THE LIMIT AS $p \to +\infty$ OF THE FIRST EIGENVALUE FOR THE $p$-LAPLACIAN WITH MIXED DIRICHLET AND ROBIN BOUNDARY CONDITIONS.

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Abstract. We analyze the behavior as $p \to \infty$ of the first eigenvalue of the $p$–Laplacian with mixed boundary conditions of Dirichlet-Robin type. We find a nontrivial limit that we associate to a variational principle involving $L^\infty$-norms. Moreover, we provide a geometrical characterization of the limit value as well as a description of it using optimal mass transportation techniques. Our results interpolate between the pure Dirichlet case and the mixed Dirichlet-Neumann case.

1. Introduction and description of the main results

Let $U \subset \mathbb{R}^n$ be a smooth, bounded, open and connected set. In order to consider mixed boundary conditions, we split the boundary of $U$ as $\partial U = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_1| > 0$. In this paper we deal with the first eigenvalue, that we will call $\lambda_p$, of the $p$-Laplacian with Dirichlet condition on $\Gamma_1$ and Robin condition on $\Gamma_2$ namely the smallest $\lambda$ such that there is a nontrivial solution to the following problem,

$$
\begin{cases}
-\Delta_p u = \lambda |u|^{p-2}u & \text{in } U, \\
u = 0 & \text{on } \Gamma_1, \\
|\nabla u|^{p-2} \partial_\nu u + \alpha |u|^{p-2}u = 0 & \text{on } \Gamma_2.
\end{cases}
$$

Here $\alpha$ is a non-negative parameter. Notice that when $\alpha = +\infty$, the boundary condition become $u = 0$ in all $\partial U$ (a pure Dirichlet condition) and when $\alpha = 0$ we have a mixed Dirichlet-Neumann boundary condition.

Our main goal is to compute the limit as $p \to \infty$ of this problem and look at its dependence on the parameter $\alpha$.

To start our analysis we remark that $\lambda_p$ has the following variational formulation:

$$
\lambda_p = \inf_{u \in \mathcal{X}_p} \left\{ \int_U |\nabla u|^p + \alpha^p \int_{\Gamma_2} |u|^p : u \in W^{1,p}_{\Gamma_1}(U), \|u\|_{L^p(U)} = 1 \right\}
$$

where $W^{1,p}_{\Gamma_1}(U) = \left\{ u \in W^{1,p}(U), u = 0 \text{ on } \Gamma_1 \right\}$.

Note that the infimum is attained since we assumed that $|\Gamma_1| > 0$. Also notice that if we regard $\lambda_p$ as a function of $\alpha$, $\alpha \in [0, +\infty) \to \lambda_p(\alpha)$, then $\lambda_p(\alpha)$ is non-decreasing with $\lim_{\alpha \to +\infty} \lambda_p = \lambda_{p,D}$ the first Dirichlet eigenvalue for the $p$–Laplacian in $U$. 

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We expect the limit problem of (2) as $p \to \infty$ to be
\[
\lambda_\infty = \inf_{u \in W^{1,\infty}_0(U), \|u\|_{L^\infty(\Gamma)} = 1, u \geq 0} \max \left\{ \|\nabla u\|_{L^\infty(U)}, \|u\|_{L^\infty(\Gamma_2)} \right\}
\]
where $W^{1,\infty}_0(U) = \left\{ u \in W^{1,\infty}(U), u = 0 \text{ on } \Gamma_1 \right\}$. Notice that when we let $\alpha \to +\infty$ in (3) with $\Gamma_1 = \partial U$ we obtain
\[
\lim_{\alpha \to +\infty} \lambda_\infty(\alpha) = \lambda_{\infty,D} = \inf_{u \in W^{1,\infty}_0(\Gamma_1), \|u\|_{L^\infty(\Gamma)} = 1} \|\nabla u\|_{L^\infty(U)}
\]
that is the first eigenvalue of the infinity Laplacian, $\Delta_\infty u = DuD^2uDu$ with Dirichlet boundary conditions. This value, $\lambda_{\infty,D}$, turns out to be the limit of $(\lambda_{p,D})^{1/p}$ as $p \to \infty$, see [14]. Our first result says that this kind of limit can be also computed for any nonnegative $\alpha$.

**Theorem 1.** There holds that
\[
\lim_{p \to +\infty} (\lambda_p)^{1/p} = \lambda_\infty.
\]
Moreover the positive, normalized extremals for $\lambda_p$, $u_p$ converge uniformly in $\bar{U}$ along subsequences $p_j \to \infty$ to $u \in X$ which is a minimizer for (3) and a viscosity solution to
\[
\begin{cases}
\min \{ |Du| - \lambda_\infty u, -\Delta_\infty u\} = 0 & \text{in } U, \\
u = 0 & \text{on } \Gamma_1, \\
\min\{ |Du| - \alpha u, -\partial_{\nu} u\} = 0 & \text{on } \Gamma_2.
\end{cases}
\]

Our next goal is to characterize this limit value $\lambda_\infty$. The value of $\lambda_\infty$ results of the interplay between $\alpha$, the geometry of $U$ and the sets $\Gamma_1, \Gamma_2$. We consider the (possibly empty) set
\[
\mathcal{A} := \left\{ x \in \bar{U}, \, d(x, \Gamma_1) \geq \frac{1}{\alpha} + d(x, \Gamma_2) \right\}.
\]
Notice that if $\mathcal{A} = \emptyset$ then the set
\[
\mathcal{A}' := \left\{ x \in \bar{U}, \, d(x, \Gamma_1) = \frac{1}{\alpha} + d(x, \Gamma_2) \right\}
\]
is also not empty. Indeed the function $f(x) = \frac{1}{\alpha} + d(x, \Gamma_2) - d(x, \Gamma_1)$ is continuous, less or equal to 0 on $\mathcal{A}$, and greater or equal to 0 if $d(x, \Gamma_1) << 1$ (we are using here the fact that $U$ is connected to apply the mean value theorem). Our next result gives a geometrical characterization of $\lambda_\infty$.

**Theorem 2.** It holds that
\[
\lambda_\infty = \begin{cases}
\min_{x \in \mathcal{A}} \frac{1}{\frac{1}{\alpha} + d(x, \Gamma_2)}, & \text{if } \mathcal{A} = \emptyset, \\
\min_{x \in \mathcal{A}'} \frac{1}{\frac{1}{\alpha} + d(x, \Gamma_2)}, & \text{if } \mathcal{A} \neq \emptyset.
\end{cases}
\]

Notice that when $\alpha = +\infty$, which corresponds to pure Dirichlet boundary conditions on the whole $\partial U$, then $\mathcal{A} = \mathcal{A}' = \emptyset$ and we recover the result of [14], $\lambda^{-1}_\infty = \lambda^{-1}_{\infty,D} = \max_{x \in \bar{U}} d(x, \partial U)$. In the case of Neumann boundary conditions i.e. $\Gamma_1 = \emptyset$ and $\alpha = 0$ then $\mathcal{A} = \emptyset$ and $d(x, \Gamma_1) = d(x, \emptyset) = +\infty$ for any $x \in \bar{U}$ so that $\lambda_\infty = 0$ which is consistent with the fact the 1st eigenvalue of $\Delta_p$ with Neumann boundary conditions is 0.
We will first give a simple proof in the case where $U$ is convex by using a test-function argument based proof which we were not able to extend to the general case. In fact the result for an arbitrary connected domain will be a consequence of an optimal mass transport formulation of $\lambda_\infty$ that we now introduce.

To continue our analysis we have to recall some notions and notations from optimal mass transport theory. Recall that the Monge-Kantorovich distance $W_1(\mu, \nu)$ between two probability measures $\mu$ and $\nu$ over $\bar{U}$ is defined by

\[ W_1(\mu, \nu) = \max_{v \in W^{1, \infty}(U), \|\nabla v\|_\infty \leq 1} \int_U v (d\mu - d\nu). \]

Recently the authors in [7] relate $\lambda_\infty, D$ with the Monge-Kantorovich distance $W_1$. They proved that

\[ \lambda_\infty^{-1, D} = \max_{\mu \in P(U)} W_1(\mu, P(\partial U)), \]

where $P(U)$ and $P(\partial U)$ denotes the set of probability measures over $\bar{U}$ and $\partial U$. Notice that the maximum is easily seen to be reached at $\delta_x$ where $x \in U$ is a most inner point.

In our case we are also able to give a characterization for $\lambda_\infty$ in terms of a maximization problem involving $W_1$ but this time we get an extra term involving the total variation of a measure on $\Gamma_2$.

**Theorem 3.** It holds that

\[ \frac{1}{\lambda_\infty} = \max_{\sigma \in P(\bar{U})} \inf_{\nu \in P(\partial U)} \left\{ W_1(\sigma, \nu) + \frac{1}{\alpha} \nu(\Gamma_2) \right\}. \]

Moreover, the measures $\nu_p^{p-1} dx$ weakly converge (up to a subsequence) as $p \to +\infty$ to a probability measure $f_\infty$ which attains the maximum in (7).

Notice that when $\alpha = +\infty$, which corresponds to Dirichlet boundary conditions, then we recover the result of [7], who showed that (6) holds.

As a corollary of this characterization in terms of optimal transportation, we can extend the result stated in Theorem 2 for the value of $\lambda_\infty$ to the case where $U$ is not convex. We prefer to present our results in this order (even if Theorem 2 is not initially proved in its full generality) for readability of the whole paper (the proof of Theorem 2 in the convex case is much simpler).

Let us end the introduction with a brief description of the previous bibliography and the main ideas and techniques used to prove our results. First, as by now classical results, we mention that the limit as $p \to \infty$ of the first eigenvalue $\lambda_{p,D}$ of the $p$-Laplacian with Dirichlet boundary condition was studied in [15], [14] (see also [4] for an anisotropic version). For its dependence with respect to the domain we refer to [17]. The limit operator that appears here, the infinity-Laplacian is given by the limit as $p \to \infty$ of the $p-$Laplacian, in the sense that solutions to $\Delta_p v_p = 0$ with a Dirichlet data $v_p = f$ on $\partial \Omega$ converge as $p \to \infty$ to the solution to $\Delta_\infty v = 0$ with $v = f$ on $\partial \Omega$ in the viscosity sense (see [2], [3] and [8]). This operator appears naturally when one considers absolutely minimizing Lipschitz extensions in $\Omega$ of a boundary data $f$ (see [1], [2], and [13]).
The case of a Steklov boundary condition (here the eigenvalue appears in the boundary condition) has also been investigated recently. Indeed in [11] (see also [16] for a slightly different problem) it is studied the behaviour as \( p \to +\infty \) of the so-called variational eigenvalues \( \lambda_{k,p,S} \), \( k \geq 1 \), of the \( p \)-Laplacian with a Steklov boundary condition. In particular it is proved that

\[
\lim_{p \to +\infty} \lambda_{1,p,S}^{1/p} = 1 \quad \text{and} \quad \lambda_{2,\infty,S} := \lim_{p \to +\infty} \lambda_{2,p,S}^{1/p} = \frac{2}{\text{diam}(U, \mathbb{R}^n)},
\]

where here \( \text{diam}(U, \mathbb{R}^n) \) denotes the diameter of \( U \) for the usual Euclidean distance in \( \mathbb{R}^n \).

For pure Neumann eigenvalues, we quote [10] and [19]. In those references it is considered the limit for the second eigenvalue (the first one is zero). It is proved that in this case \( \lambda_\infty := \lim_{p \to +\infty} \lambda_2^{1/p} = 2/\text{diam}(U) \), where \( \text{diam}(U) \) denotes the diameter of \( U \) with respect to the geodesic distance in \( U \). In addition, the regularity of \( \lambda_\infty \) as a function of the domain \( U \) is studied in [19] and in [10] it is proved that there are no nonzero eigenvalues below \( \lambda_\infty \), so that \( \lambda_\infty \) is indeed the first nontrivial eigenvalue for the infinity-Laplacian with Neumann boundary conditions.

Concerning ideas and methods used in the proofs we use classical variational ideas to obtain the limit of \((\lambda_p)^{1/p}\) and viscosity techniques to find the limit PDE problem. We use viscosity techniques as in [14] (we refer to [8] for the definition of a viscosity solution). The characterization of \( \lambda_\infty \) given in Theorem 2 follows using cones as test functions in the variational formulation. Finally, mass transport techniques (we refer to [20]) and gamma-convergence of functionals are used to show the more general characterization of \( \lambda_\infty \) given in Theorem 3, see [7] and [19] for similar arguments in different contexts.

The paper is organized as follows. In Section 2 we deal with the limit as \( p \to \infty \) and prove Theorem 1. In Section 3 we prove Theorem 2 that characterizes \( \lambda_\infty \) in geometrical terms in the cases of a convex domain \( U \). In Section 4 we use optimal transport ideas to obtain Theorem 3. As a corollary, we eventually prove Theorem 2 for a general connected domain in the last section.

2. Proof of Theorem 1

For the proof of Theorem 1 we will use the following lemma.

**Lemma 1.** For any \( f, g \in L^\infty(U) \) there holds

\[
\lim_{p \to +\infty} \left( \|f\|_{L^p(U)} + \|g\|_{L^p(U)} \right)^{\frac{1}{p}} = \max \{ \|f\|_{L^\infty(U)}, \|g\|_{L^\infty(U)} \}.
\]

**Proof.** The result is a direct consequence of the inequalities

\[
\max \{ \|f\|_{L^p(U)}, \|g\|_{L^p(U)} \} \leq \|f\|_{L^p(U)} + \|g\|_{L^p(U)} \\
\leq 2 \max \{ \|f\|_{L^p(U)}, \|g\|_{L^p(U)} \}.
\]
In fact, from the previous inequalities, we get
\[
\lim_{p \to +\infty} \max \{ \| f \|_{L^p(U)}, \| g \|_{L^p(U)} \} \\
\leq \lim_{p \to +\infty} \left( \| f \|_{L^p(U)} + \| g \|_{L^p(U)} \right)^{\frac{1}{p}} \\
\leq \lim_{p \to +\infty} 2^{\frac{1}{p}} \max \{ \| f \|_{L^p(U)}, \| g \|_{L^p(U)} \}.
\]
We conclude using that
\[
\lim_{p \to +\infty} \| f \|_{L^p(U)} = \| f \|_{L^\infty(U)}
\]
and
\[
\lim_{p \to +\infty} \| g \|_{L^p(U)} = \| g \|_{L^\infty(U)}.
\]

Now let us proceed with the proof of Theorem 1.

**Proof of Theorem 1.** Let \( u \in X \) then \( u \in \cap_p X_p \). From the variational characterization of \( \lambda_p \) we have
\[
(\lambda_p)^{1/p} \leq \frac{1}{\| u \|_{L^p(U)}} \left( \int_U |\nabla u|^p + \alpha \int_{\Gamma_2} |u|^p \right)^{1/p}.
\]
Hence, using the previous Lemma we get
\[
\limsup_{p \to \infty} (\lambda_p)^{1/p} \leq \max \left\{ \| \nabla u \|_{L^\infty(U)}, \alpha \| u \|_{L^\infty(\Gamma_2)} \right\}
\]
for any \( u \in X \). Therefore, we conclude that
\[
\limsup_{p \to \infty} (\lambda_p)^{1/p} \leq \lambda_\infty.
\]
In addition, we get that, for \( u_p \) an eigenfunction associated to \( \lambda_p \) in \( X_p \) it holds that
\[
\limsup_{p \to \infty} \| \nabla u_p \|_{L^p(U)} \leq \lambda_\infty.
\]
Therefore, we have that \( \{ u_p \} \) is uniformly bounded (independently of \( p \)) in \( W^{1,p}(U) \). Then, for any fixed \( q \) we obtain
\[
\| \nabla u_p \|_{L^q(U)} \leq \| \nabla u_p \|_{L^p(U)} |U|^\frac{p-q}{pq} \leq C
\]
with \( C \) independent of \( p \). Hence, by a diagonal procedure, we can extract a subsequence \( p_j \to \infty \) such that
\[
u_{p_j} \to u
\]
uniformly in \( U \) and weakly in every \( W^{1,q}(U) \), \( q \in \mathbb{N} \). This limit \( u \) verifies that
\[
\| \nabla u \|_{L^q(U)} \leq \limsup_{p \to \infty} \| \nabla u_p \|_{L^p(U)} \leq \limsup_{p \to \infty} \| \nabla u_p \|_{L^p(U)} |U|^\frac{p-q}{pq} \leq \lambda_\infty |U|^\frac{1}{q}
\]
and then we get
\[
\| \nabla u \|_{L^\infty(U)} \leq \lambda_\infty.
\]
Moreover, we have
\[
\alpha \| u_p \|_{L^q(\Gamma_2)} \leq \left( \alpha^p \| u_p \|_{L^p(\Gamma_2)} |\Gamma_2|^\frac{p-q}{pq} \right)^{1/p} \leq \left( \lambda_p |\Gamma_2|^\frac{p-q}{pq} \right)^{1/p},
\]
then
\[ \alpha \| u \|_{L^q(\Gamma_2)} \leq \limsup_{p \to \infty} \alpha \| u_p \|_{L^q(\Gamma_2)} \leq \limsup_{p \to \infty} \alpha \| u_p \|_{L^q(\Gamma_2)} \leq \lambda_\infty |\Gamma_2|^\frac{1}{q} \]
and we conclude that
\[ \alpha \| u \|_{L^\infty(\Gamma_2)} \leq \lambda_\infty. \]
Hence
\[ \max \left\{ \| \nabla u \|_{L^\infty(U)}, \alpha \| u \|_{L^\infty(\Gamma_2)} \right\} \leq \lambda_\infty. \]
Now, we only have to observe that form the uniform convergence we get that
\[ u \in X, \]
and then we conclude that \( u \) is a minimizer of (3). In addition, our previous calculations show that
\[ \lambda_\infty \leq \liminf_{p \to \infty} (\lambda_p)^{\frac{1}{p}}. \]

Now, concerning the equation verified by the limit of \( u_p, \ u \), we have that, from the fact that \( u_p \) are viscosity solutions to \( \Delta_p u = \lambda_p |u|^{p-2}u \) and that \( (\lambda_p)^{1/p} \) converges to \( \lambda_\infty \) we conclude as in [14] that the limit \( u \) is a viscosity solution to
\[ \min \left\{ |D u| - \lambda_\infty u, -\Delta u \right\} = 0. \]
That \( u = 0 \) on \( \Gamma_1 \) is immediate from uniform convergence in \( \bar{U} \) and the fact that \( u_p \) verify the same condition.

On \( \Gamma_2 \) we have
\[ |\nabla u|^{p-2} \partial_n u + \alpha^p |u|^{p-2}u = 0, \]
therefore, passing to the limit in the viscosity sense as done in [12] we obtain
\[ \min \{ |D u| - \alpha u, -\partial_n u \} = 0. \]
This ends the proof. \( \square \)

3. PROOF OF THEOREM 2 FOR CONVEX DOMAINS.

Along this section we assume that \( U \) is convex.

Proof of Theorem 2. Using the variational characterization (3) proved in the previous section, we estimate \( \lambda_\infty \) from above by using as test-function a truncated cone of the form
\[ u(x) = \left( 1 - a|x - x_0| \right)_{+} \]
where \( a > 0 \) and \( x_0 \in \bar{U} \). Then
\[ u \equiv 0 \text{ on } \Gamma_1 \quad \text{iff} \quad a \geq \frac{1}{d(x_0, \Gamma_1)} \]
\[ \| \nabla u \|_{L^\infty(U)} = a, \]
and
\[ \| u \|_{L^\infty(\Gamma_2)} = \left( 1 - ad(x_0, \Gamma_2) \right)_{+}. \]
It follows that
\[ \lambda_\infty \leq \inf \max \{ a, \alpha[1 - ad(x_0, \Gamma_2)]_{+} \} \]
where the infimum is taken over all the \( x_0 \in \bar{U} \) and \( a > 0 \) such that \( a \geq 1/d(x_0, \Gamma_1) \). Examining the two possibilities for the max, we obtain easily the upper bound for \( \lambda_\infty \).
To prove the lower bound we argue as follows: for any \( x_0 \in \bar{U} \), and any Lipschitz function \( u \in X \) with \( u(x_0) = 1 \), we have
\[
1 \geq \|u\|_{L^\infty(\Gamma_2)} \geq \left(1 - \|\nabla u\|_{d(x_0,\Gamma_2)}\right) + .
\]
Thus
\[
\lambda_\infty \geq \inf \max \left\{\|\nabla u\|_{L^\infty(U)}, \alpha \left(1 - \|\nabla u\|_{L^\infty(U)}d(x_0,\Gamma_2)\right)\right\}_+
\]
where the infimum is taken over all \( u \in W^{1,\infty}(\bar{U}) \) such that \( u = 0 \) in \( \Gamma_1 \), \( \|u\|_{L^\infty(U)} = 1 \) and \( \|\nabla u\|_{L^\infty(U)} \geq \frac{1}{d(x_0,\Gamma_1)} \) for any \( x_0 \in \{u = 1\} \). From this point the argument conclude as for the previous case just analyzing the possibilities for the max. \( \square \)

4. PROOF OF THEOREM 3

The proof follows the lines of [7] (see also [19] for the pure Neumann boundary case).

We begin rewriting the variational formulation (2) of \( \lambda_p \) as
\[
1 = \sup \left\{ \int_U |u|^p : u \in W^{1,p}_1(U) \text{ s.t. } \int_U |\nabla u|^p + \alpha \int_{\Gamma_2} |u|^p = \lambda_p \right\}.
\]
We are thus lead to consider the functions \( G_p : C(\bar{U}) \times M(\bar{U}) \to \mathbb{R}, p \geq 1 \), defined by
\[
G_p(v,\sigma) = \begin{cases} -\int v \, d\sigma & \text{if } v \in W^{1,p}_1(U), \int_U |\nabla v|^p + \alpha \int_{\Gamma_2} |v|^p \leq \lambda_p^p, \\
+\infty & \text{otherwise}, \end{cases}
\]
and \( \sigma \in L^{p'}(U) \int_U |\sigma|^{p'} \leq 1, \)

Notice that the pair \((u_p, u_p^{-1} \, dx)\) is an extremal for \( G_p \) so that \( \min G_p = -1 \). Indeed for any admissible pair \((v,\sigma) \in W^{1,p}_1(U) \times L^{p'}(U)\), we have
\[
-G_p(v,\sigma) = \int_U v \sigma \leq \|v\|_p \|\sigma\|_{p'} \leq \lambda_p^{-1/p} \left(\int_U |\nabla v|^p + \alpha \int_{\Gamma_2} |v|^p\right)^{1/p}
\]
\[
\leq 1 = \int_U u_p^p
\]
(we used successively Hölder’s inequality, the definition of \( \lambda_p \) and the fact that \( v \) is admissible). In view of Lemma 1, we introduce the formal limit functional \( G_\infty : C(\bar{U}) \times M(\bar{U}) \to \mathbb{R} \) of the \( G_p \) by
\[
G_\infty(v,\sigma) = \begin{cases} -\int v \, d\sigma & \text{if } v \in W^{1,\infty}_1(U), \max\{\|\nabla u\|_\infty, \alpha \|v\|_{L^\infty(\Gamma_2)}\} \leq \lambda_\infty, \\
+\infty & \text{otherwise,} \end{cases}
\]
and \(|\sigma|(\bar{U}) \leq 1, \)

The convergence of the functionals \( G_p \) to \( G_\infty \) can be justified using the notion of \( \Gamma \)-convergence. Recall that a sequence of functionals \( F_n : X \to [0, +\infty] \) defined over a metric space \( X \) is said to \( \Gamma \)-converge to a functional \( F_\infty : X \to [0, +\infty] \) if the following two conditions hold:
Then $v$ is admissible for $F_n$ if for any $x \in X$, there exists a sequence $(x_n)_n \subset X$ converging to $x$ such that $F(x) \geq \limsup F(x_n)$.

An easy but important consequence of the definition, that we will use later, is the fact that if $x_n$ is a minimizer of $F_n$ then every cluster point of the sequence $(x_n)$ is a minimizer of $F_\infty$. We refer e.g. to [6] and [9] for a detailed account on $\Gamma$-convergence.

**Proposition 4.1.** The functionals $G_p$ $\Gamma$-converge as $p \to +\infty$ to $G_\infty$.

**Proof.** The proof is very similar to [7] (see also [19] for the pure Neumann boundary case). We briefly sketch it for the reader’s convenience.

Assume that $(v_p, \sigma_p) \in C(U) \times M(U)$ converge to $(v, \sigma)$. We have to prove that

$$
\liminf_{p \to +\infty} G_p(v_p, \sigma_p) \geq G(v, \sigma).
$$

We can assume that $G_p(v_p, \sigma_p) < \infty$. Then we have

$$
\int_U v_p \sigma_p \, dx - \int_U v \, d\sigma = \int_U (v_p - v) \sigma_p \, dx + \int_U v (\sigma_p \, dx - d\sigma) \to 0
$$

as $p \to +\infty$. Indeed the first integral on the right hand side can be bounded by $\|v_p - v\|_\infty \|\sigma_p\|_{\nu'} |U|^{\frac{1}{\nu'}} = o(1)$. Independently

$$
\int_U |\sigma_p| \, dx + o(1) \leq \|\sigma_p\|_{\nu'} |U|^{\frac{1}{\nu'}} + o(1) \leq 1 + o(1)
$$

so that $\int_U |\sigma| \leq 1$. Moreover taking limit in $\alpha \|v_p\|_{L^\nu(\Gamma_2)} \leq \lambda_p$ yields $\alpha \|v\|_{L^\nu(\Gamma_2)} \leq \lambda_\infty$. Eventually, for any $\phi \in L^\nu(U, \mathbb{R}^n)$ such that $\|\phi\|_{\nu'} \leq 1$ we have

$$
\int_U \phi \nabla v_p \, dx = - \int_U v \, \text{div} \phi \, dx = - \int_U v_p \, \text{div} \phi \, dx + o(1) = \int_U \phi \nabla v_p \, dx + o(1)
$$

$$
\leq \|\nabla v_p\|_\nu + o(1) \leq \frac{1}{\lambda_p} + o(1) = \lambda_\infty + o(1),
$$

where the $o(1)$ does not depend on $\phi$. Taking the supremum over all such $\phi$ we obtain $\|\nabla v\|_\nu \leq \lambda_\infty + o(1)$, so that $\|\nabla v\|_\infty \leq \lambda_\infty$. It follows that $(v, \sigma)$ is admissible for $G_\infty$.

We now fix a pair $(v, \sigma)$ admissible for $G_\infty$. We have to find some pair $(v_p, \sigma_p)$ admissible for $G_p$ which converges to $(v, \sigma)$ and such that

$$
\limsup_{p \to +\infty} G_p(v_p, \sigma_p) \leq G_\infty(v, \sigma).
$$

We define

$$
v_p = \frac{\lambda_p^{\frac{1}{p}}}{\lambda_\infty(|U| + |\Gamma_2|)^{\frac{1}{p}}} v.
$$

Then $v_p \in W^{1,p}(U)$, $v_p \to v$ uniformly, and

$$
\int_U |\nabla v_p|^p + \alpha^p \int_{\Gamma_2} |v_p|^p \leq \lambda_p^p.
$$

In order to define $\sigma_p$ by regularizing $\sigma$ by convolution, we first need to adjust a little. Let $\vec{n}$ be the unit inner normal vector to $U$ that we extend in a smooth way to $\mathbb{R}^n$ with compact support in a neighborhood of $\partial U$. We
consider $T_\varepsilon : \bar{U} \to \bar{U}_{2\varepsilon} := \{ x \in \bar{U}, \text{dist}(x, \partial U) \geq 2\varepsilon \}$ defined by $T_\varepsilon (x) = x + 2\varepsilon n$. Let $\sigma_\varepsilon = T_\varepsilon \# \sigma$ be the push-forward of $\sigma$ by $T_\varepsilon$ i.e. $\int f \, d\sigma_\varepsilon = \int f \circ T_\varepsilon \, d\sigma$ for any $f \in C(\bar{U}_{2\varepsilon})$. Observe that $\text{supp} \, \sigma_\varepsilon \subset \bar{U}_{2\varepsilon}$ and also that $\int |\sigma_\varepsilon| \leq 1$ since
\[
\int |\sigma_\varepsilon| = \sup_{\|\phi\|_{L^\infty(\bar{U}_{2\varepsilon})} \leq 1} \int \phi \, d\sigma_\varepsilon = \sup_{\|\phi\|_{L^\infty(\bar{U}_{2\varepsilon})} \leq 1} \int \phi \circ T_\varepsilon \, d\sigma \\
\leq \int d|\sigma| \leq 1.
\]
Moreover
\[
\sigma_\varepsilon \rightharpoonup \sigma \quad \text{weakly in the sense of measure.}
\]
Indeed for any $\phi \in C(\bar{U})$,
\[
|\int \phi \, d\sigma_\varepsilon - \int \phi \, d\sigma| \leq \int |\phi(x + 2\varepsilon n) - \phi(x)| \, d\sigma(x) = o(1)
\]
since the integrand goes to 0 uniformly in $x \in \bar{U}$. Denote by $\rho_\varepsilon$ the usual mollifying functions (i.e. $\rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon)$ where $\rho$ is a smooth function compactly supported in the unit ball of $\mathbb{R}^n$ with $\int \rho = 1$). Then
\[
\rho_\varepsilon \ast \sigma_\varepsilon - \sigma_\varepsilon \rightharpoonup 0 \quad \text{weakly in the sense of measure}
\]
This follows from the fact that $\|\phi \ast \rho_\varepsilon - \phi\|_{L^\infty(\bar{U}_{2\varepsilon})} \to 0$ for any $\phi \in C(\bar{U})$. Hence
\[
(9) \quad \rho_\varepsilon \ast \sigma_\varepsilon \rightharpoonup \sigma \quad \text{weakly in the sense of measure.}
\]
We now regularize $\sigma_\varepsilon$ considering
\[
\tilde{\sigma}_\varepsilon := \sigma_\varepsilon \ast \tilde{\rho}_\varepsilon \in C^\infty(U)
\]
with
\[
\tilde{\rho}_\varepsilon := \frac{\rho_\varepsilon}{\|\rho_\varepsilon\|_{p'}}, \quad \varepsilon = 1/p.
\]
Then $\|\rho_\varepsilon\|_{p'} \to 1$ since $\|\rho_\varepsilon\|_{p'} = \varepsilon^{-n/p} \|\rho\|_{p'} \to \|\rho\|_1 = 1$. It then follows that $\tilde{\sigma}_\varepsilon \rightharpoonup \sigma$. Moreover $\tilde{\sigma}_\varepsilon$ is admissible for $G_p$ since, by Holder inequality and recalling (9),
\[
\|\tilde{\sigma}_\varepsilon\|_{p'}^{p'} \leq \left( \int |\sigma_\varepsilon| \right)^{p'/(p'-1)} \int \tilde{\rho}_\varepsilon(x-y) \, d|\sigma_\varepsilon|(y) = \|\tilde{\rho}_\varepsilon\|_{p'}^{p'} \left( \int |\sigma_\varepsilon| \right)^{p'-1} \leq 1.
\]
It follows that $(\sigma_\varepsilon, v_\varepsilon)$ is admissible for $G_p$ and converge to $(v, \sigma)$. As before we have $G_p(v_\varepsilon, \sigma_\varepsilon) \to G_{\infty}(v, \sigma)$. \hfill \square

Recall that from Theorem 1, $u_p$ converge in $C(\bar{U})$ up to a subsequence to some $u_\infty \in C(\bar{U})$, $\|u\|_\infty = 1$. Moreover, up to a subsequence, the measures $u_p^{-1} \, dx$ converge weakly to some probability measure $\sigma_\infty$. Indeed since $\bar{U}$ is compact, it suffices, according to Prokhorov theorem, to show that
\[
\lim_{p \to +\infty} \int_{\bar{U}} u_p^{-1} \, dx = 1.
\]
This follows from
\[
\int_{\bar{U}} u_p^{-1} \, dx \leq \|u_p\|_p |U|^{1/p} \to 1
\]
and, for $p > n$,
\[
1 = \int_G u_p^{-1} u_p \, dx \leq \|u_p\|_{\infty} \int_G u_p^{-1} \, dx = (1 + o(1)) \int_G u_p^{-1} \, dx.
\]

As a consequence of the $\Gamma$-convergence of $G_p$ to $G_\infty$ and the fact that $(u_p, u_p^{-1} \, dx)$ is a minimizer of $G_p$, we obtain that $(\mu_\infty, \sigma_\infty)$ is a minimizer of $G_\infty$ with $G_\infty(\mu_\infty, \sigma_\infty) = \lim_{p \to +\infty} G_p(u_p, u_p^{-1} \, dx) = -1$. Since $\sigma_\infty \in P(U)$ and $u_\infty$ is an extremal for $\lambda_\infty$, we can thus write
\[
1 = \max \left\{ \int v \, d\sigma; \ v \in W^{1, \infty}_{\Gamma_1}(U), \ \max \left\{ \|\nabla v\|_{\infty}, \|\alpha\|_{L^\infty(\Gamma_2)} \right\} = \lambda_\infty, \ \sigma \in P(U) \right\}
\]
i.e.
\[
\lambda_\infty^{-1} = \max \left\{ \int v \, d\sigma; \ v \in W^{1, \infty}_{\Gamma_1}(U), \ \max \left\{ \|\nabla v\|_{\infty}, \|\alpha\|_{L^\infty(\Gamma_2)} \right\} = 1, \ \sigma \in P(U) \right\}
\]

An approximation argument shows that we can replace $W^{1, \infty}_{\Gamma_1}(U)$ by $C^1(U) \cap C_{\Gamma_1}(\bar{U})$ where $C_{\Gamma_1}(\bar{U}) = \{ u \in C(\bar{U}) : u = 0 \text{ on } \Gamma_1 \}$.

**Proposition 4.2.** Given $v \in W^{1, \infty}_{\Gamma_1}(U)$, \( \max \{ \|\nabla v\|_{\infty}, \|\alpha\|_{L^\infty(\Gamma_2)} \} \leq 1 \), there exist $v_k \in C^1(U) \cap C_{\Gamma_1}(\bar{U})$, \( \max \{ \|\nabla v_k\|_{\infty}, \|\alpha\|_{L^\infty(\Gamma_2)} \} \leq 1 \), such that $v_k \to v$ uniformly in $\bar{U}$.

**Proof.** The proof uses ideas from [7]. We first extend $v$ in a neighborhood of $\partial U$ by antisymmetric reflection across $\partial U$ so that the extended function $\bar{v}$ is Lipschitz with $\|\nabla \bar{v}\|_{\infty} = \|\nabla v\|_{\infty} \leq 1$. We then apply the same method as in [7] consisting in introducing the function $\theta_\varepsilon(t) = (t - \text{sgn}(t) \varepsilon)1_{|t| \geq \varepsilon}$ and then regularizing $\theta_\varepsilon \circ \bar{v}$ by convolution with the usual mollifying functions. Observe that $\|\nabla (\theta_\varepsilon \circ \bar{v})\|_{\infty} \leq \|\nabla \bar{v}\|_{\infty} \leq 1$ and that $\theta_\varepsilon \circ \bar{v} = 0$ in the $\varepsilon$-neighborhood $\{ x \in \mathbb{R}^n, \text{dist}(x, \Gamma_1) < \varepsilon \}$ of $\Gamma_1$ since $\bar{v}$ is 1-Lipschitz. Note also that $|\theta(t)| = (|t| - \varepsilon)_+1_{|t| \geq \varepsilon}$ so that on $\Gamma_2$, $|\theta_\varepsilon \circ \bar{v}| \leq (\alpha^{-1} - \varepsilon)_+$. Hence $|\theta_\varepsilon \circ \bar{v}| \leq \alpha^{-1}$ in the $\varepsilon$-neighborhood of $\Gamma_2$. It follows from these three comments that the regularizing of $\theta_\varepsilon \circ \bar{v}$ is adequate.

Denoting by $\text{Res} : C(\bar{U}) \to C(\Gamma_2)$ the restriction operator, $Au = \nabla u$ the derivation operator with domain $C^1(U)$, and $B(R)$ the closed ball of radius $R$ centered at 0 in $C(\bar{U})$, $B = B(1)$, we can rewrite (10) as
\[
\frac{1}{\lambda_\infty} = \max_{\sigma \in P(U)} \max_{u \in C(\bar{U})} \left\{ (\sigma, u) - (\chi_{B(1/\alpha)} \circ \text{Res})(u) - (\chi_B \circ A)(u) - \chi_{C_{\Gamma_1}(U)}(u) \right\},
\]

Recalling the definition of the Legendre transform, we eventually obtain
\[
\frac{1}{\lambda_\infty} = \max_{\sigma \in P(U)} \left( (\chi_{B(1/\alpha)} \circ \text{Res}) + (\chi_B \circ A) + \chi_{C_{\Gamma_1}(U)} \right)^* (\sigma).
\]

The inf-convolution $f \square g$ of two proper lower semi-continuous (lsc) convex functions $f, g : E \to \mathbb{R}$ ($E$ denotes a normed space - we will take $E = C(\bar{U})$)
here) is defined by \((f \square g)(x) = \inf_{y \in E} f(y) + g(x - y)\). This operation is commutative and associative. Moreover, \(f \square g\) is a proper lsc convex function with domain \(\text{Dom}(f) + \text{Dom}(g)\), and its Legendre transform is \((f \square g)^* = f^* + g^*\). Eventually, if 0 belongs to the interior of \(\text{Dom}(f) - \text{Dom}(g)\) then \((f + g)^* = f^* \square g^*\) (see [18][section 3.9 p42]). This last assumption is trivially satisfied here since any neighborhood of 0 in \(C(\bar{U})\) is contained in \(C^1(\bar{U}) + C(\bar{U})\).

We can thus rewrite (11) as

\[
\frac{1}{\lambda_\infty} = \max_{\sigma \in \mathcal{P}(\bar{U})} \left( (\chi_{B(1/\alpha)} \circ \text{Res})^* \square (\chi_B \circ A)^* \square \chi^*_{\Gamma_1(U)}(\sigma) \right)
\]

(12)  

where the inf is taken over all triple of measures \(\mu_1, \mu_2, \mu_3 \in M(\bar{U})\) such that \(\sigma = \mu_1 + \mu_2 + \mu_3\). To pursue further we need to compute the various Legendre transforms involved in this expression. This is the content of the next proposition.

**Proposition 4.3.** There holds for \(\mu \in M(\bar{U})\),

\[
\chi^*_{\Gamma_1(U)}(\mu) = \begin{cases} 
0 & \text{if } \text{supp } \mu \subset \Gamma_1 \\
+\infty & \text{otherwise}
\end{cases}
\]

and

\[
(\chi_B \circ A)^*(\mu) = \inf \left\{ \int_{\mathcal{O}} |\sigma| : \sigma \in M(\bar{U}, \mathbb{R}^n) \text{ s.t. } \text{div } \sigma = \mu \text{ in } \mathcal{D}'(\mathbb{R}^n) \right\}
\]

(14)  

where the inf is taken over all measures \(\mu \in M(\bar{U})\) such that \(\sigma = \mu_1 + \mu_2 + \mu_3\). To pursue further we need to compute the various Legendre transforms involved in this expression. This is the content of the next proposition.

Moreover,

\[
(\chi_{B(1/\alpha)} \circ \text{Res})^*(\mu) = \begin{cases} 
\frac{1}{\alpha} |\mu|(\Gamma_2) & \text{if } \text{supp } \mu \subset \Gamma_2 \\
+\infty & \text{otherwise}
\end{cases}
\]

(15)  

**Proof.** These computations are more or less classical. We sketch them here for the reader’s convenience.

First, the definition of the Legendre transform gives

\[
\chi^*_{\Gamma_1(U)}(\mu) = \sup_{u \in C(\bar{U})} (\mu(u) - \chi_{\Gamma_1(U)}(u))
\]

(16)  

from which we deduce (13).

We now prove (14). The second equality in (14) is well-known. It remains to prove the first one. We recall the following result concerning the Legendre transform: if \(E\) and \(F\) are two normed space, \(L : E \rightarrow F\) linear with domain \(\text{Dom}(L)\) and \(f : E \rightarrow \mathbb{R}\) is convex, consider the function \((LF)(y) = \inf \{ f(x) : x \in \text{Dom}(L) \text{ s.t. } Lx = y \}\), \(y \in F\). Then \(Lf\) is convex with \((Lf)^* = f^* \circ L^*\) in the domain \(\text{Dom}(L^*)\) of the adjoint \(L^* : F^* \rightarrow E^*\) of \(L\).
Thus the inf in (14) can be written as $\inf_{\text{Dom }\mu} M(u) = \mu$ and noticing that $\chi_B$ is convex lsc (because $B$ is convex and closed), so that $\chi_B^\ast = \chi_B$, we obtain $\chi_B \ast A = (A \ast \chi_B^\ast)^\ast$. Observe that $A^\ast = A$ on $\text{Dom}(A)$ so that $\chi_B \ast A = (A \ast \chi_B^\ast)^\ast$ on $\text{Dom}(A)$.

Observe that $A \ast \chi_B^\ast$, which is the r.h.s of (14), is lsc for the weak convergence (and thus also for the strong i.e. total variation convergence) in the sense that if $\mu_n, \mu \in M(\bar{U})$ verify $\mu_n \rightarrow \mu$ weakly then

$$\liminf_{n \rightarrow +\infty} (A \ast \chi_B^\ast)(\mu_n) \geq (A \ast \chi_B^\ast)(\mu).$$

Indeed we can assume that $(A \ast \chi_B^\ast)(\mu_n) \leq C\text{ste}$. Then taking $\sigma_n \in M(\bar{U}, \mathbb{R}^n)$ s.t. $-\text{div }\sigma_n = \mu_n$ and $A \ast \chi_B^\ast(\mu_n) = \int |\sigma_n| + o(1)$, we have $\int |\sigma_n| \leq C$. Then applying Prokhorov theorem to $\sigma_n^\ast$ and $\sigma^\ast$, we have, up to a subsequence, that $\sigma_n \rightarrow \sigma$ weakly. In particular $-\text{div }\sigma = \mu$ and $\liminf_{n \rightarrow +\infty} \int |\sigma_n| \geq \int |\sigma| \geq (A \ast \chi_B^\ast)(\sigma)$ from which we deduce the result.

We thus have that $A \ast \chi_B^\ast$ is convex lsc so that $A \ast \chi_B = (A \ast \chi_B^\ast)^\ast$. Hence $(\chi_B \ast A)^\ast = A \ast \chi_B^\ast$ which is exactly (14).

The proof of (15) is similar. We have as before that for any $\mu \in M(\bar{U})$,

$$(\chi_B^\ast(\mu_n)) = (\chi_B^\ast(\mu_n)) = \inf \{\chi_B^\ast(\mu_n) : \mu \in \text{C}(\bar{U}), C(\Gamma_2) \}$$

with $\text{C}(U) \rightarrow \text{C}(\Gamma_2)$ and $\text{C}(\Gamma_2)^\ast = M(\Gamma_2) \rightarrow \text{C}(U)^\ast = M(\bar{U})$ is given by

$$(\chi_B^\ast(\mu_n), v) = (\sigma, \text{Res}_{\Gamma_2}(v)) = (\sigma, \text{Res}_{\Gamma_2}(v)) = \int_{\Gamma_2} v d\sigma$$

for any $\sigma \in C(\Gamma_2)^\ast, v \in C(\bar{U})$. Moreover $\chi_B^\ast(\mu_n) \in \text{C}(\Gamma_2)^\ast$, for any $\sigma \in C(\Gamma_2)^\ast$,

$$\chi_B^\ast(\mu_n) = \sup_{v \in C(\Gamma_2)} (\sigma, v) - \chi_B^\ast(v) = \sup_{v \in C(\Gamma_2)} \int_{\Gamma_2} v d\sigma$$

Thus

$$(\chi_B^\ast(\mu_n)) = \inf \left\{ \int_{\Gamma_2} |\sigma| : \sigma \in C(\Gamma_2)^\ast \text{ s.t. } \int_{\Gamma_2} u d\sigma = \int_{\bar{U}} u d\mu \text{ for all } u \in C(\bar{U}) \right\}.$$
It follows that there cannot exist an \( A \subset \overline{U} \setminus \Gamma_2 \) s.t. \( \mu(A) \neq 0 \) i.e. \( \text{supp} \, \mu \subset \Gamma_2 \) and then \( \sigma = \mu \). Hence \( (\chi_B(1/\alpha) \circ \text{Res})^* (\mu) = \frac{1}{\alpha} |\mu| (\Gamma_2) \) if \( \text{supp} \, \mu \subset \Gamma_2 \). Otherwise there does not exist any admissible \( \sigma \) and the inf is \( +\infty \). \( \square \)

Using the previous proposition, we can rewrite (16) as
\[
\frac{1}{\lambda_{\infty}} = \max_{\sigma \in P(\overline{U})} \inf_{\mu_2} (\chi_B \circ A)^* (\mu_2) + \frac{1}{\alpha} |\mu_1| (\Gamma_2),
\]
where the inf is taken over all triple of measures \( \mu_1, \mu_2, \mu_3 \in M(\overline{U}) \) such that \( \sigma = \mu_1 + \mu_2 + \mu_3 \), \( \text{supp} \, \mu_3 \subset \Gamma_1 \), \( \text{supp} \, \mu_1 \subset \Gamma_2 \), \( \mu_2 (\overline{U}) = 0 \). Letting \( \nu = \mu_1 + \mu_3 = \sigma - \mu_2 \), we have \( |\mu_1| (\Gamma_2) = |\nu (\Gamma_2) = \nu^+(\Gamma_2) + \nu^- (\Gamma_2) \) since \( \mu_1 \) and \( \mu_3 \) have disjoint support. Moreover, since \( \mu_2 (\overline{U}) = 0 \) i.e. \( (\sigma + \nu^-)(\overline{U}) = \nu^+(\overline{U}) \), we have
\[
(\chi_B \circ A)^* (\mu_2) = (\chi_B \circ A)^* (\sigma - \nu) \\
= \inf \left\{ \int_\overline{U} |\tilde{\sigma}| : \tilde{\sigma} \in M(\overline{U}, \mathbb{R}^n) \text{ s.t. } - \text{div} \tilde{\sigma} = (\sigma + \nu^-) - \nu^+ \text{ in } D'(\mathbb{R}^n) \right\} \\
= W_1 (\sigma + \nu^-, \nu^+).
\]

We thus obtain
\[
\frac{1}{\lambda_{\infty}} = \max_{\sigma \in P(\overline{U})} \inf_{\nu \in M(\partial \overline{U}), \mu \in \partial \overline{U} = 1} W_1 (\sigma + \nu, \nu_2) + \frac{1}{\alpha} \nu^+(\Gamma_2) + \frac{1}{\alpha} \nu^- (\Gamma_2).
\]
To conclude the proof of (7), it suffices to verify that the inf can be taken over non-negative \( \nu \). This is a consequence of the following Proposition:

**Proposition 4.4.** For any \( \sigma \in P(\overline{U}) \),
\[
\inf_{\nu_1, \nu_2 \in M_+(\partial \overline{U}), \nu_2 (\partial \overline{U}) = \nu_1 (\partial \overline{U}) + 1} W_1 (\sigma + \nu_1, \nu_2) = \inf_{\nu \in P(\partial \overline{U})} W_1 (\sigma, \nu).
\]

The proof of this lemma is based on the following lemma:

**Lemma 2.** Consider probability measures \( \mu_{\epsilon}, \mu \in P(\mathbb{R}^n) \) such that
\[
\lim_{\epsilon \to 0} W_1 (\mu_{\epsilon}, \mu) = 0,
\]
and a subset \( A \subset P(\mathbb{R}^n) \) compact w.r.t the convergence in distance \( W_1 \). Then \( \lim_{\epsilon \to 0} W_1 (\mu_{\epsilon}, A) = W_1 (\mu, A) \) where \( W_1 (\mu, A) = \inf_{\nu \in A} W_1 (\mu, \nu) \).

Observe that the compactness assumption is satisfied for \( A = P(\mathbb{K}) \) where \( \mathbb{K} \subset \mathbb{R}^n \) is compact in view of Prokhorov theorem and the fact that \( W_1 \) metrizes the weak convergence in \( P(\mathbb{K}) \) (because \( \mathbb{K} \) is bounded).

**Proof of lemma 2.** Consider \( \nu_5 \in A \) s.t. \( \lim_{\delta \to 0} W_1 (\nu_5, \mu) = W_1 (\mu, A) \). Then passing to the limit in \( W_1 (\mu_{\epsilon}, A) \leq W_1 (\mu_{\epsilon}, \nu_5) \) yields \( \lim_{\epsilon \to 0} W_1 (\mu_{\epsilon}, A) \leq W_1 (\mu, \nu_5) \) for any \( \delta \), so that \( \limsup_{\epsilon \to 0} W_1 (\mu_{\epsilon}, A) \leq W_1 (\mu, A) \).

To prove the opposite inequality we consider \( \nu_6 \in A \) such that \( W_1 (\mu_{\epsilon}, \nu_6) = W_1 (\mu_{\epsilon}, A) + o(1) \). Since \( A \) is compact, we can assume up to a subsequence that there exists \( \nu \in A \) s.t. \( W_1 (\nu, \nu) \to 0 \). Since \( W_1 (\mu_{\epsilon}, \mu) \to 0 \), we obtain
\[
\lim_{\epsilon \to 0} W_1 (\mu_{\epsilon}, A) = \lim_{\epsilon \to 0} W_1 (\mu_{\epsilon}, \nu_6) = W_1 (\mu, \nu) \geq W_1 (\mu, A)
\]
which ends the proof of the lemma. \( \square \)

We now prove Proposition 4.4.
Proof of Proposition 4.4. The $\leq$ inequality is clear (take $\nu_1 = 0$). To prove the opposite inequality, we first assume that $\text{supp } \sigma \subset U$. Given any $\nu_1, \nu_2$, any transfer plan $\pi \in \Pi(\sigma + \nu_1, \nu_2)$ (i.e. $\pi \in P(U)$ has marginals $\sigma + \nu_1$ and $\nu_2$) can be written as

$$\pi = \tilde{\pi} + \pi,$$

for some decomposition $\nu_2 = \tilde{\nu}_2 + \bar{\nu}_2$ with $\tilde{\nu}_2, \bar{\nu}_2 \in M_+(\partial U)$, $\tilde{\nu}_2(\partial U) = 1$, $\bar{\nu}_2(\partial U) = \nu_1(\partial U)$. It follows that

$$W_1(\sigma + \nu_1, \nu_2) = \inf_{\pi \in \Pi(\sigma + \nu_1, \nu_2)} \int_{\bar{U} \times U} d(x, y) \, d\pi(x, y)$$

$$= \inf_{\nu_2 = \bar{\nu}_2 + \tilde{\nu}_2, \tilde{\pi} \in \Pi(\sigma, \tilde{\nu}_2), \pi \in \Pi(\nu_1, \tilde{\nu}_2)} \int_{\bar{U} \times U} d(x, y) \, d\tilde{\pi}(x, y)$$

$$+ \int_{\bar{U} \times U} d(x, y) \, d\bar{\pi}(x, y)$$

$$\geq \inf_{\nu_2 = \bar{\nu}_2 + \tilde{\nu}_2} W_1(\sigma, \bar{\nu}_2) + W_1(\nu_1, \tilde{\nu}_2)$$

Then

$$\inf_{\nu_1, \nu_2 \in M_+(\partial U), \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma + \nu_1, \nu_2)$$

$$\geq \inf_{\nu_1, \nu_2 \in M_+(\partial U), \nu_2(\partial U) = \nu_1(\partial U) + 1} \inf_{\nu_2 = \bar{\nu}_2 + \tilde{\nu}_2} W_1(\sigma, \bar{\nu}_2) + W_1(\nu_1, \tilde{\nu}_2)$$

which is clearly greater or equal than $\inf_{\nu_2 \in P(\partial U)} W_1(\sigma, \bar{\nu}_2)$. This proves the $\geq$ inequality when $\text{supp } \sigma \subset U$.

In the general case we have $\text{supp } \sigma \subset \bar{U}$. We consider $\sigma_\varepsilon = T_\varepsilon^* \sigma$ the push-forward of $\sigma$ under $T_\varepsilon(x) = \kappa + \varepsilon \bar{n}$ where $\bar{n}$ denote some smooth extension of the unit exterior normal to a neighborhood of $\partial U$. Then $\text{supp } \sigma_\varepsilon \subset U$ so that

$$\inf_{\nu_1, \nu_2 \in M_+(\partial U), \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma_\varepsilon + \nu_1, \nu_2) = W_1(\sigma_\varepsilon, P(\partial U)).$$

To pass to the limit as $\varepsilon \to 0$, we use Lemma 2. Just notice that $\sigma_\varepsilon \to \sigma$ weakly as measure i.e. $W_1(\sigma_\varepsilon, \sigma) \to 0$ since $U$ is bounded, and $A = P(\partial U)$ is compact for the weak convergence. We then have $W_1(\sigma_\varepsilon, P(\partial U)) \to W_1(\sigma, P(\partial U))$. Observe also that the first part of the proof of Proposition 4.4, which does not use the compactness assumption, yields

$$\limsup_{\varepsilon \to 0} \inf_{\nu_1, \nu_2 \in M_+(\partial U), \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma_\varepsilon + \nu_1, \nu_2)$$

$$\leq \inf_{\nu_1, \nu_2 \in M_+(\partial U), \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma + \nu_1, \nu_2).$$

The result follows. \qed

To end the proof of theorem 3, we verify that the max in (7) is attained by $f_\infty$, the weak limit as $p \to +\infty$ of the measures $f_p = u_p^{p-1} \, dx$ (which exists up to a subsequence). Notice that $u_p$ is the unique minimizer of the functional $F_p : W^{1,p}_{\Gamma_1}(U) \to \mathbb{R}$ defined by

$$F_p(u) = \frac{1}{p\lambda_p} \int_U |\nabla u|^p + \frac{\alpha^p}{p\lambda_p} \int_{\Gamma_2} |u|^p - (f_p, u).$$
Indeed the associated Euler-Lagrange equation, which has a unique solution since $F_p$ is strictly convex, is the equation $\Delta_p u = \lambda_p f_p$ with the boundary conditions of (1), which admits $u_p$ as a solution.

Writing $F_p$ as

$$F_p(u) = \int_U \left| \frac{\nabla u}{p/\lambda_p} \right|^p + \int_{\Gamma_2} \left| \frac{\alpha u}{p/\lambda_p} \right|^p - (f_p, u),$$

we can prove, as in Proposition 4.1, that $F_p \Gamma$-converge as $p \to +\infty$ to the functionnal $F_\infty : C(\bar{U}) \to \mathbb{R}$ defined by

$$F_\infty(u) = \begin{cases} -(f_\infty, u), & \text{if } u \in W^{1,\infty}_1(U), \|\nabla u\|_\infty \leq \lambda_\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Since

$$\inf F_p = F_p(u_p) = \frac{1}{p} - 1,$$

we obtain that

$$F_\infty(u_\infty) = \inf F_\infty = \lim_{p \to +\infty} \inf F_p = -1.$$  

Hence

$$-1 = \min \left\{ - (f_\infty, u) + \chi_{B(1/\alpha)}(u_{|\Gamma_1} / \lambda_\infty) + \chi_B(\nabla u / \lambda_\infty) + \chi_{C_{T_1}(U)}(u) \right\},$$

i.e.

$$\frac{-1}{\lambda_\infty} = \min \left\{ - (f_\infty, u) + \chi_{B(1/\alpha)}(u_{|\Gamma_1}) + \chi_B(\nabla u) + \chi_{C_{T_1}(U)}(u) \right\},$$

Then

$$\frac{1}{\lambda_\infty} = \max_{u \in C(U)} \left\{ (f_\infty, u) - \chi_{B(1/\alpha)}(u_{|\Gamma_1}) - \chi_B(\nabla u) - \chi_{C(U)}u \right\}$$

$$= \left( (\chi_{B(1/\alpha)} \circ Res) + (\chi_B \circ A) + \chi_{C_{T_1}(U)} \right)^*(f_\infty).$$

Since $f_\infty \in P(\partial U)$, we obtain in view of (11) that $f_\infty$ is extremal in (7).

5. PROOF OF THEOREM 2 FOR CONNECTED DOMAINS.

Let $\phi(\sigma, \nu) = W_1(\sigma, \nu) + \frac{1}{\alpha} \nu(\Gamma_2)$, $\sigma, \nu \in P(\partial U)$. Since $W_1$ is convex in $(\sigma, \nu)$ (see e.g. [20][thm. 4.8]), we see that $\phi$ is convex. It easily follows that the function $\Phi(\sigma) = \inf_{\nu \in P(\partial U)} \phi(\sigma, \nu)$, $\sigma \in P(\bar{U})$ is also convex. Indeed given $\sigma_1, \sigma_2 \in P(\bar{U})$ and any $\nu_1, \nu_2 \in P(\bar{U})$, we have

$$\Phi(t\sigma_1 + (1 - t)\sigma_2) \leq \phi(t\sigma_1 + (1 - t)\sigma_2, tv_1 + (1 - t)v_2) \leq t\phi(\sigma_1, v_1) + (1 - t)\phi(\sigma_2, v_2).$$

The result follows taking the infimum in $\nu_1, \nu_2$.

Since $\Phi$ is convex, it attains its maximum at an extreme point of the convex compact $P(\bar{U})$ i.e. at some Dirac mass $\delta_x$, $x \in U$:

$$\frac{1}{\lambda_\infty} = \max_{x \in U} \inf_{\nu \in P(\partial U)} W_1(\delta_x, \nu) + \frac{1}{\alpha} \nu(\Gamma_2).$$

It is well-known that $W_1(\delta_x, \nu) = \int_U d(x, y) \, d\nu(y)$ for any $x \in \bar{U}$. This follows from the fact that the unique $\pi \in P(\bar{U} \times \bar{U})$ with marginals $\delta_x$ and...
ν is π = δx ⊗ ν. Indeed such a π must have support in \{x\} × supp ν so that for any A, B ⊂ \bar{U}, π(A × B) = 0 = (δx ⊗ ν)(A × B) if x ∉ A, and if x ∈ A,
\[ π(A × B) = π(\{x\} × B) = π(X × B) = ν(B) = (δx ⊗ ν)(A × B). \]

Given x ∈ \bar{U}, we consider x_1 ∈ Γ_1 and x_2 ∈ Γ_2 such that d(x, x_i) = d(x, x_i), i = 1, 2. We write ν ∈ P(\partial U) as ν = ν_1 + ν_2 where ν_i = ν|_{Γ_i}, i = 1, 2. Then
\[
W_1(δx, ν) = \int_{\partial U} d(x, y) \, dν(y) = \int_{Γ_1} d(x, y) \, dν_1(y) + \int_{Γ_2} d(x, y) \, dν_2(y) ≥ d(x, Γ_1)ν_1(Γ_1) + d(x, Γ_2)ν_2(Γ_2) = W_1(δx_1, βδx_1 + (1 − β)δx_2),
\]
where β = ν_1(Γ_1). We thus have
\[
\frac{1}{\lambda_∞} = \max_{x ∈ \bar{U}} \inf_{0 ≤ β ≤ 1} βd(x, Γ_1) + (1 − β)d(x, Γ_2) + \frac{1 − β}{α}.
\]
We deduce theorem 2 noticing that for any x ∈ \bar{U}, the inf in β is
\[
\begin{cases}
d(x, Γ_2) + \frac{1}{α} & \text{if } d(x, Γ_1) − d(x, Γ_2) − \frac{1}{α} ≥ 0 \quad \text{i.e. } x ∈ A \\
d(x, Γ_1) & \text{otherwise.}
\end{cases}
\]

REFERENCES


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