ANISOTROPIC p,q-LAPLACIAN EQUATIONS WHEN p GOES TO 1

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Abstract. In this paper we prove a stability result for an anisotropic elliptic problem. More precisely, we consider the Dirichlet problem for an anisotropic equation, which is as the p–Laplacian equation with respect to a group of variables and as the q–Laplacian equation with respect to the other variables $(1 < p < q)$, with datum f belonging to a suitable Lebesgue space. For this problem, we study the behaviour of the solutions as p goes to 1, showing that they converge to a function u, which is almost everywhere finite, regardless of the size of the datum f. Moreover, we prove that this u is the unique solution of a limit problem having the 1–Laplacian operator with respect to the first group of variables.

Furthermore, the regularity of the solutions to the limit problem is studied and explicit examples are shown.

Résumé Dans cet article nous démontrons un résultat de stabilité pour un problème elliptique et anisotrope. Plus précisément, nous considérons le problème de Dirichlet pour un'équation anisotropee qui est an équation p–Laplacien par rapport at un group des variables et une équation q–Laplacien par rapport a les autres avec un donnée f qui est dans an opportun espace de Lebesgue. Pour cet probléme, nous etudions le comportement des solutions quand p tend à 1 et nous montròns qu'elles convergent à une fonction u qui est presque partout finie, quelque soit la taille du donnée f.

De plus, nous montròns que cette function u est l'uniquee solution d'un probléme limit avec l'operator 1–Laplacian par rapport au premier group des variables.

En fin la regularité des solutions du probléme limit est étudiée et explicite exemples sont montrés.

1. Introduction

Our aim is to study the Dirichlet problem for an anisotropic elliptic equation which is as the 1–Laplacian equation in some directions (say x) and as the q–Laplacian equation in the others (say y), that is:

$$(1.1)\qquad \begin{cases} -\mathrm{div}_x\Big(\dfrac{D_x u}{|D_x u|}\Big) - \mathrm{div}_y\big(|\nabla_y u|^{q-2}\nabla_y u\big) = f(x,y)\,, & \text{in } \Omega\,;\\ u = 0\,, & \text{on } \partial\Omega\,. \end{cases}$$

Here $(x,y) \in \Omega = \Xi \times \Upsilon$ with Ξ and Υ bounded open subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$ respectively and the subindexes denote differentiation with respect to x and y respectively.

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We also assume that Ξ has a Lipschitz boundary. Concerning the right-hand side, we assume that f belongs to $L^r(\Omega)$ with

$$r = \min\Big\{\frac{N+K}{1+(K/q')}, q'\Big\}. \tag{1.2}$$

To handle equation (1.1), we have to give a notion of solution and then consider a suitable functional framework. Adapting the well-known definition of solution for the 1-Laplacian equation (see [3]), we consider an anisotropic subspace of $BV(\Omega)$, which consists, roughy speaking, of functions such that $D_x u$ is a Radon measure and $\nabla_y u$ belongs to the Lebesgue space $L^q(\Omega)$. Obviously any notion of solution have to give sense to the quotient $\frac{D_x u}{|D_x u|}$ where, in general, $D_x u$ is not a function but a Radon measure. To this aim our definition (see Section 4) is based on a vector field $\zeta \in L^\infty(\Omega;\mathbb{R}^N)$ satisfying $\|\zeta\|_\infty \le 1$, $-\text{div}_x\zeta - \text{div}_y\big(|\nabla_y u|^{q-2}\nabla_y u\big) = f$ in $\mathcal{D}'(\Omega)$ and $(\zeta, D_x u) = |D_x u|$. Observe that, formally, $\|\zeta\|_\infty \le 1$ and $(\zeta, D_x u) = |D_x u|$ imply $\zeta = \frac{D_x u}{|D_x u|}$. The meaning of $(\zeta, D_x u)$ relies on an anisotropic extension of the theory of L^∞–divergence–measure vector fields by Anzellotti [2] and by Chen–Frid [13, 14, 15].

In this paper we prove the existence, uniqueness and regularity of such a solution to problem (1.1), as well as we present explicit examples. To this end, we consider approximate problems of the form

$$\begin{cases} -\text{div}_x\big(|\nabla_x u_p|^{p-2}\nabla_x u_p\big) - \text{div}_y\big(|\nabla_y u_p|^{q-2}\nabla_y u_p\big) = f\,, & \text{in } \Omega\,; \\ u_p = 0\,, & \text{on } \partial\Omega\,; \end{cases} \tag{1.3}$$

where $1 < p, q < \infty$, and then we study the behaviour as p goes to 1 of the solutions u_p. Thus, we may assume without lost of generality that $p < \min\{q, N\}$; moreover we may also assume $p < \frac{Nq}{q-K}$ if $q > K$, and $\frac{Np}{N-p} < q$ if $\frac{N}{N-1} < q$.

We prove that the approximate solutions u_p converge to a BV–function u that turns out to be a solution to equation (1.1).

Formally (1.1) is the limit problem of (1.3) as p goes to 1. Solutions to this limit problem may be seen as critical points in the anisotropic subset of BV described above of the functional defined by

$$J[u] = \int_\Omega |D_x u| + \frac{1}{q}\int_\Omega |\nabla_y u|^q - \int_\Omega f u\,.$$

However, if we try to show that J is bounded from below, then we will need to consider a datum f small enough. Instead, in this paper we prove that the limit problem (1.1) has a solution for all f regardless of its size, whether large or small.

A similar approach has been used to study the isotropic version of problem (1.1), where the differential operator is replaced by $-\text{div}\big(|\nabla u|^{-1}\nabla u\big)$. In such a case there is no stability result for solutions to p–Laplacian equation as p goes to 1, in the sense that solutions of the p–Laplacian equation converge to a function that can be infinity on a set of positive measure when the datum f is large enough (see [26, 27] for particular data and [30, 31] for more general data). Moreover, there is no uniqueness of the solution to the limit problem. Equation (1.1) shares some features with its isotropic version, as shown in Subsection 4.3. Indeed, the solution is trivial (identically 0) when the considered datum f is small enough. This situation occurs until the vector field ζ satisfies $\|\zeta\|_\infty = 1$. After that the two equations differ. When the norm of the datum increases, in the isotropic problem it is not possible to find a vector field satisfying $\|\zeta\|_\infty \le 1$ and solutions blow up, while in the anisotropic problem the extra term $\mathrm{div}_y\big(|\nabla_y u|^{q-2}\nabla_y u\big)$ absorbs the excess and a finite solution can always be obtained.

Anisotropic problems have been studied by many years. Recently the number of papers devoted to these kind of problems has increased. We refer, for example, to [5, 6, 9, 16, 17, 20, 21, 22, 23, 28, 29, 32, 33]. We also point out that anisotropic problems appear in connections with some problems in Physics [11, 18, 19, 25], in Biology [5, 6, 7], and in Image Processing [34].

The plan of this paper is as follows. After introducing our precise hypotheses and notation, in Section 2 we study our functional setting: we discuss two crucial inequalities and extend the Anzellotti theory of L^∞–divergence–measure vector fields to the anisotropic case, giving sense to $(\zeta, D_x u)$ and obtaining a Green's formula. In Section 3, we begin by studying the asymptotic behaviour of the sequence (u_p) of approximate solutions to problem (1.3). As $p \to 1$, we get a limit function u and a vector field ζ which is the weak limit of $|\nabla_x u_p|^{p-2}\nabla_x u_p$. In Section 4 we introduce our notion of solution and we prove the existence result stated in in Theorem 4.2, which consists in proving that the limit function u above is a solution to (1.1). Our uniqueness result is established in Theorem 4.3. We also show in Theorem 4.5 a regularity result when more regular data are considered. Finally, we show some explicit examples of solutions to equation (1.1) regardless of the size of the datum, in which it is seen how the extra term $\mathrm{div}_y\big(|\nabla_y u|^{q-2}\nabla_y u\big)$ absorbs the excess when $\|\zeta\|_\infty$ reaches 1.

2. Crucial tools

2.1. **Notation and Inequalities.** Recall from the introduction that we denote by Ξ and Υ bounded open subsets of $\mathbb{R}^N$ and $\mathbb{R}^K$, respectively. We assume that Ξ has a Lipschitz boundary, so that we may handle a unit vector field (denoted by ν_x) normal to $\partial\Xi$ and exterior to Ξ, defined $\mathcal{H}^{N-1}$-a.e. on $\partial\Xi$, where $\mathcal{H}^{N-1}$ denotes the $(N-1)$-dimensional Hausdorff measure. Let $\Omega = \Xi \times \Upsilon \subset \mathbb{R}^{N+K}$. If $u : \Omega \to \mathbb{R}$ is a regular enough function, we will denote

$$\nabla_x u = \Big(\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}, \ldots, \frac{\partial u}{\partial x_N}\Big) \quad \text{and} \quad \nabla_y u = \Big(\frac{\partial u}{\partial y_1}, \frac{\partial u}{\partial y_2}, \ldots, \frac{\partial u}{\partial y_K}\Big).$$

Thus, the gradient of u reads as $\nabla u = (\nabla_x u, \nabla_y u)$.

If $z : \Omega \to \mathbb{R}^{N+K}$ is a smooth vector field, we will write $\zeta = (z_1, z_2, \ldots, z_N)$ and $\lambda = (z_{N+1}, z_{N+2} \ldots, z_{N+K})$, so that $z = (\zeta, \lambda)$. Then we will denote

$$\mathrm{div}_x\zeta = \sum_{i=1}^{N} \frac{\partial z_i}{\partial x_i} \quad \text{and} \quad \mathrm{div}_y\lambda = \sum_{i=1}^{K} \frac{\partial z_{N+i}}{\partial y_i}\,, \tag{2.4}$$

and this yields $\mathrm{div}\, z = \mathrm{div}_x\zeta + \mathrm{div}_y\lambda$.

Throughout this paper we will denote by $W_0^{1,(p,q)}(\Omega)$, with $1 < p, q < \infty$, the anisotropic Sobolev space defined as the closure of the space $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{(p,q)} = \|\nabla_x u\|_p + \|\nabla_y u\|_q$. A function u belonging to $W_0^{1,(p,q)}(\Omega)$ satisfies $\nabla_x u \in L^p(\Omega; \mathbb{R}^N)$ and $\nabla_y u \in L^q(\Omega; \mathbb{R}^K)$. Moreover for almost all $x \in \Xi$ the function $y \mapsto u(x, y)$ belongs to $W_0^{1,q}(\Upsilon)$.

We will denote $BV^{(q)}(\Omega)$, with $1 < q < \infty$, the anisotropic subspace of $BV(\Omega)$ consisting of those functions u satisfying that $D_x u$ is a Radon measure and $\nabla_y u$ belongs to $L^q(\Omega; \mathbb{R}^K)$ in such a way that for almost all $x \in \Xi$ the function $y \mapsto$

$u(x,y)$ belongs to $W_0^{1,q}(\Upsilon)$. Some remarks concerning this space are in order. As in the corresponding isotropic space (see for instance [1]), we may prove that, for a fixed $u \in BV^{(q)}(\Omega)$,

$$\int_\Omega |D_x u| = \sup\left\{ \int_\Omega u \,\mathrm{div}_x \phi \ :\ \phi \in C_0^1(\Omega;\mathbb{R}^N),\ \|\phi\|_\infty \le 1 \right\}.$$

Note that each $\phi \in C_0^1(\Omega;\mathbb{R}^N)$ defines a linear functional

$$u \mapsto \int_\Omega u \,\mathrm{div}_x \phi\,,$$

which is continuous in $L^1(\Omega)$. Hence, the functional defined by

$$u \mapsto \int_\Omega |D_x u| \tag{2.5}$$

is lower semicontinuous with respect to the convergence in $L^1(\Omega)$. In the same way, we may see that each $\varphi \in C_0^1(\Omega)$, with $\varphi \ge 0$, defines a functional

$$u \mapsto \int_\Omega \varphi |D_x u|\,,$$

which is lower semicontinuous in $L^1(\Omega)$. Furthermore, we may handle another lower semicontinuous functional which takes into account the value of u on the boundary. Indeed, extend $u \in BV^{(q)}(\Omega)$ to a larger domain $\Xi' \times \Upsilon$, with $u \equiv 0$ outside of Ω, and consider the total variation of the extended function. The divergence theorem gives

$$\int_{\Xi'\times\Upsilon} u \,\mathrm{div}_x \phi = -\int_\Omega u \,\mathrm{div}_x \phi + \int_{\partial\Xi\times\Upsilon} u\,\phi\cdot\nu_x \, d\mathcal{H}^{N+K-1}$$

for all $\phi \in C_0^1(\Xi'\times\Upsilon;\mathbb{R}^N)$. Hence, we deduce that the functional

$$u \mapsto \int_\Omega |D_x u| + \int_{\partial\Xi\times\Upsilon} |u| \, d\mathcal{H}^{N+K-1}\,, \tag{2.6}$$

is lower semicontinuous in $L^1(\Omega)$.

Obtaining a priori estimates for solutions of (1.3) depends on the following two inequalities. The first one is a Sobolev type inequality, whose proof can be found in [32] (see also [23]). The second one is a Poincaré type inequality, for which we give a proof below (see also [23]). Observe that, if $p < N$, then the first one is better when $\frac{Np}{N-p} > q$.

Theorem 2.1. *Let $(p,q)^* = \frac{N+K}{(N/p)+(K/q)-1}$. Then $W_0^{1,(p,q)}(\Omega) \hookrightarrow L^{(p,q)^*}(\Omega)$ with continuous embedding and there exists a positive constant $S_{(p,q)}$ (only depending on p, q, N and K) such that*

$$\|u\|_{(p,q)^*} \le S_{(p,q)} \left(\prod_{i=1}^N \left\| \frac{\partial u}{\partial x_i} \right\|_p^{\frac{1}{N+K}} \right) \left(\prod_{i=1}^K \left\| \frac{\partial u}{\partial y_i} \right\|_q^{\frac{1}{N+K}} \right) \le S_{(p,q)} \|\nabla_x u\|_p^{\frac{N}{N+K}} \|\nabla_y u\|_q^{\frac{K}{N+K}}\,, \tag{2.7}$$

for all $u \in W_0^{1,(p,q)}(\Omega)$.

Remark 2.2. Following the proof given in [32] an estimate of the constant $S_{(p,q)}$ can be made. Indeed, we may take $S_{(p,q)} = \frac{N+K-1}{(N/p)+(K/q)-1}$. We explicitly remark that with this choice $\lim_{p\to 1} S_{(p,q)} = S_{(1,q)}$.

Theorem 2.3. *Let D denote the diameter of Υ and let $u \in BV^{(q)}(\Omega)$ be fixed. Then the following inequality holds*

$$\int_\Omega |u(x,y)|^q \, dx\, dy \le D^q \int_\Omega |\nabla_y u(x,y)|^q \, dx\, dy\,. \tag{2.8}$$

PROOF: Fix $x \in \Xi$ and choose a direction in Υ, say that of y_1. Then there is a closed interval I of length D and open $\Upsilon_1 \subset \mathbb{R}^{K-1}$ such that $\Upsilon \subset I \times \Upsilon_1$. To be more precise, if $(y_1, y_2, \dots, y_K) \in \Upsilon$, then $y_1 \in I$ and $(y_2, \dots, y_K) \in \Upsilon_1$. Next, extend the function u to $\Xi \times I \times \Upsilon_1$: $u \equiv 0$ outside of $\Xi \times \Upsilon$. For almost every $(y_2, y_3, \dots, y_K) \in \Upsilon_1$, the function $t \mapsto u(x; t, y_2, \cdots, y_K)$ is absolutely continuous and vanishes in the extremes of I. Applying Poincare's inequality to this function we obtain

$$\begin{aligned}\int_I |u(x;t,y_2,y_3,\dots,y_K)|^q \, dt &\le D^q \int_I \Big| \frac{\partial u}{\partial y_1}(x;t,y_2,y_3,\dots,y_K)\Big|^q \, dt \\ &\le D^q \int_I \big|\nabla_y u(x;t,y_2,y_3,\dots,y_K)\big|^q \, dt\,.\end{aligned}$$

Integrating over $\Xi \times \Upsilon_1$, we end the proof. ■

2.2. **Anisotropic Anzellotti's theory.** In order to give sense to our notion of solution, we have to define certain pairings between vectors fields and derivatives of a BV–function, and to prove a Green's formula. Throughout this subsection, we take $z = (\zeta, \lambda)$ with $\zeta \in L^\infty(\Omega; \mathbb{R}^N)$ and $\lambda \in L^{q'}(\Omega; \mathbb{R}^K)$, satisfying $\operatorname{div} z \in L^r(\Omega)$. On the other hand, we assume that $u \in BV^{(q)}(\Omega)$.

We begin by defining three distributions on Ω. For every $\varphi \in C_0^\infty(\Omega)$, we write

$$\langle (z, Du), \varphi\rangle = -\int_\Omega u\,\varphi \operatorname{div} z - \int_\Omega u z\cdot\nabla\varphi \tag{2.9}$$

$$\langle (\lambda, \nabla_y u), \varphi\rangle = \int_\Omega \varphi\, \lambda\cdot\nabla_y u \tag{2.10}$$

$$(\zeta, D_x u) = (z, Du) - (\lambda, \nabla_y u)\,. \tag{2.11}$$

Since the third distribution is the sum of the other two and $(\lambda, \nabla_y u)$ is a function, if we prove that (z, Du) is a Radon measure, so is $(\zeta, D_x u)$.

Following [2], we have the following result.

Proposition 2.4. *(1) For every $\varphi \in C_0^\infty(\Omega)$, it holds*

$$|\langle (z, Du), \varphi\rangle| \le \|\varphi\|_\infty \Big[\|\zeta\|_\infty |D_x u|(\Omega) + \|\lambda\|_{q'} \|\nabla_y u\|_q\Big]\,.$$

(2) For every open set $U \subset \Omega$ and every $\varphi \in C_0^\infty(U)$, we have

$$|\langle (\zeta, D_x u), \varphi\rangle| \le \|\varphi\|_\infty \|\zeta\|_{L^\infty(U)} \int_U |D_x u|\,.$$

Therefore, (z, Du) and $(\zeta, D_x u)$ are Radon measures, and $|(\zeta, D_x u)| \le \|\zeta\|_\infty |D_x u|$.

In order to go on, we need the following anisotropic Meyer–Serrin theorem.

Proposition 2.5. *For each $u \in BV^{(q)}(\Omega)$ there exists a sequence (u_n) in $W^{1,1}(\Omega) \cap C^\infty(\Omega)$ such that*

$$\begin{aligned} &(1)\ u_n \to u \quad \text{in } L^{r'}(\Omega)\,,\\ &(2)\ \int_\Omega |\nabla_x u_n| \to |D_x u|(\Omega)\,,\\ &(3)\ \nabla_y u_n \to \nabla_y u \quad \text{in } L^q(\Omega). \end{aligned}$$

Moreover, since $\partial\Xi$ is Lipschitz–continuous, we can find u_n satisfying

$$u_n\big|_{\partial\Xi\times\Upsilon} = u\big|_{\partial\Xi\times\Upsilon}\,.$$

Proof: Fixed $\delta > 0$, we claim the existence of a function $u_\delta \in W^{1,1}(\Omega) \cap C^\infty(\Omega)$ such that

$$\int_\Omega |u - u_\delta|^{r'} < \delta^{r'} \quad \int_\Omega |\nabla_y u_n - \nabla_y u|^q < \delta^q\,, \quad \text{and} \quad \int_\Omega |\nabla_x u_\delta| \le |D_x u|(\Omega) + \delta\,.$$

In order to prove this claim, we denote by Ω_k a sequence of open sets defined as in the proof of the Meyers–Serrin theorem (see for instance [1], p. 122). Consider also a partition of unity subordinate to this covering: $\phi_k \in C_0^\infty(\Omega)$ such that $\operatorname{supp}\phi_k \subset \Omega_k$, $0 \le \phi \le 1$ and $\sum_{k=0}^\infty \phi_k(x) = 1$ for all $x \in \Omega$. Moreover, let $(\rho_n)_n$ be a sequence of positive symmetric mollifiers. Finally, let $(\delta_k)_k$ be a sequence of positive numbers satisfying $\sum_{k=1}^\infty \delta_k < \delta$. Now, for each $k \in \mathbb{N}$, we can find $\epsilon_k > 0$ such that

$$\operatorname{supp}\rho_{\epsilon_k} * (\phi_k u) \subset \Omega_k\,, \qquad \int_\Omega |\rho_{\epsilon_k} * (\phi_k u) - \phi_k u|^{r'} < \delta_k^{r'}\,,$$

$$\int_\Omega |\rho_{\epsilon_k} * (u\nabla_x\phi_k) - u\nabla_x\phi_k| < \delta_k\,, \quad \text{and} \quad \int_\Omega |\rho_{\epsilon_k} * \nabla_y(u\phi_k) - \nabla_y(u\phi_k)|^q < \delta_k^q\,.$$

Letting $u_\delta = \sum_{k=0}^\infty \rho_{\epsilon_k} * (u\,\phi_k)$, we next follow the steps of the proof of [1], p. 123 to conclude the result. ■

Proposition 2.6. *Let $(u_n)_n$ be a sequence in $W^{1,1}(\Omega) \cap C^\infty(\Omega)$ which converges to u as in the above Proposition 2.5. Then*

$$\int_\Omega (\zeta, \nabla_x u_n) \to \int_\Omega (\zeta, D_x u) \qquad \textit{and} \qquad \int_\Omega \lambda\cdot\nabla_y u_n \to \int_\Omega \lambda\cdot\nabla_y u\,.$$

Proof: For any $\varphi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} &|\langle(\zeta, \nabla_x u_n), \varphi\rangle - \langle(\zeta, D_x u), \varphi\rangle|\\ &\qquad \le |\langle(z, \nabla u_n), \varphi\rangle - \langle(z, Du), \varphi\rangle| + \Big|\int_\Omega \varphi\,\lambda\cdot\nabla_y u_n - \int_\Omega \varphi\,\lambda\cdot\nabla_y u\Big|. \end{aligned}$$

Therefore the first assertion follows from the analogous result proved by Anzellotti (see [2]), and from the strong convergence $\nabla_y u_n \to \nabla_y u$ in $L^q(\Omega)$.

The second assertion is also a straightforward consequence of the strong convergence $\nabla_y u_n \to \nabla_y u$ in $L^q(\Omega)$. ■

Now we prove a Green's formula for function belonging to $BV^{(q)}(\Omega)$.

Theorem 2.7. *Let Ξ be an open subset of $\mathbb{R}^N$ with Lipschitz boundary and let Υ be an open subset of $\mathbb{R}^K$. Denote $\Omega = \Xi \times \Upsilon \subset \mathbb{R}^{N+K}$*

If $z=(\zeta,\lambda)$ satisfies $\zeta\in L^\infty(\Omega;\mathbb{R}^N)$, $\lambda\in L^{q'}(\Omega;\mathbb{R}^K)$, and $\operatorname{div} z\in L^r(\Omega)$, then for every $u\in BV^{(q)}(\Omega)$ the following formula holds

$$\int_\Omega u\operatorname{div} z+\int_\Omega(\zeta,D_xu)+\int_\Omega\lambda\cdot\nabla_y u=\int_{\partial\Xi\times\Upsilon}u[\zeta,\nu_x]\,d\mathcal{H}^{N+K-1}\,.$$

Proof: Consider a sequence $(u_n)_n$ in $W^{1,1}(\Omega)\cap C^\infty(\Omega)$ which converges to u as in Proposition 2.5. Applying Green's formula to each u_n, we obtain

$$\int_\Omega u_n\operatorname{div}_x\zeta+\int_\Omega\zeta\cdot\nabla_x u_n=\int_{\partial\Xi\times\Upsilon}u_n[\zeta,\nu_x]\,d\mathcal{H}^{N+K-1}\,,$$

$$\int_\Omega u_n\operatorname{div}_y\lambda+\int_\Omega\lambda\cdot\nabla_y u_n=0\,.$$

Hence,

$$\int_\Omega u_n\operatorname{div} z+\int_\Omega z\cdot\nabla u_n=\int_{\partial\Xi\times\Upsilon}u_n[\zeta,\nu_x]\,d\mathcal{H}^{N+K-1}\,. \tag{2.12}$$

We next study the convergence of each term in the above equality. Since $u_n\to u$ in $L^{r'}(\Omega)$, we get

$$\lim_{n\to\infty}\int_\Omega u_n\operatorname{div} z=\int_\Omega u\operatorname{div} z\,. \tag{2.13}$$

By applying Proposition 2.4, we deduce that

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega z\cdot\nabla u_n&=\lim_{n\to\infty}\int_\Omega\zeta\cdot\nabla_x u_n+\lim_{n\to\infty}\int_\Omega\lambda\cdot\nabla_y u_n\\&=\int_\Omega(\zeta,D_xu)+\int_\Omega\lambda\cdot\nabla_y u\,.\end{aligned} \tag{2.14}$$

Finally, since $u_n\big|_{\partial\Xi\times\Upsilon}=u\big|_{\partial\Xi\times\Upsilon}$,

$$\lim_{n\to\infty}\int_{\partial\Xi\times\Upsilon}u_n[\zeta,\nu_x]\,d\mathcal{H}^{N+K-1}=\int_{\partial\Xi\times\Upsilon}u[\zeta,\nu_x]\,d\mathcal{H}^{N+K-1}\,. \tag{2.15}$$

Therefore, on account of (2.13), (2.14) and (2.15), Theorem 2.7 follows by letting $n\to\infty$ in (2.12). ■

3. Behaviour of u_p as p goes to 1

Let $u_p\in W_0^{1,(p,q)}(\Omega)$ be the unique solution to the anisotropic elliptic equation

$$\begin{cases}-\operatorname{div}_x\big(|\nabla_x u_p|^{p-2}\nabla_x u_p\big)-\operatorname{div}_y\big(|\nabla_y u_p|^{q-2}\nabla_y u_p\big)=f, & \text{in }\ \Omega\\ u_p=0, & \text{on }\ \partial\Omega\,,\end{cases} \tag{3.16}$$

where $1<p,q<\infty$ and f belongs to $L^r(\Omega)$ with r as in (1.2). This means that the following equality holds

$$\int_\Omega|\nabla_x u_p|^{p-2}\nabla_x u_p\cdot\nabla_x\varphi+\int_\Omega|\nabla_y u_p|^{q-2}\nabla_y u_p\cdot\nabla_y\varphi=\int_\Omega f\varphi\,, \tag{3.17}$$

for any $\varphi\in W_0^{1,(p,q)}(\Omega)$. Existence of a unique solution u_p to (3.16) can be easily obtained minimizing the functional

$$G[u]=\frac1p\int_\Omega|\nabla_x u|^p+\frac1q\int_\Omega|\nabla_y u|^q-\int_\Omega fu$$

in the space $W_0^{1,(p,q)}(\Omega)$. Indeed, first we note that

$$r \geq r_p := \min\left\{\frac{N+K}{(N/p')+(K/q')+1}, q'\right\}$$

so, by (1.2), $f \in L^{r_p}(\Omega)$. Observe also that every minimizing sequence is bounded in $W_0^{1,(p,q)}(\Omega)$ and as a consequence we obtain a minimizing sequence which is weakly convergent to some u in $W_0^{1,(p,q)}(\Omega)$. Applying Theorems 2.1 and 2.3, one deduces that that sequence weakly converges to u in $L^{r_p'}(\Omega)$. Finally, the lower–semicontinuity of the gradient terms $\frac{1}{p}\int_\Omega |\nabla_x u|^p + \frac{1}{q}\int_\Omega |\nabla_y u|^q$ and the continuity of $\int_\Omega fu$ imply that u is a minimizer of the functional G.

In what follows, with abuse of notation, we will say that u_p is a sequence and we will consider subsequences of it, as p goes to 1.

Theorem 3.1. *Let u_p be a solution to* (3.16) *for any $1 < p < \infty$. Then there exist $u \in BV(\Omega) \cap L^{r'}(\Omega)$ and a subsequence of u_p, not relabelled, such that as p goes to* 1*,*

$$\nabla_x u_p \rightharpoonup D_x u \quad \text{*–weakly in the sense of measures;} \tag{3.18}$$

$$\nabla_y u_p \rightharpoonup \nabla_y u \quad \text{weakly in } L^q(\Omega;\mathbb{R}^K)\,; \tag{3.19}$$

$$u_p \to u \quad \text{a.e. in } \Omega\,; \tag{3.20}$$

$$u_p \to u \quad \text{in } L^m(\Omega) \quad \text{for } 1 \leq m < r'\,. \tag{3.21}$$

Proof: Taking u_p as test function in (3.17), we obtain

$$\int_\Omega |\nabla_x u_p|^p + \int_\Omega |\nabla_y u_p|^q = \int_\Omega f u_p \leq \|f\|_r \|u_p\|_{r'}\,. \tag{3.22}$$

Our aim is to obtain an inequality as

$$\int_\Omega |\nabla_x u_p|^p + \int_\Omega |\nabla_y u_p|^q \leq M\,, \tag{3.23}$$

being M a positive constant that does not depend on p. To get this estimate, first recall that

$$r' = \max\left\{\frac{N+K}{N-1+(K/q)}, q\right\}.$$

So that two possibilities have to be taken into account.

We begin by considering the case when $r' = \frac{N+K}{N-1+(K/q)}$, that is, $q \leq \frac{N}{N-1}$. Applying Theorem 2.1, we obtain

$$\int_\Omega |\nabla_x u_p|^p + \int_\Omega |\nabla_y u_p|^q \leq S_{(p,q)}\|f\|_r \|\nabla_x u\|_p^{\frac{N}{N+K}} \|\nabla_y u\|_q^{\frac{K}{N+K}}\,.$$

Then Young's inequality implies

$$\int_\Omega |\nabla_x u_p|^p + \int_\Omega |\nabla_y u_p|^q \leq \frac{(N/p')+(K/q')}{N+K}(S_{(p,q)}\|f\|_r)^{\frac{N+K}{(N/p')+(K/q')}} + \frac{N}{p(N+K)}\|\nabla_x u\|_p^p + \frac{K}{q(N+K)}\|\nabla_y u\|_q^q\,,$$

from where it follows that

$$\left(\frac{N}{p'}+K\right)\int_\Omega |\nabla_x u_p|^p + \left(N+\frac{K}{q'}\right)\int_\Omega |\nabla_y u_p|^q \leq \left(\frac{N}{p'}+\frac{K}{q'}\right)(S_{(p,q)}\|f\|_r)^{\frac{N+K}{(N/p')+(K/q')}}\,.$$

Thus, since $K > \frac{K}{q'}$ and $N > \frac{N}{p'}$, this yields

$$\int_\Omega |\nabla_x u_p|^p + \int_\Omega |\nabla_y u_p|^q \le (S_{(p,q)}\|f\|_r)^{\frac{N+K}{(N/p')+(K/q')}} .$$

Hence, we have obtained an inequality as (3.23) since $\frac{N+K}{(N/p')+(K/q')} \le \frac{N+K}{K}q'$ and $\lim_{p\to 1} S_{(p,q)} = S_{(1,q)}$.

We now turn to analyze the case when $r' = q$, that is, $q > \frac{N}{N-1}$. If we take $\epsilon = 1/(2D^q)$, then Young's inequality and Theorem 2.3 imply

$$\|f\|_r\|u_p\|_q \le \epsilon \int_\Omega |u_p|^q + C(\epsilon)\|f\|_r^{q'} \le \frac{1}{2}\int_\Omega |\nabla_y u_p|^q + C(\epsilon)\|f\|_r^{q'} .$$

Thus (3.22) becomes

$$\int_\Omega |\nabla_x u_p|^p + \frac{1}{2}\int_\Omega |\nabla_y u_p|^q \le C(\|f\|_r, D, q) \tag{3.24}$$

and so inequality (3.23) is also obtained in the second case.

Applying Young's inequality, it follows from (3.23) that

$$\begin{aligned} \int_\Omega |\nabla u_p| &\le \int_\Omega |\nabla_x u_p| + \int_\Omega |\nabla_y u_p| \\ &\le \frac{1}{p}\int_\Omega |\nabla_x u_p|^p + \frac{p-1}{p}|\Omega| + \frac{1}{q}\int_\Omega |\nabla_y u_p|^q + \frac{q-1}{q}|\Omega| \le M + 2|\Omega| , \end{aligned} \tag{3.25}$$

for p small enough. Hence, u_p is bounded in $BV(\Omega)$ and we may find $u \in BV(\Omega)$ satisfying

$$\nabla_x u_p \rightharpoonup D_x u \quad \text{*–weakly in the sense of measures;} \tag{3.26}$$

$$\nabla_y u_p \rightharpoonup D_y u \quad \text{*–weakly in the sense of measures;;} \tag{3.27}$$

$$u_p \to u \quad \text{in } L^1(\Omega) \text{ and a.e. in } \Omega . \tag{3.28}$$

Since, by (3.23), the sequence $\nabla_y u_p$ is bounded in $L^q(\Omega)$, it follows that, actually, $\nabla_y u_p \rightharpoonup \nabla_y u$ weakly in $L^q(\Omega)$. Moreover, Theorem 2.1 implies that u_p is bounded in $L^{\frac{N+K}{N+(K/q)-1}}(\Omega)$ and Theorem 2.3 implies that u_p is bounded in $L^q(\Omega)$, so that it is bounded in $L^{r'}(\Omega)$. Therefore, $u \in L^{r'}(\Omega)$ and it follows from (3.28) that $u_p \to u$ in $L^m(\Omega)$, where $1 \le m < r'$. ∎

Remark 3.2. We explicitely remark that inequality (2.7) is usually written replacing the right-hand side with the norm of the anisotropic Sobolev space (that do not contain a product), i.e.,

$$\|u\|_{(p,q)^*} \le S_{(p,q)}\Big(\|\nabla_x u\|_p + \|\nabla_y u\|_q\Big) \qquad u \in W^{1,(p,q)}(\Omega) .$$

Nevertheless, this inequality is not suitable for our purposes, since from here we can not obtain the a priori estimates (3.23). Indeed, taking u_p as test function in (3.17) and using the previous Sobolev inequality, we get

$$\int_\Omega |\nabla_x u_p|^p + \int_\Omega |\nabla_y u_p|^q \le S_{(p,q)}^{p'}\|f\|_r^{p'} + \frac{p'}{q'}S_{(p,q)}^{q'}\|f\|_r^{q'} ,$$

from where we are not able to deduce the a priori estimate (3.23) since $p' \to +\infty$, when $p \to 1$.

Theorem 3.3. *Under the same assumptions of Theorem 3.1, there exist $z = (\zeta, \lambda)$ with $\zeta \in L^\infty(\Omega;\mathbb{R}^N)$ and $\lambda \in L^{q'}(\Omega;\mathbb{R}^K)$ and a subsequence of u_p, not relabelled, satisfying*

$$\|\zeta\|_\infty \le 1\,, \quad \lambda = |\nabla_y u|^{q-2}\nabla_y u$$

and

$$|\nabla_x u_p|^{p-2}\nabla_x u_p \rightharpoonup \zeta \quad \text{weakly in } L^s(\Omega;\mathbb{R}^N) \quad \forall s<\infty\,,$$
$$\nabla_y u_p \to \nabla_y u \quad \text{in } L^q(\Omega;\mathbb{R}^K)\,.$$

as p goes to 1.

Proof: It follows from the fundamental inequality (3.23) that there exists $z = (\zeta, \lambda)$ where $\zeta \in L^s(\Omega;\mathbb{R}^N)$, for all $s<\infty$, and $\lambda \in L^{q'}(\Omega;\mathbb{R}^K)$ such that, up to subsequences, $|\nabla_x u_p|^{p-2}\nabla_x u_p \rightharpoonup \zeta$ weakly in $L^s(\Omega)$, and $|\nabla_y u_p|^{q-2}\nabla_y u_p \rightharpoonup \lambda$ weakly in $L^{q'}(\Omega)$.

To prove that $\zeta \in L^\infty(\Omega;\mathbb{R}^N)$ and $\|\zeta\|_\infty \le 1$ we may follow the same arguments of [3], which we sketch for the sake of completeness. Fixed $k>0$, define

$$B_{p,k} = \{x \in \Omega : |\nabla u_p(x)| > k\}\,.$$

As a consequence of (3.23) we have that

$$|B_{p,k}| \le \frac{C}{k^p} \qquad \text{for every } p>1, k>0. \tag{3.29}$$

The same inequality (3.23) implies that $(|\nabla_x u_p|^{p-2}\nabla_x u_p \chi_{B_{p,k}})$ is bounded in any $L^s(\Omega;\mathbb{R}^N)$ with $s<\infty$. Thus, there is some $g_k \in L^1(\Omega,\mathbb{R}^N)$ such that

$$|\nabla_x u_p|^{p-2}\nabla_x u_p \chi_{B_{p,k}} \rightharpoonup g_k$$

weakly in $L^1(\Omega,\mathbb{R}^N)$ as $p\to 1$. Now for any $\phi \in L^\infty(\Omega,\mathbb{R}^N)$ with $\|\phi\|_\infty \le 1$, we easily prove that

$$\left|\int_\Omega |\nabla_x u_p|^{p-2}\nabla_x u_p \cdot \phi \chi_{B_{p,k}}\right| \le \frac{C}{k}.$$

Letting p goes to 1, we get that

$$\left|\int_\Omega g_k \cdot \phi\right| \le \frac{C}{k}$$

holds for all $\phi \in L^\infty(\Omega,\mathbb{R}^N)$ with $\|\phi\|_\infty \le 1$. By duality, we obtain

$$\int_\Omega |g_k| \le \frac{C}{k}\,. \tag{3.30}$$

On the other hand, we also have that

$$\left||\nabla_x u_p|^{p-2}\nabla_x u_p \chi_{\Omega\setminus B_{p,k}}\right| \le k^{p-1} \quad \text{for any } p>1,$$

Taking the limit as p tends to 1, we obtain that $|\nabla u_p|^{p-2}\nabla u_p \chi_{\Omega\setminus B_{p,k}}$ weakly converges in $L^1(\Omega,\mathbb{R}^N)$ to some function $f_k \in L^1(\Omega,\mathbb{R}^N)$ such that $\|f_k\|_\infty \le 1$. Hence, we may write $\zeta = f_k + g_k$ with $\|f_k\|_\infty \le 1$ and g_k satisfying (3.30), for all $k>0$. It follows that $\zeta = \lim_{k\to\infty} f_k$ in $L^1(\Omega;\mathbb{R}^N)$ and so $\|\zeta\|_\infty \le 1$.

To prove the strong convergence of the gradients $\nabla_y u_p$ to $\nabla_y u$, we will compute

$$\lim_{p\to 1}\int_\Omega \left(|\nabla_y u_p|^{q-2}\nabla_y u_p - |\nabla_y u|^{q-2}\nabla_y u\right)\cdot \nabla_y(u_p - u) = 0\,. \tag{3.31}$$

Observe that, by (3.19), we already have

$$\lim_{p\to\infty}\int_\Omega |\nabla_y u|^{q-2}\nabla_y u\cdot\nabla_y(u_p-u)=0\,. \tag{3.32}$$

To handle the remaining terms, we consider $\epsilon>0$ and $v\in C^\infty(\Omega)$ such that

$$\int_\Omega |f|\,|u-v|+\Big|\int_\Omega|\nabla_x v|-|D_x u|(\Omega)\Big|<\epsilon\,. \tag{3.33}$$

Taking u_p-v as test function in (3.17), it yields

$$\int_\Omega|\nabla_x u_p|^p-\int_\Omega|\nabla_x u_p|^{p-2}\nabla_x u_p\cdot\nabla_x v+\int_\Omega|\nabla_y u_p|^{q-2}\nabla_y u_p\cdot\nabla_y(u_p-v)$$
$$=\int_\Omega f(u_p-v)\,.$$

If we apply Young's inequality and let p goes to 1 applying the lower semicontinuity of (2.5), we obtain

$$|D_x u|(\Omega)-\int_\Omega\zeta\cdot\nabla_x v+\limsup_{p\to1}\int_\Omega|\nabla_y u_p|^{q-2}\nabla_y u_p\cdot\nabla_y(u_p-v)\le\int_\Omega f(u-v)\,.$$

Since

$$\Big|\int_\Omega\zeta\cdot\nabla_x v\Big|\le\|\zeta\|_\infty\int_\Omega|\nabla_x v|\le\int_\Omega|\nabla_x v|,$$

it follows that

$$\limsup_{p\to1}\int_\Omega|\nabla_y u_p|^{q-2}\nabla_y u_p\cdot\nabla_y(u_p-v)\le\int_\Omega|f||u-v|+\int_\Omega|\nabla_x v|-|D_x u|(\Omega)<\epsilon\,,$$

by (3.33). Now the arbitrariness of $\epsilon>0$ implies

$$\limsup_{p\to1}\int_\Omega|\nabla_y u_p|^{q-2}\nabla_y u_p\cdot\nabla_y(u_p-v)\le0\,.$$

From it and (3.32) we deduce that

$$\limsup_{p\to1}\int_\Omega\big(|\nabla_y u_p|^{q-2}\nabla_y u_p-|\nabla_y u|^{q-2}\nabla_y u\big)\cdot\nabla_y(u_p-u)\le0\,.$$

Since the integrand is non–negative, we get (3.31). Once (3.31) has been proved, we apply the same argument of [12] (see also [10]) and passing to a subsequence, if necessary, we deduce that $\nabla_y u_p$ converge pointwise to $\nabla_y u$ in Ω.

Therefore, $\lambda=|\nabla_y u|^{q-2}\nabla_y u$, that is,

$$|\nabla_y u_p|^{q-2}\nabla_y u_p\rightharpoonup|\nabla_y u|^{q-2}\nabla_y u\quad\text{weakly in }L^{q'}(\Omega;\mathbb{R}^K)\,. \tag{3.34}$$

As a consequence of (3.19), (3.34) and (3.31), we obtain

$$\lim_{p\to1}\int_\Omega|\nabla_y u_p|^q=\lim_{p\to1}\int_\Omega|\nabla_y u|^{q-2}\nabla_y u\cdot\nabla_y(u_p-u)+\lim_{p\to1}\int_\Omega|\nabla_y u_p|^{q-2}\nabla_y u_p\cdot\nabla_y u$$
$$=\int_\Omega|\nabla_y u|^q\,.$$

From this convergence and (3.19), we deduce $\nabla_y u_p\to\nabla_y u$ in $L^q(\Omega;\mathbb{R}^K)$. ■

4. Main results

In this Section, we begin by introducing the definition of a solution to problem (1.1) and then we prove existence, uniqueness and regularity results for such a solution

Definition 4.1. *We say that $u \in BV^{(q)}(\Omega)$ is a solution to* (1.1) *if the following conditions hold:*

There exists $z = (\zeta, \lambda)$ with $\zeta \in L^\infty(\Omega; \mathbb{R}^N)$ and $\lambda \in L^{q'}(\Omega; \mathbb{R}^K)$ satisfying

$$\|\zeta\|_\infty \le 1 \text{ and } \lambda = |\nabla_y u|^{q-2}\nabla_y u\,; \tag{4.35}$$

$$-\operatorname{div} z = f \quad \text{in } \mathcal{D}'(\Omega)\,; \tag{4.36}$$

$$[\zeta, \nu_x] \in \operatorname{sign}(-u) \quad \mathcal{H}^{N+K-1}\text{–a.e. on } \partial\Xi \times \Upsilon\,; \tag{4.37}$$

$$(\zeta, D_x u) \quad \text{is a Radon measure and} \quad (\zeta, D_x u) = |D_x u|\,. \tag{4.38}$$

By applying Green's formula given by Theorem 2.7, one can easily deduce the following variational formulation of problem (1.1): the identity

$$\begin{aligned}\int_\Omega |D_x u| - \int_\Omega (\zeta, D_x v) + \int_\Omega |\nabla_y u|^{q-2}\nabla_y u \cdot \nabla_y (u - v) \\ = \int_\Omega f(u-v) - \int_{\partial\Xi\times\Upsilon} |u|\, d\mathcal{H}^{N+K-1} - \int_{\partial\Xi\times\Upsilon} v[\zeta, \nu_x]\, d\mathcal{H}^{N+K-1}\end{aligned} \tag{4.39}$$

holds for every $v \in BV^{(q)}(\Omega)$.

4.1. **Existence and uniqueness.** We have the following existence result.

Theorem 4.2. *There exists, at least, a solution to problem* (1.1).

Proof: First apply Theorem 3.1 to get $u \in BV(\Omega) \cap L^{r'}(\Omega)$ that, by Theorem 3.3, satisfies

$$\lim_{p\to 1} \int_\Xi \int_\Upsilon |\nabla_y u_p(x,y) - \nabla_y u(x,y)|^q dy\, dx = 0\,.$$

Then there exists a sequence p_n satisfying $p_n > 1$, $\lim_{n\to\infty} p_n = 1$ and

$$\lim_{n\to\infty} \int_\Upsilon |\nabla_y u_{p_n}(x,y) - \nabla_y u(x,y)|^q dy = 0\,,$$

for almost all $x \in \Xi$. We may assume, without loss of generality, that for those $x \in \Xi$ each function $y \mapsto u_{p_n}(x,y)$ belongs to $W_0^{1,q}(\Upsilon)$. Hence, for almost all $x \in \Xi$, the function $y \mapsto u(x,y)$ belongs to $W_0^{1,q}(\Upsilon)$ and so $u \in BV^{(q)}(\Omega)$. Moreover, from Theorem 3.3, we obtain $z = (\zeta, \lambda)$ satisfying (4.35) in Definition 4.1.

We next proceed to prove the other three conditions of Definition 4.1.

Proof of (4.36)*:* Taking $\varphi \in C_0^\infty(\Omega)$ as test function in (3.17) we get

$$\int_\Omega |\nabla_x u_p|^{p-2}\nabla_x u_p \cdot \nabla_x \varphi + \int_\Omega |\nabla_y u_p|^{q-2}\nabla_y u_p \cdot \nabla_y \varphi = \int_\Omega f\varphi\,.$$

Letting p goes to 1, by Theorem 3.3, we obtain

$$\int_\Omega \zeta \cdot \nabla_x \varphi + \int_\Omega \lambda \cdot \nabla_y \varphi = \int_\Omega f\varphi\,.$$

Proof of (4.38)*:* We now choose $T_k(u_p)\,\varphi$, with $\varphi \in C_0^\infty(\Omega)$ and $\varphi \geq 0$, as test function in (3.17). Then

$$\int_\Omega \varphi|\nabla_x T_k(u_p)|^p + \int_\Omega T_k(u_p)|\nabla_x u_p|^{p-2}\nabla_x u_p \cdot \nabla_x \varphi + \int_\Omega \varphi|\nabla_y T_k(u_p)|^q + \int_\Omega T_k(u_p)|\nabla_y u_p|^{q-2}\nabla_y u_p \cdot \nabla_y \varphi = \int_\Omega f T_k(u_p)\varphi\,. \tag{4.40}$$

Applying Young's inequality, we get

$$\int_\Omega \varphi|\nabla_x T_k(u_p)| \leq \frac{1}{p}\int_\Omega \varphi|\nabla_x T_k(u_p)|^p + \frac{p-1}{p}\int_\Omega \varphi\,,$$

so that it follows from (4.40) that

$$\int_\Omega \varphi|\nabla_x T_k(u_p)| + \frac{1}{p}\int_\Omega T_k(u_p)|\nabla_x u_p|^{p-2}\nabla_x u_p \cdot \nabla_x \varphi + \frac{1}{p}\int_\Omega \varphi|\nabla_y T_k(u_p)|^q + \frac{1}{p}\int_\Omega T_k(u_p)|\nabla_y u_p|^{q-2}\nabla_y u_p \cdot \nabla_y \varphi \leq \frac{1}{p}\int_\Omega f T_k(u_p)\varphi + \frac{p-1}{p}\int_\Omega \varphi\,.$$

In order to pass to the limit in the first term on the left hand–side, we may apply the lower semicontinuity of the functional $u \mapsto \int_\Omega \varphi|D_x u|$, obtaining

$$\int_\Omega \varphi|D_x T_k(u)| + \int_\Omega T_k(u)\zeta \cdot \nabla_x \varphi + \int_\Omega \varphi|\nabla_y T_k(u)|^q + \int_\Omega T_k(u)\lambda \cdot \nabla_y \varphi \leq \int_\Omega f T_k(u)\varphi\,.$$

Letting now $k \to \infty$, we get

$$\int_\Omega \varphi|D_x u| + \int_\Omega u\,\zeta \cdot \nabla_x \varphi + \int_\Omega \varphi|\nabla_y u|^q + \int_\Omega u\lambda \cdot \nabla_y \varphi \leq \int_\Omega f u\varphi = -\int_\Omega (\operatorname{div} z)u\varphi\,.$$

It follows from Green's formula that

$$\int_\Omega \varphi|D_x u| + \int_\Omega \varphi|\nabla_y u|^q \leq \langle(\zeta, D_x u), \varphi\rangle + \int_\Omega \varphi\,\lambda \cdot \nabla_y u\,.$$

By the definition of λ, we get $\int_\Omega \varphi|D_x u| \leq \langle(\zeta, D_x u), \varphi\rangle$ for all $\varphi \in C_0^\infty(\Omega)$ satisfying $\varphi \geq 0$. Hence,

$$|D_x u| \leq (\zeta, D_x u) \qquad \text{as measures.}$$

The equality follows since

$$(\zeta, D_x u) \leq \|\zeta\|_\infty |D_x u| \leq |D_x u|\,.$$

Proof of (4.37)*:* Considering u_p as test function in (3.17), we get

$$\int_\Omega |\nabla_x u_p|^p + \int_\Omega |\nabla_y u_p|^q = \int_\Omega f u_p\,.$$

By applying Young's inequality and the lower semicontinuity of the functional (2.6) we obtain

$$\int_\Omega |D_x u| + \int_{\partial\Xi\times\Upsilon} |u|\, d\mathcal{H}^{N+K-1} + \int_\Omega |\nabla_y u|^q \le \int_\Omega f u\,. \tag{4.41}$$

Now, $f = -\mathrm{div}\, z$ in $\mathcal{D}'$ and Green's formula imply

$$\int_\Omega f u = -\int_\Omega \mathrm{div}\,(z) u = \int_\Omega (\zeta, D_x u) + \int_\Omega \lambda\cdot\nabla_y u - \int_{\partial\Xi\times\Upsilon} u[\zeta,\nu_x]\, d\mathcal{H}^{N+K-1}\,.$$

Therefore, it follows from (4.41) that

$$\begin{aligned}\int_\Omega |D_x u| + \int_{\partial\Xi\times\Upsilon} |u|\, d\mathcal{H}^{N+K-1} + \int_\Omega |\nabla_y u|^q \\ \le \int_\Omega (\zeta, D_x u) + \int_\Omega \lambda\cdot\nabla_y u - \int_{\partial\Xi\times\Upsilon} u[\zeta,\nu_x]\, d\mathcal{H}^{N+K-1}\,.\end{aligned}$$

Since $|D_x u| = (\zeta, D_x u)$ and $\int_\Omega |\nabla_y u|^q = \int_\Omega \lambda\cdot\nabla_y u$, it yields

$$\int_{\partial\Xi\times\Upsilon} (|u| + u[\zeta,\nu_x])\, d\mathcal{H}^{N+K-1} \le 0\,.$$

Thus we obtain $|u|+u[\zeta,\nu_x] = 0$ $\mathcal{H}^{N+K-1}$–a.e. in $\partial\Xi\times\Upsilon$ and so $[\zeta,\nu_x]\in\mathrm{sign}\,(-u)$, $\mathcal{H}^{N+K-1}$–a.e. in $\partial\Xi\times\Upsilon$. ■

Now we prove the uniqueness result.

Theorem 4.3. *There exists, at most, a solution to problem* (1.1).

Proof: Suppose that u_1 and u_2 are two solutions to problem (1.1). Thus, there exist $z_1 = (\zeta_1,\lambda_1)$ and $z_2 = (\zeta_2,\lambda_2)$ satisfying (4.35)–(4.38). Taking u_2 as test function in the variational formulation (4.39) corresponding to u_1, it yields

$$\begin{aligned}\int_\Omega |D_x u_1| - \int_\Omega (\zeta_1, D_x u_2) + \int_\Omega |\nabla_y u_1|^{q-2}\nabla_y u_1\cdot\nabla_y(u_1-u_2) \\ = \int_\Omega f(u_1-u_2) - \int_{\partial\Omega} |u_1|\, d\mathcal{H}^{N+K-1} - \int_{\partial\Omega} [\zeta_1,\nu_x]u_2\, d\mathcal{H}^{N+K-1}\,.\end{aligned}$$

Analogously, we obtain

$$\begin{aligned}\int_\Omega |D_x u_2| - \int_\Omega (\zeta_2, D_x u_1) + \int_\Omega |\nabla_y u_2|^{q-2}\nabla_y u_2\cdot\nabla_y(u_2-u_1) \\ = \int_\Omega f(u_2-u_1) - \int_{\partial\Omega} |u_2|\, d\mathcal{H}^{N+K-1} - \int_{\partial\Omega} [\zeta_2,\nu_x]u_1\, d\mathcal{H}^{N+K-1}\,.\end{aligned}$$

Adding both equalities, we deduce

$$\begin{aligned}&\int_\Omega |D_x u_1| - \int_\Omega (\zeta_2, D_x u_1) + \int_\Omega |D_x u_2| - \int_\Omega (\zeta_1, D_x u_2) \\ &\quad + \int_\Omega \big(|\nabla_y u_1|^{q-2}\nabla_y u_1 - |\nabla_y u_2|^{q-2}\nabla_y u_2\big)\cdot\nabla_y(u_1-u_2) \\ &\quad = -\int_{\partial\Omega} |u_2|\, d\mathcal{H}^{N+K-1} - \int_{\partial\Omega} [\zeta_1,\nu_x]u_2\, d\mathcal{H}^{N+K-1} \\ &\qquad - \int_{\partial\Omega} |u_1|\, d\mathcal{H}^{N+K-1} - \int_{\partial\Omega} [\zeta_2,\nu_x]u_1\, d\mathcal{H}^{N+K-1}\,.\end{aligned} \tag{4.42}$$

It follows from $\|\zeta_1\|_\infty \le 1$ and $\|\zeta_2\|_\infty \le 1$ that

$$\int_\Omega |D_x u_1| - \int_\Omega (\zeta_2, D_x u_1) \ge 0\,, \quad \int_\Omega |D_x u_2| - \int_\Omega (\zeta_1, D_x u_2) \ge 0\,,$$

$$\int_{\partial\Omega} |u_2|\, d\mathcal{H}^{N+K-1} + \int_{\partial\Omega} [\zeta_1, \nu_x] u_2\, d\mathcal{H}^{N+K-1} \ge 0\,,$$

$$\int_{\partial\Omega} |u_1|\, d\mathcal{H}^{N+K-1} + \int_{\partial\Omega} [\zeta_2, \nu_x] u_1\, d\mathcal{H}^{N+K-1} \ge 0\,,$$

and so (4.42) becomes

$$\int_\Omega \big(|\nabla_y u_1|^{q-2}\nabla_y u_1 - |\nabla_y u_2|^{q-2}\nabla_y u_2\big)\cdot \nabla_y (u_1 - u_2) \le 0\,.$$

Hence, since the integrand is nonnegative,

$$\big(|\nabla_y u_1|^{q-2}\nabla_y u_1 - |\nabla_y u_2|^{q-2}\nabla_y u_2\big)\cdot \nabla_y (u_1 - u_2) = 0 \quad \text{a.e. in } \Omega\,,$$

and as a consequence $\nabla_y u_1 = \nabla_y u_2$ a.e. in Ω. Finally, applying Theorem 2.3, we conclude $u_1 = u_2$ a.e. in Ω, as desired. ■

Remark 4.4. If there is no direction where the operator is a q–Laplacian with $q > 1$, it can not be expected a uniqueness result. Assume, to simplify, that u is a regular solution to the problem

$$\begin{cases} -\text{div}\Big(\dfrac{Du}{|Du|}\Big) = f\,, & \text{in } \Omega\,;\\ u = 0, & \text{on } \partial\Omega\,; \end{cases} \tag{4.43}$$

and $h \in C^1(\mathbb{R},\mathbb{R})$ is strictly increasing and satisfies $h(0) = 0$, then $v = h(u)$ is also a solution to (4.43). Hence uniqueness in general does not hold (see also [4], p. 61).

4.2. **Regularity.** The following regularity result holds.

Theorem 4.5. *Let $f \in L^m(\Omega)$, with $m > r$. We also assume that $m < N + \frac{K}{q}$ when $q < \frac{N}{N-1}$. Then $u \in L^s(\Omega)$, where*

$$s = \max\Big\{\frac{m'(N+K)(q-1)}{\big((N-1)q+K\big)m' - (N+K)}, m(q-1)\Big\}\,.$$

Proof: We are going to prove that the sequence u_p is bounded in $L^s(\Omega)$. To this end, we will follow the arguments in [8]. Let $k \ge 1$ and consider, as test function in (3.17),

$$v = \begin{cases} 1\,, & \text{if } |u_p| \ge k\,;\\ (|u_p| - k)\,\text{sign}\, u_p\,, & \text{if } k-1 \le |u_p| < k\,;\\ 0\,, & \text{if } |u_p| < k-1\,. \end{cases}$$

We get

$$\int_{\{k-1\le|u_p|<k\}} |\nabla_x u_p|^p + \int_{\{k-1\le|u_p|<k\}} |\nabla_y u_p|^q \le \int_{\{|u_p|\ge k-1\}} |f|\,. \tag{4.44}$$

Assume first that $q \ge \frac{N}{N-1}$, so that $q' = r$ and our datum belongs to $L^m(\Omega)$ with $m > q'$. Thus, by (4.44), we have

$$\int_{\{k-1\le|u_p|<k\}} |\nabla_y u_p|^q \le \int_{\{|u_p|\ge k-1\}} |f|\,, \qquad \text{for all } k \ge 1\,. \tag{4.45}$$

Consider a parameter $\gamma > 1$ to be determined. Then, by Theorem 2.3, we get

$$\int_\Omega |u_p|^{\gamma q} \le C \int_\Omega \big|\nabla_y |u_p|^\gamma\big|^q = C \int_\Omega |u_p|^{q(\gamma-1)} |\nabla_y u_p|^q$$
$$= C \sum_{k=1}^\infty \int_{\{k-1 \le |u_p| < k\}} |u_p|^{q(\gamma-1)} |\nabla_y u_p|^q \le C \sum_{k=1}^\infty \int_{\{k-1 \le |u_p| < k\}} k^{q(\gamma-1)} |\nabla_y u_p|^q \,.$$

By applying (4.45) in each term of the right hand side, one deduces

$$\int_\Omega |u_p|^{\gamma q} \le C \sum_{k=1}^\infty \int_{\{|u_p| \ge k-1\}} k^{q(\gamma-1)} |f| = C \sum_{k=1}^\infty \sum_{h=k}^\infty k^{q(\gamma-1)} \int_{\{h-1 \le |u_p| < h\}} |f| \,.$$

Changing the order of summation and using $\sum_{k=1}^h k^{q(\gamma-1)} \le C h^{q(\gamma-1)+1}$, we have

$$\int_\Omega |u_p|^{\gamma q} \le C \sum_{h=1}^\infty h^{q(\gamma-1)+1} \int_{\{h-1 \le |u_p| < h\}} |f|$$
$$= C \sum_{h=1}^\infty \int_{\{h-1 \le |u_p| < h\}} (1+|u_p|)^{q(\gamma-1)+1} |f| = C \int_\Omega (1+|u_p|)^{q(\gamma-1)+1} |f|$$
$$\le C \left(\int_\Omega |f|^m \right)^{1/m} \left(\int_\Omega (1+|u_p|)^{\big(q(\gamma-1)+1\big)m'} \right)^{1/m'} \,.$$

If we take γ satisfying $\gamma q = \big(q(\gamma-1)+1\big)m'$, then we obtain $\gamma = \frac{m}{q'} > 1$ (since $m' < q$) and an estimate of u_p in $L^{\gamma q}(\Omega)$. Since $\gamma q = m(q-1)$, we are done.

Assume now that $q < \frac{N}{N-1}$. It implies $q < \frac{N+K}{N+(K/q)-1} = r'$. The proof follows the same lines as above but applying Theorem 2.1 instead of Theorem 2.3. We only point out the differences.

Consider parameters $\gamma_p > 1$ to be determined. Then, by Theorem 2.3 and Young's inequality, we get

$$\left(\int_\Omega |u_p|^{\gamma_p \frac{N+K}{(N/p)+(K/q)-1}} \right)^{\frac{(N/p)+(K/q)-1}{(N/p)+(K/q)}}$$
$$\le S_{(p,q)} \left(\int_\Omega \big|\nabla_x |u_p|^{\gamma_p}\big|^p \right)^{\frac{N/p}{(N/p)+(K/q)}} \left(\int_\Omega \big|\nabla_y |u_p|^{\gamma_p}\big|^q \right)^{\frac{K/q}{(N/p)+(K/q)}}$$
$$\le S_{(p,q)} \left[\int_\Omega \big|\nabla_x |u_p|^{\gamma_p}\big|^p + \int_\Omega \big|\nabla_y |u_p|^{\gamma_p}\big|^q \right]$$
$$\le S_{(p,q)} \gamma_p^q \sum_{k=1}^\infty k^{q(\gamma_p-1)} \left[\int_{\{k-1 \le |u_p| < k\}} |\nabla_x u_p|^p + \int_{\{k-1 \le |u_p| < k\}} |\nabla_y u_p|^q \right] .$$

Now we apply (4.44) and perform similar computations as those done in the previous case to get

$$\left(\int_\Omega |u_p|^{\gamma_p \frac{N+K}{(N/p)+(K/q)-1}} \right)^{\frac{(N/p)+(K/q)-1}{(N/p)+(K/q)}} \le S_{(p,q)} \gamma_p^q \|f\|_m \left(\int_\Omega (1+|u_p|)^{\big(q(\gamma_p-1)+1\big)m'} \right)^{1/m'} \,. \tag{4.46}$$

We remark that $\frac{1}{m'} < \frac{N+(K/q)-1}{N+(K/q)}$ since $m < N + \frac{K}{q}$. Thus, we may consider p small enough to satisfy $\frac{1}{m'} < \frac{(N/p)+(K/q)-1}{(N/p)+(K/q)}$. If we take γ_p satisfying $\gamma_p \frac{N+K}{(N/p)+(K/q)-1} = \big(q(\gamma_p - 1) + 1\big)m'$, then we obtain

$$\gamma_p = \frac{m'(q-1)}{qm' - \frac{N+K}{(N/p)+(K/q)-1}}\,.$$

Observe that γ_p is bounded by a constant not depending on p. Hence, we have

$$\int_\Omega |u_p|^{\gamma_p \frac{N+K}{(N/p)+(K/q)-1}} \le C\,,$$

where C depends on f through its m–norm, and on p through the parameter γ_p, the Sobolev constant $S_{(p,q)}$ and the exponent $\frac{(N/p)+(K/q)-1}{(N/p)+(K/q)}$. Therefore, we may let p goes to 1 and get an estimate of u in a Lebesgue space:

$$\lim_{p\to 1} \gamma_p \frac{N+K}{(N/p)+(K/q)-1} = \frac{m'(q-1)(N+K)}{qm'(N+(K/q)-1)-(N+K)} = s\,.$$

Now some remarks are in order. Observe that $N - q(N-1) > 0$ since $q < \frac{N}{N-1}$. So that, we obtain

$$N + \frac{K}{q} < \frac{N+K}{N-q\big(N-1\big)}\,.$$

Since $m < \frac{N+K}{N-q(N-1)}$, one deduces $qm' - \frac{N+K}{N+(K/q)-1} > 0$. As a consequence, $\lim_{p\to 1}\gamma_p > 1$ if and only if $m > \frac{N+K}{(K/q')+1}$. Since this last inequality holds, we have really improved the regularity of our solution. Finally, we point out that this improvement needs that the inequalities $\frac{N+K}{(K/q')+1} < m < N + \frac{K}{q}$ hold, and it is easy to see that we indeed have $\frac{N+K}{(K/q')+1} < N + \frac{K}{q}$. ∎

4.3. **Examples.** We take $\Omega = \Xi\times\Upsilon$, with $\Xi = B_1(0)$ in $\mathbb{R}^N$ ($N \ge 2$) and $\Upsilon = B_1(0)$ in $\mathbb{R}^K$, and we will assume throughout this subsection that $q > \frac{N}{N-1}$, so $r = q' < N$. Our aim is to show examples of problems (1.1) having solutions of the form

$$u(x,y) = a(x)b(y).$$

Consider f_1 a positive radial decreasing function belonging to the Marcinkiewicz space $L^{N,\infty}(\Xi)$ and satisfying $\|f_1\|_{L^{N,\infty}(\Xi)} \le 1$. Let $a \in W_0^{1,1}(\Xi) \cap L^\infty(\Xi)$ be a solution to

$$\begin{cases} -\mathrm{div}\Big(\dfrac{Da}{|Da|}\Big) = f_1\,, & \text{in } \Xi\,;\\ a = 0\,, & \text{on } \partial\Xi\,. \end{cases} \tag{4.47}$$

Thus, a is a radial nonnegative function and there exists $\overline{\zeta} \in L^\infty(\Xi;\mathbb{R}^N)$ satisfying

1. $\|\overline{\zeta}\|_\infty \le 1$,
2. $-\mathrm{div}\,\overline{\zeta} = f_1$ in $\mathcal{D}'(\Xi)$,
3. $(\overline{\zeta}, Da) = |Da|$ as measures on Ξ.

We refer to [30], Section 3, for a detailed discussion of all matters concerning radial solutions to problem (4.47).

Now let $f_2 \in L^{q'}(\Upsilon)$, with $f_2 \ge 0$, and let $b \in W_0^{1,q}(\Upsilon)$ be the unique solution to

$$\begin{cases} -\Delta_q b = f_2\,, & \text{in } \Upsilon\,;\\ b = 0\,, & \text{on } \partial\Upsilon\,. \end{cases} \tag{4.48}$$

It is straightforward that $b \geq 0$ in Υ.

Let us check that $u(x,y) = a(x)b(y)$ is a solution to (1.1) with datum

$$f(x,y) = f_1(x) + f_2(y)a(x)^{q-1} \in L^r(\Omega)\,. \tag{4.49}$$

To this end, define $\zeta(x,y) = \overline{\zeta}(x)$, $\lambda(x,y) = a(x)^{q-1}|\nabla b(y)|^{q-2}\nabla b(y)$ and $z = (\zeta, \lambda)$. Then $\zeta \in L^\infty(\Omega; \mathbb{R}^N)$ with $\|\zeta\|_\infty = \|\overline{\zeta}\|_\infty \leq 1$, and $\lambda(x,y) \in L^{q'}(\Omega; \mathbb{R}^K)$ with $\lambda = |\nabla_y u|^{q-2}\nabla_y u$. Moreover,

$$-\mathrm{div}_x \zeta = f_1(x) \quad \text{and} \quad -\mathrm{div}_y \lambda = a(x)^{q-2} f_2(y) \quad \text{in } \mathcal{D}'(\Omega)\,,$$

so that $-\mathrm{div}\, z = f$ in the sense of distributions.

To show that $(\zeta, D_x u)$ is a Radon measure on Ω, take first $\varphi(x,y) = \phi(x)\psi(y)$, where $\phi(x) \in C_0^\infty(\Xi)$ and $\psi(y) \in C_0^\infty(\Upsilon)$. Then

$$\begin{aligned}\langle(\zeta, D_x u), \varphi)\rangle &= -\int_\Omega a(x)b(y)\phi(x)\psi(y)\mathrm{div}\,\overline{\zeta}(x) - \int_\Omega a(x)b(y)\psi(y)\overline{\zeta}(x)\cdot\nabla\phi(x) \\ &= \Big(\int_\Upsilon b\,\psi\Big)\Big[-\int_\Xi a\,\phi\,\mathrm{div}\,\overline{\zeta} - \int_\Xi a\,\overline{\zeta}\cdot\nabla\phi\Big] = \Big(\int_\Upsilon b\,\psi\Big)\langle(\overline{\zeta}, Da), \phi\rangle\,.\end{aligned}$$

Let us denote $C_0(\Omega) = \{\varphi \in C(\overline{\Omega}) \,:\, \varphi\big|_{\partial\Omega} = 0\}$, and let us write $C_0(\Xi)$ and $C_0(\Upsilon)$ with a similar meaning. Since $C_0^\infty(\Xi)$ is uniformly dense in $C_0(\Xi)$ and $C_0^\infty(\Upsilon)$ is uniformly dense in $C_0(\Upsilon)$, it follows that

$$\langle(\zeta, D_x u), \varphi\rangle = \Big(\int_\Upsilon b\psi\Big)\langle(\overline{\zeta}, Da), \phi\rangle\,, \tag{4.50}$$

for all $\varphi(x,y) = \phi(x)\psi(y)$, with $\phi(x) \in C_0(\Xi)$ and $\psi(y) \in C_0(\Upsilon)$. Thus, by linearity, we continuously extend $(\zeta, D_x u)$ to functions $\varphi \in C_0(\Omega)$ which can be written as $\varphi(x,y) = \sum_{i=1}^n \phi_i(x)\psi_i(y)$ with $\phi_i \in C_0(\Xi)$ and $\psi_i \in C_0(\Upsilon)$. Further, appealing to a variant of the Stone–Weierstrass Theorem, we may continuously extend $(\zeta, D_x u)$ to $C_0(\Omega)$, so that $(\zeta, D_x u)$ is a Radon measure.

We also deduce from (4.50) that if B_Ξ is a Borel subset of Ξ and B_Υ is a Borel subset of Υ, then $(\zeta, D_x u)(B_\Xi \times B_\Upsilon) = (\overline{\zeta}, Da)(B_\Xi)\int_{B_\Upsilon} b$. Since we also have $|D_x u|(B_\Xi \times B_\Upsilon) = |Da|(B_\Xi)\int_{B_\Upsilon} b$, we conclude that $(\zeta, D_x u) = |D_x u|$ as measures.

Let us study now some special choices of f_1 and f_2. First consider

$$f_1(x) = \lambda\frac{N-1}{|x|}, \quad \text{with } 0 < \lambda < 1.$$

It is well–known that $a(x) = 0$ for all $x \in \Xi$ is a solution of (4.47), with $\overline{\zeta}(x) = \lambda\frac{x}{|x|}$. So that the solution to (1.1) with datum

$$f(x,y) = \lambda\frac{N-1}{|x|}$$

is given by $u(x,y) = 0$, whatever datum $f_2(y)$ be considered in (4.49). We explicitly observe that in this case $\|\overline{\zeta}\|_{L^\infty} = \lambda < 1$.

Now let us consider

$$f_1(x) = \frac{N-1}{|x|}.$$

Then $a(x) = 1 - |x|$ is a solution to (4.47) with $\overline{\zeta}(x) = \frac{x}{|x|}$. Thus, taking $f_2(y) = 0$ in (4.49), the solution to (1.1) is also given by $u(x,y) = 0$.

On the other hand, taking

$$f_1(x) = \frac{N-1}{|x|} \quad \text{and} \quad f_2(y) \neq 0\,,$$

the solution to (1.1) is given by

$$u(x,y) = (1-|x|)\, b(y),$$

which is nontrivial since $b(y)$ is nontrivial. So the datum

$$f(x,y) = \lambda \frac{N-1}{|x|} + f_2(y), \quad \text{with } 0 < \lambda < 1\,,$$

produces the trivial solution for any choice of f_2, and the datum

$$f(x,y) = \frac{N-1}{|x|} + f_2(y)(1-|x|)^{q-1},$$

which is larger than $\frac{N-1}{|x|}$, gives a nontrivial solution $u(x,y)$ as well. In others words, once the vector field ζ satisfies $\|\zeta\|_\infty > 1$, the solution to (1.1) becomes non trivial. Roughly speaking, data large enough produce an excess which have to be absorbed only by the term $-\mathrm{div}_y\big(|\nabla_y u|^{q-1}\nabla_y u\big)$.

Since we may consider every $f_2 \in L^{q'}(\Upsilon)$, it follows that we may start from a datum $f(x,y)$ with norm $\|f\|_r$ as large as we want.

Furthermore, the above argument shows that if we take $f(x,y) = \lambda\frac{N-1}{|x|}$, with $\lambda > 1$, as datum, we can not expect the solution to problem (1.1) to be a product of two functions with separate variables.

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