AN APPLICATION OF THE MAXIMUM PRINCIPLE TO DESCRIBE THE LAYER BEHAVIOR OF LARGE SOLUTIONS AND RELATED PROBLEMS

JORGE GARCÍA-MELIÁN, JULIO D. ROSSI, AND JOSÉ C. SABINA DE LIS [1]

ABSTRACT. This work is devoted to the analysis of the asymptotic behavior of positive solutions to some problems of variable exponent reaction-diffusion equations, when the boundary condition goes to infinity (large solutions). Specifically, we deal with the equations $\Delta u = u^{p(x)}$, $\Delta u = -m(x)u + a(x)u^{p(x)}$ where $a(x) \geq a_0 > 0$, $p(x) \geq 1$ in Ω, and $\Delta u = e^{p(x)}$ where $p(x) \geq 0$ in Ω. In the first two cases p is allowed to take the value 1 in a whole subdomain $\Omega_c \subset \Omega$, while in the last case p can vanish in a whole subdomain $\Omega_c \subset \Omega$. Special emphasis is put in the layer behavior of solutions on the interphase $\Gamma_i := \partial\Omega_c \cap \Omega$. A similar study of the development of singularities in the solutions of several logistic equations is also performed. For example, we consider $-\Delta u = \lambda m(x)u - a(x)u^{p(x)}$ in Ω, $u = 0$ on $\partial\Omega$, being $a(x)$ and $p(x)$ as in the first problem. Positive solutions are shown to exist only when the parameter λ lies in certain intervals: bifurcation from zero and from infinity arises when λ approaches the boundary of those intervals. Such bifurcations together with the associated limit profiles are analyzed in detail. For the study of the layer behavior of solutions the introduction of a suitable variant of the well-known maximum principle is crucial.

1. Introduction

Let $\mathcal{L}$ be a second order elliptic operator without zero order term

$$\mathcal{L}u = -\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i}$$

with, say, bounded measurable coefficients a_{ij}, b_i defined in a bounded domain $\Omega \subset \mathbb{R}^N$, which satisfies a strict ellipticity condition under the form

$$\sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \tag{1.1}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^N$ and a fixed positive μ (see [12]). The well-known maximum principle –or more properly the "weak" maximum principle– states that, provided a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies

$$\mathcal{L}u \geq 0 \tag{1.2}$$

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[1] Corresponding author.

in Ω, then its infimum in Ω coincides with the corresponding infimum when restricted to $\partial\Omega$. That is,

$$\inf_{\Omega} u = \inf_{\partial\Omega} u. \tag{1.3}$$

Alternatively, relation (1.3) holds after changing infimum by supremum if the sign in inequality (1.2) is reversed (see [12] and the classical reference on this subject, [20]).

Unfortunately, and as it is also well-known, this nice property may fail when the operator $\mathcal{L}$ is complemented with a zero order term, i.e.,

$$(\mathcal{L} + c)\, u = \mathcal{L}u + cu,$$

where $c \in L^\infty(\Omega)$. However, useful weak versions of the maximum principle still survive for the perturbed operator $\mathcal{L} + c$ and a precise characterization of these variants can be given. Namely, those expressed in terms of the principal Dirichlet eigenvalue λ_1 of $\mathcal{L} + c$ in Ω (we will review all these features in Section 2).

In this work we provide a sort of weak alternative to the maximum principle (1.2)–(1.3). Our main objective will be then to employ such result to study different layer behaviors associated to the formation of "large" solutions, and to analyze the developing of "infinite plateaus" in the solutions of certain reaction-diffusion problems (see [1], [15], and specially [4], [21] for recent surveys on large solutions).

To be more precise, the simple version of the maximum principle which is enough for addressing the kind of applications we have in mind here is the following.

Theorem 1. *Let*

$$(\mathcal{L} + c)u = -\sum_{i,j=1}^{N} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial u}{\partial x_i} + c(x)u$$

be a strictly elliptic operator, that is $\mathcal{L}$ verifies (1.1)*, with coefficients $a_{ij}, b_i, c \in C^\alpha(\overline{\Omega})$ defined in a $C^{2,\alpha}$ bounded domain $\Omega \subset \mathbb{R}^N$. Assume in addition that its principal (Dirichlet) eigenvalue satisfies*

$$\lambda_1 = \lambda_1(\mathcal{L} + c) > 0.$$

Then,

i) *The boundary value problem,*

$$\begin{cases} (\mathcal{L} + c)\,\psi(x) = 0 & x \in \Omega \\ \quad \psi(x) = 1 & x \in \partial\Omega, \end{cases}$$

has a unique positive solution $\psi \in C^{2,\alpha}(\overline{\Omega})$.

ii) *(Maximum principle, restricted version) Suppose that $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies*

$$(\mathcal{L} + c)u \geq 0,$$

in Ω. Then

$$u(x) \geq \{\inf_{\partial\Omega} u\}\psi(x), \tag{1.4}$$

for all $x \in \Omega$.

The symmetric statement also holds: if $(\mathcal{L}+c)u \leq 0$ then $u(x) \leq \{\sup_{\partial\Omega} u\}\psi(x)$.

Remarks 1.

a) A definition of the principal eigenvalue $\lambda_1(\mathcal{L} + c)$ of the operator $\mathcal{L} + c$ in the domain Ω is given in Section 2.

b) If one deals with strong solutions instead of classical ones, the smoothness of both coefficients and solutions in Theorem 1 can be relaxed (see Remak 4-c)).

As for the applications of this maximum principle we first deal with the limit behavior as $n \to \infty$ of the solution $u = u_n$ to the semilinear problem

$$\begin{cases} \Delta u(x) = f(x, u(x)) & x \in \Omega \\ u(x) = n & x \in \partial\Omega. \end{cases} \tag{1.5}$$

Assume that $\Omega \subset \mathbb{R}^N$ is bounded and $C^{2,\alpha}$, $f \in C^\alpha(\overline{\Omega} \times \mathbb{R})$ is non decreasing in u and diverges to infinity as $u \to \infty$ with a suitable strength. Namely, $f(x, u) \geq h(u)$ being $h \in C(\mathbb{R})$ an increasing function such that the "Keller-Osserman's" condition is fulfilled

$$\int_a^\infty \left(\int_0^s h(u)\, du \right)^{-1/2} ds < \infty, \tag{1.6}$$

for a certain large $a > 0$. Then the solution u_n to (1.5) converges in $C^2(\Omega)$ to a so-called "large" solution u, that is, a solution to the boundary blow-up problem

$$\begin{cases} \Delta u = f(x, u) & x \in \Omega \\ u = \infty & x \in \partial\Omega. \end{cases} \tag{1.7}$$

In fact, references [15], [19] introduce a first systematic study of (1.7) under this approach.

Since its appearance in different fields as Riemannian geometry or population dynamics, large solutions have deserved much interest in the recent pde's literature (see the comprehensive reviews in [21] and [4]). In [7], [9] and [10] it is discussed –among other aspects– the effect of losing condition (1.6) on a whole subdomain $\Omega_c \subset \Omega$. This analysis was essentially performed for the two kind of nonlinearities: $f(x, u) = u^{p(x)}$, $f(x, u) = e^{p(x)u}$.

To summarize the relevant issues of those papers that have implications in the present work and to describe the new features here enclosed, it is convenient to introduce a simplifying hypothesis on the structure of the critical subdomain $\Omega_c \subset \Omega$.

In the different problems we are analyzing, it will be understood by "critical" the failure in Ω_c of a certain structural condition which is essential for the existence of solutions (for example, the Keller-Osserman's condition in the case of large solutions). It will be supposed that Ω_c and Ω are bounded and $C^{2,\alpha}$ domains such that $\Gamma_b := \partial\Omega_c \cap \partial\Omega$, if nonempty, consists of the union of components of $\partial\Omega$. Equivalently, $\Gamma_i := \partial\Omega_c \cap \Omega$ is closed, and thus is separated away from $\partial\Omega$. These conditions on Ω_c together with $\Omega_c \neq \Omega$, will be referred to as hypothesis (H) in the sequel. At this early stage we anticipate that much of the contributions of the present research is concerned with the behavior of solutions in the inner boundary Γ_i of the critical region Ω_c.

Some properties of the power nonlinearity are listed in the next statement, which is consequence of the analysis in [9] (see also [7] for a related logistic type nonlinearity).

Theorem 2 ([9]). *Consider the problem*

$$\begin{cases} \Delta u(x) = u(x)^{p(x)} & x \in \Omega \\ u(x) = n & x \in \partial\Omega. \end{cases} \tag{1.8}$$

where $p \in C^\alpha(\overline{\Omega})$ *is positive,* $\Omega_c \subset \Omega$ *are bounded* $C^{2,\alpha}$ *domains,* Ω_c *satisfies* (H) *and where*

$$\overline{\Omega}_c = \overline{\{x \in \Omega : p(x) \leq 1\}}.$$

Then the following properties hold true.

i) *Problem* (1.8) *admits for every* $n \in \mathbb{N}$ *a unique solution* $u_n \in C^{2,\alpha}(\overline{\Omega})$.
ii) *If* $\overline{\Omega}_c \subset \Omega$ *then* u_n *converges in* $C^{2,\alpha}(\Omega)$ *to the minimal solution of the boundary blow problem*

$$\begin{cases} \Delta u(x) = u(x)^{p(x)} & x \in \Omega \\ u(x) = \infty & x \in \partial\Omega. \end{cases}$$

iii) *Assume that* $\Gamma_b = \partial\Omega_c \cap \partial\Omega \neq \emptyset$. *Then,* $\lim u_n = \infty$ *uniformly on compacts of* $\Omega_c \cup \Gamma_b$. *In addition,* u_n *converges in* $C^{2,\alpha}(\Omega \setminus \overline{\Omega}_c)$ *to a solution of* $\Delta u = u^{p(x)}$.

Theorem 2 says that problem (1.7) with $f(x,u) = u^{p(x)}$ does not admit a solution if the critical zone Ω_c touches some component of $\partial\Omega$. In this case u_n develops an "infinite core" in Ω_c as n grows. On the other hand, u_n keeps finite in $\Omega^+ = \Omega \setminus \overline{\Omega}_c$ and one wonders which boundary condition is satisfied by its limit u. Note that u solves the equation in Ω^+. This discussion leads to a natural important problem: how does u_n behaves on the interphase Γ_i when $n \to \infty$?

Our first result gives an answer to this question.

Theorem 3. *Under the hypotheses of Theorem 2-iii) the solution* u_n *to* (1.8) *satisfies*

$$\lim u_n = \infty$$

uniformly on Γ_i *and hence such limit is also uniform in* $\overline{\Omega}_c$. *In particular,* $u = \lim_n u_n$ *in* Ω^+ *solves the boundary blow-up problem* $\Delta u = u^{p(x)}$ *in* Ω^+, $u = \infty$ *on* $\Gamma_i \cup \Gamma^+$, $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$.

Remark 2. Similar conclusions on large solutions for a larger class of "variable exponent" reactions are analyzed at the end of Section 5 (see Theorem 10).

Let us introduce next the corresponding features for the exponential nonlinearity $f(x,u) = e^{p(x)u}$. First we state some known facts extracted from [10].

Theorem 4 ([10]). *Assume that the domains* $\Omega_c \subset \Omega$ *are bounded, of class* $C^{2,\alpha}$ *and that* Ω_c *satisfies* (H). *Let* $p \in C^{\alpha}(\overline{\Omega})$ *be a nonnegative function,* $p \not\equiv 0$, *such that*

$$\overline{\Omega}_c = \overline{\{x \in \Omega : p(x) = 0\}}.$$

Then, problem

$$\begin{cases} \Delta u(x) = e^{p(x)u(x)} & x \in \Omega \\ u(x) = n & x \in \partial\Omega. \end{cases} \tag{1.9}$$

admits for each $n \in \mathbb{N}$ *a classical solution* $u_n \in C^{2,\alpha}(\overline{\Omega})$. *Moreover,*
i) *If* $\Gamma_b = \emptyset$ *and thus* $\overline{\Omega}_c \subset \Omega$ *then* u_n *converges in* $C^{2,\alpha}(\Omega)$ *to the minimal solution to* $\Delta u = e^{p(x)u}$ *in* Ω, $u = \infty$ *on* $\partial\Omega$.
ii) *If on the contrary* $\Gamma_b \neq \emptyset$ *then* $u_n \to \infty$ *uniformly in compacts of* $\Omega_c \cup \Gamma_b$ *while* u_n *converges to a finite solution of the equation in* $\Omega^+ = \Omega \setminus \overline{\Omega}_c$.

The behavior of the solution u_n to (1.9) at Γ_i was furthermore studied in [10] under the restrictive assumption that $p(x)$ vanishes at Γ_i according to

$$p(x) = o(d(x)), \tag{1.10}$$

as $d(x) \to 0$ where $d(x) = \mathrm{dist(x,\Gamma_i)}$. Our second result just removes this restriction (compare with Lemma 3.4 in [3]).

Theorem 5. *Assume the hypotheses of Theorem 4-ii) and let u_n be the solution to* (1.9)*. Then,*

$$\lim u_n = \infty$$

uniformly on the interphase Γ_i. Moreover, $u_n \to u$ in $C^{2,\alpha}(\Omega^+)$ to the minimal solution to $\Delta u = e^{p(x)u}$ with boundary condition $u = \infty$ on $\partial\Omega^+ = \Gamma_i \cup \Gamma^+$, where $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$.

Now we change the scenario and turn our attention to population dynamics. Specifically, we deal with the problem,

$$\begin{cases} -\Delta u(x) = \lambda m(x)u(x) - a(x)u^p(x) & x \in \Omega \\ \ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1.11}$$

where $p > 1$ is constant, λ is a parameter and coefficients $m, a \in C^\alpha(\overline{\Omega})$, $a(x) \geq 0$, $a \not\equiv 0$. When $m = 1$ and $a(x) > 0$ in Ω (or plainly, $a = 1$) (1.11) constitutes the classical logistic model where u is the equilibrium population density of the species, λ its the natural growth rate and a stands for the medium carrying capacity.

In problem (1.11) we are assuming that the critical domain $\Omega_c \subset \Omega$ is defined as

$$\overline{\Omega}_c = \overline{\{x \in \Omega : a(x) = 0\}}. \tag{1.12}$$

Thus a large amount of resources is available for the species u in Ω_c. On the other hand, the coefficient $m(x)$ in (1.11) may change sign. Under these assumptions, a detailed analysis of problem (1.11) was performed in [5]. We use the notation $\lambda_1^Q(-\Delta + q)$ to designate the principal Dirichlet eigenvalue of $(-\Delta + q)u = -\Delta u + qu$ in a domain $Q \subset \mathbb{R}^N$ (Section 2).

Theorem 6 ([5])**.** *Under the previous assumptions on coefficients a, m suppose $\Omega_c \subset \Omega$ are $C^{2,\alpha}$ bounded domains, Ω_c being defined by* (1.12)*.*

Then (1.11) *admits a positive solution if and only if*

$$\lambda_1^\Omega(-\Delta - \lambda m) < 0 < \lambda_1^{\Omega_c}(-\Delta - \lambda m). \tag{1.13}$$

In this case, the positive solution is unique. Moreover, let $(a, b) \subset \mathbb{R}$ be a component of the set defined through (1.13) *and let u_λ be the solution associated to every $\lambda \in (a, b)$. Then*

i) *u_λ bifurcates from zero at $\lambda = \lambda_0 \in \{a, b\}$ provided $\lambda_1^\Omega(-\Delta - \lambda m)_{|\lambda=\lambda_0} = 0$, i.e.*

$$\lim_{\lambda \to \lambda_0} u_\lambda = 0 \qquad \text{in } C^{2,\alpha}(\overline{\Omega}).$$

ii) *u_λ bifurcates from infinity at $\lambda = \lambda^* \in \{a, b\}$ if $\lambda_1^{\Omega_c}(-\Delta - \lambda m)_{|\lambda=\lambda^*} = 0$ in the sense that*

$$\lim_{\lambda \to \lambda^*} |u_\lambda|_{C(\overline{\Omega})} = \infty. \tag{1.14}$$

The singular behavior of u_λ in (1.14) posed the problem of giving an explanation on how singularities are developed by u_λ so that it ceases to exist when λ crosses λ^*. To answer this question was the main objective of [11], [18] which only dealt with the logistic problem corresponding to (1.11) with $m = 1$. The existence range (1.13) reads in this case

$$\lambda_1^\Omega := \lambda_1^\Omega(-\Delta) < \lambda < \lambda_1^{\Omega_c} := \lambda_1^{\Omega_c}(-\Delta).$$

Bifurcation from infinity occurs at $\lambda = \lambda_1^{\Omega_c}$. What is more important:

$$\lim_{\lambda \to \lambda_1^{\Omega_c}} u_\lambda = \infty,$$

uniformly in $\overline{\Omega}_c$ provided that the extra condition

$$a(x) = o(d(x)) \qquad \text{as } d(x) \to 0, \tag{1.15}$$

holds, where $d(x) = \mathrm{dist}(\mathrm{x}, \Gamma_\mathrm{i})$. Furthermore

$$\lim_{\lambda \to \lambda_1^{\Omega_c}} u_\lambda = u \qquad \text{in } C^{2,\alpha}(\Omega^+)$$

where u defines either the minimal solution to the blow-up problem $-\Delta u = \lambda u - a(x)u^p$ in Ω^+, $u = \infty$ on $\partial\Omega^+$ if $\overline{\Omega}^+ \subset \Omega$ or either the minimal solution to $-\Delta u = \lambda u - a(x)u^p$ in Ω^+ and boundary conditions $u = \infty$ on Γ_i, $u = 0$ on $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$.

It should be mentioned that some of these features of u_λ were discovered also in [3] where, however, authors succeeded in suppressing restriction (1.15).

In our next result we extend the preceding analysis to problem (1.11) in the case where the coefficient m exhibits both signs. Notice that a consequence of losing the condition $m > 0$ is that u_λ ceases to be monotone in λ which is an important information in the analysis in [11] and [18]. On the other hand, and as in [3], we are also removing the restriction (1.15).

Theorem 7. *Suppose $a, m \in C^\alpha(\overline{\Omega})$, $a \geq 0$, $a \neq 0$, $m \neq 0$, while the subdomain $\Omega_c \subset \Omega$ is defined through* (1.12) *and satisfies* (H). *For every λ satisfying* (1.13) *let $u_\lambda \in C^{2,\alpha}(\overline{\Omega})$ designate the positive solution to* (1.11). *Then,*

a) *If $m = 0$ in Ω_c the set* (1.13) *of "existence" values of λ is either a single interval $(\bar\lambda_{-2}, \bar\lambda_{-1})$, $\bar\lambda_{-1} < 0$ (alternatively, $(\bar\lambda_1, \bar\lambda_2)$, $\bar\lambda_1 > 0$) or is the union of two intervals $(\bar\lambda_{-2}, \bar\lambda_{-1}) \cup (\bar\lambda_1, \bar\lambda_2)$, $\bar\lambda_{-1} < 0 < \bar\lambda_1$. In all of these cases $\bar\lambda_{\pm 2} = \pm\infty$.*

b) *If $m \neq 0$ in Ω_c, inequalities* (1.13) *define either a finite interval $(\bar\lambda_1, \bar\lambda_2) \subset \mathbb{R}^+$ (alternatively, $(\bar\lambda_{-2}, \bar\lambda_{-1}) \subset \mathbb{R}^-$) or is the union of two intervals $(\bar\lambda_{-2}, \bar\lambda_{-1}) \cup (\bar\lambda_1, \bar\lambda_2)$, $\bar\lambda_{-1} < 0 < \bar\lambda_1$. In the latter case one (but not both at the same time !) of the $\bar\lambda_{\pm 2}$ could be infinite.*

c) *u_λ bifurcates from zero at the values $\lambda = \bar\lambda_{\pm 1}$. Furthermore, if $\bar\lambda_2 = \infty$ (respectively, $\bar\lambda_{-2} = -\infty$) then $\lim_{\lambda\to\infty} |u_\lambda|_{C(\overline{\Omega})} = \infty$ ($\lim_{\lambda\to-\infty} |u_\lambda|_{C(\overline{\Omega})} = \infty$).*

d) *Assume $\bar\lambda_2$ is finite. Then*

$$\lim_{\lambda\to\bar\lambda_2} u_\lambda = \infty$$

uniformly in $\Omega_c \cup \Gamma_i$. Furthermore, there exists a subsequence $\lambda_{n'}$ to every sequence $\lambda_n \to \bar\lambda_2$ such that $\lim u_{\lambda_{n'}} = u^*$ in $C^{2,\alpha}(\Omega^+)$, $\Omega^+ = \Omega \setminus \overline{\Omega}_c$ where $u = u^*$ defines a solution either to the problem

$$\begin{cases} -\Delta u(x) = \bar\lambda_2 m(x)u(x) - a(x)u^p(x) & x \in \Omega^+ \\ u(x) = \infty & x \in \partial\Omega^+, \end{cases} \tag{1.16}$$

if $\overline{\Omega}^+ \subset \Omega$, or to the problem

$$\begin{cases} -\Delta u(x) = \bar\lambda_2 m(x)u(x) - a(x)u^p(x) & x \in \Omega^+ \\ u(x) = \infty & x \in \partial\Gamma_i \\ u(x) = 0 & x \in \partial\Gamma^+, \end{cases} \tag{1.17}$$

provided $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$ is nonempty. Moreover, if $a(x)$ satisfies the decay condition

$$a(x) \sim C_0 d(x)^\gamma, \tag{1.18}$$

as $d(x) = dist(x, \Gamma_i) \to 0$, for certain positive constants C_0, γ, then the stronger convergence $\lim_{\lambda\to\bar\lambda_2} u_\lambda = u$ holds in $C^{2,\alpha}(\Omega^+)$, where u is the unique *solution to either* (1.16) *or* (1.17).

Of course, the same conclusions hold true at $\lambda = \bar{\lambda}_{-2}$ *provided this value is finite.*

Remark 3. It should be remarked that $\lambda = \bar{\lambda}_{\pm 1}$ provide with the so-called principal eigenvalues of the weighted eigenvalue problem

$$\begin{cases} -\Delta u(x) = \lambda m(x) u(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega. \end{cases}$$

At least one of these numbers always exists (see [16] for a detailed analysis on this issue).

Finally, we study a nonlinear problem that is an extension of

$$\begin{cases} -\Delta u(x) = \lambda u(x) - a(x) u^{p(x)} & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1.19}$$

where λ is a parameter, $a, p \in C^{\alpha}(\overline{\Omega})$ with a positive in $\overline{\Omega}$ while the variable reaction order p satisfies $p(x) \geq 1$, $p(x) \not\equiv 1$ in Ω. Solutions to (1.19) constitute equilibrium states of a reaction-diffusion process governed by a reaction characterized by a variable stoichiometric exponent $p(x)$.

In the framework of this problem the critical domain Ω_c is defined as

$$\overline{\Omega}_c = \overline{\{x \in \Omega : p(x) = 1\}}, \tag{1.20}$$

where it is understood that Ω_c is a nonempty $C^{2,\alpha}$ bounded subdomain of Ω.

Under the precedent conditions, problem (1.19) was proposed and studied in detail in [17]. We are now simplifying the proof of the basic facts of that work and at the same time extending its scope in several issues. First, by adding –in the line of problem (1.11)– a sign-indefinite weight $m(x)$ in front of λ (which causes the loss of monotonicity of u_λ in λ). Second, by allowing more general configurations in Ω_c (in [17], $\Omega^+ = \Omega \setminus \overline{\Omega}_c$ is constrained to satisfy $\overline{\Omega}^+ \subset \Omega$). Last, but not least, we provide the proof of a missing step in [17]. Theorem 1 is just in the kernel of this point and has to do with the behavior of solutions on the interphase Γ_i (Remark 5-b)).

Our results are collected in the following statement.

Theorem 8. *Consider the problem*

$$\begin{cases} -\Delta u(x) = \lambda m(x) u(x) - a(x) u(x)^{p(x)} & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{1.21}$$

where coefficients $a(x) > 0$ *and* $p(x) \geq 1$ *satisfy the preceding hypotheses,* Ω_c *is defined in* (1.20) *and satisfies (H), while* $m \in C^{\alpha}(\overline{\Omega})$ *is arbitrary regarding sign. Then, the following properties hold.*

i) *Problem* (1.21) *admits at most one positive (classical) solution. Such a solution exists if and only if*

$$\lambda_1^{\Omega}(-\Delta - \lambda m + a\chi_{\Omega_c}) < 0 < \lambda_1^{\Omega_c}(-\Delta - \lambda m + a), \tag{1.22}$$

where χ_{Ω_c} *is the characteristic function of* Ω_c*. The solution to* (1.21) *is denoted by* u_λ.

ii) *If* $m = 0$ *in* Ω_c*, the existence region* (1.22) *for* λ *is a single interval* $(\bar{\lambda}_{-2}(m), \bar{\lambda}_{-1}(m))$ *(respectively,* $(\bar{\lambda}_1(m), \bar{\lambda}_2(m))$*) or is either a union* $(\bar{\lambda}_{-2}(m), \bar{\lambda}_{-1}(m)) \cup (\bar{\lambda}_1(m), \bar{\lambda}_2(m))$ *where in all cases* $\bar{\lambda}_{-1}(m) < 0 < \bar{\lambda}_1(m)$ *and* $\bar{\lambda}_{\pm 2} = \pm\infty$*. If otherwise,* $m \neq 0$ *in* Ω_c *then the set* (1.22) *is either a finite interval* $(\bar{\lambda}_1(m), \bar{\lambda}_2(m))$ *(respectively,* $(\bar{\lambda}_{-2}(m), \bar{\lambda}_{-1}(m))$*) or splits as* $(\bar{\lambda}_{-2}(m), \bar{\lambda}_{-1}(m)) \cup (\bar{\lambda}_1(m), \bar{\lambda}_2(m))$*. In all cases* $\bar{\lambda}_{-1}(m) < 0 < \bar{\lambda}_1(m)$ *while in the latter option one (but not both) of the values* $\bar{\lambda}_{\pm 2}$ *could be infinite.*

iii) *u_λ bifurcates from zero at $\lambda = \bar\lambda_1(m)$ (respectively, $\lambda = \bar\lambda_{-1}(m)$). In addition $|u_\lambda|_{C(\overline{\Omega})} \to \infty$ as $\lambda \to \bar\lambda_2(m)$ (respectively, $\lambda \to \bar\lambda_{-2}(m)$) provided $\bar\lambda_2(m) = \infty$ ($\bar\lambda_{-2}(m) = -\infty$).*

iv) *Assume $\bar\lambda_2(m)$ is finite. Then*

$$\lim_{\lambda\to\bar\lambda_2(m)} u_\lambda = \infty$$

uniformly on compacts of $\Omega_c \cup \Gamma_i$. Furthermore, for any sequence $\lambda_n \to \bar\lambda_2$ there exists a subsequence $\lambda_{n'}$ and a solution $u^* \in C^{2,\alpha}(\Omega^+)$ to the blow-up problem

$$\begin{cases} -\Delta u(x) = \bar\lambda_2(m)m(x)u(x) - a(x)u(x)^{p(x)} & x \in \Omega^+ \\ u(x) = \infty & x \in \partial\Omega^+, \end{cases} \tag{1.23}$$

if $\overline{\Omega}^+ \subset \Omega$, or either to the mixed blow-up problem

$$\begin{cases} -\Delta u(x) = \bar\lambda_2(m)m(x)u(x) - a(x)u(x)^{p(x)} & x \in \Omega^+ \\ u(x) = \infty & x \in \Gamma_i \\ u(x) = 0 & x \in \Gamma^+, \end{cases} \tag{1.24}$$

if $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$ is nonempty, such that

$$u_{\lambda_{n'}} \to u^*,$$

in $C^{2,\alpha}(\Omega^+)$.

v) *If $\bar\lambda_2(m)$ is finite and in addition $m \geq 0$ in Ω^+ then $u_\lambda \to u$ in $C^{2,\alpha}(\Omega^+)$ where u is the minimal solution to* (1.23) *if $\overline{\Omega}^+ \subset \Omega$, or either the minimal solution to* (1.24) *provided $\Gamma^+ \neq \emptyset$.*

Analogous features as in iv)-v) hold true as $\lambda \to \bar\lambda_{-2}(m)$ provided $\lambda = \bar\lambda_{-2}(m)$ is finite.

This work is organized as follows: Section 2 contains the material on the maximum principle; boundary blow-up problems are considered in Sections 3; population dynamics (problem (1.11)) is studied in Section 4 and finally, Section 5 is devoted to the variable exponent problem (1.21).

2. On the maximum principle

As we have mentioned in the introduction, a well-known and weaker version of the maximum principle is as follows: if $u \in C^2(\Omega) \cap C(\overline{\Omega})$ satisfies $(\mathcal{L} + c)u \geq 0$ in Ω ($\mathcal{L}$ as in Section 1), with $c \in L^\infty$ and nonnegative in Ω, then

$$\inf_\Omega u \geq \inf_{\partial\Omega} u^- \tag{2.1}$$

(see [12]). On the other hand, the most immediate example, say $(\mathcal{L} + c)u = -u'' + 4u$ in $\Omega = (-\pi/2, \pi/2)$ with $u(x) = \sin(2x) + 2$ satisfying $(\mathcal{L} + c)u \geq 0$, shows that (1.2)-(1.3) is false for arbitrary zero order perturbations of the operator $\mathcal{L}$.

Nevertheless, such simple statement is enough to ensure the existence of a principal (Dirichlet) eigenvalue of $\mathcal{L} + c$. That means a real value of λ such that the problem

$$\begin{cases} (\mathcal{L} + c)u(x) = \lambda u(x) & x \in \Omega \\ u(x) = 0 & x \in \partial\Omega \end{cases}$$

admits a positive solution. More precisely, a unique such principal eigenvalue exits – regardless the sign of c – provided that Ω is of class $C^{2,\alpha}$ and $a_{ij}, b_i, c \in C^\alpha(\overline{\Omega})$ (see [14], [16], and weaker versions in [2], [4], [6]). Furthermore, such eigenvalue –which will be denoted as $\lambda_1^\Omega(\mathcal{L}+c)$– is simple, increasing in c and decreasing with respect Ω, i.e. $\Omega_0 \subset \Omega$, $\Omega_0 \neq \Omega$ implies

$$\lambda_1^\Omega(\mathcal{L}+c) < \lambda_1^{\Omega_0}(\mathcal{L}+c).$$

As additional relevant features to be used in this work, it also holds that $\lambda_1^\Omega(\mathcal{L}+c)$ is concave in c ([2], [14], [16]), the function $E(t) := \lambda_1^\Omega(\mathcal{L}+tc)$ is real analytic for $t \in \mathbb{R}$ and $\lambda_1^\Omega(\mathcal{L}+c)$ is continuous and smooth with respect to smooth perturbations of the domain Ω ([14], [16] and [13]).

A still weaker result than the previous ones which is often termed as the maximum principle is the following: a function $u \in C^2(\Omega) \cap C(\overline{\Omega})$ turns out to be nonnegative in Ω when $(\mathcal{L}+c)u \geq 0$ in Ω together with $u \geq 0$ on the boundary $\partial\Omega$. For instance $c \geq 0$ implies, via (2.1), the maximum principle in this sense. On the other hand, such statement, which largely suffices for most of applications –e. g. the comparison principle– is elegantly characterized by the principal eigenvalue as the following result asserts ([2], [14], [16], [4] and also [6] for its extension to the p-Laplacian operator).

Theorem 9. *Suppose that Ω is a class $C^{2,\alpha}$ bounded domain, $\mathcal{L}+c$ satisfies* (1.1) *and has coefficients in $C^\alpha(\overline{\Omega})$. Then, the following statements are equivalent.*

i) $\lambda_1^\Omega(\mathcal{L}+c) > 0$.
ii) *Every $u \in C^2(\Omega) \cap C(\overline{\Omega})$ such that $(\mathcal{L}+c)u \geq 0$ in Ω and $u \geq 0$ on $\partial\Omega$ satisfies $u \geq 0$ in Ω.*

However, an elementary example shows that (2.1) could be false if it is only assumed that $\lambda_1^\Omega(\mathcal{L}+c) > 0$. In fact, consider $(\mathcal{L}+c)u = -u'' - \varepsilon u$, $0 < \varepsilon < 1$ in $\Omega = (-\pi/2, \pi/2)$. If $u(x) = x^2 - k$ and $k \geq \frac{\pi^2}{4} + \frac{2}{\varepsilon}$ then $(\mathcal{L}+c)u \geq 0$ but (2.1) fails.

The spirit of Theorem 1 is just to partially recover (1.2)-(1.3) under the assumptions of Theorem 9. As will be seen later, this result will be instrumental in our applications.

Proof of Theorem 1. The positivity of λ_1 implies that the problem

$$\begin{cases} (\mathcal{L}+c)u(x) = f(x) & x \in \Omega \\ u_{|\partial\Omega} = 0, \end{cases}$$

admits, for each $f \in C^\alpha(\overline{\Omega})$, a unique solution in $C^{2,\alpha}(\overline{\Omega})$ ([12]). This shows the existence of ψ which is, by Theorem 9, nonnegative. By choosing M large we find $(\mathcal{L}+(c+M))\psi \geq 0$ with $c+M \geq 0$ and the strong maximum principle ([12]) yields $\psi > 0$ in $\overline{\Omega}$.

To show ii) we use ideas from [20] (see Theorem 10, Chap. 2). Introduce the new unknown v as $u = \psi v$. Then,

$$\mathcal{L}_1 v \geq 0$$

in Ω with $\mathcal{L}_1 v = -\sum_{ij} a_{ij}\partial_{ij} v + \sum_i \{b_i - \frac{2}{\psi}\sum_j a_{ij}\partial_j\psi\}\partial_i v$. Then, the maximum principle (1.2)-(1.3) implies

$$\inf_\Omega v = \inf_{\partial\Omega} v,$$

which is (1.4). □

Remarks 4.

a) A direct application of the strong maximum principle shows that

$$u(x) = \{\inf_{\partial\Omega} u\}\psi(x)$$

holds for all $x \in \Omega$ if such equality is attained somewhere in Ω.

b) Assume that $c \geq 0$, $c \neq 0$ in Theorem 1. Then $0 < \psi < 1$ in Ω and $\frac{\partial \psi}{\partial \nu} > 0$ in $\partial\Omega$, being ν the outward unit normal at $\partial\Omega$. In this case, setting $\underline{u} = \inf_{\partial\Omega} u$ with $u \neq \underline{u}\psi$ and $u(\bar{x}) = \underline{u}$ for $\bar{x} \in \partial\Omega$ one finds

$$\frac{\partial u}{\partial \nu} < \underline{u}\frac{\partial \psi}{\partial \nu},$$

at $x = \bar{x}$. For $\underline{u} \leq 0$ this is, of course, Hopf's Lemma ([12]).

c) In the framework of strong solutions a lower degree of smoothness in the coefficients of $\mathcal{L} + c$ is required for Theorem 1 to hold. In fact, suppose that (1.1) holds with $a_{ij} \in C(\overline{\Omega})$, $b_i \in L^\infty(\Omega)$, $c \in L^\infty(\Omega)$ while $\Omega \subset \mathbb{R}^N$ is bounded and C^2. Then, the operator $\mathcal{L} + c$ has a principal eigenvalue $\lambda_1^\Omega(\mathcal{L} + c)$ with a positive associated eigenfunction $\phi \in W^{2,q}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ for all $q > 1$, $0 < \beta < 1$. In addition, $\lambda_1^\Omega(\mathcal{L} + c)$ exhibits the same features as in the classical setting (see [2]).

The positivity of $\lambda_1^\Omega(\mathcal{L} + c)$ characterizes the maximum principle for strong solutions. Namely, $(\mathcal{L} + c)u \geq 0$ in Ω for $u \in W^{2,q}_{loc}(\Omega) \cap C(\overline{\Omega})$, $q \geq N$, and $u \geq 0$ on $\partial\Omega$ implies that $u \geq 0$ in Ω (see [2] where the regularity of Ω is considerably relaxed).

Under the preceding conditions and assuming $\lambda_1^\Omega(\mathcal{L} + c) > 0$, Theorem 1 now states that if $u \in W^{2,q}_{loc}(\Omega) \cap C(\overline{\Omega})$, $q \geq N$, satisfies

$$(\mathcal{L} + c)u \geq 0 \qquad \text{in } \Omega,$$

then

$$u(x) \geq \{\inf_{\partial\Omega} u\}\psi(x) \qquad x \in \Omega,$$

where ψ is the solution to

$$\begin{cases} (\mathcal{L} + c)\,\psi(x) = 0 & x \in \Omega \\ \psi(x) = 1 & x \in \partial\Omega. \end{cases}$$

Note that ψ is positive in $\overline{\Omega}$ with $\psi \in W^{2,q}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ for all $q > 1$, $0 < \beta < 1$.

3. Boundary blow-up problems

In this section we give the proofs of Theorems 3 and 5 concerning boundary blow-up problems and large solutions.

Proof of Theorem 3. Let $u_n \in C^{2,\alpha}(\overline{\Omega})$ be the solution to

$$\begin{cases} \Delta u(x) = u(x)^{p(x)} & x \in \Omega \\ u(x) = n & x \in \partial\Omega. \end{cases}$$

It follows from Theorem 2 that $u_n \to \infty$ uniformly in compacts of $\Omega_c \cup \Gamma_b$. In fact, a direct proof of this fact runs as follows: $u_n \geq n w_n$ where $w = w_n$ solves the problem

$$\begin{cases} \Delta w(x) = n^{p(x)-1}(w(x))^{p(x)} & x \in \Omega_c \\ w(x) = 1 & x \in \Gamma_b \\ w(x) = 0 & x \in \Gamma_i. \end{cases}$$

The assertion is then a consequence of the fact that $w_n \to w_0$ in $C^{1,\beta}(\overline{\Omega})$ where w_0 is the positive strong solution to

$$\begin{cases} \Delta w(x) = \chi_{\{p=1\}}(w(x))^{p(x)} & x \in \Omega_c \\ w(x) = 1 & x \in \Gamma_b \\ w(x) = 0 & x \in \Gamma_i. \end{cases}$$

To describe the behavior of u_n in $\Omega^+ = \Omega \setminus \overline{\Omega}_c$ choose a component Ω_0^+ of Ω^+. Then $u_n \le \hat{u}$ in Ω_0^+ where $u = \hat{u} \in C^{2,\alpha}(\Omega_0^+)$ is the minimal solution to

$$\begin{cases} \Delta u(x) = u(x)^{p(x)} & x \in \Omega_0^+ \\ u(x) = \infty & x \in \partial\Omega_0^+, \end{cases}$$

the existence of such solution is proved in [9]. Now, the increasing character of u_n together with that estimate is enough to ensure that u_n converges in $C^2(\Omega)$ towards a solution u of the equation. Regarding the boundary conditions, it is evident that $u = \infty$ on $\partial\Omega_0^+ \cap \partial\Omega$ if such part of $\partial\Omega_0^+$ is nonempty. Our main objective now is to show that

$$\lim u_n = \infty,$$

uniformly on each component $\Gamma_{0,i}$ of $\partial\Omega_0 \cap \Omega$. Being Γ_i the disjoint union of such components, this proves Theorem 3. To fulfill this objective we use the approach in [3]. Let $x_n \in \Gamma_{0,i}$ be such that

$$\underline{u}_n := u_n(x_n) = \inf_{\Gamma_{0,i}} u_n.$$

Let us prove that $\overline{u}_n \to \infty$. Otherwise, $\overline{u}_n = O(1)$. Assume this and proceed with the following constructions. First, consider the strip

$$U^+ = \{x \in \Omega^+ : \mathrm{dist(x,\Gamma_{0,i})} < \eta\}$$

where $\eta > 0$ is small enough. In U^+ consider the problems

$$\begin{cases} \Delta u(x) = u(x)^{p(x)} & x \in U^+ \\ u(x) = u_n(x) & x \in \partial U^+, \end{cases} \qquad \begin{cases} \Delta u(x) = u(x)^{p(x)} & x \in U^+ \\ u(x) = u_n(x) & x \in \partial U^+ \cap \Omega^+ \\ u(x) = \underline{u}_n & x \in \Gamma_{0,i}. \end{cases}$$

Both of them have a unique positive solution, u_n is just the solution of the first one while we set $u := \tilde{u}_n$ the solution of the latter one. We have that $u_n \ge \tilde{u}_n$ and therefore,

$$\frac{\partial u_n}{\partial \nu}(x_n) \le \frac{\partial \tilde{u}_n}{\partial \nu}(x_n), \tag{3.1}$$

for all n, where ν is the outward unit normal to U^+ on $\Gamma_{0,i}$. Since both $\overline{u}_n$ and $u_n(\cdot)_{|\partial U^+ \cap \Omega^+}$ are bounded, it follows that $\tilde{u}_n = O(1)$ in $C^{2,\alpha}(\overline{U}^+)$. Thus, from (3.1) we get that $\frac{\partial u_n}{\partial \nu}(x_n)$ is bounded above

$$\frac{\partial u_n}{\partial \nu}(x_n) \le C. \tag{3.2}$$

Now we proceed to estimate $\frac{\partial u_n}{\partial \nu}(x_n)$ from below. To this end set

$$U^- = \{x \in \Omega_c : \mathrm{dist}(x, \Gamma_{0,i}) < \eta\},$$

with $\eta > 0$ so small as to have $\overline{U}^- \setminus \Gamma_{0,i} \subset \Omega_c$. Pick $M \ge 1$ such that $M u_1 \ge 1$ in U^-. Then $w = M u_n$ solves

$$\begin{cases} \Delta w(x) = a(x)(w(x))^{p(x)} & x \in U^- \\ w(x) = M u_n(x) & x \in \partial U^-, \end{cases}$$

with $a(x) = M^{p(x)-1}$. Hence, $w = M u_n$ satisfies

$$\begin{cases} -\Delta w(x) + \|a\| w(x) \ge 0 & x \in U^- \\ w(x) = M u_n(x) & x \in \partial U^-, \end{cases}$$

with $\|a\| = \sup_{U^-} a$. Since $\lambda_1^{U^-}(-\Delta + \|a\|) > 0$, then Theorem 1 applied in U^- yields

$$\inf_{U^-} v_n = \inf_{\partial U^-} v_n \tag{3.3}$$

where $v_n = Mu_n/\psi$ (see Theorem 1 for the definition of ψ). Since $v_n \to \infty$ uniformly on $\partial U^- \cap \Omega_c$ we get

$$\inf_{U^-} v_n = \inf_{\Gamma_{0,i}} v_n = v_n(x_n) = M\underline{u}_n.$$

We now prove that

$$\lim \frac{\partial v_n}{\partial \nu}_{|x=x_n} = \infty, \tag{3.4}$$

where $\nu = \nu(x)$ is the *inner* unit normal to U^- at $x \in \Gamma_{0,i}$. Once (3.4) is proved we conclude

$$\lim M\frac{\partial u_n}{\partial \nu}(x_n) = \lim \underline{u}_n \frac{\partial \psi}{\partial \nu}(x_n) + \lim \frac{\partial v_n}{\partial \nu}(x_n) = \infty,$$

since $\underline{u}_n = O(1)$. This contradicts the previous estimate of such derivative, (3.2). Therefore

$$\lim u_n = \infty$$

uniformly on $\Gamma_{0,i}$ and, for the same reasons, on the whole of Γ_i.

Now to show (3.4) consider the family of annuli

$$\mathcal{A}_n = \{x : \frac{\eta}{8} < |x - y_n| < \frac{\eta}{4}\}, \qquad y_n = x_n + \frac{\eta}{4}\nu(x_n).$$

After decreasing η if necessary, it holds that $\mathcal{A}_n \subset U^-$ while $\partial\mathcal{A}_n \cap \Gamma_{0,i} = \{x_n\}$. For immediate use set $\partial_1\mathcal{A}_n = \{x : |x - y_n| = \frac{\eta}{8}\}$, $\partial_2\mathcal{A}_n = \{x : |x - y_n| = \frac{\eta}{4}\}$ the components of $\partial\mathcal{A}_n$. Observe that $v = v_n$ satisfies

$$\begin{cases} \mathcal{L}v(x) \geq 0 & x \in \mathcal{A}_n \\ v(x) = v_n(x) & x \in \partial\mathcal{A}_n, \end{cases}$$

with $\mathcal{L}v = -\Delta v - 2\psi^{-1}\nabla\psi\nabla v$. Moreover,

$$\inf_{\mathcal{A}_n} v_n = v_n(x_n) = M\underline{u}_n,$$

since $\inf_{\partial_1\mathcal{A}_n} v_n \to \infty$.

As in [12], we introduce now the usual barrier $h(r) = e^{\theta((\frac{\eta}{4})^2 - r^2)} - 1$, $r = |x - y_n|$, where $\theta > 0$ can be chosen independent of n so that

$$\mathcal{L}h < 0 \qquad x \in \mathcal{A}_n,$$

for every n. Thus,

$$\mathcal{L}(v_n - M\underline{u}_n - k_n h) \geq 0 \qquad x \in \mathcal{A}_n,$$

for $k_n \geq 0$. By setting

$$k_n = \frac{\inf_{\partial_1\mathcal{A}_n} v_n - M\underline{u}_n}{h(\eta/8)}$$

we finally obtain,

$$v_n \geq M\underline{u}_n + k_n h \qquad x \in \mathcal{A}_n,$$

since such inequality holds on $\partial\mathcal{A}_n$. Therefore,

$$\lim \frac{\partial v_n}{\partial \nu}(x_n) \geq \lim k_n \left(-h'\left(\frac{\eta}{4}\right)\right) = \infty.$$

Finally, once it has been shown that $u_n \to \infty$ uniformly on Γ_i this uniformity propagates to $\overline{\Omega}_c$. In fact, one obtains from (3.3) that such convergence is uniform in a closed strip $\{x \in \Omega_c : \mathrm{dist}(x, \Gamma_i) \leq \eta\}$. □

Now we prove Theorem 5.

Proof of Theorem 5. Let $\Gamma_{0,i}$ any component of Γ_i which, for simplicity, will be denoted as Γ_i. To prove the uniform limit $\lim u_n = \infty$ on Γ_i we set,

$$\inf_{\Gamma_i} u_n = u_n(x_n) := \underline{u}_n \qquad x_n \in \Gamma_i$$

and assume, arguing by contradiction, that $\underline{u}_n = O(1)$. We then proceed with a similar construction as in Theorem 3. We introduce $U^+ = \{x \in \Omega^+ : \mathrm{dist}(x, \Gamma_i) < \eta\}$, $\eta > 0$ small, where we consider the problems

$$\begin{cases} \Delta u(x) = e^{p(x)u(x)} & x \in U^+ \\ u(x) = u_n(x) & x \in \partial U^+, \end{cases} \qquad \begin{cases} \Delta u(x) = e^{p(x)u(x)} & x \in U^+ \\ u(x) = u_n(x) & x \in \partial U^+ \cap \Omega^+ \\ u(x) = \underline{u}_n & x \in \Gamma_i. \end{cases}$$

Thus we get

$$u_n(x) \geq \tilde{u}_n(x) \qquad x \in U^+,$$

where $u = \tilde{u}_n$ stands for the solution to the latter problem. That is why

$$\frac{\partial u_n}{\partial \nu} \leq \frac{\partial \tilde{u}_n}{\partial \nu}$$

at $x = x_n$, with ν the outward unit normal. Since u_n stays bounded on $\partial U^+ \cap \Omega^+$ the $C^{2,\alpha}(\overline{U}^+)$ norm of $\tilde{u}_n$ is bounded and this means that the normal derivative $\partial u_n / \partial \nu$ evaluated at $x = x_n$ remains bounded from above.

On the other hand, $u = u_n$ solves

$$\begin{cases} \Delta u(x) = 1 & x \in U^- \\ u(x) = u_n(x) & x \in \partial U^-, \end{cases}$$

with $U^- = \{x \in \Omega_c : \mathrm{dist}(x, \Gamma_i) < \eta\}$ and $\eta > 0$ small enough. If $z = \phi(x)$ stands for the solution of $-\Delta z = 1$ in U^- with $z_{|\partial U^-} = 0$, then $v = v_n := u_n + \phi$ is the solution to

$$\begin{cases} \Delta v(x) = 0 & x \in U^- \\ v(x) = u_n(x) & x \in \partial U^-. \end{cases}$$

Thus

$$\inf_{U^-} v_n = v_n(x_n) = \underline{u}_n$$

for large n, since $v_n \to \infty$ uniformly on $\partial U^- \cap \Omega_c$. By introducing now the family $\mathcal{A}_n$ of annuli constructed in the proof of Theorem 3 and employing the same barrier $h(r)$ we get

$$u_n \geq \underline{u}_n + k_n h - \phi \qquad x \in \mathcal{A}_n,$$

with $k_n \to \infty$. This in turns implies that $\frac{\partial u_n}{\partial \nu}_{|x=x_n} \to \infty$ which contradicts the previous estimate. Therefore $u_n \to \infty$ uniformly on Γ_i. Finally,

$$\inf_{U^-}\{u_n + \phi\} \geq \inf_{\Gamma_i} u_n$$

implies that $u_n \to \infty$ in the whole of $\overline{U}^-$ which, combined with Theorem 4, yields $u_n \to \infty$ uniformly in $\overline{\Omega}_c$. □

4. A Population dynamics problem with a two-signed weight

This section is devoted to prove Theorem 7.

Proof of Theorem 7. Let us begin with a)-b) and recall that $m \neq 0$. We are only considering b) since the analysis of a) is much simpler.

If $m \geq 0$, $m \neq 0$ in Ω_c then $\lambda_0(t) = \lambda_1^{\Omega_c}(-\Delta - tm)$ is real-analytic, non increasing and concave in $\mathbb{R}$ (Section 2). In addition $\lambda_0(0) > 0$. Therefore, $\lambda_0(t)$ is decreasing with $\lim_{t\to\infty} \lambda_0(t) = -\infty$ since the derivative of $\lambda_0(t)$ must be negative somewhere. Thus $\{\lambda_0(t) > 0\} = (-\infty, \bar{t}_2)$ with $\bar{t}_2 > 0$. Symmetrically, $\{\lambda_0(t) > 0\} = (\bar{t}_{-2}, \infty)$ with $\bar{t}_{-2} < 0$ if $m \leq 0$, $m \neq 0$ in Ω_c, while $\lambda_0(t)$ must vanish at $t = \bar{t}_{\pm 2}$.

Suppose now that m takes the two signs in Ω_c. By taking a ball $B^+ \subset \overline{B}^+ \subset \{m > 0\} \cap \Omega_c$ and observing that

$$\lambda_1^{\Omega_c}(-\Delta - tm) < \lambda_1^{B^+}(-\Delta - tm) \qquad t \in \mathbb{R},$$

it follows that $\lim_{t\to\infty} \lambda_0(t) = -\infty$. An analogous argument using a ball $B^- \subset \overline{B}^- \subset \{m < 0\} \cap \Omega_c$ gives $\lim_{t\to-\infty} \lambda_0(t) = -\infty$. Thus

$$\{\lambda_0(t) > 0\} = (\bar{t}_{-2}, \bar{t}_2) \qquad t_{-2} < 0 < \bar{t}_2,$$

since $\lambda_0(0) > 0$. In addition, $\lambda_0(t)$ vanishes at $t = \bar{t}_{\pm 2}$.

By setting $\lambda(t) = \lambda_1^{\Omega}(-\Delta - tm)$, the previous arguments prove that $\{\lambda(t) < 0\} = (-\infty, \bar{t}_1)$ with $\bar{t}_1 > 0$ if $m \geq 0$ (respectively, $\{\lambda(t) < 0\} = (\bar{t}_{-1}, \infty,)$, $\bar{t}_{-1} < 0$ when $m \leq 0$ in Ω) or, provided m exhibits both signs in Ω,

$$\{\lambda(t) < 0\} = (-\infty, \bar{t}_{-1}) \cup (\bar{t}_1, \infty),$$

with $\bar{t}_{-1} < 0 < \bar{t}_1$. In all cases $\lambda(t)$ vanishes at $t = \bar{t}_{\pm 1}$. Finally, the assertions in b) follow from the preceding discussion and the fact that $\lambda(t) < \lambda_0(t)$ in $\mathbb{R}$.

Regarding c), and as was quoted in Theorem 6-i), we refer to [5] for the proof that u_λ bifurcates from zero at $\lambda = \bar{\lambda}_{\pm 1}$ (see also Section 5). On the other hand, suppose that $\bar{\lambda}_2 = \infty$. According to the preceding discussion this means that $m \leq 0$ in Ω_c while m attains a positive value somewhere in Ω^+. Thus, a ball $B^+ \subset \overline{B}^+ \subset \Omega^+$ can be found so that $m \geq m_0 > 0$ in B^+. Let us denote as $v = v_\lambda$ the unique positive solution to

$$\begin{cases} -\Delta v(x) = \lambda m(x) v(x) - a(x)(v(x))^p & x \in B^+ \\ v_{|\partial B^+} = 0. \end{cases}$$

Then we have, on one hand, that $u_\lambda \geq v_\lambda$ in B^+ for large λ. On the other hand, it is well-known that (see the proof of Theorem 8)

$$\varliminf_{\lambda\to\infty} \lambda^{-\frac{1}{p-1}} v_\lambda(x) > 0 \qquad x \in B^+.$$

Thus, it follows that $|u_\lambda|_{C(\overline{\Omega})} \to \infty$ as $\lambda \to \infty$.

Let us assume next that, say $\bar{\lambda}_2 < \infty$. Then, it was shown in [5] that $|u_\lambda|_{C(\overline{\Omega})} \to \infty$ as $\lambda \to \bar{\lambda}_2$. We are going now quite beyond this result and show that

$$u_\lambda \to \infty \qquad \text{as} \;\; \lambda \to \bar{\lambda}_2, \tag{4.1}$$

uniformly on compacts of Ω_c. We first prove the existence of a function $v \in C^1(\overline{\Omega})$, $v(x) > 0$ in Ω, $\frac{\partial v}{\partial \nu} < 0$ on $\partial\Omega$, ν being the outer unit normal, such that

$$u_\lambda \geq v \qquad x \in \Omega, \tag{4.2}$$

for all $\bar{\lambda}_2 - \delta \le \lambda \le \bar{\lambda}_2$ with $\delta > 0$ small. In fact, by continuity a certain positive μ_0 exists such that $\lambda_1^{\Omega}(-\Delta - \lambda m) \le -\mu_0$ in $[\bar{\lambda}_2 - \delta, \bar{\lambda}_2]$. Setting $\phi_1 = \phi_1(\cdot, \lambda)$ the principal (positive) eigenfunction associated to $\lambda_1^{\Omega}(-\Delta - \lambda m)$ normalized so that $|\phi_1|_{C(\overline{\Omega})} = 1$, then it be can checked that $\underline{u} = \varepsilon \phi_1$ is a subsolution to (1.11) with $\lambda \in [\bar{\lambda}_2 - \delta, \bar{\lambda}_2]$ provided

$$\varepsilon^{p-1} \sup_{\Omega} a \le \mu_0.$$

Now, the continuity of ϕ_1 with respect to λ, when regarded in $C^1(\overline{\Omega})$, and the fact that $\frac{\partial \phi_1}{\partial \nu} < 0$ for every $\lambda \in [\bar{\lambda}_2 - \delta, \bar{\lambda}_2]$ allow us to find $v \in C^1(\overline{\Omega})$, $0 < v < \phi_1$ in Ω for all $\lambda \in [\bar{\lambda}_2 - \delta, \bar{\lambda}_2]$, with $\frac{\partial v}{\partial \nu} < 0$. To complete the proof of the assertion observe that (1.11) admits arbitrarily large supersolutions in the range $\lambda_1^{\Omega_c}(-\Delta - \lambda m) > 0$ (see [5] or Section 5). Therefore, inequality (4.2) follows.

To show (4.1) we deal with the case $\Gamma_b \ne \emptyset$ since the case $\overline{\Omega}_c \subset \Omega$ is much simpler. To begin with, take $\lambda_n \to \bar{\lambda}_2$, $\bar{\lambda}_2 - \delta \le \lambda_n \le \bar{\lambda}_2$, put $u_n = u_{\lambda_n}$ and observe that $u = u_n$ solves

$$\begin{cases} -\Delta u(x) - \lambda_n m(x) u(x) = 0 & x \in \Omega_c \\ u(x) = 0 & x \in \Gamma_b \\ u(x) = u_n(x) & x \in \Gamma_i. \end{cases}$$

We claim the existence of $\eta_n > 0$, $\eta_n \to 0$, such that $\lambda_1^{U_{\eta_n}}(-\Delta - \lambda_n m) = 0$ where, for $t > 0$ we set $U_t = \{x \in \Omega : \operatorname{dist}(x, \Omega_c) < t\}$. Then, taking φ the positive eigenfunction associated to $\lambda_1^{\Omega_c}(-\Delta - \bar{\lambda}_2 m)$, $|\varphi|_{C(\overline{\Omega})} = 1$, and designating by φ_n the corresponding eigenfunction associated to $\lambda_1^{U_{\eta_n}}(-\Delta - \lambda_n m)$ it follows that $\varphi_n \to \varphi$ in $C^{2,\alpha}(\overline{\Omega}_c)$ (see, for instance [13] for a proof of this convergence).

Let us choose $K \subset \Omega_c \cup \Gamma_b$, compact, and any $M > 0$. We will follow the approach in [3] to show that

$$u_n(x) \ge \theta M \varphi(x) \qquad x \in K, \tag{4.3}$$

$\theta = \frac{1}{2} \inf_K \varphi$, for large n. This certainly proves (4.1). In fact, set $\Omega_{c,\eta} = \{x \in \Omega_c : \operatorname{dist}(x, \Gamma_i) > \eta\}$ and select $\eta > 0$ small such that $K \subset \Omega_{c,\eta}$. If we take $\Gamma_{i,\eta} = \{x \in \Omega_c : \operatorname{dist}(x, \Gamma_i) = \eta\} = \partial\Omega_{c,\eta} \cap \Omega_c$, we can find η small such that

$$M \sup_{\Gamma_{i,\eta}} \varphi \le \frac{1}{2} \inf_{\Gamma_{i,\eta}} v.$$

Thus we find that

$$M \sup_{\Gamma_{i,\eta}} \varphi_n \le \inf_{\Gamma_{i,\eta}} v.$$

for $n \ge n_M$. Moreover

$$M \varphi_n(x) \ge \frac{1}{2} \inf_K \varphi \qquad x \in K, \tag{4.4}$$

for $n \ge n_{K,M}$. We next observe that $u = u_n$ satisfies

$$\begin{cases} -\Delta u(x) - \lambda_n m(x) u(x) = 0 & x \in \Omega_{c,\eta} \\ u(x) = 0 & x \in \Gamma_b \\ u(x) = u_n(x) & x \in \Gamma_{i,\eta}, \end{cases} \tag{4.5}$$

meanwhile $\hat{u}_n = M \varphi_n$ is clearly a subsolution to (4.5). Since $\lambda_1^{\Omega_{c,\eta}}(-\Delta - \lambda_n m) > 0$ for all n, direct comparison gives

$$u_n(x) \ge M \varphi_n(x) \qquad x \in \Omega_{c,\eta},$$

which, combined with (4.4) provides (4.3).

As for the claim, take $\eta_0 > 0$ small such that $\lambda_1^{U_{\eta_0}}(-\Delta - \bar{\lambda}_2 m) < 0$. The continuity of the principal eigenvalue on the zero order term (Section 2) shows that $\lambda_1^{U_{\eta_0}}(-\Delta - \lambda m) < 0$ for all $\bar{\lambda}_2 - \delta < \lambda < \bar{\lambda}_2$, $\delta > 0$ small. On the other hand, $\lambda_1^{U_t}(-\Delta - \lambda m)$ is smooth on $t \in [0, \eta_0]$ (see [13] and close results in [18]). That is why, for fixed λ_n close to $\bar{\lambda}_2$, a value η_n with $0 < \eta_n < \eta_0$ exists such that $\lambda_1^{U_{\eta_n}}(-\Delta - \lambda_n m) = 0$, what proves the claim.

We now show that $\lim_{\lambda \to \bar{\lambda}_2} u_\lambda = \infty$ uniformly on Γ_i. To this end let us prove this limit uniformly on each component of Γ_i. For simplicity we denote again Γ_i such component, we take $\lambda_n \to \bar{\lambda}_2$, set $u_n = u_{\lambda_n}$ and suppose that

$$\underline{u}_n = \inf_{\Gamma_i} u_n = u_n(x_n) \to \infty \qquad x_n \in \Gamma_i,$$

to get a contradiction. Using the same arguments as in Section 2 we observe that

$$v_n(x) \le u_n(x) \qquad x \in U^+, \qquad U^+ = \{x \in \Omega^+ : \operatorname{dist}(x, \Gamma_i) < \eta\},$$

$\eta > 0$ small, where $v = v_n$ solves

$$\begin{cases} -\Delta v(x) = \lambda_n m(x) v(x) - a(x)(v(x))^p & x \in U^+ \\ v(x) = \underline{u}_n & x \in \Gamma_i \\ v(x) = u_n(x) & x \in \partial U^+ \cap \Omega^+. \end{cases} \tag{4.6}$$

Then we get

$$\frac{\partial u_n}{\partial \nu} \le \frac{\partial v_n}{\partial \nu} \qquad \text{at } x = x_n,$$

with ν the outer unit normal to U^+. Furthermore it holds that,

$$\frac{\partial u_n}{\partial \nu}(x_n)$$

is bounded above. This follows from the fact that $|v_n|_{C^{2,\alpha}(\overline{U}^+)}$ is bounded. In fact such assertion is a consequence of Schauder's estimates applied to (4.6), after obtaining bounds of $|u_n|_{C^{2,\alpha}(\overline{U}^+)}$ near $\partial U^+ \cap \Omega^+$, and further getting bounds of $|v_n|_{C(\overline{U}^+)}$ and $|v_n|_{C^{\alpha}(\overline{U}^+)}$. To estimate u_n we notice that $u_n \le w_n$ with $w = w_n$ the solution to

$$\begin{cases} -\Delta w(x) = \bar{\lambda}_2 m^+(x) w(x) - a(x)(w(x))^p & x \in \Omega^+ \\ w(x) = u_n(x) & x \in \partial \Omega^+. \end{cases}$$

Thus $u_n \le u_\infty$ in U^+ where $u_\infty \in C^{2,\alpha}(\Omega^+)$ is the minimal solution to

$$\begin{cases} -\Delta u(x) = \bar{\lambda}_2 m^+(x) u(x) - a(x)(u(x))^p & x \in \Omega^+ \\ u(x) = \infty & x \in \partial \Omega^+. \end{cases} \tag{4.7}$$

This gives us L^∞_{loc} estimates of u_n which in turns lead to estimates in $C^{2,\alpha}(\overline{Q})$ for every subdomain $Q \subset \overline{Q} \subset \Omega^+$ (see for instance [8]). On the other hand, the estimate $v_n \le v$ with v the solution of

$$\begin{cases} -\Delta v(x) = \bar{\lambda}_2 m^+(x) v(x) - a(x)(v(x))^p & x \in U^+ \\ v(x) = \sup \underline{u}_n & x \in \Gamma_i \\ v(x) = u_\infty(x) & x \in \partial U^+ \cap \Omega^+, \end{cases}$$

proves the boundedness of $|v_n|_{C(\overline{U}^+)}$. Going back to (4.6) we can now use the standard $W^{2,q}$ estimates to get that $|v_n|_{C^{1,\alpha}(\overline{U}^+)} \le C$, C a constant, for all $0 < \beta < 1$, and that is just what we wanted to obtain.

Our objective now is to show the divergence of the normal derivative $\frac{\partial u_n}{\partial \nu}_{|x=x_n}$. Accordingly, we take $U^- = \{x \in \Omega_c : \text{dist}(x, \Gamma_i) < \eta\}$, $\eta > 0$ small enough, and observe that $u = u_n$ solves,

$$\begin{cases} -\Delta u(x) - \lambda_n m(x) u(x) = 0 & x \in U^- \\ u(x) = u_n(x) & x \in \partial U^-. \end{cases}$$

In order to apply Theorem 1 notice that $\lambda_1^{U^-}(-\Delta - \lambda_n m) > 0$ and define $\psi = \psi_n$ the solution to

$$\begin{cases} -\Delta \psi(x) - \lambda_n m(x) \psi(x) = 0 & x \in U^- \\ \psi(x) = 1 & x \in \partial U^-. \end{cases}$$

The fact that $\lambda_1^{U^-}(-\Delta - \bar{\lambda}_2 m) > 0$ and the continuity of $\lambda_1^{U^-}(-\Delta - \lambda m) > 0$ on λ allow to conclude the uniform boundedness of ψ_n in the norm of $C^{2,\alpha}(\overline{U}^-)$. On the other hand Theorem 1 ensures that

$$w_n = \frac{u_n}{\psi_n}$$

satisfies

$$\inf_{U^-} w_n = w_n(x_n) = \underline{u}_n, \tag{4.8}$$

for large n due to the divergence of w_n when regarded on $\partial U^- \cap \Omega_c$. We have in addition that $w = w_n$ solves

$$\begin{cases} \mathcal{L}_n w(x) = 0 & x \in U^- \\ w(x) = u_n(x) & x \in \partial U^-, \end{cases}$$

where the operator $\mathcal{L} w = -\Delta w - 2\frac{\nabla \psi_n}{\psi_n} \nabla w$ has its coefficients uniformly bounded in $C^\alpha(\overline{U}^-)$. The same argument as in the proof of Theorem 3 allows us to write

$$w_n(x) \ge \underline{u}_n + \hat{k}_n h(|x - y_n|) \qquad \frac{\eta}{8} < |x - y_n| < \frac{\eta}{4},$$

with $y_n = x_n + (\eta/4)\nu(x_n)$, ν the inner unit normal to U^-, $h = h(r)$ is the uniform barrier introduced in Theorem 3 and $\hat{k}_n$ is a positive sequence that can be chosen so that $\hat{k}_n \to \infty$.

The inequality above leads to the fact that $\frac{\partial w_n}{\partial \nu}(x_n) \to \infty$ and finally

$$\underline{\lim} \frac{\partial u_n}{\partial \nu}(x_n) \ge \lim \frac{\partial w_n}{\partial \nu}(x_n) + \underline{\lim}\, \underline{u}_n \frac{\partial \psi_n}{\partial \nu}(x_n) = \infty.$$

This is the contradiction that we look for and hence the proof that $u_\lambda \to \infty$ uniformly on Γ_i is concluded. Notice that this divergence propagates to the whole of $\overline{U}^-$ through (4.8). This means that $u_\lambda \to \infty$ uniformly in compacts of $\Omega_c \cup \Gamma_i$.

To finish the proof of Theorem 7 let us analyze the behavior of u_λ in Ω^+. Assume for simplicity that $\overline{\Omega}_c \subset \Omega$ (and thus $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega = \emptyset$). Estimating u_n by the solution $u = u_\infty$ of (4.7) we get –as was already shown– $C^{2,\alpha}_{loc}$ estimates of u_n in Ω^+. Thus, a subsequence can be extracted from u_n (we still use u_n to designate such subsequence) such that $u_n \to u^*$ in $C^2(U^+)$ and so $u = u^*$ solves

$$-\Delta u(x) = \bar{\lambda}_2 m(x) u(x) - a(x)(u(x))^p$$

in Ω^+. However we assert that for every $M > 0$ it holds that

$$u^*(x) \ge u_M(x) \qquad x \in \Omega^+, \tag{4.9}$$

where $u = u_M$ is the solution to

$$\begin{cases} -\Delta u(x) = \bar{\lambda}_2 m(x)u(x) - a(x)(u(x))^p & x \in \Omega^+ \\ u(x) = M & x \in \partial\Omega^+. \end{cases}$$

In fact, it can be shown that $u_M = \lim u_{M,n}$ in $C^{2,\alpha}(\overline{\Omega}^+)$ where $u = u_{M,n}$ is, in turn, the solution to

$$\begin{cases} -\Delta u(x) = \lambda_n m(x)u(x) - a(x)(u(x))^p & x \in \Omega^+ \\ u(x) = M & x \in \partial\Omega^+. \end{cases}$$

Since $u_n \geq u_{M,n}$ for large n we readily conclude (4.9). But this implies that

$$u^* \geq u$$

where u is the minimal solution to

$$\begin{cases} -\Delta u(x) = \bar{\lambda}_2 m(x)u(x) - a(x)(u(x))^p & x \in \Omega^+ \\ u(x) = \infty & x \in \partial\Omega^+, \end{cases} \tag{4.10}$$

whose existence is well-known (see [8]). Thus u^* solves the blow-up problem (4.10). Moreover, it is also known that under condition (1.18), problem (4.10) only admits a unique solution and so $u^* = u$ (see [8]). This implies in particular that the whole sequence u_n converges to u in $C^{2,\alpha}(\Omega^+)$. Finally, to deal with the case $\Gamma^+ \neq \emptyset$ we only need to add the boundary condition $u = 0$ on Γ^+ in all of the involved auxiliary boundary value problems. □

5. A reaction-diffusion problem involving a variable reaction

Proof of Theorem 8. Let us begin with the existence and uniqueness questions raised in i). Uniqueness can be obtained by using, among several approaches, the sweeping principle (see [22]). Namely, let $u_1, u_2 \in C^{2,\alpha}(\overline{\Omega})$ be positive solutions to (1.21) and set $t^* \geq 1$ the infimum among the values $t \geq 1$ such that $tu_2 \geq u_1$. By the strong maximum principle and the fact that tu_2 defines a supersolution when $t \geq 1$, it can be shown that $t^* = 1$. This shows that $u_2 \geq u_1$ and the complementary inequality is obtained in the same way.

To show the necessity of (1.22), the existence of a positive solution $u \in C^{2,\alpha}(\overline{\Omega})$ implies that,

$$\int_\Omega |\nabla u|^2 - \lambda m u^2 + a\chi_{\Omega_c} u^2 < 0.$$

The variational characterization of the first eigenvalue implies

$$\lambda_1^\Omega(-\Delta - \lambda m + a\chi_{\Omega_c}) < 0.$$

On the other hand, such solution u is a classical positive supersolution to $-\Delta u - \lambda m u + au = 0$ in Ω_c which is positive in $\Gamma_i = \partial\Omega_c \cap \Omega$. As it is well-known ([5], [16], [6]) this implies the positivity of $\lambda_1^{\Omega_c}(-\Delta - \lambda m + a)$.

Concerning the existence of solutions under (1.22), we proceed in a direct way instead of using the involved perturbation approach developed in [17] (which is in addition restricted to the case $m = 1$ and $\overline{\Omega}^+ \subset \Omega$). The crucial point is to find a subsolution. Notice that the natural candidate, namely the eigenfunction $\hat{\phi}$ to

$$(-\Delta - \lambda m + a\chi_{\Omega_c})\hat{\phi} = \hat{\sigma}\hat{\phi} \qquad x \in \Omega, \qquad \hat{\phi}_{|\partial\Omega} = 0,$$

$\hat{\sigma} = \lambda_1^\Omega(-\Delta - \lambda m + a\chi_{\Omega_c})$, must be ruled out. Indeed the standard choice $\underline{u} = \varepsilon\hat{\phi}$ requires

$$-\hat{\sigma} \geq a(\varepsilon\hat{\phi})^{p(x)-1}$$

in Ω. However, and no matter how small $\varepsilon > 0$ is taken, such inequality can not be given for granted at Γ_i since $p = 1$ there.

To avoid this problem and construct a subsolution we introduce $Q = B(\Omega_c, \delta) \cap \Omega = \{x \in \Omega : \mathrm{dist}(x, \Omega_c) < \delta\}$ with $\delta > 0$ small and observe that with a suitable election of δ we have

$$\sigma := \lambda_1^{\Omega}(-\Delta - \lambda m + a\chi_Q) < 0.$$

Setting $\phi \in W^{2,q}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$, $q > 1$, $0 < \beta < 1$ arbitrary, the positive eigenfunction associated to σ

$$(-\Delta - \lambda m + a\chi_Q)\phi = \sigma\phi \qquad x \in \Omega, \qquad \phi_{|\partial\Omega} = 0,$$

$|\phi|_{C^1(\overline{\Omega})} = 1$ (see Remark 4-c)), then $\underline{u} = \varepsilon\phi$ with $\varepsilon > 0$ small, gives a subsolution to (1.21). In fact, we need

$$(-\Delta - \lambda m + a\chi_{\Omega_c})\underline{u} \le a\chi_{\Omega^+}\underline{u}^{p(x)} \qquad x \in \Omega. \tag{5.1}$$

In Ω_c this means

$$(-\Delta - \lambda m + a\chi_{\Omega_c})\underline{u} \le 0,$$

which holds due to the negativity of σ. In $Q \setminus \Omega_c$ (5.1) reads as

$$-\sigma + a\chi_{Q\setminus\Omega_c} \ge a\chi_{Q\setminus\Omega_c}\underline{u}^{p(x)-1},$$

which is certainly true provided $0 < \varepsilon \le 1$. In $\Omega \setminus Q$ (5.1) is equivalent to

$$-\sigma \ge a\underline{u}^{p(x)-1}. \tag{5.2}$$

Such inequality is achieved for ε small either if $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega$ is empty or provided that $p(x) > 1$ for all $x \in \Gamma^+$ if $\Gamma^+ \ne \emptyset$. However, in the latter case (5.2) could become false if $p(\bar{x}) = 1$, $\bar{x} \in \Gamma^+$ and $x \to \bar{x}$. To tackle the problem if this is the case we redefine the subsolution. By introducing

$$\Omega' = \{x \in \Omega : \mathrm{dist}(x, \Gamma^+) > \delta_1\},$$

$\delta_1 > 0$ can be taken so small as to have

$$\sigma' := \lambda_1^{\Omega'}(-\Delta - \lambda m + a\chi_Q) < 0.$$

If ϕ' stands for the corresponding associated positive eigenfunction in Ω', $|\phi'|_{C(\overline{\Omega}')} = 1$, and $\bar{\phi}'$ is its extension by zero to Ω, then $\underline{u} = \varepsilon\bar{\phi}' \in H^1(\Omega) \cap C^\alpha(\overline{\Omega})$ defines a weak subsolution (which still enables us to get a classical solution to (1.21)). Indeed, we succeed now in getting (5.2) by setting $\varepsilon > 0$ small. This finishes the construction of the subsolution.

To find out a comparable supersolution we observe, following [5], that a positive large enough function $v \in C^1(\overline{\Omega})$, $v_{|\partial\Omega} = 0$, can be found so that

$$\lambda_1^{\Omega}(-\Delta - \lambda m + a\chi_{\Omega_c} + a\chi_{\Omega^+}v^{p(x)-1}) > 0.$$

Taking $\psi \in W^{2,q}(\Omega) \cap C^{1,\beta}(\overline{\Omega})$ its associated positive eigenfunction with $|\psi|_{C(\overline{\Omega})} = 1$, then $\overline{u} = M\psi$ defines a supersolution provided $M \ge 1$ is taken so large as to have $M\psi \ge v$.

The proof of assertion ii) follows exactly with the same arguments used in the corresponding proof of a) and b) in Theorem 5. Thus, let us show assertion iii) and begin by the bifurcation at, say $\lambda = \bar{\lambda}_1$. First, we have that there exists $C > 0$ such that

$$|u_\lambda|_{C(\overline{\Omega})} \le C,$$

for $\bar{\lambda}_1 < \lambda \le \bar{\lambda}_1 + \delta$. This is a consequence of the existence of arbitrarily small subsolutions and the existence of a uniform supersolution $\overline{u}_1$ in that interval. In fact, since

$$\lambda_1^{\Omega_c}(-\Delta - \bar{\lambda}_1 m + a) > 0$$

the previous construction proves the existence of $\overline{u}_1$ such that $\overline{u}_{1|\partial\Omega} = 0$ together with

$$-\Delta\overline{u}_1 - \bar{\lambda}_1 m\overline{u}_1 + a\overline{u}_1{}^{p(x)} \geq \theta\overline{u}_1 \qquad x \in \Omega,$$

for a certain $\theta > 0$. Thus,

$$-\Delta\overline{u}_1 - \lambda m\overline{u}_1 + a\overline{u}_1{}^{p(x)} \geq 0 \qquad x \in \Omega,$$

if $0 < (\lambda - \bar{\lambda}_1)\sup|m| \leq \theta$. Once we have L^∞ bounds of u_λ as $\lambda \to \bar{\lambda}_1$ we are progressively getting, as it is well-known, uniform bounds in $W^{2,q}(\Omega)$, $C^{1,\beta}(\overline{\Omega})$ and finally in $C^{2,\alpha}(\overline{\Omega})$. Then, after passing through subsequences if necessary, and taking into account the nonexistence of positive solutions at $\lambda = \bar{\lambda}_1$ one concludes that $u_\lambda \to 0$ in $C^{2,\alpha}(\overline{\Omega})$ as $\lambda \to \bar{\lambda}_1$.

Next, we describe the limit as, $\lambda \to \infty$, that is, when $\bar{\lambda}_2(m) = \infty$ (notice that this requires $m \leq 0$ in Ω_c together with the positiveness of m somewhere in Ω). Thus, there exists a ball $B^+ \subset \Omega^+$ where m is bounded away from zero. Consider the problem

$$\begin{cases} -\Delta v(x) = \lambda m(x)v(x) - a(x)(v(x))^{p(x)} & x \in B^+ \\ v_{|\partial B^+} = 0. \end{cases} \tag{5.3}$$

Our previous study shows that (5.3) admits a unique positive solution $v = v_\lambda$ for all $\lambda > \bar{\lambda}_{1,B^+}$, $t = \bar{\lambda}_{1,B^+}$ being the unique root of $\lambda_1^{B^+}(-\Delta - tm) = 0$. Moreover, $v_\lambda \geq w_\lambda$ where $w = w_\lambda(x)$ is the positive solution to

$$\begin{cases} -\Delta w(x) = \lambda m_0 w(x) - \|a\| g(w(x)) & x \in B^+ \\ w_{|\partial B^+} = 0, \end{cases} \tag{5.4}$$

where $m_0 = \inf_{B^+} m$, $\|a\| = |a|_{C(\overline{B}^+)}$, $g(v) = v^{p_0}$ if $0 \leq v \leq 1$, $g(v) = v^{p_1}$ for $v > 1$ and $p_0 = \inf_{B^+} p$, $p_1 = \sup_{B^+} p$ $(p_0, p_1 > 1)$. Problem (5.4) admits a unique positive solution for $\lambda > \lambda_1^{B^+}/m_0$. Using $\underline{w} = A\varphi_B$ as a subsolution (φ_B the positive eigenfunction associated to $\lambda_1^{B^+}(-\Delta)$, $|\varphi_B|_{C(\overline{B}^+)} = 1$) it follows that

$$u_\lambda(x_0) \geq w_\lambda(x_0) \geq A(\lambda)$$

where x_0 is the center of B^+ and $A(\lambda) \sim (\lambda m_0/\|a\|)^{\frac{1}{p_1-1}}$ as $\lambda \to \infty$. Thus, we conclude $\lim_{\lambda\to\infty} |u_\lambda|_{C(\overline{\Omega})} = \infty$.

Let us discuss next the behavior of u_λ when say $\lambda \to \bar{\lambda}_2(m)$ and provided that $\bar{\lambda}_2(m) < \infty$ (we are dropping the explicit dependence on m in the notation for short). We are beginning with the divergence towards infinity in $\Omega_c \cup \Gamma_i$ where for completeness we are directly dealing with the case $\Gamma_b \neq \emptyset$. As a first fact, the existence of a uniform subsolution $\underline{u}$ for $\bar{\lambda}_2 - \delta_2 \leq \lambda \leq \bar{\lambda}_2$ is ensured. In fact, keeping the notation introduced in the construction of a subsolution, $\lambda_1^\Omega(-\Delta - \bar{\lambda}_2 m + a\chi_{\Omega_c}) < 0$ implies

$$\bar{\sigma}_2 := \lambda_1^\Omega(-\Delta - \bar{\lambda}_2 m + a\chi_Q) < 0,$$

for $\delta > 0$ small. Using $\underline{u} = \varepsilon\phi_2$, ϕ_2 the normalized positive eigenfunction associated to $\bar{\sigma}_2$ we see that the condition for a subsolution is

$$0 \leq -(-\Delta - \lambda m + a\chi_{\Omega_c})\underline{u} = \{-\bar{\sigma}_2 - (\bar{\lambda}_2 - \lambda)m\}\underline{u},$$

in Ω_c,

$$a\chi_{Q\setminus\Omega_c}\underline{u}^{p(x)} \leq -(-\Delta - \lambda m + a\chi_{\Omega_c})\underline{u} = \{-\bar{\sigma}_2 - (\bar{\lambda}_2 - \lambda)m\}\underline{u} + a\chi_{Q\setminus\Omega_c}\underline{u},$$

in $Q \setminus \Omega_c$ and

$$a\underline{u}^{p(x)} \leq -(-\Delta - \lambda m + a\chi_{\Omega_c})\underline{u} = \{-\bar{\sigma}_2 - (\bar{\lambda}_2 - \lambda)m\}\underline{u},$$

in $\Omega \setminus Q$. Assuming that $p > 1$ on Γ^+, if $\Gamma^+ \neq \emptyset$, it is clear that such conditions are satisfied if both $\varepsilon > 0$ and $\bar{\lambda}_2 - \bar{\lambda}$ are small. If $p = 1$ at some point of Γ^+ the subsolution can be modified according to the alternative way presented above, getting again a uniform subsolution in that case.

According to the lines of the corresponding proof in Theorem 7 we show that $u_\lambda \to \infty$ as $\lambda \to \bar{\lambda}_2$ uniformly in compacts of Ω_c. Thus, let $\lambda_n \to \bar{\lambda}_2$, $\bar{\lambda}_2 - \delta_2 \leq \lambda_n < \bar{\lambda}_2$, put $u_n = u_{\lambda_n}$ and observe that $u = u_n$ solves

$$\begin{cases} -\Delta u(x) - \lambda_n m(x)u(x) + au(x) = 0 & x \in \Omega_c \\ u(x) = u_n(x) & x \in \partial\Omega_c. \end{cases} \tag{5.5}$$

Setting $U_n = \{x \in \Omega : \operatorname{dist}(x, \Omega_c) < \eta_n\}$ and arguing as in Section 4 a positive sequence $\eta_n \to 0$ can found such that $\lambda_1^{U_n}(-\Delta - \lambda_n m + a) = 0$ for each n. If $\varphi_n(\cdot, a)$ stands for the corresponding associated positive and normalized eigenfunction then $\varphi_n(\cdot, a) \to \varphi(\cdot, a)$ in $C^{2,\alpha}(\overline{\Omega}_c)$, with φ the eigenfunction associated to $\lambda_1^{\Omega_c}(-\Delta - \bar{\lambda}_2 m + a)$.

Then, given a compact $K \subset \Omega_c$ and $M > 0$ arbitrary, a small $\eta > 0$ can be found so that $K \subset \Omega_{c,\eta} := \{x \in \Omega_c : \operatorname{dist}(x, \partial\Omega_c) > \eta\}$ together with

$$\sup_{\partial\Omega_{c,\eta}} M\varphi_n(\cdot, a) \leq \inf_{\partial\Omega_{c,\eta}} \underline{u},$$

for $n \geq n_0$, where $\underline{u}$ is the uniform subsolution. Since $\tilde{u}_n(x) := M\varphi_n(x, a)$ is a subsolution to (5.5), now regarded in $\Omega_{c,\eta}$, then

$$u_n(x) \geq M\varphi_n(x, a) \qquad x \in \Omega_{c,\eta}.$$

In particular, for $0 < \theta < 1$ fixed,

$$u_n(x) \geq \theta M\varphi(x, a) \geq \theta M \inf_K \varphi(\cdot, a) \qquad x \in K,$$

for $n \geq n_1$. Since M is arbitrary this means that $u_n \to \infty$ uniformly in K.

As for the limit behavior of u_λ in Γ_i as $\lambda \to \bar{\lambda}_2$ we choose a component, still called Γ_i, and introduce strips $U^+ \subset \Omega^+$, $U^- \subset \Omega_c$ with small thickness $\eta > 0$, $U^+ = \{x \in \Omega^+ : \operatorname{dist}(x, \Gamma_i) < \eta\}$, $U^- = \{x \in \Omega_c : \operatorname{dist}(x, \Gamma_i) > \eta\}$ (see Section 4). Being

$$\hat{u}_n := \inf_{\Gamma_i} u_n = u_n(x_n) \qquad x_n \in \Gamma_i,$$

and supposing that $\hat{u}_n = O(1)$ as $n \to \infty$, we will obtain again a contradiction.

We first observe that $u_n \leq v_n$ in U^+ where $v = v_n$ is the positive solution to

$$\begin{cases} -\Delta v(x) = \lambda_n m(x)v(x) - a(x)(v(x))^{p(x)} & \text{in } U^+ \\ v(x) = \hat{u}_n & x \in \Gamma_i \\ v(x) = u_n(x) & x \in \partial U^+ \cap \partial\Omega^+. \end{cases} \tag{5.6}$$

On the other hand u_n is uniformly bounded in compact subsets of Ω^+. In fact, for every ball $B \subset \overline{B} \subset \Omega^+$, $u_n \leq u_B$ where $u = u_B$ is the minimal blow-up solution to

$$-\Delta u = \bar{\lambda}_2\{\sup_\Omega m^+\}u - \{\inf_\Omega a\}f(u) \qquad \text{in } B, \qquad u_{|\partial B} = \infty,$$

with $f(u) = u^{p_1}$ if $u \leq 1$, $f(u) = u^{p_0}$ for $u > 1$ and $p_0 = \inf_B p$, $p_1 = \sup_B p$. As was shown in Section 4, this gives $C^{2,\alpha}$ uniform bounds for u_n in every subdomain $\Omega' \subset \overline{\Omega}' \subset \Omega^+$.

Now, Schauder's estimates applied to problem (5.6) ensure us that v_n is bounded in $C^{2,\alpha}(\overline{U}^+)$. Finally, since

$$\frac{\partial u_n}{\partial \nu}(x_n) \le \frac{\partial v_n}{\partial \nu}(x_n),$$

being ν the outer unit normal at Γ_i, we conclude that $\frac{\partial u_n}{\partial \nu}(x_n)$ is bounded above. However, we use again Theorem 1 to show that such boundedness is not possible. In fact, $u = u_n$ is the solution to

$$\begin{cases} -\Delta u(x) - \lambda_n m(x) u(x) + a u(x) = 0 & x \in U^- \\ u(x) = u_n(x) & x \in \partial U^-. \end{cases} \tag{5.7}$$

The fact that $\lambda_1^{U^-}(-\Delta - \lambda_n m + a)$ is bounded away from zero implies that the sequence ψ_n, $\psi = \psi_n$ being defined as the solution to

$$\begin{cases} -\Delta \psi(x) - \lambda_n m(x)\psi(x) + a\psi(x) = 0 & x \in U^- \\ \psi(x) = 1 & x \in \partial U^-, \end{cases}$$

is bounded in $C^{2,\alpha}(\overline{U}^-)$. Taking into account that $u_n \to \infty$ on $\partial U^- \cap \Omega_c$, Theorem 1 implies that

$$w_n(x) := \frac{u_n(x)}{\psi_n(x)} \ge u_n(x_n) = \hat{u}_n \qquad x \in U^-. \tag{5.8}$$

On the other hand $w = w_n$ solves the problem

$$\begin{cases} \mathcal{L}_n w = 0 & \text{in } U^- \\ w = u_n(\cdot) & \text{on } \partial U^-. \end{cases}$$

where $\mathcal{L}_n w = -\Delta w - 2\frac{\nabla \psi_n}{\psi_n}\nabla w$. Proceeding as in Section 4 we arrive at

$$\lim \frac{\partial w_n}{\partial \nu}(x_n) = \infty,$$

ν being this time the inner unit normal to U^- at Γ_i. The boundedness of ψ_n together with its derivatives up to order two implies that $\frac{\partial u_n}{\partial \nu}(x_n) \to \infty$ and we achieve the desired contradiction. Finally, being $w_n(x) \ge \hat{u}_n$ in $\overline{U}^-$ we also get that $u_\lambda \to \infty$ uniformly in compacts of $\Omega_c \cup \Gamma_i$ as $\lambda \to \bar{\lambda}_2$.

To finish the proof of Theorem 8 we study the limit behavior of u_λ in Ω^+ as $\lambda \to \bar{\lambda}_2$. Taking $\lambda_n \to \bar{\lambda}_2$ and setting $u_n = u_{\lambda_n}$ the preceding $C^{2,\alpha}$ estimates of u_n allow us to select a subsequence $u_{n'}$ such that $u_{n'} \to u^*$ in $C^{2,\alpha}(\Omega^+)$. In particular, $u = u^*$ solves the equation in (1.21) for $\lambda = \bar{\lambda}_2$.

On the other hand, assume for completeness that $\Gamma^+ \ne \emptyset$. Then, classical L^p estimates allow us to conclude that $u_{n'} \to u^*$ in $C^{2,\alpha}(\Omega^+ \cup \Gamma^+)$ and thus $u^* = 0$ on Γ^+. Let us show in addition that $u^* = \infty$ on Γ_i in the sense that $u^*(x) \to \infty$ as $\text{dist}(x, \Gamma_i) \to 0$. A first remark is that for every $M > 0$, the problem

$$\begin{cases} -\Delta u(x) = \bar{\lambda}_2 m(x) u(x) - a(x) u(x)^{p(x)} & x \in \Omega^+ \\ u(x) = M & x \in \Gamma_i \\ u(x) = 0 & x \in \Gamma^+. \end{cases}$$

admits a unique positive solution $u = u_M \in C^{2,\alpha}(\overline{\Omega}^+)$ (details are omitted for brevity). Similarly, a unique positive solution $u = u_{M,n} \in C^{2,\alpha}(\overline{\Omega}^+)$ to

$$\begin{cases} -\Delta u(x) = \lambda_n m(x)u(x) - a(x)u(x)^{p(x)} & x \in \Omega^+ \\ u(x) = M & x \in \Gamma_i \\ u(x) = 0 & x \in \Gamma^+, \end{cases}$$

exists for all n. In addition, Schauder's estimates imply $u_{M,n} \to u_M$ in $C^{2,\alpha}(\overline{\Omega}^+)$. Now observe that $u_n \geq u_{M,n}$ for n large and thus,

$$u^*(x) \geq u_M(x) \qquad x \in \Omega^+,$$

for all $M > 0$. This means that $u^* \geq u$, where $u \in C^{2,\alpha}(\overline{\Omega}^+)$ is the minimal solution to (1.24), and iv) is proved.

Finally, if $m \geq 0$ in Ω^+ then $u_n \leq \tilde{u}_n \leq u$ in Ω^+ (u the minimal solution to (1.24)) where $u = \tilde{u}_n$ is the solution to

$$\begin{cases} -\Delta u(x) = \bar{\lambda}_2 m(x)u(x) - a(x)u(x)^{p(x)} & x \in \Omega^+ \\ u(x) = u_n(x) & x \in \Gamma_i \\ u(x) = 0 & x \in \Gamma^+. \end{cases}$$

Therefore, $u^* \leq u$ and finally $u^* = u$. The uniqueness in the limit of the subsequence then implies that $u_\lambda \to u$ as $\lambda \to \bar{\lambda}_2$. This finishes the proof of Theorem 8. □

Remarks 5.

a) The framework of Theorem 7 allows $\Gamma^+ = \partial\Omega^+ \cap \partial\Omega \neq \emptyset$ and moreover that $p = 1$ somewhere on Γ^+. As was opportunely remarked in the course of the proof, the conclusions still hold true under such extreme conditions.

b) The proof of point iv) given in [17] for the case $m = 1$ (and $\overline{\Omega}^+ \subset \Omega$) can not be carried over to the case where m is two-signed since the monotonicity of u_λ in λ is a crucial element in the arguments in [17]. In addition, the classical weak maximum principle (1.2)–(1.3) is improperly used in [17] to conclude that u_n, regarded as a solution to problem (5.7), directly satisfies $u_n(x) \geq \hat{u}_n$ (see Theorem 7.1). To the best of our knowledge, the right assertion is the alternative estimate (5.8) which is a consequence of Theorem 1. The same defect seems to be exhibited in the proof of the corresponding fact in [3] where, nevertheless, (1.2)–(1.3) is still in use due to the superharmonic character of u_n in Ω_c (Lemma 3.3). However, such approach can not be employed to handle our problem (1.11) where $m(x)$ is a two-signed coefficient.

Our last statement provides an extension to the framework of equation (1.21) of the results on large solutions contained in Theorems 2 and 3.

Theorem 10. *Let $\Omega_c \subset \Omega \subset \mathbb{R}^N$ be $C^{2,\alpha}$ bounded domains, Ω_c satisfying (H), $m, a, p \in C^\alpha(\overline{\Omega})$, $a(x) > 0$, $p(x) \geq 1$ in $\overline{\Omega}$ while*

$$\overline{\Omega}_c = \overline{\{x \in \Omega : p(x) = 1\}}.$$

Then the following properties hold.

i) *Problem*

$$\begin{cases} -\Delta u(x) = m(x)u(x) - a(x)u(x)^{p(x)} & x \in \Omega \\ u(x) = n & x \in \partial\Omega, \end{cases} \tag{5.9}$$

admits a positive solution for all $n \in \mathbb{N}$ if and only if

$$\lambda_1^{\Omega_c}(-\Delta - m + a) > 0. \tag{5.10}$$

In such case the solution is unique and lies in $C^{2,\alpha}(\overline{\Omega})$.

ii) *If $\Gamma_b = \partial\Omega_c \cap \partial\Omega \neq \emptyset$ then problem*

$$\begin{cases} -\Delta u(x) = m(x)u(x) - a(x)u(x)^{p(x)} & x \in \Omega \\ u = \infty & x \in \partial\Omega, \end{cases} \tag{5.11}$$

does no admit any solution. Moreover, if (5.10) *holds and $u = u_n(x)$ stands for the solution to* (5.9) *then*

$$\lim u_n = \infty$$

uniformly in $\overline{\Omega}_c$. Moreover, u_n converges in $C^{2,\alpha}(\Omega^+)$ to the minimal solution $u = u^*(x)$ of the blow-up problem (5.11) *when considered in Ω^+.*

iii) *If, on the contrary, $\Omega_c \subset \overline{\Omega}_c \subset \Omega$ then condition* (5.10) *is necessary and sufficient for the existence of a minimal solution $u \in C^{2,\alpha}(\Omega)$ to* (5.11). *Such solution is provided by the limit of u_n in $C^{2,\alpha}(\Omega)$.*

Proof. The necessity of condition (5.10) in i) arises from the existence of a positive solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$ to equation in (5.9). In fact, u defines a strict positive supersolution in Ω_c and hence the positivity of $\lambda_1^{\Omega_c}(-\Delta - m + a)$ ([16], [5], [6]).

Suppose now that (5.10) holds. We are assuming the more adverse conditions to construct a finite positive supersolution $\overline{u}$. Namely, $\Gamma^+ \neq \emptyset$ and $p(\bar{x}) = 1$ at some $\bar{x} \in \Gamma^+$. We first extend m, a to $Q := B(\Omega, \delta) = \{x \in \Omega : \operatorname{dist}(x, \Omega) < \delta\}$, $\delta > 0$ small, as C^α functions such that $a(x) \geq a_0 > 0$. In addition, a function $p_1 \in C^\alpha(\overline{Q})$ is chosen such that $p_1(x) \leq p(x)$ for $x \in Q$, $p_1 > 1$ in Ω^+ but $p_1 = 1$ in $Q \setminus \Omega$. We set in addition:

$$U = \{x \in \mathbb{R}^N : \operatorname{dist}(x, \Gamma^+) < \delta\}$$

and denote $U^+ = U \cap \Omega^+$, $U_c = U \setminus \overline{U}^+$. Since the measure of U_c goes to zero with $\delta \to 0$, we can conclude that

$$\lambda_1^{U_c}(-\Delta - m + a) > 0.$$

Therefore, the construction in the proof of Theorem 8 provides the existence of a positive function $\overline{u}_1 \in W^{2,q}(U) \cap C^{1,\beta}(\overline{U})$ such that

$$-\Delta\overline{u}_1 \geq m(x)\overline{u}_1 - a(x)\overline{u}_1{}^{p_1(x)} \qquad x \in U,$$

with $\overline{u}_1 = 0$ on ∂U. In the same spirit we introduce $V = B(\Omega_c, \delta)$, $V^+ = V \cap \Omega^+$, $V_c = V \setminus \overline{V}^+$. For $\delta > 0$ small we achieve again

$$\lambda_1^{V_c}(-\Delta - m + a) > 0,$$

and get a positive function $\overline{u}_2 \in W^{2,q}(V) \cap C^{1,\beta}(\overline{V})$,

$$-\Delta\overline{u}_2 \geq m(x)\overline{u}_2 - a(x)\overline{u}_2{}^{p_1(x)} \qquad x \in V,$$

$\overline{u}_{2|\partial V} = 0$. We finally take

$$\overline{u}_0(x) = \begin{cases} \overline{u}_1(x) & x \in \Omega, \quad \operatorname{dist}(x, \Gamma^+) \leq \frac{\delta}{2} \\ \varphi(x) & x \in \Omega_\delta^+ \\ \overline{u}_2(x) & x \in \Omega, \quad \operatorname{dist}(x, \Omega_c) \leq \frac{\delta}{2}, \end{cases}$$

where $\Omega^+_\delta = \{x \in \Omega^+ : \mathrm{dist}(x, \partial\Omega^+) > \frac{\delta}{2}\}$ and φ is an smooth extension of both $\overline{u}_1$ and $\overline{u}_2$ to Ω^+_δ such that $\varphi > 0$ in $\overline{\Omega}^+_\delta$. It is easily checked that $\overline{u} = M\overline{u}_0$ satisfies, for large $M \geq 1$,

$$-\Delta\overline{u} \geq m(x)\overline{u} - a(x)\overline{u}^{p_1(x)} \qquad x \in \Omega^+_\delta.$$

Therefore $\overline{u}$ defines a strong supersolution to $-\Delta u = m(x)u - a(x)u^{p_1(x)}$ in Ω and since M can be taken so large as to have $\overline{u} \geq 1$ we finally conclude

$$-\Delta\overline{u} \geq m(x)\overline{u} - a(x)\overline{u}^{p(x)} \qquad \text{in } \Omega.$$

This, in turn, allows to solve (5.9) for all $n \in \mathbb{N}$.

Let us now show ii) and assume that $u \in C^2(\Omega)$ solves (5.11) with $u(x) > 0$ for $x \in \Omega$. Then, part i) implies that

$$\lambda_1^{\widetilde{\Omega}_{c,\delta}}(-\Delta - m + a) > 0,$$

$\widetilde{\Omega}_{c,\delta} = \{x \in \Omega_c : \mathrm{dist}(x, \Gamma_b) > \delta\}$, for all $\delta > 0$ small. Thus, by continuity, $\lambda_1^{\Omega_c}(-\Delta - m + a) \geq 0$. However, $\lambda_1^{\Omega_c}(-\Delta - m + a) = 0$ is ruled out, for taking ϕ the positive eigenfunction associated to $\lambda_1^{\Omega_c}(-\Delta - m + a)$ and choosing an arbitrary $M \geq 1$, a small $\delta = \delta(M)$ exists such that $u \geq M\phi$ on $\partial\widetilde{\Omega}_{c,\delta}$. By comparison we get

$$u \geq M\phi \qquad \text{in } \widetilde{\Omega}_{c,\delta}.$$

This is not possible due to the arbitrariness of M. Therefore,

$$\lambda_1^{\Omega_c}(-\Delta - m + a) > 0.$$

This is just (5.10) and in particular implies the existence of the positive solution $u = \psi \in C^{2,\alpha}(\overline{\Omega}_c)$ to

$$\begin{cases} -\Delta u(x) = m(x)u(x) - a(x)u(x) & x \in \Omega_c \\ u(x) = 0 & x \in \Gamma_i \\ u(x) = 1 & x \in \Gamma_b. \end{cases}$$

By comparison in $\widetilde{\Omega}_{c,\delta}$ and letting $\delta \to 0+$ it follows that $u \geq n\psi$ in Ω_c for every $n \in \mathbb{N}$, what is not possible.

Regarding the second part of ii) notice that (5.10) allows us to solve (5.9) for all $n \in \mathbb{N}$. Our previous argument and comparison show that its solution u_n satisfies

$$u_n(x) \geq n\psi(x) \qquad x \in \overline{\Omega}_c.$$

This means that $u_n \to \infty$ uniformly on compacts of $\Omega_c \cup \Gamma_b$. The proof of the fact that $u_n \to \infty$ uniformly on Γ_i mimics, via Theorem 1, the corresponding argument given in Theorem 3 (Section 3). Details are therefore omitted for the sake of brevity. The proof of the convergence of u_n towards the minimal solution to $-\Delta u = m(x)u - a(x)u^{p(x)}$ in Ω^+ is also skipped for the same reasons (let us point out that it is enough to use the ideas used in the similar situation in Theorem 8).

To show iii) first observe that the existence of a positive solution to (5.11) directly implies (5.10). Conversely, such condition ensures the existence of the solution $u = u_n$ to (5.9) for all $n \in \mathbb{N}$. To show that u_n converges to the minimal solution u to (5.11) in $C^{2,\alpha}(\Omega)$ it is enough, due to the increasing character of u_n, to show that u_n is bounded on compact subsets of Ω. To this end first notice that $u_n \leq \tilde{u}$ in $\Omega \setminus B(\Omega_c, \frac{\delta}{2})$, $\delta > 0$ small, where $u = \tilde{u}$ is the minimal solution to

$$\begin{cases} -\Delta u(x) = m(x)u(x) - a(x)u(x)^{p(x)} & x \in \Omega \setminus B(\Omega_c, \frac{\delta}{2}) \\ u = \infty & x \in \partial\{\Omega \setminus B(\Omega_c, \frac{\delta}{2})\}. \end{cases}$$

In fact, the finiteness of $\tilde{u}$ follows from the fact that $p > 1$ in $\Omega \setminus B(\Omega_c, \frac{\delta}{2}) \subset \Omega^+$. We next consider $Q = B(\Omega, \delta)$, $\delta > 0$, and notice that $\partial Q = \{x \in \Omega^+ : \mathrm{dist}(x, \Gamma_i) = \delta\}$ (recall $\overline{\Omega}_c \subset \Omega$ in the present case). In addition and by continuity

$$\lambda_1^Q(-\Delta - m + a\chi_{\Omega_c}) > 0,$$

for small $\delta > 0$. Moreover, a smooth function $a_\varepsilon = a_\varepsilon(x)$ defined in Q can be found such that sup $a_\varepsilon = \overline{\Omega}_c$, $a_\varepsilon \le a\chi_{\Omega_c}$ and $a_\varepsilon \to a$ uniformly on compacts of Ω_c as $\varepsilon \to 0$. Thus, for small ε we get

$$\lambda_1^Q(-\Delta - m + a_\varepsilon) > 0. \tag{5.12}$$

We now observe that $u = u_n$ satisfies

$$-\Delta u - mu + a_\varepsilon u \le -\Delta u - mu + a\chi_{\Omega_c} u = -a\chi_{Q\setminus\Omega_c} \le 0 \qquad \text{in } Q.$$

In view of (5.12) we can apply Theorem 1 in the domain Q to get

$$u_n(x) \le \{\sup_{\partial Q} u_n\}\psi(x) \qquad x \in Q, \tag{5.13}$$

with ψ satisfying

$$\begin{cases} -\Delta\psi(x) - m(x)\psi(x) + a_\varepsilon\psi(x) = 0 & x \in Q \\ \psi_{|\partial Q} = 1. \end{cases}$$

Since $\partial Q \subset \Omega \setminus B(\Omega_c, \frac{\delta}{2})$ we have that u_n is uniformly bounded on ∂Q and from (5.13) we achieve both the boundedness of u_n in Q and in Ω. This finishes the proof. □

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Departamento de Análisis Matemático, Universidad de La Laguna, C/ Astrofísico Francisco Sánchez s/n, 38271 – La Laguna, Spain and, Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atómica, Molecular y Fotónica, Universidad de La Laguna, C/ Astrofísico Francisco Sánchez s/n, 38203 – La Laguna , Spain

E-mail address: jjgarmel@ull.es

Departamento de Matemática, FCEyN UBA, Ciudad Universitaria, Pab 1 (1428), Buenos Aires, Argentina.

E-mail address: jrossi@dm.uba.ar

Departamento de Análisis Matemático, Universidad de La Laguna, C/ Astrofísico Francisco Sánchez s/n, 38271 – La Laguna, Spain and, Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atómica, Molecular y Fotónica, Universidad de La Laguna, C/ Astrofísico Francisco Sánchez s/n, 38203 – La Laguna , Spain

E-mail address: josabina@ull.es