u^*\) is the tangential derivative. Then, since \(|\nu(x)| \leq 1\) and \(|\nabla u^*(x)| \leq 1\) for \(\mu\)-a.e \(x \in \Omega\), from (5.35) and (5.31), we obtain that

\[\nu(x) = \nabla u^*(x) \quad \text{and} \quad |\nabla u^*(x)| = 1 \quad \mu\text{-a.e } x \in \Omega.\]

Therefore, we have

\[(5.36) \quad \begin{cases} -\text{div}(\mu \nabla u^*) = f^+ - f^- & \text{in } D'(\Omega), \\ |\nabla u^*(x)| = 1 & \mu\text{-a.e } x \in \Omega. \end{cases}\]

Now, by the regularity results given in [12] (see also [1] and [13]), since \(f^+, f^- \in L^\infty(\Omega)\), we have that the transport density \(\mu \in L^\infty(\Omega)\).

Consequently we conclude that the density transport of Evans-Gangbo is represented by

\[a = \pi_1 \#\sigma_+,\]

for any \(\sigma_+\) obtained as in (5.27).

**Example 5.6.** By Example 3.2, we know that there is not optimal transport map between \(f^+ = 2\chi_{[0,1]}\) and \(f^- = \chi_{[-2,0]}\) for the cost function \(d_\varepsilon\), \(\varepsilon = 1/2^n\), \(n = 1, 2, ...,\) and that

\[\inf\{\mathcal{F}_\varepsilon(T) : T \in \mathcal{A}(f^+, f^-)\} = \min\{\mathcal{K}_\varepsilon(\mu) : \mu \in \pi(f^+, f^-)\}\]

\[= \sup\{\mathcal{P}_{f^+, f^-}(u) : u \in K_\varepsilon\} = 3,\]

being the supremum attained at \(u_\varepsilon^*(x) = \sum_j \varepsilon_j \chi_{(\varepsilon_j, \varepsilon_j + \varepsilon]}(x)\). Also we have that the following antisymmetric measure satisfies the above conditions (5.14), (5.15) and (5.16),

\[\sigma_+^\varepsilon(x, y) = \sum_{i=1}^{2^n} 2(2^n - i + 1)\chi_{(\frac{i-1}{2^n} + \varepsilon, \frac{i}{2^n}]}(x)\delta_{y = x + \frac{1}{2^n}} - \sum_{i=1}^{2^n} 2(2^n - i + 1)\chi_{(\frac{i-2}{2^n} - \varepsilon, \frac{i}{2^n}]}(x)\delta_{y = x + \frac{1}{2^n}} \]

\[+ \sum_{i=1}^{2^{2^n-1}} (2 \cdot 2^n - i)\chi_{(\frac{i-1}{2^{2^n-1}}, \frac{i}{2^{2^n-1}})}(x)\delta_{y = x + \frac{1}{2^{2^n-1}}} - \sum_{i=1}^{2^{2^n-1}} (2 \cdot 2^n - i)\chi_{(\frac{i-2}{2^{2^n-1}} - \varepsilon, \frac{i}{2^{2^n-1}})}(x)\delta_{y = x + \frac{1}{2^{2^n-1}}}.\]
Then, for $\xi \in C^1_c((-2, 1))$ we have
\[
\int \int \xi(x) d \left( \sigma_{1/2^n}(x, y) \right)
= \sum_{i=1}^{2^n} 2(2^n - i + 1) \int \int (\xi(x) - \xi(y)) \chi_{(\frac{i-1}{2^n}, \frac{i}{2^n})}(x) \delta_{[y=x-\frac{1}{2^n}]}
+ \sum_{i=1}^{2^n - 1} (2 \cdot 2^n - i) \int \int (\xi(x) - \xi(y)) \chi_{(\frac{i}{2^n}, \frac{i+1}{2^n})}(x) \delta_{[y=x-\frac{1}{2^n}]}
= \sum_{i=1}^{2^n} 2(2^n - i + 1) \int \int (\xi(x) - \xi(y)) \chi_{(\frac{i}{2^n}, \frac{i+1}{2^n})}(x) \delta_{[y=x-\frac{1}{2^n}]}
+ \sum_{i=1}^{2^n - 1} (2 \cdot 2^n - i) \int \int (\xi(x) - \xi(y)) \chi_{(\frac{i-1}{2^n}, \frac{i}{2^n})}(x) \delta_{[y=x-\frac{1}{2^n}]},
\]
Hence
\[
\lim_{n \to \infty} \int \int \xi(x) d \left( \sigma_{1/2^n}(x, y) \right) = \int_0^1 2(1-x)\xi'(x)dx - \int_0^1 4(1 - x)\xi'(-2x)dx
= \int_0^1 \xi'(x)a(x)dx = \int_{-2}^1 \xi'(x)a(x)u^*(x)dx,
\]
where
\[
a(x) = \begin{cases} 
2 + x & \text{if } -2 < x < 0, \\
2 - 2x & \text{if } 0 < x < 1,
\end{cases}
\]
and
\[
u^*(x) = x.
\]
Observe that for $\xi \in C^1_c((-2, 1))$,
\[
\int_{-2}^1 \xi'(x)a(x)u^*(x)dx = 2 \int_{-2}^1 \xi(x) dx - \int_{-2}^0 \xi(x) dx = \int_{-2}^1 \xi(x)(f'(x) - f'(x)) dx.
\]

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