

**ON THE BEHAVIOUR OF SOLUTIONS TO THE DIRICHLET  
PROBLEM FOR THE  $p(x)$ -LAPLACIAN WHEN  $p(x)$  GOES TO 1  
IN A SUBDOMAIN**

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ABSTRACT. In this paper we prove a stability result for some classes of elliptic problems involving variable exponents. More precisely, we consider the Dirichlet problem for an elliptic equation in a domain  $\Omega$ , which is the  $p$ -Laplacian equation,  $-\operatorname{div}(|\nabla u|^{p-2}\nabla u) = f$ , in a subdomain  $\Omega_1$  of  $\Omega$  and the Laplace equation,  $-\Delta u = f$ , in its complementary (that is, our equation involves a  $p(x)$ -Laplacian with a discontinuous exponent  $p(x)$ ). We assume that the right-hand side  $f$  belongs to  $L^\infty(\Omega)$ . For this problem, we study the behaviour of the solutions as  $p$  goes to 1, showing that they converge to a function  $u$ , which is almost everywhere finite when the size of the datum  $f$  is small enough. Moreover, we prove that this  $u$  is a solution of a limit problem involving the 1-Laplacian operator in  $\Omega_1$ . We also discuss uniqueness under a favorable geometry.

1. INTRODUCTION.

Let us consider a bounded domain (open and connected)  $\Omega$  of  $\mathbb{R}^N$  and  $\Omega_1, \Omega_2$  a partition of it into two subdomains, that is,  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , with  $\Gamma = \partial\Omega_1 \cap \Omega = \partial\Omega_2 \cap \Omega$ ; three cases for the relative locations of the domains are possible, namely:  $\partial\Omega_1 \cap \partial\Omega = \emptyset$  and  $\Gamma = \partial\Omega_1$  (that is,  $\overline{\Omega_1}$  lies in the interior of  $\Omega$ );  $\partial\Omega_2 \cap \partial\Omega = \emptyset$  and  $\Gamma = \partial\Omega_2$  ( $\overline{\Omega_2}$  lies in the interior of  $\Omega$ );  $\partial\Omega_1 \cap \partial\Omega \neq \emptyset \neq \partial\Omega_2 \cap \partial\Omega$  and  $\partial\Omega_1 \neq \Gamma \neq \partial\Omega_2$  (the closure of both subdomains do not lie in the interior of  $\Omega$ ). We will assume that all the involved domains are Lipschitz continuous and that the interface  $\Gamma$  is smooth and it satisfies  $\mathcal{H}^{N-1}(\Gamma) < \infty$ , where  $\mathcal{H}^{N-1}$  denotes the  $(N-1)$ -dimensional Hausdorff measure.

Our aim is to study the Dirichlet problem for an elliptic equation which is the 1-Laplacian equation in  $\Omega_1$  and the usual Laplace equation in  $\Omega_2$ . To this end we introduce the variable exponent  $l(x)$  defined by

$$(1.1) \quad l(x) = \begin{cases} 1, & \text{if } x \in \Omega_1; \\ 2, & \text{if } x \in \Omega_2; \end{cases}$$

and we consider the Dirichlet problem

$$(1.2) \quad \begin{cases} -\operatorname{div}(|\nabla u|^{l(x)-2}\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

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where the right-hand side  $f$  belongs to  $L^\infty(\Omega)$ . Observe that the problem involves the  $p(x)$ -Laplacian, that is, the operator  $\Delta_{l(x)}u = \operatorname{div}(|\nabla u|^{l(x)-2}\nabla u)$ , with a discontinuous exponent  $l(x)$ . Common hypotheses in the literature of  $p(x)$ -Laplacian type equations are that  $p(x)$  is a continuous function (with a modulus of continuity) and, moreover, the variable exponent  $p(x)$  is usually assumed to be bounded away from 1. To the best of our knowledge, this paper is the first attempt to analyze a problem where the exponent  $p(x)$  is discontinuous and equal to 1 in some part of the domain.

Note that the solution must solve  $-\Delta_1 u = -\operatorname{div}(\frac{Du}{|Du|}) = f$  in  $\Omega_1$  and  $-\Delta u = f$  in  $\Omega_2$ . Hence, to handle this equation, (1.2), we have to give a notion of solution and then consider a suitable functional framework. Adapting the definition of solution for the 1-Laplacian equation given in [4], we look for a solution to problem (1.2) in the set of measurable functions in  $\Omega$  which belongs to  $BV(\Omega)$  and to  $H^1(\Omega_2)$ . Obviously our notion of solution needs to give sense to the quotient  $\frac{Du}{|Du|}$  in  $\Omega_1$ , where, in general,  $Du$  is not a function but a Radon measure, while it is a solution of the Laplace equation with datum  $f$  in  $\Omega_2$ . To this aim our definition of solution (see Section 4) is based on a vector field  $z \in L^\infty(\Omega_1; \mathbb{R}^N)$  satisfying  $\|z\|_\infty \leq 1$ ,  $-\operatorname{div} z = f$  in  $\mathcal{D}'(\Omega_1)$  and  $(z, Du) = |Du|$ . Observe that, formally,  $\|z\|_\infty \leq 1$  and  $(z, Du) = |Du|$  imply  $z = \frac{Du}{|Du|}$  in  $\Omega_1$ . To give sense to  $(z, Du)$  we use the theory of  $L^\infty$ -divergence-measure vector fields due to Anzellotti [7] and Chen-Frid [8, 9, 10].

In this paper, for small right-hand side  $f$ , we prove the existence and uniqueness of such a solution to problem (1.2), as well as we present explicit examples. Our main result reads as follows:

**Theorem 1.1.** *There exists a constant  $C$  depending on the domains,  $\Omega_1, \Omega_2$ , such that, if  $\|f\|_{L^\infty(\Omega)} < C$ , then there is, at least, a solution to problem (1.2).*

*Moreover, if  $\mathcal{H}^{N-1}(\partial\Omega_2 \cap \partial\Omega) \neq 0$ , then the problem has a unique solution in the sense that two solutions  $u_1, u_2$  must agree in  $\Omega_2$ .*

REMARK 1.2. The fact that  $l(x) = 2$  in  $\Omega_2$  is used only to simplify the presentation of our results. From the proofs it is clear that we can consider more general exponents of the form

$$l(x) = \begin{cases} 1, & \text{if } x \in \Omega_1; \\ q(x), & \text{if } x \in \Omega_2; \end{cases}$$

as long as  $q(x)$  is greater than 1.

Our strategy to find a solution to problem (1.2) is to consider a limit of a “sequence” of approximating problems involving the so-called  $p(x)$ -laplacian operator. This means that we consider the variable exponent  $p : \Omega \rightarrow [p, 2]$  which is the measurable function defined by

$$(1.3) \quad p(x) = \begin{cases} p, & \text{if } x \in \Omega_1; \\ 2, & \text{if } x \in \Omega_2; \end{cases}$$

with  $1 < p < 2$ , and we study the behaviour, as  $p$  goes to 1, of the solutions  $u_p$  to the Dirichlet problem

$$(1.4) \quad \begin{cases} -\operatorname{div}(|\nabla u_p|^{p(x)-2}\nabla u_p) = f(x), & \text{in } \Omega; \\ u_p = 0, & \text{on } \partial\Omega. \end{cases}$$

It is well-known (see [13, 17]) that a weak solution  $u_p$  to the Dirichlet problem (1.4) is a function belonging to the so-called generalized Sobolev space  $W_0^{1,p(x)}(\Omega)$

such that the following identity holds true

$$(1.5) \quad \int_{\Omega} |\nabla u_p|^{p(x)-2} \nabla u_p \cdot \nabla v = \int_{\Omega} f v, \quad v \in W_0^{1,p(x)}(\Omega).$$

We prove that the approximate solutions  $u_p$  converge to a  $BV$ -function  $u$  that turns out to be a solution to equation (1.2) according to our definition (given in Section 4).

A similar approach has been used to study the problem (1.4) (and its limit as  $p \rightarrow 1$ ), when the variable exponent  $p(x)$  is constant in the whole  $\Omega$ , i.e.  $p(x) = p$ . In such a case there is no stability result for solutions to  $p$ -Laplacian equation as  $p$  goes to 1, in the sense that solutions of the  $p$ -Laplacian equation converge to a function that can be infinity on a set of positive measure when the datum  $f$  is large enough (see [15, 16] for particular data and [19, 20] for more general data, see also [12]). Furthermore, there is no uniqueness of the solution of the limit problem. On the contrary, in [18] we consider an anisotropic operator that is as the  $q$ -Laplacian ( $q > 1$ ) in just one direction to find a unique solution to the limit equation for each datum  $f$  in a suitable Lebesgue space (regardless of the size of  $f$ ). Moreover, this unique solution is the limit, as  $p$  goes to 1, of solutions to anisotropic equations which are equal to the  $p$ -Laplacian in some directions and as the  $q$ -Laplacian in the others. In view of these results one may wonder if the fact that the equation in (1.2) contains a part in which the usual Laplacian occurs, plays a role in the existence of solutions or in their uniqueness. As we can see from our main result, Theorem 1.1, existence is also restricted to small data  $f$  (as in the case that  $p(x) = p$  is constant in  $\Omega$ ), but uniqueness holds (in a weak sense) under a favorable position of  $\Omega_2$  in  $\Omega$  (as in the anisotropic case).

Elliptic problems involving variable exponents became popular a few years ago in relation to applications to elasticity and electrorheological fluids. Meanwhile, the underlying functional analytical tools have been extensively developed and new applications, for example to image processing, have kept the subject as the focus of an intensive research activity. For general references on the  $p(x)$ -Laplacian we refer to [11] and [14] that include a large bibliography, and [17]. The delicate regularity properties of  $p(x)$ -harmonic functions have been established in [1] and [2].

The rest of this paper is organized as follows: in Section 2 we study our functional setting and we recall a few properties of the generalized Sobolev spaces; in Section 3, we begin by studying the asymptotic behaviour of the sequence  $u_p$  of approximate solutions to problem (1.4), as  $p \rightarrow 1$ , we get a limit function  $u$  and a vector field  $z$  which is the weak limit of  $|\nabla u_p|^{p-2} \nabla u_p$  in  $\Omega_1$ ; in Section 4 we introduce our notion of solution and we prove the existence result stated in Theorem 1.1, which consists in proving that the limit function  $u$  above is a solution to (1.2), finally we prove the uniqueness result and compute explicit examples.

## 2. PRELIMINARY RESULTS.

Throughout this paper, we will denote by  $|E|$  the Lebesgue measure of a measurable set  $E \subset \mathbb{R}^N$  and by  $\mathcal{H}^{N-1}(E)$  its  $N - 1$ -dimensional Hausdorff measure. We denote by  $\Omega$  an open bounded and connected subset of  $\mathbb{R}^N$  and  $\Omega_1, \Omega_2$  are subdomains of  $\Omega$  satisfying  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , with  $\Gamma = \partial\Omega_1 \cap \Omega = \partial\Omega_2 \cap \Omega$ . We assume that  $\Omega$ ,  $\Omega_1$  and  $\Omega_2$  are Lipschitz continuous and we denote by  $\nu_i$  the unit

outward normal to  $\Omega_i$ ,  $i = 1, 2$ . We also assume that the interface  $\Gamma$  is smooth and satisfies  $\mathcal{H}^{N-1}(\Gamma) < \infty$ .

Some remarks on our functional setting are in order. A function  $u \in L^1(\Omega)$  whose gradient  $Du$  in the sense of distributions is a vector valued Radon measure with finite total variation in  $\Omega$  is called a function of bounded variation. The class of such functions will be denoted by  $BV(\Omega)$ . It is well-known that the functional

$$(2.6) \quad u \mapsto \int_{\Omega_1} |Du| + \int_{\partial\Omega_1 \cap \partial\Omega} |u| d\mathcal{H}^{N-1} + \int_{\Gamma} |u - g| d\mathcal{H}^{N-1}, \quad g \in L^1(\Gamma),$$

is lower semi-continuous in  $BV(\Omega_1)$  with respect to the  $L^1(\Omega_1)$ -convergence.

Next we will recall some well-known facts about the generalized Lebesgue and Sobolev spaces (see for example [13, 17]). Let us consider a bounded domain  $\Omega$  of  $\mathbb{R}^N$  and let  $p : \Omega \rightarrow (1, +\infty)$  be a measurable function. The so-called ‘‘generalized Lebesgue space’’  $L^{p(x)}(\Omega)$  is the class of all measurable functions  $u$  on  $\Omega$  such that

$$\int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} < +\infty,$$

for some  $\lambda > 0$ . If the variable exponent  $p(x)$  satisfies the condition

$$(2.7) \quad 1 < p^- = \text{ess inf}_{x \in \Omega} p(x) \leq \text{ess sup}_{x \in \Omega} p(x) = p^+ < +\infty,$$

then the space  $L^{p(x)}(\Omega)$  is a Banach space endowed with the norm

$$(2.8) \quad \|u\|_p = \inf \left\{ \lambda > 0 : \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} \leq 1 \right\}.$$

Since  $p(x) > 1$ , the space  $L^{p(x)}(\Omega)$  is reflexive and its dual space is  $L^{p'(x)}(\Omega)$ , where

$$\frac{1}{p'(x)} = 1 - \frac{1}{p(x)}, \quad \text{a.e. } x \in \Omega.$$

If  $p_1(x), p_2(x)$  are measurable functions which satisfy condition (2.7), then

$$(2.9) \quad L^{p_2(x)}(\Omega) \subset L^{p_1(x)}(\Omega),$$

if, and only if,  $p_1(x) \leq p_2(x)$  a.e. in  $\Omega$ ; in this case the embedding is continuous.

The generalized Hölder inequality holds

$$(2.10) \quad \int_{\Omega} |f(x)g(x)| \leq 2 \|f\|_{L^{p(x)}(\Omega)} \|g\|_{L^{p'(x)}(\Omega)},$$

for any  $f \in L^{p(x)}(\Omega)$  and  $g \in L^{p'(x)}(\Omega)$ . The generalized Sobolev space  $W^{1,p(x)}(\Omega)$  with  $p(x)$  satisfying (2.7) is the set of measurable functions  $u \in L^{p(x)}(\Omega)$  such that  $|\nabla u| \in L^{p(x)}(\Omega)$ , i.e.  $W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\}$ . It is a reflexive and separable Banach space equipped with the norm

$$\|u\|_{W^{1,p(x)}(\Omega)} = \|u\|_{L^{p(x)}(\Omega)} + \|\nabla u\|_{L^{p(x)}(\Omega)}.$$

As a consequence of the embedding (2.9), if  $p_1(x), p_2(x)$  are measurable functions satisfying (2.7) and  $p_1(x) \leq p_2(x)$  a.e. in  $\Omega$ , then  $W^{1,p_2(x)}(\Omega) \subset W^{1,p_1(x)}(\Omega)$ , continuously. Finally  $W_0^{1,p(x)}(\Omega)$  is defined as the closure of  $C_0^\infty(\Omega)$  in  $W^{1,p(x)}(\Omega)$ .

Sobolev’s embedding theorems are well-known (cf. [14]) when  $p(x)$  is a continuous function. Nevertheless, in our framework, these results do not apply and we have to prove the corresponding Sobolev embedding theorem.

**Proposition 2.1.** *Let  $1 < p < 2$  and let  $p(x)$  be the following step function*

$$(2.11) \quad p(x) = \begin{cases} p & \text{if } x \in \Omega_1, \\ 2 & \text{if } x \in \Omega_2, \end{cases}$$

*then there exist a positive constant  $c_1$  such that*

$$\begin{aligned} \frac{1}{2} \left[ \left( \int_{\Omega_1} |v|^p \right)^{1/p} + \left( \int_{\Omega_2} |v|^2 \right)^{1/2} \right] &\leq \|v\|_{L^{p(x)}(\Omega)} \\ &\leq c_1 \left[ \left( \int_{\Omega_1} |v|^p \right)^{1/p} + \left( \int_{\Omega_2} |v|^2 \right)^{1/2} \right] \end{aligned}$$

*for every  $v \in L^{p(x)}(\Omega)$ .*

REMARK 2.2. Note that one can choose  $c_1$  independent of  $p$  as long as  $p$  remains bounded. For our purposes we let  $p \in (1, 2]$ .

*Proof.* Let  $v$  be a function belonging to  $L^{p(x)}(\Omega)$  and let

$$\mu = \left( \int_{\Omega_1} |v|^p \right)^{1/p} + \left( \int_{\Omega_2} |v|^2 \right)^{1/2}.$$

We have

$$\mu^p \geq \int_{\Omega_1} |v|^p, \quad \mu^2 \geq \int_{\Omega_2} |v|^2$$

and then, for every  $\alpha > 0$ ,

$$\int_{\Omega_1} \frac{|v|^p}{(\alpha\mu)^p} + \int_{\Omega_2} \frac{|v|^2}{(\alpha\mu)^2} \leq \frac{1}{\alpha^p} + \frac{1}{\alpha^2}.$$

Given  $\bar{\alpha} > 0$  such that  $\frac{1}{\bar{\alpha}^p} + \frac{1}{\bar{\alpha}^2} = 1$ , by definition of norm we obtain

$$(2.12) \quad \|v\|_{L^{p(x)}} \leq \bar{\alpha}\mu = \bar{\alpha} \left[ \left( \int_{\Omega_1} |v|^p \right)^{1/p} + \left( \int_{\Omega_2} |v|^2 \right)^{1/2} \right].$$

On the other hand, for every  $\beta > 0$ ,

$$(2.13) \quad \int_{\Omega_1} \frac{|v|^p}{\beta^p} \leq \int_{\Omega_1} \frac{|v|^p}{\beta^p} + \int_{\Omega_2} \frac{|v|^2}{\beta^2}$$

which implies

$$\left( \int_{\Omega_1} |v|^p \right)^{1/p} \leq \|v\|_{L^{p(x)}(\Omega)}.$$

Analogously,

$$\left( \int_{\Omega_2} |v|^2 \right)^{1/2} \leq \|v\|_{L^{p(x)}(\Omega)},$$

and then adding the two inequalities we get

$$(2.14) \quad \left( \int_{\Omega_1} |v|^p \right)^{1/p} + \left( \int_{\Omega_2} |v|^2 \right)^{1/2} \leq 2\|v\|_{L^{p(x)}(\Omega)}.$$

■

**Proposition 2.3.** *Let  $\Omega$  be a bounded domain in  $\mathbb{R}^N$  with boundary  $\partial\Omega$  smooth enough. If  $\Gamma \subset \partial\Omega$  satisfies  $\mathcal{H}^{N-1}(\Gamma) \neq 0$ , then for each  $p \in (1, +\infty)$  there exists a constant  $C_p > 0$  such that*

$$(2.15) \quad \left( \int_{\Omega} |v|^p \right)^{1/p} + \left( \int_{\Omega} |\nabla v|^p \right)^{1/p} \leq C_p \left[ \int_{\Gamma} |v| + \left( \int_{\Omega} |\nabla v|^p \right)^{1/p} \right]$$

for all  $v \in W^{1,p}(\Omega)$ . Moreover, these constants do not degenerate as  $p$  goes to 1, and so a constant can be chosen independently on  $p$  (when  $p$  is close to 1).

*Proof.* The existence, for each  $p \in (1, +\infty)$ , of a constant  $C_p$  satisfying (2.15) is well-known (see for instance [21], pp 247–248).

Let us prove the second assertion. Assume to get a contradiction that constants  $C_p$  degenerate as  $p$  goes to 1. Then there exist  $v_p \in W^{1,p}(\Omega)$  satisfying

$$(1) \quad \left( \int_{\Omega} |v_p|^p \right)^{1/p} + \left( \int_{\Omega} |\nabla v_p|^p \right)^{1/p} = 1.$$

$$(2) \quad \lim_{p \rightarrow 1} \int_{\Gamma} |v_p| + \left( \int_{\Omega} |\nabla v_p|^p \right)^{1/p} = 0.$$

Observe that it follows that

$$(2.16) \quad \lim_{p \rightarrow 1} \int_{\Omega} |v_p|^p = 1.$$

Since, by Hölder's inequality,

$$\int_{\Omega} |v_p| + \int_{\Omega} |\nabla v_p| \leq |\Omega|^{1-(1/p)} \left( \int_{\Omega} |v_p|^p \right)^{1/p} + |\Omega|^{1-(1/p)} \left( \int_{\Omega} |\nabla v_p|^p \right)^{1/p} \leq 1 + |\Omega|,$$

we may find  $v \in BV(\Omega)$  such that, up to “subsequences”,

$$v_p \rightharpoonup v, \quad \text{weakly}^* \text{ in } BV(\Omega).$$

As a consequence, we have

$$(2.17) \quad v_p \rightarrow v, \quad \text{strongly in } L^\alpha(\Omega), \text{ for every } 1 < \alpha < \frac{N}{N-1}$$

and

$$\int_{\Omega} |Dv| \leq \liminf_{p \rightarrow 1} \int_{\Omega} |\nabla v_p| \leq \liminf_{p \rightarrow 1} |\Omega|^{1-(1/p)} \left( \int_{\Omega} |\nabla v_p|^p \right)^{1/p} = 0.$$

Hence,  $Dv = 0$  in  $\Omega$  and so  $v$  is constant in  $\Omega$ . Since

$$\int_{\Gamma} |v| \leq \liminf_{p \rightarrow 1} \int_{\Gamma} |v_p| \leq \liminf_{p \rightarrow 1} |\Gamma|^{1-(1/p)} \left( \int_{\Gamma} |v_p|^p \right)^{1/p} = 0,$$

it follows that this constant must be 0. Therefore,  $v = 0$ . So that (2.17) becomes

$$v_p \rightarrow 0, \quad \text{strongly in } L^{\frac{N}{N-1}}(\Omega).$$

It implies, for  $p$  close to 1, that

$$\int_{\Omega} |v_p|^p \leq |\Omega|^{1-\frac{(N-1)p}{N}} \left( \int_{\Omega} |v_p|^{\frac{N}{N-1}} \right)^{\frac{(N-1)p}{N}} \leq |\Omega| \left( \int_{\Omega} |v_p|^{\frac{N}{N-1}} \right)^{\frac{(N-1)p}{N}}.$$

Thus,  $\lim_{p \rightarrow 1} \int_{\Omega} |v_p|^p = 0$  and it contradicts (2.16). ■

**Proposition 2.4.** *Let  $p(x)$  be the function defined in Proposition 2.1. Then there exists a positive constant  $A_p$  such that*

$$(2.18) \quad \|v\|_{L^{p(x)}(\Omega)} \leq A_p \|\nabla v\|_{L^{p(x)}(\Omega)}$$

for every  $v \in W_0^{1,p(x)}(\Omega)$ .

REMARK 2.5. The constant  $A_p$  appearing in (2.18) can be chosen independent of  $p$  as long as  $p$  is bounded, say  $p \in (1, 2]$ . Indeed, one has to observe that in the following proof the constant depends essentially on the constants appearing in the previous proposition and on the best constant for the Sobolev trace embedding  $W^{1,p}(D) \hookrightarrow L^p(\partial D)$ . We explicitly remark that these constants do not degenerate as  $p \rightarrow 1$  (see [6] for the trace embedding). In the sequel, we will denote

$$A = \limsup_{p \rightarrow 1} A_p .$$

*Proof.* Let us distinguish three cases according to the relative location of the subdomains  $\Omega_1$  and  $\Omega_2$ .

FIRST CASE:  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , and  $\Gamma = \partial\Omega_1$ . In this case  $\partial\Omega_1 \cap \partial\Omega = \emptyset$ , that is  $\bar{\Omega}_1$  lies in the interior of  $\Omega$ . Let  $v \in W_0^{1,p(x)}(\Omega)$ , then

$$(2.19) \quad \left( \int_{\Omega_2} |v|^2 \right)^{1/2} \leq C \left( \int_{\Omega_2} |\nabla v|^2 \right)^{1/2}, \quad \int_{\Gamma} |v|^2 \leq C \int_{\Omega_2} |\nabla v|^2.$$

Moreover

$$\left( \int_{\Gamma} |v|^p \right)^{1/p} \leq |\Gamma|^{(1/p)-(1/2)} \left( \int_{\Gamma} |v|^2 \right)^{1/2} \leq C \left( \int_{\Omega_2} |\nabla v|^2 \right)^{1/2},$$

and

$$(2.20) \quad \begin{aligned} \left( \int_{\Omega_1} |v|^p \right)^{1/p} &\leq C \left( \int_{\Gamma} |v|^p \right)^{1/p} + C \left( \int_{\Omega_1} |\nabla v|^p \right)^{1/p} \\ &\leq C \left( \int_{\Omega_2} |\nabla v|^2 \right)^{1/2} + C \left( \int_{\Omega_1} |\nabla v|^p \right)^{1/p}. \end{aligned}$$

Adding (2.19) and (2.20) we obtain

$$\left( \int_{\Omega_1} |v|^p \right)^{1/p} + \left( \int_{\Omega_2} |v|^2 \right)^{1/2} \leq C \left( \int_{\Omega_1} |\nabla v|^p \right)^{1/p} + C \left( \int_{\Omega_2} |\nabla v|^2 \right)^{1/2}$$

and the claim is proved.

SECOND CASE:  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , and  $\Gamma = \partial\Omega_2$ . In this case  $\bar{\Omega}_2$  lies in the interior of  $\Omega$ . We have

$$(2.21) \quad \left( \int_{\Omega_1} |v|^p \right)^{1/p} \leq C \left( \int_{\Omega_1} |\nabla v|^p \right)^{1/p}, \quad \int_{\Gamma} |v|^p \leq C \int_{\Omega_1} |\nabla v|^p,$$

and

$$\left( \int_{\Omega_2} |v|^2 \right)^{1/2} \leq C \left( \int_{\Gamma} |v| \right) + C \left( \int_{\Omega_2} |\nabla v|^2 \right)^{1/2}.$$

Since

$$\int_{\Gamma} |v| \leq |\Gamma|^{1-(1/p)} \left( \int_{\Gamma} |v|^p \right)^{1/p} \leq C \left( \int_{\Omega_1} |\nabla v|^p \right)^{1/p},$$

then

$$(2.22) \quad \left( \int_{\Omega_2} |v|^2 \right)^{1/2} \leq C \left( \int_{\Omega_1} |\nabla v|^p \right)^{1/p} + C \left( \int_{\Omega_2} |\nabla v|^2 \right)^{1/2},$$

and the claim follows adding (2.21) and (2.22).

THIRD CASE:  $\Omega = \Omega_1 \cup \Omega_2 \cup \Gamma$ , and  $\partial\Omega_1 \cap \partial\Omega \neq \emptyset \neq \partial\Omega_2 \cap \partial\Omega$ . In this last case the claim follows by adding the following two inequalities

$$\left( \int_{\Omega_2} |v|^2 \right)^{1/2} \leq C \left( \int_{\Omega_2} |\nabla v|^2 \right)^{1/2}, \quad \left( \int_{\Omega_1} |v|^p \right)^{1/p} \leq C \left( \int_{\Omega_1} |\nabla v|^p \right)^{1/p}.$$

This ends the proof.  $\blacksquare$

We conclude this section with the Green formula for a vector field  $F \in L^2(\Omega; \mathbb{R}^N)$  such that  $\operatorname{div} F \in L^{2N/(N+2)}(\Omega)$ . This vector field defines a functional

$$[F, \nu] : W^{1/2,2}(\partial\Omega) \rightarrow \mathbb{R}$$

by the expression

$$(2.23) \quad [F, \nu](u) = \int_{\Omega} w \operatorname{div} F + \int_{\Omega} \nabla w \cdot F,$$

where  $w \in H^1(\Omega)$  verifies  $w|_{\partial\Omega} = u$ . This definition does not depend on the function  $w$  we have chosen. Indeed, suppose that  $w_1, w_2 \in H^1(\Omega)$  satisfy  $w_i|_{\partial\Omega} = u$ ,  $i = 1, 2$ . Then  $w_1 - w_2 \in H_0^1(\Omega)$  and so

$$\int_{\Omega} (w_1 - w_2) \operatorname{div} F + \int_{\Omega} \nabla(w_1 - w_2) \cdot F = 0.$$

We point out that the functional defined by (2.23) is continuous. To see it consider the function  $w \in H^1(\Omega)$  whose trace is  $u$  and minimizes the norm  $\|w\|_{H^1(\Omega)}$ . Now the desired continuity is as a consequence of the following computations

$$\begin{aligned} |[F, \nu](u)| &\leq \int_{\Omega} |w| |\operatorname{div} F| + \int_{\Omega} |\nabla w| |F| \\ &\leq \left[ \left( \int_{\Omega} |\operatorname{div} F|^{2N/(N+2)} \right)^{(N+2)/2N} + \left( \int_{\Omega} |F|^2 \right)^{1/2} \right] \|w\|_{H^1(\Omega)} \\ &\leq C(F, N, \Omega) \|u\|_{W^{1/2,2}(\partial\Omega)}. \end{aligned}$$

Hence,  $[F, \nu] \in (W^{1/2,2}(\partial\Omega))^*$  and it is an extension of a distribution in  $\mathbb{R}^N$ , namely the one given by

$$[F, \nu](\varphi) = \int_{\Omega} \varphi \operatorname{div} F + \int_{\Omega} \nabla \varphi \cdot F \quad \varphi \in C_0^\infty(\mathbb{R}^N).$$

The support of this distribution lies on  $\partial\Omega$ . From now on, we will write

$$\int_{\partial\Omega} w [F, \nu] d\mathcal{H}^{N-1} = [F, \nu](w|_{\partial\Omega}), \quad w \in H^1(\Omega).$$

With this notation, (2.23) becomes

$$\int_{\partial\Omega} w [F, \nu] d\mathcal{H}^{N-1} = \int_{\Omega} w \operatorname{div} F + \int_{\Omega} \nabla w \cdot F, \quad w \in H^1(\Omega),$$



and it is in this sense that Green's formula holds true. When the trace  $w|_{\partial\Omega}$  vanishes on  $\partial\Omega \setminus \Gamma$ , we will write

$$\int_{\Gamma} w [F, \nu] d\mathcal{H}^{N-1},$$

instead of

$$\int_{\partial\Omega} w [F, \nu] d\mathcal{H}^{N-1}.$$

### 3. CONVERGENCE OF $u_p$ AS $p$ GOES TO 1.

In this section we want to study the behaviour as  $p$  goes to 1 of the solution  $u_p$  to problem (1.4). Our first step is to prove that such a problem has a unique weak solution.

**Proposition 3.1.** *For every  $f \in L^\infty(\Omega)$ , problem (1.4) has a unique weak solution.*

*Proof.* Consider the functional in  $W_0^{1,p(x)}(\Omega)$  defined by

$$I[u] = \frac{1}{p} \int_{\Omega_1} |\nabla u|^p + \frac{1}{2} \int_{\Omega_2} |\nabla u|^2 - \int_{\Omega} f u.$$

It is straightforward that this functional is strictly convex and lower semicontinuous with respect to the weak convergence. Moreover, it is bounded from below, since it is bigger (up to a constant) than the functional

$$J(u) = \frac{1}{p} \int_{\Omega} |\nabla u|^p - \int_{\Omega} f u.$$

Therefore  $I$  has a unique minimum in  $W_0^{1,p(x)}(\Omega)$ , which is the unique weak solution to (1.4). ■

In what follows, with abuse of notation, we will say that  $u_p$  is a sequence and we will consider subsequences of it, as  $p$  goes to 1.

**Proposition 3.2.** *If  $f \in L^\infty(\Omega)$  and*

$$(3.24) \quad 2A \max\{\sqrt{|\Omega_2|}, 1\} \|f\|_{L^\infty(\Omega)} < 1,$$

*then there exists a function  $u \in BV(\Omega) \cap H^1(\Omega_2)$  and a vector field  $z \in L^\infty(\Omega_1; \mathbb{R}^N)$  such that, up to subsequences,*

$$(3.25) \quad u_p \rightarrow u \quad \text{strongly in } L^1(\Omega) \text{ and a.e. in } \Omega,$$

$$(3.26) \quad |\nabla u_p|^{p-2} \nabla u_p \rightharpoonup z \quad \text{weakly in } L^q(\Omega_1; \mathbb{R}^N) \text{ for all } 1 \leq q < +\infty,$$

*with*

$$(3.27) \quad \|z\|_{L^\infty(\Omega_1; \mathbb{R}^N)} \leq 1.$$

*Proof. Step 1. A priori estimates.* Let  $u_p$  be the weak solution to problem (1.4). We want to prove an a priori estimate for the gradients of  $u_p$  in the space  $L^{p(x)}(\Omega; \mathbb{R}^N)$ . Let us denote  $\lambda_p$  the norm of  $\nabla u_p$  in such a space. Then it holds

$$(3.28) \quad \int_{\Omega} \left| \frac{\nabla u_p}{\lambda_p} \right|^{p(x)} = 1.$$

Choosing  $u_p/\lambda_p$  as test function in (1.5), we get

$$(3.29) \quad \frac{1}{\lambda_p} \int_{\Omega} |\nabla u_p|^{p(x)} = \frac{1}{\lambda_p} \int_{\Omega} f u_p.$$

By generalized Holder's inequality (2.10) and inequality (2.18), we get

$$\begin{aligned} \frac{1}{\lambda_p} \int_{\Omega} |\nabla u_p|^{p(x)} &\leq \frac{1}{\lambda_p} \|f\|_{L^\infty(\Omega)} \int_{\Omega} |u_p| \\ &\leq \frac{2}{\lambda_p} \|f\|_{L^\infty(\Omega)} \|1\|_{L^{p'(x)}(\Omega)} \|u_p\|_{L^{p(x)}(\Omega)} \\ &\leq \frac{2A_p}{\lambda_p} \|f\|_{L^\infty(\Omega)} \|1\|_{L^{p'(x)}(\Omega)} \|\nabla u_p\|_{L^{p(x)}(\Omega; \mathbb{R}^N)} \\ &\leq 2A_p \|f\|_{L^\infty(\Omega)} \|1\|_{L^{p'(x)}(\Omega)}. \end{aligned}$$

On the other hand, by (3.28), we can estimate the left-hand side of (3.29) in the following way

$$(3.30) \quad \frac{1}{\lambda_p} \int_{\Omega} |\nabla u_p|^{p(x)} = \lambda_p^{p-1} \int_{\Omega_1} \left| \frac{\nabla u_p}{\lambda_p} \right|^p + \lambda_p \int_{\Omega_2} \left| \frac{\nabla u_p}{\lambda_p} \right|^2 \geq \min\{\lambda_p^{p-1}, \lambda_p\}.$$

Combining (3.29) and (3.30), we obtain

$$(3.31) \quad \min\{\lambda_p^{p-1}, \lambda_p\} \leq 2A_p \|f\|_{L^\infty(\Omega)} \|1\|_{L^{p'(x)}(\Omega)}.$$

We claim that

$$(3.32) \quad \limsup_{p \rightarrow 1} \|1\|_{L^{p'(x)}(\Omega)} = \max\{\sqrt{|\Omega_2|}, 1\}.$$

Indeed, denoted  $\tau_p = \|1\|_{L^{p'(x)}(\Omega)}$ , by definition of norm (2.8), we get

$$\frac{|\Omega_1|}{\tau_p^{p'}} + \frac{|\Omega_2|}{\tau_p^{2'}} = 1.$$

Let  $\tau$  a limit point of  $\tau_p$  as  $p$  goes to 1. The inequality

$$\frac{|\Omega_1|}{\tau^{p'}} \leq 1$$

implies that  $1 \leq \tau < +\infty$ . Moreover if  $\tau > 1$ , we deduce that  $\lim_{p \rightarrow 1} \tau_p = \sqrt{|\Omega_2|}$ . This yields (3.32).

Now, since (3.24) holds, then  $\min\{\lambda_p^{p-1}, \lambda_p\} < 1$ , for  $p \leq p_0$ , and hence, from the definition of  $\lambda_p$ , we obtain the following a priori estimate

$$(3.33) \quad \|\nabla u_p\|_{L^{p(x)}(\Omega; \mathbb{R}^N)} < 1.$$

*Step 2. Convergence of  $u_p$  and of the gradients.* By classical Holder's inequality, Proposition 2.1 and the a priori estimate (3.33), we deduce

$$\begin{aligned} \int_{\Omega} |\nabla u_p| &\leq |\Omega_1|^{1-\frac{1}{p}} \left( \int_{\Omega_1} |\nabla u_p|^p \right)^{\frac{1}{p}} + |\Omega_2|^{\frac{1}{2}} \left( \int_{\Omega_2} |\nabla u_p|^2 \right)^{\frac{1}{2}} \\ (3.34) \quad &\leq C \max\{|\Omega_1|^{1-\frac{1}{p}}, |\Omega_2|^{\frac{1}{2}}\} \|\nabla u_p\|_{L^{p(x)}(\Omega; \mathbb{R}^N)} \\ &\leq C(1 + 1 + |\Omega_2|^{\frac{1}{2}}) \end{aligned}$$

for any  $p \leq p_0$ . In an analogous way, we obtain

$$(3.35) \quad \left( \int_{\Omega} |\nabla u_p|^2 \right)^{1/2} \leq c_p \|\nabla u_p\|_{L^{p(x)}(\Omega; \mathbb{R}^N)} \leq c_1 + 1.$$

This implies that there exist a subsequence still denoted by  $u_p$  and a function  $u \in BV(\Omega) \cap H^1(\Omega_2)$  such that

$$\begin{cases} \nabla u_p \rightharpoonup Du & \text{weakly* in the sense of measures in } \Omega_1, \\ \nabla u_p \rightharpoonup \nabla u & \text{weakly in } L^2(\Omega_2; \mathbb{R}^N), \\ u_p \rightarrow u & \text{strongly in } L^1(\Omega), \\ u_p \rightarrow u & \text{a.e. in } \Omega. \end{cases}$$

Now we prove (3.26). For every  $q$  such that  $1 \leq q < p'$ , by Proposition 2.1 and our a priori estimate (3.33), we have

$$(3.36) \quad \begin{aligned} \int_{\Omega_1} |\nabla u_p|^{(p-1)q} &\leq \frac{1}{c_2} |\Omega_1|^{1-\frac{(p-1)q}{p}} \|\nabla u\|_{L^{p(x)}(\Omega; \mathbb{R}^N)} \\ &\leq \frac{1}{c_2} |\Omega_1|^{1-\frac{(p-1)q}{p}} \leq \frac{1}{c_2} (|\Omega_1| + 1), \end{aligned}$$

for any  $p \leq p_1$ .

It yields that, for any  $q$  fixed, the sequence  $|\nabla u_p|^{p-2} \nabla u_p$  is bounded in the space  $L^q(\Omega_1; \mathbb{R}^N)$  and then there exists  $z_q \in L^q(\Omega_1; \mathbb{R}^N)$  such that, up to subsequences,

$$|\nabla u_p|^{p-2} \nabla u_p \rightharpoonup z_q \quad \text{in } L^q(\Omega_1; \mathbb{R}^N) \quad \text{for all } 1 \leq q < +\infty.$$

Moreover, by a diagonal argument we can find a limit  $z$  that does not depend on  $q$ , that is,

$$(3.37) \quad |\nabla u_p|^{p-2} \nabla u_p \rightharpoonup z \quad \text{in } L^q(\Omega_1; \mathbb{R}^N) \quad \text{for all } 1 \leq q < +\infty.$$

Now by (3.36) we deduce

$$\| |\nabla u_p|^{p-2} \nabla u_p \|_{L^q(\Omega_1; \mathbb{R}^N)} \leq \left( \frac{1}{c_2} (|\Omega_1| + 1) \right)^{1/q}.$$

Therefore, by the lower semicontinuity of the norm, we have

$$(3.38) \quad \|z\|_{L^q(\Omega_1; \mathbb{R}^N)} \leq \left( \frac{1}{c_2} (|\Omega_1| + 1) \right)^{1/q} \quad \text{for all } 1 \leq q < +\infty.$$

Hence, we obtain that  $z$  belongs to  $L^\infty(\Omega; \mathbb{R}^N)$  and

$$\|z\|_{L^\infty(\Omega_1; \mathbb{R}^N)} = \lim_{q \rightarrow +\infty} \|z\|_{L^q(\Omega_1; \mathbb{R}^N)} \leq 1,$$

that is, (3.27) is proved. ■

#### 4. MAIN RESULTS.

In this Section, we will study the limit problem. For the sake of simplicity, in what follows we will deal with the case  $\partial\Omega_1 \cap \partial\Omega \neq \emptyset \neq \partial\Omega_2 \cap \partial\Omega$ , the existence of solutions in the other cases can be handled analogously using Proposition 2.4. Note that for our uniqueness result we need a condition on the location of the subdomains, namely,  $\mathcal{H}^{N-1}(\partial\Omega_2 \cap \partial\Omega) \neq 0$ .

We begin by introducing the definition of solution to the limit problem

$$(4.39) \quad \begin{cases} -\Delta_{l(x)} u = f, & \text{in } \Omega; \\ u = 0, & \text{on } \partial\Omega; \end{cases}$$

where

$$(4.40) \quad l(x) = \begin{cases} 1, & \text{if } x \in \Omega_1; \\ 2, & \text{if } x \in \Omega_2. \end{cases}$$

Let  $u$  be a function belonging to  $BV(\Omega)$  and let  $z$  be a vector field belonging to  $L^\infty(\Omega_1; \mathbb{R}^N)$  such that  $\operatorname{div} z$ , in the sense of distributions, belongs to  $L^\infty(\Omega_1)$ , i.e.

$$\langle \operatorname{div} z, \varphi \rangle = - \int_{\Omega_1} z \cdot \nabla \varphi,$$

for all  $\varphi \in C_0^\infty(\Omega_1)$ . Then we define the distribution  $(z, Du) : C_0^\infty(\Omega_1) \rightarrow \mathbb{R}$  by

$$(4.41) \quad \langle (z, Du), \varphi \rangle = - \int_{\Omega_1} u z \cdot \nabla \varphi - \int_{\Omega_1} u \varphi \operatorname{div} z,$$

for every  $\varphi \in C_0^\infty(\Omega_1)$ . Since  $u \in BV(\Omega) \subset L^{N',1}(\Omega_1)$ ,  $\varphi \in C_0^\infty(\Omega_1)$ ,  $z \in L^\infty(\Omega_1; \mathbb{R}^N)$  and  $\operatorname{div} z \in L^\infty(\Omega_1)$ , all terms in (4.41) make sense.

Moreover, as in [7] (see also [4] pp. 126–127) we may define the weak trace of the exterior normal component of  $z$  on  $\partial\Omega_1$ , which will be denoted by  $[z, \nu_1]$ . Then a generalized Green's formula holds, relating the function  $[z, \nu_1]$  and the measure  $(z, Du)$ :

$$(4.42) \quad \int_{\Omega_1} u \operatorname{div} z + \int_{\Omega_1} (z, Du) = \int_{\partial\Omega_1} [z, \nu_1] u \, d\mathcal{H}^{N-1}.$$

We point out that  $u \in BV(\Omega)$  and so jump discontinuities are allowed. Thus, function  $u$  may have a jump discontinuity on  $\Gamma$  and so different traces on  $\Gamma$ . To distinguish these different values, we will write the restriction of  $u$  to  $\Omega_i$  as  $u|_{\Omega_i}$ ,  $i = 1, 2$ .

**Definition 4.1.** *We say that a function  $u \in BV(\Omega) \cap H^1(\Omega_2)$  is a solution to (4.39) if the following conditions hold:*

*There exists  $z \in L^\infty(\Omega_1; \mathbb{R}^N)$  satisfying*

$$(4.43) \quad \|z\|_{L^\infty(\Omega_1; \mathbb{R}^N)} \leq 1;$$

$$(4.44) \quad -\operatorname{div} z = f \quad \text{in } \mathcal{D}'(\Omega_1), \quad -\Delta u = f \quad \text{in } \mathcal{D}'(\Omega_2);$$

$$(4.45) \quad \int_{\Omega_1} z \cdot \nabla \varphi + \int_{\Omega_2} \nabla u \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in C_0^\infty(\Omega);$$

$$(4.46) \quad (z, Du) = |Du| \quad \text{as measures in } \Omega_1;$$

$$(4.47) \quad [z, \nu_1] = [\nabla u|_{\Omega_2}, \nu_1] \quad \mathcal{H}^{N-1}\text{-a.e. on } \Gamma;$$

$$(4.48) \quad u|_{\Omega_2} = 0 \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega_2 \cap \partial\Omega;$$

$$(4.49) \quad [z, \nu_1] \in \operatorname{sign}(-u|_{\Omega_1}) \quad \mathcal{H}^{N-1}\text{-a.e. on } \partial\Omega_1 \cap \partial\Omega;$$

$$(4.50) \quad [z, \nu_1] \in \operatorname{sign}(u|_{\Omega_2} - u|_{\Omega_1}) \quad \mathcal{H}^{N-1}\text{-a.e. on } \Gamma.$$

REMARK 4.2. The following weak formulation holds: if  $u \in BV(\Omega) \cap H^1(\Omega_2)$  is a solution to (4.39) in the sense of Definition 4.1, then

$$(4.51) \quad \int_{\Omega_1} |Du| - \int_{\Omega_1} (z, Dw) + \int_{\partial\Omega_1 \cap \partial\Omega} |u| \, d\mathcal{H}^{N-1} + \int_{\partial\Omega_1 \cap \partial\Omega} w [z, \nu_1] \, d\mathcal{H}^{N-1} + \int_{\Omega_2} \nabla u \cdot \nabla (u - w) = \int_{\Omega} f(u - w)$$

for every  $w \in BV(\Omega) \cap H^1(\Omega_2)$  with  $w = 0$  on  $\partial\Omega \cap \partial\Omega_2$ .

Now we are ready to prove the following existence result.

**Theorem 4.3.** *If  $f \in L^\infty(\Omega)$  verifies (3.24), then there exists, at least, a solution to problem (4.39).*

*Proof. Proof of (4.44)-(4.45).* Let  $z$  be the vector field given in (3.26), and let  $u_p$  be the solution to (1.4), then

$$(4.52) \quad \int_{\Omega_1} |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \varphi + \int_{\Omega_2} \nabla u_p \cdot \nabla \varphi = \int_{\Omega} f \varphi, \quad \forall \varphi \in C_0^\infty(\Omega).$$

and also

$$(4.53) \quad \int_{\Omega_1} |\nabla u_p|^{p-2} \nabla u_p \nabla \varphi = \int_{\Omega_1} f \varphi, \quad \forall \varphi \in C_0^\infty(\Omega_1),$$

and

$$(4.54) \quad \int_{\Omega_2} \nabla u_p \nabla \varphi = \int_{\Omega_2} f \varphi, \quad \forall \varphi \in C_0^\infty(\Omega_2).$$

Letting  $p \rightarrow 1$  in all the above equations, we obtain (4.45)-(4.44).

**Proof of (4.46).** By Anzellotti's theory,

$$|(z, Du)| \leq \|z\|_\infty |Du|$$

and, since  $\|z\|_\infty \leq 1$ , we have

$$(4.55) \quad (z, Du) \leq |Du|, \quad \text{as measures in } \Omega_1.$$

Now we prove that in fact equality holds in (4.55). Consider  $u_p \phi$ , with  $\phi \geq 0$ ,  $\phi \in C_0^\infty(\Omega_1)$  as test function. We obtain

$$(4.56) \quad \int_{\Omega_1} \phi |\nabla u_p|^p + \int_{\Omega_1} u_p |\nabla u_p|^{p-2} \nabla u_p \cdot \nabla \phi = \int_{\Omega_1} f u_p \phi.$$

On the other hand, by Young inequality, we have

$$(4.57) \quad \int_{\Omega_1} \phi |\nabla u_p| \leq \frac{1}{p} \int_{\Omega_1} \phi |\nabla u_p|^p + \frac{p-1}{p} \int_{\Omega_1} \phi.$$

Combining (4.56)-(4.57) and letting  $p$  go to 1, we deduce

$$(4.58) \quad \int_{\Omega_1} \phi |Du| + \int_{\Omega_1} u z \cdot \nabla \phi \leq \int_{\Omega_1} f u \phi.$$

On the other hand, we have

$$(4.59) \quad \int_{\Omega_1} f u \phi = \int_{\Omega_1} \phi (z, Du) + \int_{\Omega_1} u z \cdot \nabla \phi.$$

Combining (4.58) and (4.59), we deduce

$$\int_{\Omega_1} \phi |Du| \leq \int_{\Omega_1} \phi (z, Du)$$

This proves the claim.

**Proof of (4.47).** From (4.45) and the generalized Green formula we have that for every  $\varphi \in C_0^\infty(\Omega)$ ,

$$- \int_{\Omega_1} \varphi \operatorname{div} z + \int_{\partial\Omega_1} \varphi [z, \nu_1] d\mathcal{H}^{N-1} - \int_{\Omega_2} \varphi \operatorname{div} \nabla u + \int_{\partial\Omega_2} \varphi [\nabla u, \nu_2] d\mathcal{H}^{N-1} = \int_{\Omega} f \varphi.$$

Then taking into account (4.44) and the fact that  $\varphi \in C_0^\infty(\Omega)$  we obtain

$$\int_{\Gamma} \varphi [z, \nu_1] + \int_{\Gamma} \varphi [\nabla u|_{\Omega_2}, \nu_2] = 0$$

for every  $\varphi \in C_0^\infty(\Omega)$ . Having in mind that  $\nu_1 = -\nu_2$  on  $\Gamma$ , it gives (4.47).

**Proof of (4.48).** This is a straightforward consequence of the following facts:  $u_p = 0$  on  $\partial\Omega_2 \cap \partial\Omega$  for all  $p$ ;  $u_p \rightharpoonup u$  weakly in  $H^1(\Omega_2)$  and the embedding  $H^1(\Omega_2) \hookrightarrow L^2(\partial\Omega_2 \cap \partial\Omega)$  is continuous.

**Proof of (4.49) and (4.50).** Denote by  $g_p$  the trace of  $u_p$  on  $\Gamma$ , and by  $g$  the trace of  $u|_{\Omega_2}$  on  $\Gamma$ . It follows from  $u_p \rightharpoonup u|_{\Omega_2}$  weakly in  $H^1(\Omega_2)$  that  $g_p \rightarrow g$  strongly in  $L^2(\Gamma)$ . Taking  $u_p$  as test function in (1.5) and using Young's inequality we get

$$(4.60) \quad \begin{aligned} \int_{\Omega_1} |\nabla u_p| + \frac{1}{p} \int_{\Omega_2} |\nabla u_p|^2 \\ \leq \frac{1}{p} \int_{\Omega_1} |\nabla u_p|^p + \frac{p-1}{p} |\Omega_1|^{1-1/p} + \frac{1}{p} \int_{\Omega_2} |\nabla u_p|^2 \\ = \frac{1}{p} \int_{\Omega} f u_p + \frac{p-1}{p} |\Omega_1|^{1-1/p} \end{aligned}$$

In other words, we have

$$(4.61) \quad \begin{aligned} \int_{\Omega_1} |\nabla u_p| + \int_{\partial\Omega_1 \cap \partial\Omega} |u_p| d\mathcal{H}^{N-1} + \int_{\Gamma} |u_p - g| d\mathcal{H}^{N-1} + \frac{1}{p} \int_{\Omega_2} |\nabla u_p|^2 \\ \leq \frac{1}{p} \int_{\Omega} f u_p + \frac{p-1}{p} |\Omega_1|^{1-1/p} + \int_{\Gamma} |g_p - g| d\mathcal{H}^{N-1}. \end{aligned}$$

Having in mind the lower semi-continuity of the functional (2.6), we may let  $p$  go to 1, obtaining

$$\int_{\Omega_1} |Du| + \int_{\partial\Omega_1 \cap \partial\Omega} |u|_{\Omega_1} d\mathcal{H}^{N-1} + \int_{\Gamma} |u|_{\Omega_1} - g| d\mathcal{H}^{N-1} + \int_{\Omega_2} |\nabla u|^2 \leq \int_{\Omega} f u.$$

Taking into account that  $u|_{\Omega_2} = g$  on  $\Gamma$  on the left-hand side and applying the generalized Green formula on the right-hand, we get

$$(4.62) \quad \begin{aligned} |Du|(\Omega_1) + \int_{\partial\Omega_1 \cap \partial\Omega} |u|_{\Omega_1} d\mathcal{H}^{N-1} + \int_{\Gamma} |u|_{\Omega_1} - u|_{\Omega_2}| d\mathcal{H}^{N-1} + \int_{\Omega_2} |\nabla u|^2 \\ \leq \int_{\Omega_1} f u + \int_{\Omega_2} f u \\ = \int_{\Omega_1} (z, Du) - \int_{\partial\Omega_1} u|_{\Omega_1} [z, \nu_1] d\mathcal{H}^{N-1} + \int_{\Omega_2} |\nabla u|^2 - \int_{\Gamma} u|_{\Omega_2} [\nabla u|_{\Omega_2}, \nu_2] d\mathcal{H}^{N-1}. \end{aligned}$$

Since  $|Du|(\Omega_1) = \int_{\Omega_1} (z, Du)$  and  $[z, \nu_1] = -[\nabla u|_{\Omega_2}, \nu_2]$ , we can simplify the remainders terms, so (4.62) becomes

$$\begin{aligned} \int_{\partial\Omega_1 \cap \partial\Omega} |u|_{\Omega_1} + u|_{\Omega_1} [z, \nu_1] d\mathcal{H}^{N-1} + \\ \int_{\Gamma} |u|_{\Omega_1} - u|_{\Omega_2}| + (u|_{\Omega_1} - u|_{\Omega_2}) [z, \nu_1] d\mathcal{H}^{N-1} \leq 0, \end{aligned}$$

which prove (4.49) and (4.50).  $\blacksquare$

Now we prove the uniqueness of the solution to the limit problem.

**Theorem 4.4.** *If  $\mathcal{H}^{N-1}(\partial\Omega_2 \cap \partial\Omega) \neq 0$ , then problem (4.39) has a unique solution in the sense that two solutions  $u_1, u_2$  must agree in  $\Omega_2$ .*

*Proof.* Let us assume that  $u_1, u_2$  are two solutions to problem (4.39). By Remark 4.2, we can use  $u_2$  as test function in the weak formulation written for  $u_1$  and  $u_1$  as test function in the weak formulation written for  $u_2$ . Adding the two equations, we have

$$\begin{aligned} & \int_{\Omega_1} |Du_1| + \int_{\Omega_1} |Du_2| - \int_{\Omega_1} (z_1, Du_2) - \int_{\Omega_1} (z_2, Du_1) + \int_{\partial\Omega \cap \partial\Omega_1} |u_1| d\mathcal{H}^{N-1} \\ & + \int_{\partial\Omega \cap \partial\Omega_1} |u_2| d\mathcal{H}^{N-1} + \int_{\partial\Omega \cap \partial\Omega_1} u_2[z_1; \nu] d\mathcal{H}^{N-1} + \int_{\partial\Omega \cap \partial\Omega_1} u_1[z_2; \nu] d\mathcal{H}^{N-1} \\ & + \int_{\Omega_2} |\nabla(u_1 - u_2)|^2 \leq 0. \end{aligned}$$

Since, we have

$$\int_{\Omega_1} (z_1, Du_2) \leq \int_{\Omega_1} |Du_2|, \quad \text{and} \quad \int_{\Omega_1} (z_2, Du_1) \leq \int_{\Omega_1} |Du_2|,$$

and moreover

$$-u_1[z_2; \nu] \leq |u_1|, \quad \text{and} \quad -u_2[z_1; \nu] \leq |u_2|,$$

we deduce

$$\int_{\Omega_2} |\nabla u_1 - \nabla u_2|^2 \leq 0,$$

which implies, since  $u_1 = u_2 = 0$  on  $\partial\Omega_2 \cap \partial\Omega$ , that  $u_1 = u_2$  in  $\Omega_2$ .  $\blacksquare$

REMARK 4.5. As in [19] we can prove that, if the norm of  $f$  is large enough, there is not any  $u \in BV(\Omega_1)$  which is the weak\* limit of  $u_p$ . Precisely if  $\|f\|_{W^{-1,\infty}(\Omega_1)} > 1$ , then

$$\lim_{p \rightarrow 1} \int_{\Omega_1} |\nabla u_p| = +\infty.$$

Indeed, since  $\|f\|_{W^{-1,\infty}(\Omega_1)} = \lim_{p \rightarrow 1} \|f\|_{W^{-1,p'}(\Omega_1)}$ , we may take  $\epsilon > 0$  and  $p_0 > 1$  such that  $\|f\|_{W^{-1,p'}(\Omega_1)} > 1 + \epsilon$ , for all  $p \leq p_0$ . On the other hand, if  $\varphi \in W_0^{1,p}(\Omega_1)$  with  $\|\nabla\varphi\|_{L^p(\Omega_1;\mathbb{R}^N)} \leq 1$ , then

$$\langle f, \varphi \rangle_{W^{-1,p'}(\Omega_1), W_0^{1,p}(\Omega_1)} = \int_{\Omega_1} |\nabla u_p|^{p-1} \nabla u_p \cdot \nabla \varphi \leq \left( \int_{\Omega_1} |\nabla u_p|^p \right)^{\frac{p-1}{p}}.$$

This implies

$$\|f\|_{W^{-1,p'}(\Omega_1)}^{\frac{p}{p-1}} \leq \int_{\Omega_1} |\nabla u_p|^p.$$

Therefore,

$$(1 + \epsilon)^{\frac{p}{p-1}} \leq \int_{\Omega_1} |\nabla u_p|^p$$

for  $p \leq p_0$  and then

$$\lim_{p \rightarrow 1} \int_{\Omega_1} |\nabla u_p|^p = +\infty.$$

Since for a suitable  $g \in L^\infty(\Omega_1)$ , we have  $f = \operatorname{div} g$ , the conclusion follows from

$$\int_{\Omega_1} |\nabla u_p|^p = \langle f, u_p \rangle_{W^{-1,\infty}(\Omega_1), W_0^{1,1}(\Omega_1)} = \int_{\Omega_1} g \cdot \nabla u_p \leq \|g\|_\infty \int_{\Omega_1} |\nabla u_p|.$$

Finally, to illustrate our results, we compute explicit solutions in the 1-dimensional case.

EXAMPLE 4.6. Assume that  $\Omega = ]-1, 1[$ , with  $\Omega_1 = ]-1, 0[$  and  $\Omega_2 = ]0, 1[$ . Consider  $\lambda, \Lambda > 0$  and take as right-hand side the function  $f$  defined by

$$f(x) = \begin{cases} \lambda, & \text{if } x < 0, \\ \Lambda, & \text{if } x > 0. \end{cases}$$

The solution at level  $p$  can explicitly be computed, namely,

$$u_p(x) = \begin{cases} \frac{p-1}{\lambda p} [(\lambda + k_p)^{p/(p-1)} - (-\lambda x + k_p)^{p/(p-1)}], & \text{if } x < 0, \\ -\frac{\Lambda}{2}x^2 + k_p x + \frac{\Lambda}{2} - k_p, & \text{if } x > 0, \end{cases}$$

where  $k_p \geq 0$  satisfies

$$k_p + \frac{p-1}{\lambda p} [(\lambda + k_p)^{p/(p-1)} - (k_p)^{p/(p-1)}] = \frac{\Lambda}{2}.$$

Now, assume that there is a limit function  $u(x) = \lim_{p \rightarrow 1} u_p(x)$ . If this limit is finite, we get that there exists  $k = \lim_{p \rightarrow 1} k_p$ . Then we deduce that this limit must satisfy  $\lambda + k \leq 1$  and  $k = \frac{\Lambda}{2}$ . Thus,  $\lambda + \frac{\Lambda}{2} \leq 1$  and consequently  $\Lambda \leq 2 - 2\lambda$ .

In this case, the limit function is given by

$$u(x) = \begin{cases} 0, & \text{if } x < 0, \\ -\frac{\Lambda}{2}x^2 + \frac{\Lambda}{2}x, & \text{if } x > 0, \end{cases}$$

and so

$$\begin{aligned} z(x) &= \lim_{p \rightarrow 1} |u'_p(x)|^{p-2} u'_p(x) = -\lambda x + \frac{\Lambda}{2}, & \text{if } x < 0, \\ u'(x) &= -\Lambda x + \frac{\Lambda}{2}, & \text{if } x > 0. \end{aligned}$$

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