THE LIMIT AS $p \to +\infty$ OF THE FIRST EIGENVALUE FOR THE p-LAPLACIAN WITH MIXED DIRICHLET AND ROBIN BOUNDARY CONDITIONS.

JULIO D. ROSSI AND NICOLAS SAINTIER

Abstract. We analyze the behaviour as $p \to \infty$ of the first eigenvalue of the $p-$Laplacian with mixed boundary conditions of Dirichlet-Robin type. We find a nontrivial limit that we associate to a variational principle involving L^∞-norms. Moreover, we provide a geometrical characterization of the limit value as well as a description of it using optimal mass transportation techniques. Our results interpolate between the pure Dirichlet case and the mixed Dirichlet-Neumann case.

1. Introduction and description of the main results

Let $U \subset \mathbb{R}^n$ be a smooth, bounded, open and connected set. In order to consider mixed boundary conditions, we split the boundary of U as $\partial U = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 \cap \Gamma_2 = \emptyset$ and $|\Gamma_1| > 0$. In this paper we deal with the first eigenvalue, that we will call λ_p, of the p-Laplacian with Dirichlet condition on Γ_1 and Robin condition on Γ_2 namely the smallest λ such that there is a nontrivial solution to the following problem,

$$(1) \qquad \begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } U, \\ u = 0 & \text{on } \Gamma_1, \\ |\nabla u|^{p-2} \partial_\nu u + \alpha^p |u|^{p-2} u = 0 & \text{on } \Gamma_2. \end{cases}$$

Here α is a non-negative parameter. Notice that when $\alpha = +\infty$, the boundary condition become $u = 0$ in all ∂U (a pure Dirichlet condition) and when $\alpha = 0$ we have a mixed Dirichlet-Neumann boundary condition.

Our main goal is to compute the limit as $p \to \infty$ of this problem and look at its dependence on the parameter α.

To start our analysis we remark that λ_p has the following variational formulation:

$$(2) \qquad \lambda_p = \inf_{u \in X_p} \left\{ \int_U |\nabla u|^p + \alpha^p \int_{\Gamma_2} |u|^p \, : \, u \in W^{1,p}_{\Gamma_1}(U), \, \|u\|_{L^p(U)} = 1 \right\}$$

where $W^{1,p}_{\Gamma_1}(U) = \left\{ u \in W^{1,p}(U), \, u = 0 \text{ on } \Gamma_1 \right\}$. Note that the infimum is attained since we assumed that $|\Gamma_1| > 0$. Also notice that if we regard λ_p as a function of α, $\alpha \in [0, +\infty) \to \lambda_p(\alpha)$, then $\lambda_p(\alpha)$ is non-decreasing with $\lim_{\alpha \to +\infty} \lambda_p = \lambda_{p,D}$ the first Dirichlet eigenvalue for the $p-$Laplacian in U.

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We expect the limit problem of (2) as $p \to \infty$ to be

$$\lambda_\infty = \inf_{u \in W^{1,\infty}_{\Gamma_1}(U),\, \|u\|_{L^\infty(U)}=1,\, u \geq 0} \max \Big\{ \|\nabla u\|_{L^\infty(U)},\, \alpha \|u\|_{L^\infty(\Gamma_2)} \Big\} \tag{3}$$

where $W^{1,\infty}_{\Gamma_1}(U) = \Big\{ u \in W^{1,\infty}(U),\, u = 0 \text{ on } \Gamma_1 \Big\}$. Notice that when we let $\alpha \to +\infty$ in (3) with $\Gamma_1 = \partial U$ we obtain

$$\lim_{\alpha \to +\infty} \lambda_\infty(\alpha) = \lambda_{\infty,D} = \inf_{u \in W^{1,\infty}_0(U),\, \|u\|_{L^\infty(U)}=1} \|\nabla u\|_{L^\infty(U)}$$

that is the first eigenvalue of the infinity Laplacian, $\Delta_\infty u = DuD^2uDu$ with Dirichlet boundary conditions. This value, $\lambda_{\infty,D}$, turns out to be the limit of $(\lambda_{p,D})^{1/p}$ as $p \to \infty$, see [14]. Our first result says that this kind of limit can be also computed for any nonnegative α.

Theorem 1. *There holds that*

$$\lim_{p \to +\infty} (\lambda_p)^{1/p} = \lambda_\infty.$$

Moreover the positive, normalized extremals for λ_p, u_p converge uniformly in $\bar{U}$ along subsequences $p_j \to \infty$ to $u \in X$ which is a minimizer for (3) and a viscosity solution to

$$\begin{cases} \min\{|Du| - \lambda_\infty u,\, -\Delta_\infty u\} = 0 & \text{in } U, \\ u = 0 & \text{on } \Gamma_1, \\ \min\{|Du| - \alpha u, ; -\partial_\nu u\} = 0 & \text{on } \Gamma_2. \end{cases}$$

Our next goal is to characterize this limit value λ_∞. The value of λ_∞ results of the interplay between α, the geometry of U and the sets Γ_1, Γ_2. We consider the (possibly empty) set

$$\mathcal{A} := \Big\{ x \in \bar{U},\, d(x, \Gamma_1) \geq \frac{1}{\alpha} + d(x, \Gamma_2) \Big\}.$$

Notice that if $\mathcal{A} \neq \emptyset$ then the set

$$\mathcal{A}' := \Big\{ x \in \bar{U},\, d(x, \Gamma_1) = \frac{1}{\alpha} + d(x, \Gamma_2) \Big\}$$

is also not empty. Indeed the function $f(x) = \frac{1}{\alpha} + d(x, \Gamma_2) - d(x, \Gamma_1)$ is continuous, less or equal to 0 on $\mathcal{A}$, and greater or equal to 0 if $d(x, \Gamma_1) << 1$ (we are using here the fact that U is connected to apply the mean value theorem). Our next result gives a geometrical characterization of λ_∞.

Theorem 2. *It holds that*

$$\lambda_\infty = \begin{cases} \min_{x \in \bar{U}} \dfrac{1}{d(x, \Gamma_1)}, & \text{if } \mathcal{A} = \emptyset, \\ \min_{x \in \mathcal{A}} \dfrac{1}{\frac{1}{\alpha} + d(x, \Gamma_2)} = \min_{x \in \mathcal{A}'} \dfrac{1}{\frac{1}{\alpha} + d(x, \Gamma_2)}, & \text{if } \mathcal{A} \neq \emptyset. \end{cases} \tag{4}$$

Notice that when $\alpha = +\infty$, which corresponds to pure Dirichlet boundary conditions on the whole ∂U, then $\mathcal{A} = \mathcal{A}' = \emptyset$ and we recover the result of [14], $\lambda_\infty^{-1} = \lambda_{\infty,D}^{-1} = \max_{x \in \bar{U}} d(x, \partial U)$. In the case of Neumann boundary conditions i.e. $\Gamma_1 = \emptyset$ and $\alpha = 0$ then $\mathcal{A} = \emptyset$ and $d(x, \Gamma_1) = d(x, \emptyset) = +\infty$ for any $x \in \bar{U}$ so that $\lambda_\infty = 0$ which is consistent with the fact the 1st eigenvalue of Δ_p with Neumann boundary conditions is 0.

We will first give a simple proof in the case where U is convex by using a test-function argument based proof which we were not able to extend to the general case. In fact the result for a arbitrary connected domain will be a consequence of an optimal mass transport formulation of λ_∞ that we now introduce.

To continue our analysis we have to recall some notions and notations from optimal mass transport theory. Recall that the Monge-Kantorovich distance $W_1(\mu,\nu)$ between two probability measures μ and ν over $\bar{U}$ is defined by

$$W_1(\mu,\nu) = \max_{v\in W^{1,\infty}(U),\, \|\nabla v\|_\infty \le 1} \int_U v\,(d\mu - d\nu). \tag{5}$$

Recently the authors in [7] relate $\lambda_{\infty,D}$ with the Monge-Kantorovich distance W_1. They proved that

$$\lambda_{\infty,D}^{-1} = \max_{\mu\in P(U)} W_1(\mu, P(\partial U)), \tag{6}$$

where $P(U)$ and $P(\partial U)$ denotes the set of probability measures over $\bar{U}$ and ∂U. Notice that the maximum is easily seen to be reached at δ_x where $x\in U$ is a most inner point.

In our case we are also able to give a characterization for λ_∞ in terms of a maximization problem involving W_1 but this time we get an extra term involving the total variation of a measure on Γ_2.

Theorem 3. *It holds that*

$$\frac{1}{\lambda_\infty} = \max_{\sigma\in P(\bar{U})} \inf_{\nu\in P(\partial U)} \Big\{ W_1(\sigma,\nu) + \frac{1}{\alpha}\nu(\Gamma_2)\Big\}. \tag{7}$$

Moreover, the measures $u_p^{p-1}\,dx$ weakly converge (up to a subsequence) as $p\to+\infty$ to a probability measure f_∞ which attains the maximum in (7).

Notice that when $\alpha = +\infty$, which corresponds to Dirichlet boundary conditions, then we recover the result of [7], who showed that (6) holds.

As a corollary of this characterization in terms of optimal transportation, we can extend the result stated in Theorem 2 for the value of λ_∞ to the case where U is not convex. We prefer to present our results in this order (even if Theorem 2 is not initially proved in its full generality) for readability of the whole paper (the proof of Theorem 2 in the convex case is much simpler).

Let us end the introduction with a brief description of the previous bibliography and the main ideas and techniques used to prove our results. First, as by now classical results, we mention that the limit as $p\to\infty$ of the first eigenvalue $\lambda_{p,D}$ of the p-Laplacian with Dirichlet boundary condition was studied in [15], [14] (see also [4] for an anisotropic version). For its dependence with respect to the domain we refer to [17]. The *limit operator* that appears here, the infinity-Laplacian is given by the limit as $p\to\infty$ of the $p-$Laplacian, in the sense that solutions to $\Delta_p v_p = 0$ with a Dirichlet data $v_p = f$ on $\partial\Omega$ converge as $p\to\infty$ to the solution to $\Delta_\infty v = 0$ with $v = f$ on $\partial\Omega$ in the viscosity sense (see [2], [5] and [8]). This operator appears naturally when one considers absolutely minimizing Lipschitz extensions in Ω of a boundary data f (see [1], [2], and [13]).

The case of a Steklov boundary condition (here the eigenvalue appears in the boundary condition) has also been investigated recently. Indeed in [11] (see also [16] for a slightly different problem) it is studied the behaviour as $p \to +\infty$ of the so-called variational eigenvalues $\lambda_{k,p,S}$, $k \geq 1$, of the p-Laplacian with a Steklov boundary condition. In particular it is proved that

$$\lim_{p\to+\infty} \lambda_{1,p,S}^{1/p} = 1 \quad \text{ and } \quad \lambda_{2,\infty,S} := \lim_{p\to+\infty} \lambda_{2,p,S}^{1/p} = \frac{2}{\operatorname{diam}(U,\mathbb{R}^n)},$$

where here $\operatorname{diam}(U,\mathbb{R}^n)$ denotes the diameter of U for the usual Euclidean distance in $\mathbb{R}^n$.

For pure Neumann eigenvalues, we quote [10] and [19]. In those references it is considered the limit for the second eigenvalue (the first one is zero). It is proved that in this case $\lambda_\infty := \lim_{p\to+\infty} \lambda_p^{1/p} = 2/\operatorname{diam}(U)$, where $\operatorname{diam}(U)$ denotes the diameter of U with respect to the geodesic distance in U. In addition, the regularity of λ_∞ as a function of the domain U is studied in [19] and in [10] it is proved that there are no nonzero eigenvalues below λ_∞, so that λ_∞ is indeed the first nontrivial eigenvalue for the infinity-Laplacian with Neumann boundary conditions.

Concerning ideas and methods used in the proofs we use classical variational ideas to obtain the limit of $(\lambda_p)^{1/p}$ and viscosity techniques and to find the limit PDE problem we use viscosity techniques as in [14] (we refer to [8] for the definition of a viscosity solution). The characterization of λ_∞ given in Theorem 2 follows using cones as test functions in the variational formulation. Finally, mass transport techniques (we refer to [20]) and gamma-convergence of functionals are used to show the more general characterization of λ_∞ given in Theorem 3, see [7] and [19] for similar arguments in different contexts.

The paper is organized as follows. In Section 2 we deal with the limit as $p \to \infty$ and prove Theorem 1. in Section 3 we prove Theorem 2 that characterizes λ_∞ in geometrical terms in the cases of a convex domain U. In Section 4 we use optimal transport ideas to obtain Theorem 3. As a corollary, we eventually prove Theorem 2 for a general connected domain in the last section.

2. Proof of Theorem 1

For the proof of Theorem 1 we will use the following lemma.

Lemma 1. *For any $f, g \in L^\infty(U)$ there holds*

$$\lim_{p\to+\infty} \Big(\|f\|_{L^p(U)} + \|g\|_{L^p(U)} \Big)^{\frac{1}{p}} = \max\{\|f\|_{L^\infty(U)}, \|g\|_{L^\infty(U)}\}.$$

Proof. The result is a direct consequence of the inequalities

$$\begin{aligned} &\max\{\|f\|^p_{L^p(U)}, \|g\|^p_{L^p(U)}\} \\ &\quad \leq \|f\|^p_{L^p(U)} + \|g\|^p_{L^p(U)} \\ &\qquad \leq 2\max\{\|f\|^p_{L^p(U)}, \|g\|^p_{L^p(U)}\}. \end{aligned}$$

In fact, from the previous inequalities, we get

$$\begin{aligned}\lim_{p\to+\infty} & \max\{\|f\|_{L^p(U)}, \|g\|_{L^p(U)}\} \\ & \le \lim_{p\to+\infty}\Big(\|f\|_{L^p(U)} + \|g\|_{L^p(U)}\Big)^{\frac{1}{p}} \\ & \le \lim_{p\to+\infty} 2^{\frac{1}{p}} \max\{\|f\|_{L^p(U)}, \|g\|_{L^p(U)}\}.\end{aligned}$$

We conclude using that

$$\lim_{p\to+\infty} \|f\|_{L^p(U)} = \|f\|_{L^\infty(U)}$$

and

$$\lim_{p\to+\infty} \|g\|_{L^p(U)} = \|g\|_{L^\infty(U)}.$$

□

Now let us proceed with the proof of Theorem 1.

Proof of Theorem 1. Let $u \in X$ then $u \in \cap_p X_p$. From the variational characterization of λ_p we have

$$(\lambda_p)^{1/p} \le \frac{1}{\|u\|_{L^p(U)}}\Big(\int_U |\nabla u|^p + \alpha^p \int_{\Gamma_2} |u|^p\Big)^{1/p}.$$

Hence, using the previous Lemma we get

$$\limsup_{p\to\infty}(\lambda_p)^{1/p} \le \max\left\{\|\nabla u\|_{L^\infty(U)}, \, \alpha\|u\|_{L^\infty(\Gamma_2)}\right\}$$

for any $u \in X$. Therefore, we conclude that

$$\limsup_{p\to\infty}(\lambda_p)^{1/p} \le \lambda_\infty.$$

In addition, we get that, for u_p an eigenfunction associated to λ_p in X_p it holds that

$$\limsup_{p\to\infty} \|\nabla u_p\|_{L^p(U)} \le \lambda_\infty.$$

Therefore, we have that $\{u_p\}$ is uniformly bounded (independently of p) in $W^{1,p}(U)$. Then, for any fixed q we obtain

$$\|\nabla u_p\|_{L^q(U)} \le \|\nabla u_p\|_{L^p(U)} |U|^{\frac{p-q}{qp}} \le C$$

with C independent of p. Hence, by a diagonal procedure, we can extract a subsequence $p_j \to \infty$ such that

$$u_{p_j} \to u$$

uniformly in $\overline{U}$ and weakly in every $W^{1,q}(U)$, $q \in \mathbb{N}$. This limit u verifies that

$$\|\nabla u\|_{L^q(U)} \le \limsup_{p\to\infty} \le \|\nabla u_p\|_{L^q(U)} \le \limsup_{p\to\infty} \|\nabla u_p\|_{L^p(U)} |U|^{\frac{p-q}{qp}} \le \lambda_\infty |U|^{\frac{1}{q}}$$

and then we get

$$\|\nabla u\|_{L^\infty(U)} \le \lambda_\infty.$$

Moreover, we have

$$\alpha\|u_p\|_{L^q(\Gamma_2)} \le \Big(\alpha^p \|u_p\|^p_{L^p(\Gamma_2)} |\Gamma_2|^{\frac{p-q}{q}}\Big)^{1/p} \le \Big(\lambda_p |\Gamma_2|^{\frac{p-q}{q}}\Big)^{1/p},$$

then

$$\alpha\|u\|_{L^q(\Gamma_2)} \le \limsup_{p\to\infty} \alpha\|u_p\|_{L^q(\Gamma_2)} \le \limsup_{p\to\infty} \alpha\|u_p\|_{L^q(\Gamma_2)} \le \lambda_\infty|\Gamma_2|^{\frac{1}{q}}$$

and we conclude that

$$\alpha\|u\|_{L^\infty(\Gamma_2)} \le \lambda_\infty.$$

Hence

$$\max\Big\{\|\nabla u\|_{L^\infty(U)},\, \alpha\|u\|_{L^\infty(\Gamma_2)}\Big\} \le \lambda_\infty.$$

Now, we only have to observe that form the uniform convergence we get that $u \in X$, and then we conclude that u is a minimizer of (3). In addition, our previous calculations show that

$$\lambda_\infty \le \liminf_{p\to\infty} (\lambda_p)^{\frac{1}{p}}.$$

Now, concerning the equation verified by the limit of u_p, u, we have that, from the fact that u_p are viscosity solutions to $\Delta_p u = \lambda_p|u|^{p-2}u$ and that $(\lambda_p)^{1/p}$ converges to λ_∞ we conclude as in [14] that the limit u is a viscosity solution to

$$\min\{|Du| - \lambda_\infty u,\, -\Delta_\infty u\} = 0.$$

That $u = 0$ on Γ_1 is immediate from uniform convergence in $\overline{U}$ and the fact that u_p verify the same condition.

On Γ_2 we have

$$|\nabla u|^{p-2}\partial_\nu u + \alpha^p|u|^{p-2}u = 0,$$

therefore, passing to the limit in the viscosity sense as done in [12] we obtain

$$\min\{|Du| - \alpha u, ; -\partial_\nu u\} = 0.$$

This ends the proof. □

3. Proof of Theorem 2 for convex domains.

Along this section we assume that U is convex.

Proof of Theorem 2. Using the variational characterization (3) proved in the previous section, we estimate λ_∞ from above by using as test-function a truncated cone of the form

$$u(x) = \Big(1 - a|x - x_0|\Big)_+$$

where $a > 0$ and $x_0 \in \bar{U}$. Then

$$u \equiv 0 \text{ on } \Gamma_1 \qquad \text{iff} \qquad a \ge \frac{1}{d(x_0,\Gamma_1)}$$

$$\|\nabla u\|_{L^\infty(U)} = a,$$

$$\text{and} \qquad \|u\|_{L^\infty(\Gamma_2)} = \Big(1 - ad(x_0,\Gamma_2)\Big)_+.$$

It follows that

$$\lambda_\infty \le \inf \max\{a,\, \alpha[1 - ad(x_0,\Gamma_2)]_+\}$$

where the infimum is taken over all the $x_0 \in \bar{U}$ and $a > 0$ such that $a \ge 1/d(x_0,\Gamma_1)$. Examining the two possibilities for the max, we obtain easily the upper bound for λ_∞.

To prove the lower bound we argue as follows: for any $x_0 \in \bar{U}$, and any Lipschitz function $u \in X$ with $u(x_0) = 1$, we have

$$1 \geq \|u\|_{L^\infty(\Gamma_2)} \geq \Big(1 - \|\nabla u\|_\infty d(x_0, \Gamma_2)\Big)_+.$$

Thus

$$\lambda_\infty \geq \inf \max \Big\{ \|\nabla u\|_{L^\infty(U)}, \alpha \Big(1 - \|\nabla u\|_{L^\infty(U)} d(x_0, \Gamma_2)\Big)_+ \Big\}$$

where the infimum is taken over all $u \in W^{1,\infty}(\bar{U})$ such that $u = 0$ in Γ_1, $\|u\|_{L^\infty(U)} = 1$ and $\|\nabla u\|_{L^\infty(U)} \geq \frac{1}{d(x_0,\Gamma_1)}$ for any $x_0 \in \{u = 1\}$. From this point the argument conclude as for the previous case just analyzing the possibilities for the max. □

4. Proof of theorem 3

The proof follows the lines of [7] (see also [19] for the pure Neumann boundary case).

We begin rewriting the variational formulation (2) of λ_p as

$$1 = \sup \Big\{ \int_U |u|^p \ : \ u \in W^{1,p}_{\Gamma_1}(U) \text{ s.t. } \int_U |\nabla u|^p + \alpha^p \int_{\Gamma_2} |u|^p = \lambda_p \Big\}.$$

We are thus lead to consider the functions $G_p : C(\bar{U}) \times M(\bar{U}) \to \mathbb{R}$, $p \geq 1$, defined by

$$G_p(v, \sigma) = \begin{cases} -\displaystyle\int v\, d\sigma & \text{if } v \in W^{1,p}_{\Gamma_1}(U),\ \displaystyle\int_U |\nabla v|^p + \alpha^p \int_{\Gamma_2} |v|^p \leq \lambda_p^p, \\ & \text{and } \sigma \in L^{p'}(U)\ \int_U |\sigma|^{p'} \leq 1, \\ +\infty & \text{otherwise} \end{cases}$$

Notice that the pair $(u_p, u_p^{p-1}\, dx)$ is an extremal for G_p so that $\min G_p = -1$. Indeed for any admissible pair $(v, \sigma) \in W^{1,p}_{\Gamma_1}(U) \times L^{p'}(U)$, we have

$$\begin{aligned} -G_p(v, \sigma) &= \int_U v\sigma \leq \|v\|_p \|\sigma\|_{p'} \leq \lambda_p^{-1/p} \Big(\int_U |\nabla v|^p + \alpha^p \int_{\Gamma_2} |v|^p \Big)^{1/p} \\ &\leq 1 = \int_U u_p^p \end{aligned}$$

(we used succesively Hölder's inequality, the definition of λ_p and the fact that v is admissible). In view of Lemma 1, we introduce the formal limit functional $G_\infty : C(\bar{U}) \times M(\bar{U}) \to \mathbb{R}$ of the G_p by

$$G_\infty(v, \sigma) = \begin{cases} -\displaystyle\int v\, d\sigma & \text{if } v \in W^{1,\infty}_{\Gamma_1}(U),\ \max\{\|\nabla u\|_\infty, \alpha \|v\|_{L^\infty(\Gamma_2)}\} \leq \lambda_\infty, \\ & \text{and } |\sigma|(\bar{U}) \leq 1, \\ +\infty & \text{otherwise.} \end{cases}$$

The convergence of the functionals G_p to G_∞ can be justified using the notion of Γ-convergence. Recall that a sequence of functionals $F_n : X \to [0, +\infty]$ defined over a metric space X is said to Γ-converge to a functional $F_\infty : X \to [0, +\infty]$ if the following two conditions hold:

- for every $x \in X$ and every sequence $(x_n)_n \subset X$ converging to x, $F(x) \le \liminf F(x_n)$,
  and
- for any $x \in X$, there exists a sequence $(x_n)_n \subset X$ converging to x such that $F(x) \ge \limsup F(x_n)$.

An easy but important consequence of the definition, that we will use later, is the fact that if x_n is a minimizer of F_n then every cluster point of the sequence (x_n) is a minimizer of F_∞. We refer e.g. to [6] and [9] for a detailed account on Γ-convergence.

Proposition 4.1. *The functionals G_p Γ-converge as $p \to +\infty$ to G_∞.*

Proof. The proof is very similar to [7] (see also [19] for the pure Neumann boundary case). We briefly sketch it for the reader's convenience.

Assume that $(v_p, \sigma_p) \in C(\bar{U}) \times M(\bar{U})$ converge to (v, σ). We have to prove that

$$\liminf_{p \to +\infty} G_p(v_p, \sigma_p) \ge G(v, \sigma). \tag{8}$$

We can assume that $G_p(v_p, \sigma_p) < \infty$. Then we have

$$\int_U v_p \sigma_p \, dx - \int_U v \, d\sigma = \int_U (v_p - v)\sigma_p \, dx + \int_U v \, (\sigma_p \, dx - d\sigma) \to 0$$

as $p \to +\infty$. Indeed the first integral on the right hand side can be bounded by $\|v_p - v\|_\infty \|\sigma_p\|_{p'} |U|^{\frac{1}{p}} = o(1)$. Independently

$$\int_U |\sigma| = \int_U |\sigma_p| \, dx + o(1) \le \|\sigma_p\|_{p'} |U|^{\frac{1}{p}} + o(1) \le 1 + o(1)$$

so that $\int_U |\sigma| \le 1$. Moreover taking limit in $\alpha \|v_p\|_{L^p(\Gamma_2)} \le \lambda_p$ yields $\alpha \|v\|_{L^\infty(\Gamma_2)} \le \lambda_\infty$. Eventually, for any $\phi \in L^{p'}(U, \mathbb{R}^n)$ such that $\|\phi\|_{p'} \le 1$ we have

$$\begin{aligned} \int_U \phi \nabla v \, dx = -\int_U v \operatorname{div} \phi \, dx = -\int_U v_p \operatorname{div} \phi \, dx + o(1) = \int_U \phi \nabla v_p \, dx + o(1) \\ \le \|\nabla v_p\|_p + o(1) \le \lambda_p^{\frac{1}{p}} + o(1) = \lambda_\infty + o(1), \end{aligned}$$

where the $o(1)$ does not depend on ϕ. Taking the supremum over all such ϕ we obtain $\|\nabla v\|_p \le \lambda_\infty + o(1)$, so that $\|\nabla v\|_\infty \le \lambda_\infty$. It follows that (v, σ) is admissible for G_∞.

We now fix a pair (v, σ) admissible for G_∞. We have to find some pair (v_p, σ_p) admissible for G_p which converges to (v, σ) and such that

$$\limsup_{p \to +\infty} G_p(v_p, \sigma_p) \le G_\infty(v, \sigma).$$

We define

$$v_p = \frac{\lambda_p^{\frac{1}{p}}}{\lambda_\infty (|U| + |\Gamma_2|)^{\frac{1}{p}}} v.$$

Then $v_p \in W^{1,p}(U)$, $v_p \to v$ uniformly, and $\int_U |\nabla v_p|^p + \alpha^p \int_{\Gamma_2} |v_p|^p \le \lambda_p^p$.

In order to define σ_p by regularizing σ by convolution, we first need to adjust a little. Let $\vec{n}$ be the unit inner normal vector to U that we extend in a smooth way to $\mathbb{R}^n$ with compact support in a neighborhood of ∂U. We consider $T_\varepsilon : \bar{U} \to \bar{U}_{2\varepsilon} := \{x \in \bar{U},\, dist(x, \partial U \geq 2\varepsilon\}$ defined by $T_\varepsilon(x) = x + 2\varepsilon\vec{n}$. Let $\sigma_\varepsilon = T_\varepsilon \sharp \sigma$ be the push-forward of σ by T_ε i.e. $\int f\, d\sigma_\varepsilon = \int f \circ T_\varepsilon\, d\sigma$ for any $f \in C(\bar{U}_{2\varepsilon})$. Oberve that supp $\sigma_\varepsilon \subset \bar{U}_{2\varepsilon}$ and also that $\int |\sigma_\varepsilon| \leq 1$ since

$$\begin{aligned} \int |\sigma_\varepsilon| &= \sup_{\|\phi\|_{L^\infty(U_{2\varepsilon})} \leq 1} \int \phi\, d\sigma_\varepsilon = \sup_{\|\phi\|_{L^\infty(U_{2\varepsilon})} \leq 1} \int \phi \circ T_\varepsilon\, d\sigma \\ &\leq \int d|\sigma| \leq 1. \end{aligned}$$

Moreover

$$\sigma_\varepsilon \to \sigma \qquad \text{weakly in the sense of measure.}$$

Indeed for any $\phi \in C(\bar{U})$,

$$\Big| \int \phi\, d\sigma_\varepsilon - \int \phi\, d\sigma \Big| \leq \int \Big| \phi(x + 2\varepsilon\vec{n}) - \phi(x) \Big|\, d\sigma(x) = o(1)$$

since the integrand goes to 0 uniformly in $x \in \bar{U}$. Denote by ρ_ε the usual mollifying functions (i.e. $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ where ρ is a smooth function compactly supported in the unit ball of $\mathbb{R}^n$ with $\int \rho = 1$). Then

$$\rho_\varepsilon * \sigma_\varepsilon - \sigma_\varepsilon \to 0 \qquad \text{weakly in the sense of measure}$$

This follows from the fact that $\|\phi * \rho_\varepsilon - \phi\|_{L^\infty(U_{2\varepsilon})} \to 0$ for any $\phi \in C(\bar{U})$. Hence

$$\rho_\varepsilon * \sigma_\varepsilon \rightharpoonup \sigma \qquad \text{weakly in the sense of measure.} \tag{9}$$

We now regularize σ_ε considering

$$\tilde{\sigma}_\varepsilon := \sigma_\varepsilon * \tilde{\rho}_\varepsilon \in C^\infty(U)$$

with

$$\tilde{\rho}_\varepsilon := \frac{\rho_\varepsilon}{\|\rho_\varepsilon\|_{p'}}, \qquad \varepsilon = 1/p.$$

Then $\|\rho_\varepsilon\|_{p'} \to 1$ since $\|\rho_\varepsilon\|_{p'} = \varepsilon^{-n/p}\|\rho\|_{p'} \to \|\rho_\varepsilon\|_1 = 1$. It then follows that $\tilde{\sigma}_\varepsilon \rightharpoonup \sigma$. Moreover $\tilde{\sigma}_\varepsilon$ is admissible for G_p since, by Holder inequality and recalling (9),

$$\|\tilde{\sigma}_\varepsilon\|_{p'}^{p'} \leq \Big(\int |\sigma_\varepsilon| \Big)^{\frac{1}{p-1}} \int \tilde{\rho}_\varepsilon(x-y)^{p'}\, dx\, d|\sigma_\varepsilon|(y) = \|\tilde{\rho}_\varepsilon\|_{p'}^{p'} \Big(\int |\sigma_\varepsilon| \Big)^{\frac{p}{p-1}} \leq 1.$$

It follows that $(\sigma_\varepsilon, v_p)$ is admissible for G_p and converge to (v, σ). As before we have $G_p(v_p, \sigma_\varepsilon) \to G_\infty(v, \sigma)$. □

Recall that from Theorem 1, u_p converge in $C(\bar{U})$ up to a subsequence to some $u_\infty \in C(\bar{U})$, $\|u\|_\infty = 1$. Moreover, up to a subsequence, the measures $u_p^{p-1}\, dx$ converge weakly to some probability measure σ_∞. Indeed since $\bar{U}$ is compact, it suffices, according to Prokhorov theorem, to show that

$$\lim_{p \to +\infty} \int_{\bar{U}} u_p^{p-1}\, dx = 1.$$

This follows from

$$\int_{\bar{U}} u_p^{p-1}\, dx \leq \|u_p\|_p |U|^{1/p} \to 1$$

and, for $p > n$,

$$1 = \int_{\bar{U}} u_p^{p-1} u_p \, dx \le \|u_p\|_\infty \int_{\bar{U}} u_p^{p-1} \, dx = (1 + o(1)) \int_{\bar{U}} u_p^{p-1} \, dx.$$

As a consequence of the Γ-convergence of G_p to G_∞ and the fact that $(u_p, u_p^{p-1} \, dx)$ is a minimizer of G_p, we obtain that $(u_\infty, \sigma_\infty)$ is a minimizer of G_∞ with $G_\infty(u_\infty, \sigma_\infty) = \lim_{p\to+\infty} G_p(u_p, u_p^{p-1} \, dx) = -1$. Since $\sigma_\infty \in P(\bar{U})$ and u_∞ is an extremal for λ_∞, we can thus write

$$1 = \max \quad \Big\{ \int v \, d\sigma; v \in W^{1,\infty}_{\Gamma_1}(U), \\ \max\{\|\nabla v\|_\infty, \alpha \|v\|_{L^\infty(\Gamma_2)}\} = \lambda_\infty, \\ \sigma \in P(\bar{U}) \Big\}$$

i.e.

$$\lambda_\infty^{-1} = \max \quad \Big\{ \int v \, d\sigma; v \in W^{1,\infty}_{\Gamma_1}(U), \\ \max\{\|\nabla v\|_\infty, \alpha \|v\|_{L^\infty(\Gamma_2)}\} = 1, \\ \sigma \in P(\bar{U}) \Big\} \tag{10}$$

An appoximation argument shows that we can replace $W^{1,\infty}_{\Gamma_1}(U)$ by $C^1(U) \cap C_{\Gamma_1}(\bar{U})$ where $C_{\Gamma_1}(\bar{U}) = \{u \in C(\bar{U}) : u = 0 \text{ on } \Gamma_1\}$.

Proposition 4.2. *Given $v \in W^{1,\infty}_{\Gamma_1}(U)$, $\max\{\|\nabla v\|_\infty, \alpha \|v\|_{L^\infty(\Gamma_2)}\} \le 1$, there exist $v_k \in C^1(U) \cap C_{\Gamma_1}(\bar{U})$, $\max\{\|\nabla v_k\|_\infty, \alpha \|v_k\|_{L^\infty(\Gamma_2)}\} \le 1$, such that $v_k \to v$ uniformly in $\bar{U}$.*

Proof. The proof uses ideas from [7]. We first extend v in a neighborhood of ∂U by antisymmetric reflection across ∂U so that the extended function $\bar{v}$ is Lipschitz with $\|\nabla \bar{v}\|_\infty = \|\nabla v\|_\infty \le 1$. We then apply the same method as in [7] consisting in introducing the function $\theta_\varepsilon(t) = (t - sgn(t)\varepsilon) 1_{|t|\ge\varepsilon}$ and then regularizing $\theta_\varepsilon \circ \bar{v}$ by convolution with the usual mollifying functions. Observe that $\|\nabla(\theta_\varepsilon \circ \bar{v})\|_\infty \le \|\nabla \bar{v}\|_\infty \le 1$ and that $\theta_\varepsilon \circ \bar{v} = 0$ in the ε-neighborhood $\{x \in \mathbb{R}^n, \, dist(x, \Gamma_1) < \varepsilon\}$ of Γ_1 since $\bar{v}$ is 1-Lipschitz. Note also that $|\theta(t)| = (|t| - \varepsilon) 1_{|t|\ge\varepsilon}$ so that on Γ_2, $|\theta_\varepsilon \circ \bar{v}| \le (\alpha^{-1} - \varepsilon)_+$. Hence $|\theta_\varepsilon \circ \bar{v}| \le \alpha^{-1}$ in the ε-neighborhood of Γ_2. It follows from these three comments that the regularizing of $\theta_\varepsilon \circ \bar{v}$ is adequate. □

Denoting by $Res : C(\bar{U}) \to C(\Gamma_2)$ the restriction operator, $Au = \nabla u$ the derivation operator with domain $C^1(U)$, and $B(R)$ the closed ball of radius R centered at 0 in $C(\bar{U})$, $B = B(1)$, we can rewrite (10) as

$$\frac{1}{\lambda_\infty} = \max_{\sigma \in P(\bar{U})} \max_{u \in C(\bar{U})} \Big\{ (\sigma, u) - (\chi_{B(1/\alpha)} \circ Res)(u) - (\chi_B \circ A)(u) \\ - \chi_{C_{\Gamma_1}(U)}(u) \Big\}.$$

Recalling the definition of the Legendre transform, we eventually obtain

$$\frac{1}{\lambda_\infty} = \max_{\sigma \in P(\bar{U})} \Big((\chi_{B(1/\alpha)} \circ Res) + (\chi_B \circ A) + \chi_{C_{\Gamma_1}(U)} \Big)^* (\sigma). \tag{11}$$

The inf-convolution $f \,\square\, g$ of two proper lower semi-continuous (lsc) convex functions $f, g : E \to \mathbb{R}$ (E denotes a normed space - we will take $E = C(\bar{U})$

here) is defined by $(f \,\Box\, g)(x) = \inf_{y\in E} f(y) + g(x-y)$. This operation is commutative and associative. Moreover $f \,\Box\, g$ is a proper lsc convex function with domain $\mathrm{Dom}(f) + \mathrm{Dom}(g)$, and its Legendre transform is $(f \,\Box\, g)^* = f^* + g^*$. Eventually, if 0 belongs to the interior of $\mathrm{Dom}(f) - \mathrm{Dom}(g)$ then $(f+g)^* = f^* \,\Box\, g^*$ (see [18][section3.9 p42]). This last assumption is trivially satisfied here since any neigborhood of 0 in $C(\bar U)$ is contained in $C^1(\bar U) + C(\bar U)$.

We can thus rewrite (11) as

$$\begin{aligned} \frac{1}{\lambda_\infty} &= \max_{\sigma\in P(\bar U)} \Big((\chi_{B(1/\alpha)} \circ Res)^* \,\Box\, (\chi_B \circ A)^* \,\Box\, \chi^*_{C_{\Gamma_1}(U)} \Big)(\sigma) \\ &= \max_{\sigma\in P(\bar U)} \inf \Big(\chi_{B(1/\alpha)} \circ Res)^*(\mu_1) + (\chi_B \circ A)^*(\mu_2) + \chi^*_{C_{\Gamma_1}(U)}(\mu_3) \Big), \qquad (12) \end{aligned}$$

where the inf is taken over all triple of measures $\mu_1, \mu_2, \mu_3 \in M(\bar U)$ such that $\sigma = \mu_1 + \mu_2 + \mu_3$. To pursue further we need to compute the various Legendre transforms involved in this expression. This is the content of the next proposition.

Proposition 4.3. *There holds for $\mu \in M(\bar U)$,*

$$\chi^*_{C_{\Gamma_1}(U)}(\mu) = \begin{cases} 0 & \text{if } supp\ \mu \subset \Gamma_1 \\ +\infty & otherwise \end{cases} \qquad (13)$$

and

$$\begin{aligned} (\chi_B \circ A)^*(\mu) &= \inf \Big\{ \int_{\bar U} |\sigma| : \sigma \in M(\bar U, \mathbb{R}^n) \ s.t. \ -div\ \sigma = \mu \ in\ \mathcal{D}'(\mathbb{R}^n) \Big\} \\ &= \begin{cases} W_1(\mu^+, \mu^-) \ \text{if } \mu(\bar U) = 0, \\ +\infty \ otherwise. \end{cases} \qquad (14) \end{aligned}$$

Moreover,

$$(\chi_{B(1/\alpha)} \circ Res)^*(\mu) = \begin{cases} \frac{1}{\alpha}|\mu|(\Gamma_2) & \text{if } supp\ \mu \subset \Gamma_2 \\ +\infty & otherwise. \end{cases} \qquad (15)$$

Proof. These computations are more or less classical. We sketch them here for the reader's convenience.

First, the definition of the Legendre transform gives

$$\begin{aligned} \chi^*_{C_{\Gamma_1}(U)}(\mu) &= \sup_{u\in C(\bar U)} (\mu, u) - \chi_{C_{\Gamma_1}(U)}(u) \\ &= \sup_{u\in C(\bar U),\, u=0 \text{ on } \Gamma_1} \int_{\bar U} u\, d\mu \qquad (16) \end{aligned}$$

from which we deduce (13).

We now prove (14). The second equality in (14) is well-known. It remains to prove the first one. We recall the following result concerning the Legendre transform: if E and F are two normed space, $L : E \to F$ linear with domain $Dom(L)$ and $f : E \to \mathbb{R}$ is convex, consider the function $(LF)(y) = \inf \{ f(x) \ : \ x \in Dom(L) \text{ s.t. } Lx = y \}$, $y \in F$. Then Lf is convex with $(Lf)^* = f^* \circ L^*$ in the domain $Dom(L^*)$ of the adjoint $L^* : F^* \to E^*$ of L.

Notice that the adjoint $A^* : M(\bar U) \to M(\bar U)$ of A is defined by $A^*\mu = -\mathrm{div}\,\mu$ in the weak sense (i.e. $(A^*\mu, u) = (\mu, \nabla u) = \int \nabla u\, d\mu$ for any $u \in Dom(A) = C^1(\bar U)$) with domain $Dom(A^*) = \{\mu \in M(\bar U),\, -\mathrm{div}\,\mu \in M_b(\mathbb{R}^n)\}$.

In a similar way as in (16), it can be seen that $\chi_B^*(\sigma) = \int |\sigma|$, so that the inf in (14) can be written as $(A^*\chi_B^*)(\mu)$. Then taking $f = \chi_B^*$, $L = A^*$ and noticing that χ_B is convex lsc (because B is convex and closed), so that $\chi_B^{**} = \chi_B$, we obtain $\chi_B \circ A^{**} = (A^*\chi_B^*)^*$. Observe that $A^{**} = A$ on $Dom(A)$ so that $\chi_B \circ A = (A^*\chi_B^*)^*$ on $Dom(A)$.

Observe that $A^*\chi_B^*$, which is the r.h.s of (14), is lsc for the weak convergence (and thus also for the strong i.e. total variation convergence) in the sense that if $\mu_n, \mu \in M(\bar U)$ verify $\mu_n \to \mu$ weakly then

$$\liminf_{n\to+\infty}(A^*\chi_B^*)(\mu_n) \geq (A^*\chi_B^*)(\mu).$$

Indeed we can assume that $(A^*\chi_B^*)(\mu_n) \leq Cste$. Then taking $\sigma_n \in M(\bar U, \mathbb{R}^n)$ s.t. $-\mathrm{div}\,\sigma_n = \mu_n$ and $A^*\chi_B^*(\mu_n) = \int |\sigma_n| + o(1)$, we have $\int |\sigma_n| \leq C$. Then applying Prokhorov theorem to σ_n^+ and σ^-, we have, up to a subsequence, that $\sigma_n \to \sigma$ weakly. In particular $-\mathrm{div}\,\sigma = \mu$ and $\liminf_{n\to+\infty}\int |\sigma_n| \geq \int |\sigma| \geq (A^*\chi_B^*)(\sigma)$ from which we deduce the result.

We thus have that $A^*\chi_B^*$ is convex lsc so that $A^*\chi_B^* = (A^*\chi_B^*)^{**}$. Hence $(\chi_B \circ A)^* = A^*\chi_B^*$ which is exactly (14).

The proof of (15) is similar. We have as before that for any $\mu \in M(\bar U)$,

$$(\chi_{B(1/\alpha)} \circ Res)^*(\mu) = (Res^*\chi^*_{B(1/\alpha)})(\mu) = \inf\{\chi^*_{B(1/\alpha)}(\sigma) : Res^*(\sigma) = \mu\}$$

with $Res : C(\bar U) \to C(\Gamma_2)$ and $Res^* : C(\Gamma_2)^* = M(\Gamma_2) \to C(\bar U)^* = M(\bar U)$ is given by

$$(Res^*(\sigma), v) = (\sigma, Res(v)) = (\sigma, v_{|\Gamma_2}) = \int_{\Gamma_2} v\, d\sigma$$

for any $\sigma \in C(\Gamma_2)^*$, $v \in C(\bar U)$. Moreover $\chi_{B(1/\alpha)} : C(\Gamma_2) \to \mathbb{R}$ and for any $\sigma \in C(\Gamma_2)^*$,

$$\begin{aligned}
\chi^*_{B(1/\alpha)}(\sigma) &= \sup_{v\in C(\Gamma_2)} (\sigma, v) - \chi_{B(1/\alpha)}(v) = \sup_{v\in C(\Gamma_2), \|v\|_{L^\infty(\Gamma_2)}\leq 1/\alpha} \int_{\Gamma_2} v\, d\sigma \\
&= \frac{1}{\alpha} \sup_{v\in C(\Gamma_2), \|v\|_{L^\infty(\Gamma_2)}\leq 1} \int_{\Gamma_2} v\, d\sigma \\
&= \frac{1}{\alpha}\int_{\Gamma_2} |\sigma|
\end{aligned}$$

Thus

$$\begin{aligned}
&(\chi_{B(1/\alpha)} \circ Res)^*(\mu) \\
= \;& \inf\Big\{\frac{1}{\alpha}\int_{\Gamma_2} |\sigma| : \sigma \in C(\Gamma_2)^* \text{ s.t. } \int_{\Gamma_2} u\, d\sigma = \int_{\bar U} u\, d\mu \text{ for all } u \in C(\bar U)\Big\}.
\end{aligned}$$

Consider an admissible measure σ. Then for any $A \subset \bar U$,

$$\sigma(A \cap \Gamma_2) = \int_{\Gamma 2} 1_A\, d\sigma = \int_{\bar U} 1_A\, d\mu = \mu(A).$$

It follows that there cannot exists $A \subset \bar{U}\backslash\Gamma_2$ s.t. $\mu(A) \neq 0$ i.e. supp $\mu \subset \Gamma_2$. and then $\sigma = \mu$. Hence $(\chi_{B(1/\alpha)} \circ Res)^*(\mu) = \frac{1}{\alpha}|\mu|(\Gamma_2)$ if supp $\mu \subset \Gamma_2$. Otherwise there does not exist any admissible σ and the inf is $+\infty$. □

Using the previous proposition, we can rewrite (16) as

$$\frac{1}{\lambda_\infty} = \max_{\sigma \in P(\bar{U})} \inf\,(\chi_B \circ A)^*(\mu_2) + \frac{1}{\alpha}|\mu_1|(\Gamma_2),$$

where the inf is taken over all triple of measures $\mu_1, \mu_2, \mu_3 \in M(\bar{U})$ such that $\sigma = \mu_1 + \mu_2 + \mu_3$, supp $\mu_3 \subset \Gamma_1$, supp $\mu_1 \subset \Gamma_2$, $\mu_2(\bar{U}) = 0$. Letting $\nu = \mu_1 + \mu_3 = \sigma - \mu_2$, we have $|\mu_1|(\Gamma_2) = |\nu|(\Gamma_2) = \nu^+(\Gamma_2) + \nu^-(\Gamma_2)$ since μ_1 and μ_3 have disjoint support. Moreover, since $\mu_2(\bar{U}) = 0$ i.e. $(\sigma + \nu^-)(\bar{U}) = \nu^+(\bar{U})$, we have

$$\begin{aligned} & (\chi_B \circ A)^*(\mu_2) = (\chi_B \circ A)^*(\sigma - \nu) \\ = \ & \inf \left\{ \int_{\bar{U}} |\tilde{\sigma}| : \tilde{\sigma} \in M(\bar{U}, \mathbb{R}^n) \text{ s.t. } -\operatorname{div} \tilde{\sigma} = (\sigma + \nu^-) - \nu^+ \text{ in } \mathcal{D}'(\mathbb{R}^n) \right\} \\ = \ & W_1(\sigma + \nu^-, \nu^+). \end{aligned}$$

We thus obtain

$$\frac{1}{\lambda_\infty} = \max_{\sigma \in P(\bar{U})} \inf_{\nu \in M(\partial U),\, \nu(\partial U) = 1} W_1(\sigma + \nu^-, \nu^+) + \frac{1}{\alpha}\nu^+(\Gamma_2) + \frac{1}{\alpha}\nu^-(\Gamma_2).$$

To conclude the proof of (7), it suffices to verify that the inf can be taken over non-negative ν. This is a consequence of the following Proposition:

Proposition 4.4. *For any $\sigma \in P(\bar{U})$,*

$$\inf_{\nu_1, \nu_2 \in M_+(\partial U),\, \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma + \nu_1, \nu_2) = \inf_{\nu \in P(\partial U)} W_1(\sigma, \nu).$$

The proof of this lemma is based on the following lemma:

Lemma 2. *Consider probability measures $\mu_\varepsilon, \mu \in P(\mathbb{R}^n)$ such that*

$$\lim_{\varepsilon \to 0} W_1(\mu_\varepsilon, \mu) = 0,$$

and a subset $A \subset P(\mathbb{R}^n)$ compact w.r.t the convergence in distance W_1. Then $\lim_{\varepsilon \to 0} W_1(\mu_\varepsilon, A) = W_1(\mu, A)$ where $W_1(\mu, A) = \inf_{\nu \in A} W_1(\mu, \nu)$.

Observe that the compactness assumption is satisfied for $A = P(K)$ where $K \subset \mathbb{R}^n$ is compact in view of Prokhorov theorem and the fact that W_1 metrizes the weak convergence in $P(K)$ (because K is bounded).

Proof of lemma 2. Consider $\nu_\delta \in A$ s.t. $\lim_{\delta \to 0} W_1(\nu_\delta, \mu) = W_1(\mu, A)$. Then passing to the limit in $W_1(\mu_\varepsilon, A) \leq W_1(\mu_\varepsilon, \nu_\delta)$ yields $\limsup_{\varepsilon \to 0} W_1(\mu_\varepsilon, A) \leq W_1(\mu, \nu_\delta)$ for any δ, so that $\limsup_{\varepsilon \to 0} W_1(\mu_\varepsilon, A) \leq W_1(\mu, A)$.

To prove the opposite inequality we consider $\nu_\varepsilon \in A$ such that $W_1(\mu_\varepsilon, \nu_\varepsilon) = W_1(\mu_\varepsilon, A) + o(1)$. Since A is compact, we can assume up to a subsequence that there exists $\nu \in A$ s.t. $W_1(\nu_\varepsilon, \nu) \to 0$. Since $W_1(\mu_\varepsilon, \mu) \to 0$, we obtain

$$\lim_{\varepsilon \to 0} W_1(\mu_\varepsilon, A) = \lim_{\varepsilon \to 0} W_1(\mu_\varepsilon, \nu_\varepsilon) = W_1(\mu, \nu) \geq W_1(\mu, A)$$

which ends the proof of the lemma. □

We now prove Proposition 4.4.

Proof of Proposition 4.4. The $\leq$ inequality is clear (take $\nu_1 = 0$). To prove the opposite inequality, we first assume that supp $\sigma \subset U$. Given any ν_1, ν_2, any transfer plan $\pi \in \Pi(\sigma + \nu_1, \nu_2)$ (i.e. $\pi \in P(\bar{U})$ has marginals $\sigma + \nu_1$ and ν_2) can be written as

$$\pi = \tilde{\pi} + \bar{\pi}, \qquad \tilde{\pi} \in \Pi(\sigma, \tilde{\nu}_2),\ \bar{\pi} \in \Pi(\nu_1, \bar{\nu}_2)$$

for some decomposition $\nu_2 = \tilde{\nu}_2 + \bar{\nu}_2$ with $\tilde{\nu}_2, \bar{\nu}_2 \in M_+(\partial U)$, $\tilde{\nu}_2(\partial U) = 1$, $\bar{\nu}_2(\partial U) = \nu_1(\partial U)$. It follows that

$$\begin{aligned}
W_1(\sigma + \nu_1, \nu_2) &= \inf_{\pi \in \Pi(\sigma + \nu_1, \nu_2)} \int_{\bar{U} \times \bar{U}} d(x,y)\, d\pi(x,y) \\
= & \inf_{\nu_2 = \tilde{\nu}_2 + \bar{\nu}_2,\, \tilde{\pi} \in \Pi(\sigma, \tilde{\nu}_2),\, \bar{\pi} \in \Pi(\nu_1, \bar{\nu}_2)} \int_{\bar{U} \times \bar{U}} d(x,y)\, d\tilde{\pi}(x,y) \\
& + \int_{\bar{U} \times \bar{U}} d(x,y)\, d\bar{\pi}(x,y) \\
\geq & \inf_{\nu_2 = \tilde{\nu}_2 + \bar{\nu}_2} W_1(\sigma, \tilde{\nu}_2) + W_1(\nu_1, \bar{\nu}_2)
\end{aligned}$$

Then

$$\begin{aligned}
& \inf_{\nu_1, \nu_2 \in M_+(\partial U),\, \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma + \nu_1, \nu_2) \\
\geq & \inf_{\nu_1, \nu_2 \in M_+(\partial U),\, \nu_2(\partial U) = \nu_1(\partial U) + 1} \ \inf_{\nu_2 = \tilde{\nu}_2 + \bar{\nu}_2} W_1(\sigma, \tilde{\nu}_2) + W_1(\nu_1, \bar{\nu}_2)
\end{aligned}$$

which is clearly greater or equal than $\inf_{\tilde{\nu}_2 \in P(\partial U)} W_1(\sigma, \tilde{\nu}_2)$. This proves the $\geq$ inequality when supp $\sigma \subset U$.

In the general case we have supp $\sigma \subset \bar{U}$. We consider $\sigma_\varepsilon = T_\varepsilon \sharp \sigma$ the push-forward of σ under $T_\varepsilon(x) = x - \varepsilon \vec{n}$ where $\vec{n}$ denote some smooth extension of the unit exterior normal to a neighborhood of ∂U. Then supp $\sigma_\varepsilon \subset U$ so that

$$\inf_{\nu_1, \nu_2 \in M_+(\partial U),\, \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma_\varepsilon + \nu_1, \nu_2) = W_1(\sigma_\varepsilon, P(\partial U)).$$

To pass to the limit as $\varepsilon \to 0$, we use Lemma 2. Just notice that $\sigma_\varepsilon \to \sigma$ weakly as measure i.e. $W_1(\sigma_\varepsilon, \sigma) \to 0$ since U is bounded, and $A = P(\partial U)$ is compact for the weak convergence. We then have $W_1(\sigma_\varepsilon, P(\partial U)) \to W_1(\sigma, P(\partial U))$. Observe also that the first part of the proof of Proposition 4.4, which does not use the compactness assumption, yields

$$\begin{aligned}
& \limsup_{\varepsilon \to 0} \inf_{\nu_1, \nu_2 \in M_+(\partial U),\, \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma_\varepsilon + \nu_1, \nu_2) \\
\leq & \inf_{\nu_1, \nu_2 \in M_+(\partial U),\, \nu_2(\partial U) = \nu_1(\partial U) + 1} W_1(\sigma + \nu_1, \nu_2).
\end{aligned}$$

The result follows. $\square$

To end the proof of theorem 3, we verifiy that the max in (7) is attained by f_∞, the weak limit as $p \to +\infty$ of the measures $f_p = u_p^{p-1}\, dx$ (which exists up to a subsequence). Notice that u_p is the unique minimizer of the functional $F_p : W^{1,p}_{\Gamma_1}(U) \to \mathbb{R}$ defined by

$$F_p(u) = \frac{1}{p\lambda_p} \int_U |\nabla u|^p + \frac{\alpha^p}{p\lambda_p} \int_{\Gamma_2} |u|^p - (f_p, u).$$

Indeed the associated Euler-Lagrange equation, which has a unique solution since F_p is strictly convex, is the equation $\Delta_p u = \lambda_p f_p$ with the boundary conditions of (1), which admits u_p as a solution.

Writing F_p as

$$F_p(u) = \int_U \Big|\frac{\nabla u}{p^{1/p}\lambda_p^{1/p}}\Big|^p + \int_{\Gamma_2} \Big|\frac{\alpha u}{p^{1/p}\lambda_p^{1/p}}\Big|^p - (f_p, u),$$

we can prove, as in Proposition 4.1, that F_p Γ-converge as $p \to +\infty$ to the functionnal $F_\infty : C(\bar{U}) \to \mathbb{R}$ defined by

$$F_\infty(u) = \begin{cases} -(f_\infty, u), & \text{if } u \in W^{1,\infty}_{\Gamma_1}(U), \|\nabla u\|_\infty \leq \lambda_\infty, \\ & \quad \text{and } \alpha\|u\|_{L^\infty(\Gamma_2)} \leq \lambda_\infty \\ +\infty & \text{otherwise.} \end{cases}$$

Since

$$\inf F_p = F_p(u_p) = \frac{1}{p} - 1,$$

we obtain that

$$F_\infty(u_\infty) = \inf F_\infty = \lim_{p\to+\infty} \inf F_p = -1.$$

Hence

$$-1 = \min\Big\{ - (f_\infty, u) + \chi_{B(1/\alpha)}(u_{|\Gamma_1}/\lambda_\infty) + \chi_B(\nabla u/\lambda_\infty) + \chi_{C_{\Gamma_1}(\bar{U})}(u)\Big\},$$

i.e.

$$-\frac{1}{\lambda_\infty} = \min\Big\{ - (f_\infty, u) + \chi_{B(1/\alpha)}(u_{|\Gamma_1}) + \chi_B(\nabla u) + \chi_{C_{\Gamma_1}(\bar{U})}(u)\Big\},$$

Then

$$\begin{aligned} \frac{1}{\lambda_\infty} &= \max_{u\in C(\bar{U})} \Big\{(f_\infty, u) - \chi_{B(1/\alpha)}(u_{|\Gamma_1}) - \chi_B(\nabla u) - \chi_C(u)\Big\} \\ &= \Big((\chi_{B(1/\alpha)} \circ Res) + (\chi_B \circ A) + \chi_{C_{\Gamma_1}(\bar{U})}\Big)^*(f_\infty). \end{aligned}$$

Since $f_\infty \in P(\partial U)$, we obtain in view of (11) that f_∞ is extremal in (7).

5. Proof of Theorem 2 for connected domains.

Let $\phi(\sigma, \nu) = W_1(\sigma, \nu) + \frac{1}{\alpha}\nu(\Gamma_2)$, $\sigma, \nu \in P(\partial U)$. Since W_1 is convex in (σ, ν) (see e.g. [20][thm. 4.8]), we see that ϕ is convex. It easily follows that the function $\Phi(\sigma) = \inf_{\nu\in P(\partial U)} \phi(\sigma, \nu)$, $\sigma \in P(\bar{U})$ is also convex. Indeed given $\sigma_1, \sigma_2 \in P(\bar{U})$ and any $\nu_1, \nu_2 \in P(\bar{U})$, we have

$$\begin{aligned} \Phi(t\sigma_1 + (1-t)\sigma_2) &\leq \phi(t\sigma_1 + (1-t)\sigma_2, t\nu_1 + (1-t)\nu_2) \\ &\leq t\phi(\sigma_1, \nu_1) + (1-t)\phi(\sigma_2, \nu_2). \end{aligned}$$

The result follows taking the infimum in ν_1, ν_2.

Since Φ is convex, it attains its maximum at an extreme point of the comvex compact $P(\bar{U})$ i.e. at some Dirac mass δ_x, $x \in \bar{U}$:

$$\frac{1}{\lambda_\infty} = \max_{x\in\bar{U}} \inf_{\nu\in P(\partial U)} W_1(\delta_x, \nu) + \frac{1}{\alpha}\nu(\Gamma_2).$$

It is well-known that $W_1(\delta_x, \nu) = \int_{\bar{U}} d(x, y)\, d\nu(y)$ for any $x \in \bar{U}$. This follows from the fact that the unique $\pi \in P(\bar{U} \times \bar{U})$ with marginals δ_x and

ν is $\pi = \delta_x \otimes \nu$. Indeed such a π must have support in $\{x\} \times \operatorname{supp} \nu$ so that for any $A, B \subset \bar{U}$, $\pi(A \times B) = 0 = (\delta_x \otimes \nu)(A \times B)$ if $x \notin A$, and if $x \in A$,

$$\pi(A \times B) = \pi(\{x\} \times B) = \pi(X \times B) = \nu(B) = (\delta_x \otimes \nu)(A \times B).$$

Given $x \in \bar{U}$, we consider $x_1 \in \Gamma_1$ and $x_2 \in \Gamma_2$ such that $d(x, \Gamma_i) = d(x, x_i)$, $i = 1, 2$. We write $\nu \in P(\partial U)$ as $\nu = \nu_1 + \nu_2$ where $\nu_i = \nu_{|\Gamma_i}$, $i = 1, 2$. Then

$$\begin{aligned} W_1(\delta_x, \nu) &= \int_{\partial U} d(x,y)\, d\nu(y) = \int_{\Gamma_1} d(x,y)\, d\nu_1(y) + \int_{\Gamma_2} d(x,y)\, d\nu_2(y) \\ &\geq d(x, \Gamma_1)\nu_1(\Gamma_1) + d(x, \Gamma_2)\nu_2(\Gamma_2) \\ &= W_1(\delta_{x_1}, \beta\delta_{x_1} + (1-\beta)\delta_{x_2}), \end{aligned}$$

where $\beta = \nu_1(\Gamma_1)$. We thus have

$$\frac{1}{\lambda_\infty} = \max_{x \in \bar{U}} \inf_{0 \leq \beta \leq 1} \beta d(x, \Gamma_1) + (1-\beta) d(x, \Gamma_2) + \frac{1-\beta}{\alpha}.$$

We deduce theorem 2 noticing that for any $x \in \bar{U}$, the inf in β is

$$\begin{cases} d(x, \Gamma_2) + \frac{1}{\alpha} & \text{if } d(x,\Gamma_1) - d(x,\Gamma_2) - \frac{1}{\alpha} \geq 0 \quad \text{i.e. } x \in \mathcal{A} \\ d(x, \Gamma_1) & \text{otherwise.} \end{cases}$$

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J. D. Rossi
Departamento de Matemática, FCEyN
Univ. de Buenos Aires,
C.P. 1428, Buenos Aires, Argentina.
E-mail address: `jrossi@dm.uba.ar`

N. Saintier
Departamento de Matemática, FCEyN
Univ. de Buenos Aires,
C.P. 1428, Buenos Aires, Argentina.
and
Instituto de Ciencias, Univ. Gral Sarmiento
Juan María Gutierrez 1150, Los Polvorines,
Pcia de Buenos Aires, Argentina.
E-mail address: `nsaintie@dm.uba.ar`