A limit case in non-isotropic two-phase minimization problems driven by $p-$Laplacians

by

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Abstract

In this work we study a minimization problem with two-phases where in each phase region the problem is ruled by a quasi-linear elliptic operator of $p-$Laplacian type. The problem in its variational form is as follows:

$$\min_{v}\left\{\int\limits_{\Omega\cap\{v>0\}}\left(\frac{1}{p}|\nabla v|^p+\lambda_+^p(x)+f_+(x)v\right)dx+\int\limits_{\Omega\cap\{v\le 0\}}\left(\frac{1}{q}|\nabla v|^q+\lambda_-^q(x)+f_-(x)v\right)dx\right\}.$$

Here we minimize among all admissible functions v in an appropriate Sobolev space with a prescribed boundary datum $v=g$ on $\partial\Omega$. First, we show existence of a minimizer, prove some properties, and provide an example for non-uniqueness. Moreover, we analyze the limit case where p and q go to infinity, obtaining a limiting free boundary problem governed by the $\infty-$Laplacian operator. Consequently, Lipschitz regularity for any limiting solution is obtained. Finally, we establish some weak geometric properties for solutions.

Keywords: Free boundary problems, non-isotropic two-phase problems, $\infty-$Laplacian operator.
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Contents

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1 Introduction

In the universe of the applied sciences, *phase transition problems* (or *transmission problems*) are often models which involve different media and hence they involve different analytical processes in distinct zones. Such phenomena appear in several fields as biology, material sciences, physics, etc. Moreover, its study plays an essential role, for example, for mathematical modeling of composite materials, since they deal with heterogenous media with distinct diffusive processes (cf. [9] for a reference on this subject). Finally, electromagnetic or thermodynamic processes with different diffusivity are other examples of phase transition problems.

Equations related to phase transition problems involve (in general) different diffusivity laws. Such a phenomena occurs due to different properties and distinct features of the media. Devices made of distinct materials, Multi-constituent substances and anti-plane shear deformation are some examples of such processes. Typical mathematical models of phase transition type are driven by a second order elliptic equations of the form

$$\operatorname{div}(\chi_{\Omega'}|\nabla u|^{p-2}\nabla u)+\operatorname{div}((1-\chi_{\Omega'})|\nabla u|^{q-2}\nabla u)=\mathscr{A}\chi_{\Omega'}+\mathscr{B}(1-\chi_{\Omega'}) \quad \text{in} \quad \Omega, \tag{1.1}$$

where $\Omega' \Subset \Omega$ is a subregion and $\mathscr{A}$ and $\mathscr{B}$ are constants. In such modeling, knowing the local behaviour of the associated solutions and their "transition surface", namely $\partial\Omega'$, as well as its smoothness and weak geometric properties play a crucial role in the physical-mathematical studies previously cited.

Another interesting physical motivation in the same spirit runs as follows: Consider the system composed by an iced substance submerged in a heated liquid inhomogeneous medium. In order to understand the phenomenon, we will focus our attention on the stationary case. For this very reason, when the temperature $\mathfrak{T}$ is negative, the thermodynamic process is driven by a diffusion operator associated to the iced substance. Thus, we can assume that

$$\Delta_p \mathfrak{T}(t)=0 \quad \text{inside the iced substance (for some } 1<p<\infty).$$

On the other hand, for positive temperatures, the thermodynamic process is driven by a diffusion operator associated to the exterior inhomogeneous medium. Thus,

$$\Delta_q \mathfrak{T}(t)=0 \quad \text{in the exterior liquid (for some } 1<q<\infty \text{ with } q\neq p).$$

Now, from thermodynamics' laws, an extra energy (the latent heat flow) is required in order to change the state of the matter. Mathematically speaking, this means that there exists a prescribed flux balance along the phase transition $\{\mathfrak{T}=0\}$. Precisely, there exists a mapping $\mathscr{G}_{p,q}:\mathbb{S}^N\times\mathbb{S}^N\to\mathbb{R}$ such that

$$\mathscr{G}_{p,q}(\mathfrak{T}_\nu^+,\mathfrak{T}_\nu^-)=\mathfrak{c} \quad \text{along} \quad \{\mathfrak{T}=0\},$$

where the constant $\mathfrak{c}\neq 0$ means "the latent heat flow for melting". In contrast with (1.1), in the previous physical model, the phase transition is *a priori* unknown. Moreover, it depends on the solution itself. Unifying the previous equations involved in the system, one finishes up with a nonhomogeneous elliptic equation with a measure datum:

$$\Delta_{p\chi_{\{\mathfrak{T}>0\}}+q\chi_{\{\mathfrak{T}<0\}}}\mathfrak{T}(t)=(\Delta_p\mathfrak{T}(t))\chi_{\{\mathfrak{T}>0\}}+(\Delta_q\mathfrak{T}(t))\chi_{\{\mathfrak{T}<0\}}=\mu,$$

where μ is a nonzero measure supported along the "phase transition surface" $\{\mathfrak{T}=0\}$. Particularly, μ is not absolutely continuous with respect to the Lebesgue measure. Moreover, when $\{\mathfrak{T}=0\}$ is an $(N-1)$-surface (in the measure theoretic sense), then $\mu=\mathfrak{c}\lfloor\{\mathfrak{T}=0\}$ in the sense of measures.

The main goal of present manuscript is to provide a rigorous mathematical analysis, which includes existence, regularity and some geometric properties for solutions to the phase transition problems involving free boundaries with different degenerate diffusion operators in each phase region, as happens in the previous example. Motivated by the analysis of the asymptotic behaviour of certain variational problems, we will pay special attention to the analysis under the condition that the diffusivity degrees of each operator are large enough. Intuitively, we would like to understand and describe the physical-mathematical processes when the media tend to "homogenize", i.e., as p, q diverges.

In the early 80's in [1] and [2] Alt-Caffarelli and Alt-Caffarilli-Friedman established the beginning of the study of minimizing problems with free boundaries. Since then this research area has obtained a significant development with regards to the regularity theory for solutions of such free boundary problems. In this scenario, the minimizer satisfies a PDE within an *a priori* unknown region together with a free boundary condition, and a key question consists in studying the regularity of such solution, as well as the regularity of the associated free boundary. Recall that such solutions can be one-signed (one-phase problems) or can change sign (two-phase problems). For example, a common two-phase free boundary problem is to seek for a minimizer to the variational integral

$$\text{(ACF)} \qquad \int_\Omega \left(\frac{1}{2}|\nabla u|^2 + \lambda_+\chi_{\{u>0\}} + \lambda_-\chi_{\{u\le 0\}}\right)dx$$

among all admissible functions with prescribed boundary datum. A local minimizer u_0 fulfils (in the weak sense) the Dirichlet problem

$$\begin{cases} -\Delta u_0 &= 0 \quad \text{in} \quad (\{u_0>0\}\cup\{u_0\le 0\}^\circ)\cap\Omega \\ u_0 &= g \quad \text{on} \quad \partial\Omega, \end{cases}$$

as well as the following free boundary condition

$$|\nabla u_0^+|^2 - |\nabla u_0^-|^2 = 2(\lambda_+ - \lambda_-) \quad \text{on} \quad (\partial\{u_0>0\}\cup\partial\{u_0\le 0\}^\circ)\cap\Omega$$

understood in an appropriate weak sense.

With this preliminaries in mind let us introduce our two-phase free boundary problem. Let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain, $p, q \ge 2$, $f_\pm \in L^s(\Omega)$, for $s \ge \max\left\{\frac{N}{p}, \frac{N}{q}\right\}$, $0 \le \lambda_\pm$ with $\lambda_+ \in L^p(\Omega)$, $\lambda_- \in L^q(\Omega)$, with $\lambda_+^p \ne \lambda_-^q$ (this is used to have a discontinuous flux) and $g \in W^{1,p}(\Omega)\cap W^{1,q}(\Omega)\cap L^\infty(\Omega)$, with, let us say, $g^+ \ne 0$. The purpose of this manuscript is to study the following minimization problem:

$$\textbf{(Min)} \qquad \min_{v\in\mathscr{K}_g^{(p,q)}(\Omega)} \mathfrak{J}_{p,q}[v] = \mathfrak{J}_{p,q}[u_0],$$

for the functional given by

$$(1.2) \quad \mathfrak{J}_{p,q}[v] := \int_{\Omega\cap\{v>0\}} \left(\frac{1}{p}|\nabla v|^p + \lambda_+^p(x) + f_+(x)v\right)dx + \int_{\Omega\cap\{v<0\}} \left(\frac{1}{q}|\nabla v|^q + \lambda_-^q(x) + f_-(x)v\right)dx$$

and the class of functions

$$\mathscr{K}_g^{(p,q)}(\Omega) := \left\{v \in W^{1,\min\{p,q\}}(\Omega) \;\middle|\; v^+ \in W^{1,p}(\Omega), v^- \in W^{1,q}(\Omega), v = g \text{ on } \partial\Omega \text{ in the sense of traces}\right\}.$$

In the following we will denote by $\Omega^+[u] := \{u>0\}\cap\Omega$ and $\Omega^-[u] := \{u<0\}\cap\Omega$, the positive and negative phase respectively. Now, notice that any minimizer u_0 to (**Min**) satisfies, in the weak sense, the following (p,q)-degenerate system

$$(1.3) \qquad \begin{cases} \Delta_p u_0 = f_+(x) & \text{in} \quad \Omega^+[u_0]\cap\Omega \\ \Delta_q u_0 = f_-(x) & \text{in} \quad \Omega^+[u_0]\cap\Omega \\ u_0(x) = g(x) & \text{on} \quad \partial\Omega. \end{cases}$$

Moreover, we highlight that, due to the fact that we assumed $\lambda_+^p \neq \lambda_-^q$, the potential that appears in our functional $\mathfrak{F}_0(\lambda_-,\lambda_+) := \lambda_+^p(x)\chi_{\{u_0>0\}} + \lambda_-^q(x)\chi_{\{u_0\leq 0\}}$ is discontinuous along the free boundary points, enforcing the *flux balance* across the free boundary (in $C^{1,\alpha}$ smooth pieces of the free boundary that separates the two phases)

$$\textbf{(FBC)} \qquad \mathscr{G}_{p,q}(u_\nu^+, u_\nu^-, \lambda_+, \lambda_-) := \frac{p-1}{p}(u_\nu^+(x))^p - \frac{q-1}{q}(u_\nu^-(x))^q - \lambda_+^p(x) + \lambda_-^q(x) = 0,$$

preventing any possible continuity for the gradient through free boundary. Here $u_\nu^\pm$ are respectively the normal derivatives in the inward direction to $\partial\Omega^\pm[u]$. Notice that such discontinuity phenomenon along the interface involve several technical difficulties in the treatment of these type of two-phase problems. Particularly, existence of minimizers (that we prove in Section 3) does not follow from standard methods from Calculus of Variations. In fact, the main difficulty lies in the lack of convexity of the functional defined in (1.2).

To gain some insight concerning possible configurations for (p,q) large, we are interested in the *limiting free boundary problem*, namely the asymptotic profile when p,q goes to infinity. More precisely, given a minimizer $u_{p,q}$ to (**Min**), then, we show that, up to subsequences, there exists a limit, $u_{p,q} \to \hat{u}$ uniformly when $p,q \to \infty$, that fulfils in the viscosity sense

$$(1.4) \qquad \left\{\begin{array}{rclll} -\Delta_\infty \hat{u} & = & 0 & \text{in} & \Omega^+[\hat{u}] \cup \Omega^-[\hat{u}] \cap (\Omega \setminus \operatorname{supp}(f_\pm))^\circ \\ |\nabla \hat{u}| & = & 1 & \text{in} & \Omega^+[\hat{u}] \cup \Omega^-[\hat{u}] \cap (\Omega \cap \Omega^+[f_\pm]) \\ -|\nabla \hat{u}| & = & -1 & \text{in} & \Omega^+[\hat{u}] \cup \Omega^-[\hat{u}] \cap (\Omega \cap \Omega^-[f_\pm]) \\ -\Delta_\infty \hat{u} & \gtreqqless & 0 & \text{in} & \Omega^+[\hat{u}] \cup \Omega^-[\hat{u}] \cap (\Omega \cap \partial\Omega^\pm[f_\pm] \setminus \partial\Omega^\mp[f_\pm]) \\ \hat{u}(x) & = & g(x) & \text{on} & \partial\Omega. \end{array}\right.$$

This system is complemented with a limit free boundary condition, that we deduce only formally, which depends only on how the quotient q/p behaves. We assume here that

$$\lim_{p,q\to\infty} \frac{q}{p} = \mathscr{Q} \in (0,+\infty)$$

and we obtain

$$(\infty-\textbf{FBC}) \qquad \max\left\{\hat{u}_\nu^+(x),\ \lambda_-^{\mathscr{Q}}(x)\right\} = \max\left\{(\hat{u}_\nu^-)^{\mathscr{Q}}(x),\ \lambda_+(x)\right\}.$$

The main obstacle to obtain this condition rigorously comes from the fact that solutions to (1.3) are not (in general) regular enough across the free boundary, as well as the limiting free boundary is not "smooth" enough (in an appropriate measure theoretical sense) in order to pass to the limit point-wise in (**FBC**) (cf. [3], [18] and the references therein for regularity issues).

Let us present a brief overview on minimization problems with free boundaries and their connections with our work. The minimization problem (**Min**) is related with jets flow and cavity type problems. Recall that the linear $(p=q=2)$, homogeneous $(f=0)$, one-phase version of this problem was completely studied in [1], where it is proved that minimizers are Lipschitz continuous, the expected optimal regularity. On the other hand, the two-phase counterpart of this problem yields new obstacles and local Lipschitz regularity of minimizers was proven in [2], by using the powerful Alt-Caffarelli-Friedman's monotonicity formula. Thereafter, gradient estimates (Lipschitz bounds) for two-phase cavitation type problem with bounded non-homogeneity, i.e., $p,q=2$ and $f \in L^\infty$, were established in [8] by using an almost monotonicity formula. The general degenerate jet type problems $(p=q\neq 2)$ have received a warm attention in the last decade. The homogeneous one-phase problem $(f_+=0\leq g)$ was fully studied in [11], proving optimal

Lipschitz regularity, non-degeneracy, as well as finiteness of the $(N-1)$-Hausdorff measure for the free boundary of minimizers. Latter, a general inhomogeneous two-phase minimization problem was studied in [18], where several analytic and geometric properties for minimizers and their free boundaries were established. Particularly, they state (see [18, Remark 4.2]) that we should not expect Lipschitz regularity for minimizers even if the source term is bounded. In this direction, determining whether any minimizer is Lipschitz (provided the source term is L^s, for $s>N$) has became a long-standing challenging problem in the theory of free boundary problems.

Taking into account the previous facts, our regularity result is surprising, because limits of minimizers for (**Min**) (resp. viscosity solutions of (1.4)) are Lipschitz continuous under suitable assumption on the data, see Theorem 5.2. Summarizing, the limiting problem admits a better regularity theory for solutions than its "stationary" $(p,q)-$counterpart.

As mentioned previously, another interesting aspect of our work is its connection with free transmission/transition problems, i.e., two-phases free boundary problems whose solutions are required to solve distinct PDEs, driven by distinct diffusion operators $\mathfrak{L}_+$ and $\mathfrak{L}_-$, within their positivity and negativity sets respectively. Furthermore, on the phase-transition region (the free boundary of the model) appears a *balance flux* relating the corresponding positive and negative phase like (**FBC**) (cf. [3] and [7] for excellent surveys on this subject). Finally, we stress that our analysis is related to the previous article [24], in the which it is studied a minimization problem under geometric restrictions (like optimal design type problems) with two-phases for the $p-$Laplacian as p goes to infinity. We must also quote [18], where the two-phase $p-$isotropic problem, i.e., our problem with $p=q$ fixed, was studied.

We end this introduction with a brief description of recent references concerning limits as $p\to\infty$ in different p-Laplacian type problems and their connection with some free boundary problems. Taking account the analysis of the limit of $p-$variational, one of pioneering works goes back to [6]. Precisely, they establish that for a non-negative function f, the corresponding weak solution for the p-Laplacian

$$(1.5)\qquad \left\{\begin{array}{rcll} -\Delta_p u_p(x) & = & f(x) & \text{in} \quad \Omega \\ u_p(x) & = & 0 & \text{on} \quad \partial\Omega \end{array}\right.$$

converge, up to a subsequence, to a limit u_∞, which satisfies in the viscosity sense the following problem

$$(1.6)\qquad \left\{\begin{array}{rcll} -\Delta_\infty u_\infty & = & 0 & \text{in} \quad \Omega\cap\overline{\{f>0\}}^c \\ |\nabla u_\infty| & = & 1 & \text{in} \quad \Omega\cap\{f>0\}, \end{array}\right.$$

where $\Delta_\infty v := Dv^T\cdot DvD^2v$ is the nowadays well-known *Infinity-Laplacian operator*. We also refer to [21] for a general treatment of this subject and its connection with game theory ("Tug of-war games").

One motivation to study this kind of issues comes from the *best Lipschitz extension problem* of a datum $g\in W^{1,\infty}(\partial\Omega)$. In fact, such a extension, which we will denote by $\tilde{g}$, can be obtained as limit of solutions to (1.5) provided we put $f=0$ and $u_p=g$ on the boundary. Moreover, such a function is the unique Lipschitz function with best Lipschitz constant $\mathrm{Lip}_g(\partial\Omega)$ that is also optimal in every sub-domain of Ω in the sense that $\tilde{g}=g$ on $\partial\Omega$ and

$$(\textbf{AMLE})\qquad \mathrm{Lip}_{\tilde{g}}(\Omega')\leq \mathrm{Lip}_z(\Omega')\quad \forall\ \Omega'\Subset\Omega \quad \text{such that} \quad \tilde{g}=z \quad \text{on} \quad \partial\Omega'.$$

This is known in the literature as an *Absolutely Minimizing Lipschitz Extension*, in short AMLE, a concept introduced by G. Aronsson at the end of sixties and extensively studied by several authors in the last decades (cf. [4] and [5]). Finally, (1.6) means that the $\infty-$Laplacian operator governs the Euler-Lagrange equation to such a $L^\infty-$minimization problem (**AMLE**) (cf. [15] for more details).

Regarding free boundary problems, the strategy of passing the limit as $p\to\infty$ in p-variational problems in order to obtain a non-variational limiting configuration (a problem governed by the Infinity-Laplacian operator or another non-variational limiting operator) has been successful in many contexts: In dead core type problems [13], Bernoulli type problems [20] , optimal design problems [22] and [24], obstacle type problems [23], to cite just a few examples (See also [12] for an optimal design problem and [14] for an obstacle type problem in the nonlocal scenery). Furthermore, such approach allows us to use several technical features of the corresponding p-sequential problems to their limiting points, via uniform convergence. Regularity estimates, weak geometric and measure-theoretic properties are some of the obtained features.

Finally we remark that, concerning limiting minimization problems, our results are new even for the one-phase homogeneous minimization problem, i.e., $f_{\pm} = 0 = \lambda_-$ and $g \geq 0$ (compare with [22] and [24]).

2 Preliminaries

First, let us state precisely the functional framework for our problem.

Definition 2.1 (Weak solution). We say that $u \in W^{1,p}(\Omega) \cap W^{1,q}(\Omega)$ is a weak solution to (1.3) if $u - g \in W_0^{1,p}(\Omega) \cap W_0^{1,q}(\Omega)$ and for every $\phi \in C_0^1(\Omega)$ there holds,

$$\int_{\Omega \cap \{u>0\}} \left(|\nabla u|^{p-2} \nabla u \nabla \phi - f_+ \phi\right) dx = 0 \quad \text{and} \quad \int_{\Omega \cap \{u<0\}} \left(|\nabla u|^{q-2} \nabla u \nabla \phi - f_- \phi\right) dx = 0$$

Recall that our limiting solutions will satisfy a fully nonlinear elliptic problem of degenerate type. For this reason, we introduce the concept of viscosity solution to a PDE problem like

$$\left\{\begin{array}{rcll} F(\nabla \mathfrak{h}, D^2 \mathfrak{h}) & = & f(x) & \text{in} \quad \Omega \\ \mathfrak{h}(x) & = & g(x) & \text{on} \quad \partial\Omega. \end{array}\right. \tag{2.1}$$

Notice that $F : \mathbb{R}^N \times \mathrm{Sym}(N) \to \mathbb{R}$ can be a discontinuous operator (in general such a discontinuity occurs along the critical point set). For this reason, we must introduce $F^{\natural}$ and $F_{\natural}$, respectively the upper and lower semi-continuous envelopes of F given by

$$F^{\natural}(z, M) = \limsup_{\varepsilon \to 0} \{F(w, N) \mid |z - w| + |M - N| < \varepsilon\} \quad \text{and} \quad F_{\natural}(z, M) = -(-F^{\natural})(z, M).$$

Definition 2.2 (Viscosity solution). An upper (resp. lower) semi-continuous function u defined in Ω is a viscosity sub-solution to (2.1) if $u \leq g$ and, whenever $x_0 \in \Omega$, $\phi \in C^2(\Omega)$ are such that $u - \phi$ has a local maximum (resp. minimum) at x_0, then

$$F^{\natural}(\nabla \phi(x_0), D^2 \phi(x_0)) \leq f(x_0) \quad (\text{resp. } F_{\natural}(\nabla \phi(x_0), D^2 \phi(x_0)) \geq f(x_0))$$

Finally, a continuous function u is a viscosity solution to (2.1) if it is simultaneously a viscosity super-solution and a viscosity sub-solution.

Definition 2.3. A function $u \in C(\Omega)$ is said to be a viscosity solution to

$$\max\{-\Delta_\infty v(x), |\nabla v(x)| - h(x)\} = 0 \quad \text{in} \quad \Omega$$

if: whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $v(x_0) = \phi(x_0)$ and $v(x) < \phi(x)$, when $x \neq x_0$, then

$$-\Delta_\infty \phi(x_0) \leq 0 \quad \text{or} \quad |\nabla \phi(x_0)| - h(x_0) \leq 0.$$

and whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $v(x_0) = \phi(x_0)$ and $v(x) > \phi(x)$, when $x \neq x_0$, then

$$-\Delta_\infty \phi(x_0) \geq 0 \quad \text{and} \quad |\nabla \phi(x_0)| - h(x_0) \geq 0.$$

For a complete survey about the theory of viscosity solutions and its machinery we refer read the classical reference [10]. Moreover, regarding viscosity solutions related to the Infinity-Laplacian and the p-Laplacian operator we recommend the reference [16].

The following lemma establish a relation between weak and viscosity sub and super-solutions to (1.3).

Lemma 2.4. *A continuous weak sub-solution (resp. super-solution)* $u \in W^{1,p}_{loc}(\{v>0\}) \cap W^{1,q}_{loc}(\{v<0\})$ *to* (1.3) *is a viscosity sub-solution (resp. super-solution) to*

$$\begin{cases} -\left[|\nabla u(x)|^{p-2}\Delta u(x) + (p-2)|\nabla u(x)|^{p-4}\Delta_\infty u(x)\right] &= -f_+(x) \quad in \quad \{u>0\}\cap\Omega \\ -\left[|\nabla u(x)|^{q-2}\Delta u(x) + (q-2)|\nabla u(x)|^{q-4}\Delta_\infty u(x)\right] &= -f_-(x) \quad in \quad \{u<0\}\cap\Omega \\ u(x) &= g(x) \qquad on \quad \partial\Omega. \end{cases}$$

Proof. First, let us proceed with the case of super-solutions for the equation

$$-\left[|\nabla u(x)|^{p-2}\Delta u(x) + (p-2)|\nabla u(x)|^{p-4}\Delta_\infty u(x)\right] = -f_+(x) \qquad \text{in } \{u>0\}\cap\Omega.$$

Fix $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ such that ϕ touches u by below, i.e., $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$ for $x \neq x_0$. Our goal is to show that

$$-\left[|\nabla \phi(x_0)|^{p-2}\Delta \phi(x_0) + (p-2)|\nabla \phi(x_0)|^{p-4}\Delta_\infty \phi(x_0)\right] + f_+(x_0) \geq 0.$$

From now on, we will proceed by contradiction and suppose that the above inequality does not hold. Then, by continuity there exists $0 < r \ll 1$ (small enough) such that

$$-\left[|\nabla \phi(x)|^{p-2}\Delta \phi(x) + (p-2)|\nabla \phi(x)|^{p-4}\Delta_\infty \phi(x)\right] + f_+(x) < 0,$$

provided that $x \in B_r(x_0)$. Letting

$$\Phi(x) := \phi(x) + \frac{1}{7}\mathfrak{m}, \quad \text{where} \quad \mathfrak{m} := \inf_{\partial B_r(x_0)} (u(x) - \phi(x)).$$

we observe that Φ verifies $\Psi < u$ on $\partial B_r(x_0)$, $\Phi(x_0) > u(x_0)$ and

$$-\Delta_p \Phi(x) < -f_+(x). \tag{2.2}$$

Notice that extending by zero outside $B_r(x_0)$, we may use $(\Phi - u)_+$ as a test function in (1.3) (first line). Moreover, since u is a weak super-solution, we obtain

$$\int_{\{\Phi>u\}} |\nabla u|^{p-2}\nabla u \cdot \nabla(\Phi - u)dx \geq -\int_{\{\Phi>u\}} f_+(x)(\Phi - u)dx. \tag{2.3}$$

On the other hand, multiplying (2.2) by $\Phi - u$ and integrating by parts we get

$$\int_{\{\Phi>u\}} |\nabla \Phi|^{p-2}\nabla \Phi \cdot \nabla(\Phi - u)dx < -\int_{\{\Phi>u\}} f_+(x)(\Phi - u)dx. \tag{2.4}$$

Next, subtracting (2.3) from (2.4) we obtain

$$\int_{\{\Phi>u\}} (|\nabla \Phi|^{p-2}\nabla \Phi - |\nabla u|^{p-2}\nabla u) \cdot \nabla(\Phi - u)dx < 0, \tag{2.5}$$

Finally, since the left hand side in (2.5) is bounded by below by

$$2^{-p}\int_{\{\Phi>u\}} |\nabla \Phi - \nabla u|^p dx \geq 0,$$

we conclude that $\Phi \le u$ in $B_r(x_0)$. However, this contradicts the fact that $\Phi(x_0) > u(x_0)$. This proves that u is a viscosity super-solution. Analogously, one proves that u is a viscosity sub-solution.

Finally, with a similar reasoning one can deal with the equation

$$-\left[|\nabla u(x)|^{q-2}\Delta u(x) + (q-2)|\nabla u(x)|^{q-4}\Delta_\infty u(x)\right] = -f_-(x) \quad \text{in} \quad \{u<0\}\cap\Omega.$$

$\square$

3 Existence and bounds for minimizers

In this section we will discuss existence and bounds of minimizers to (**Min**) (solutions to (1.3)). Before proving our existence theorem, let us emphasize the lack of convexity for the functional $\mathfrak{J}_{p,q}[\cdot]$. For simplicity, at this point, we are going to restrict our analysis to the case where $f_\pm = 0$ and $\lambda_\pm = 0$. Thus, fixed $j \in \mathbb{N}\setminus\{1\}$, for $k \in \{1,\cdots,j\}$ consider $\Omega = (0,j)$ and

$$u(x) = \begin{cases} 2x-1 & \text{if} \quad x\in[0,1] \\ 1 & \text{if} \quad x\in[1,j] \end{cases}$$

and

$$v(x) = \begin{cases} 2(x-k)+1 & \text{if} \quad x\in\left[k-1,\frac{2k-1}{2}\right] \\ -2(x-k)-1 & \text{if} \quad x\in\left[\frac{2k-1}{2},k\right]. \end{cases}$$

Observe that such functions take the same boundary data. An straightforward calculation shows that

$$\frac{1}{2}\left(\mathfrak{J}_{p,q}[u]+\mathfrak{J}_{p,q}[v]\right) = \frac{2^{p-1}}{p} + \frac{j2^{q-1}}{q}$$

and

$$\mathfrak{J}_{p,q}\left[\frac{u+v}{2}\right] = \frac{(j-1)2^p}{p} + \frac{2^{q-1}}{q}.$$

Therefore, we can choose constants $p \ge q \ge 2$ such that

$$\mathfrak{J}_{p,q}\left[\frac{u+v}{2}\right] > \frac{1}{2}\left(\mathfrak{J}_{p,q}[u]+\mathfrak{J}_{p,q}[v]\right)$$

provided $2^{p-q} > \frac{p}{q}\frac{j-1}{2j-3}$. This shows that $\mathfrak{J}_{p,q}[\cdot]$ is not convex. Finally, a similar argument could be applied to construct examples where the concavity inequality fails as well.

In order to tackle the previous obstacle (the lack of convexity) we will combine methods from the Calculus of Variations with theoretical measure estimates to show the existence of local minimizers (cf. [3], and [18] for a similar strategy).

Theorem 3.1 (Existence of minimizers). *Let $p,q > N$ and $f \in L^r(\Omega)$ with $\max\left\{\frac{1}{p}+\frac{1}{r},\frac{1}{q}+\frac{1}{r}\right\} \le 1$. Then, there exists at least one minimizer u_0 to* (**Min**).

Proof. Before proving our theorem, for convenience we will re-write the functional $\mathfrak{J}_{p,q}[\cdot]$ as follows:

$$\mathfrak{J}_{p,q}[v] = \int_{\Omega\cap\{v>0\}} \frac{1}{p}|\nabla v|^p + \int_{\Omega\cap\{v\le0\}} \frac{1}{q}|\nabla v|^q + \int_\Omega \left(\mathfrak{F}_0(\lambda_-,\lambda_+)[v] + fv\right)dx$$

where

$$\mathfrak{F}_0(\lambda_-,\lambda_+)[v] := \lambda_+^p(x)\chi_{\{v>0\}} + \lambda_-^q(x)\chi_{\{v\le0\}}$$

and
$$f(x) := f_+(x)\chi_{\{v>0\}} + f_-(x)\chi_{\{v\leq 0\}}.$$
Moreover, let us label
$$\mathfrak{J}_0^{(p,q)} := \inf_{v \in \mathscr{K}_g^{(p,q)}(\Omega)} \mathfrak{J}_{p,q}[v].$$

First of all, we will show that $\mathfrak{J}_0^{(p,q)}$ has a lower bound in $\mathscr{K}_g^{(p,q)}(\Omega)$. In fact, for any $v \in \mathscr{K}_g^{(p,q)}(\Omega)$, it follows according to Poincaré's inequality that there exist positive constants $\mathfrak{c}_1 = \mathfrak{c}_1(p,N,\Omega,\|f\|_{L^r}(\Omega))$ and $\mathfrak{c}_2 = \mathfrak{c}_2(q,N,\Omega,\|f\|_{L^r}(\Omega))$ such that

$$\left\{\begin{array}{lcl} \frac{1}{p}\left[\mathfrak{c}_1\left(\|v\|^p_{L^p(\{v>0\})} - \|g\|^p_{L^p(\{v>0\})}\right) - \|\nabla g\|^p_{L^p(\{v>0\})}\right] & \leq & \frac{1}{p}\|\nabla v\|^p_{L^p(\{v>0\})} \\ \frac{1}{q}\left[\mathfrak{c}_2\left(\|v\|^q_{L^q(\{v<0\})} - \|g\|^q_{L^q(\{v<0\})}\right) - \|\nabla g\|^q_{L^q(\{v<0\})}\right] & \leq & \frac{1}{q}\|\nabla v\|^q_{L^q(\{v\leq 0\})}. \end{array}\right. \tag{3.1}$$

Moreover, due to Hölder's inequality, since $r \geq N > \max\left\{\frac{p}{p-1}, \frac{q}{q-1}\right\}$ we obtain

$$\left|\int_\Omega f v\, dx\right| \leq \mathfrak{C}(N,p,q,\Omega)\max\left\{2\max\left\{\|f_\pm\|_{L^r(\Omega)}\right\}, \max\left\{\|v\|^p_{L^p(\Omega)}, \|v\|^q_{L^q(\Omega)}\right\}\right\} := \mathfrak{M}. \tag{3.2}$$

Thus, combining (3.1) and (3.2) we have

$$\begin{array}{lcl} \mathfrak{A} := -\mathfrak{C}_1 - \frac{1}{p}\left[\mathfrak{c}_1\|g\|^p_{L^p(\{v>0\})} + \|\nabla g\|^p_{L^p(\{v>0\})}\right] & \leq & \frac{1}{p}\|\nabla v\|^p_{L^p(\{v>0\})} - \mathfrak{M} \\ \mathfrak{B} := -\mathfrak{C}_2 - \frac{1}{q}\left[\mathfrak{c}_2\|g\|^q_{L^q(\{v<0\})} - \|\nabla g\|^q_{L^q(\{v<0\})}\right] & \leq & \frac{1}{q}\|\nabla v\|^q_{L^q(\{v<0\})} - \mathfrak{M} \end{array}$$

and we conclude that

$$\min\{\mathfrak{A},\mathfrak{B}\} \leq \frac{1}{p}\|\nabla v\|^p_{L^p(\{v>0\})} + \frac{1}{q}\|\nabla v\|^q_{L^q(\{v<0\})} - \mathfrak{M} \leq \mathfrak{J}_{p,q}[v]. \tag{3.3}$$

Therefore, we have checked that the functional $\mathfrak{J}_{p,q}[\cdot]$ is bounded below in $\mathscr{K}_g^{(p,q)}(\Omega)$.

Now we show existence of minimizers to $\mathfrak{J}_{p,q}[\cdot]$. Let $\{u_j\}_{j\geq 1} \subset \mathscr{K}_g^{(p,q)}(\Omega)$ be a minimizing sequence for (**Min**). For $j \gg 1$ (large enough) we have $\mathfrak{J}_{p,q}[u_j] \leq \mathfrak{J}_0^{(p,q)} + 1$. By performing similar arguments as the ones that lead to the previous equations (3.3) we obtain for $\mathfrak{s} := \min\{p,q\}$ that

$$\int_\Omega |\nabla u_j|^{\mathfrak{s}} dx \leq \mathfrak{C}\left(\|u_j\|_{L^{\mathfrak{s}}(\Omega)} + \mathfrak{J}_0^{(p,q)} + 1\right). \tag{3.4}$$

Now, using Poincaré's inequality we have

$$\mathfrak{C}\|u_j\|_{L^{\mathfrak{s}}(\Omega)} \leq \mathfrak{C}\|\nabla u_j\|_{L^{\mathfrak{s}}(\Omega)} + \|g\|_{W^{1,\mathfrak{s}}(\Omega)}. \tag{3.5}$$

Moreover, it holds that

$$\mathfrak{C}\|\nabla u_j\|_{L^{\mathfrak{s}}(\Omega)} \leq \mathfrak{C}_0 + \frac{1}{7}\|\nabla u_j\|^{\mathfrak{s}}_{L^{\mathfrak{s}}(\Omega)}. \tag{3.6}$$

Finally, combining (3.4), (3.5) and (3.6) we get that

$$\int_\Omega |\nabla u_j|^{\mathfrak{s}} dx \leq \mathfrak{C}\|g\|_{W^{1,\mathfrak{s}}(\Omega)} + \mathfrak{J}_0^{(p,q)} + 1.$$

Therefore, invoking one more time Poincaré's inequality we conclude that $\{u_j\}_{j\geq 1}$ is a bounded sequence in $\mathscr{K}_g^{(p,q)}(\Omega)$. Thus, by reflexivity, there exists u_0 such that, modulo a subsequence,

$$\begin{array}{lll} u_j \rightharpoonup u_0 & \text{in} & W^{1,\mathfrak{s}}(\Omega) \\ u_j \to u_0 & \text{in} & L^{\mathfrak{s}}(\Omega) \\ u_j \to u_0 & & \text{a.e. in } \Omega. \end{array}$$

From now on, fix $\varepsilon > 0$. By Egoroff's Theorem there exists an open set $\mathscr{V}_\varepsilon \subset \Omega$ with $\mathcal{L}^N(\Omega \setminus \mathscr{V}_\varepsilon) < \varepsilon$, such that $u_j \to u_0$ uniformly in $\mathscr{V}_\varepsilon$. Next, fixed $\varsigma > 0$, we estimate

$$\begin{array}{rcl} \displaystyle\int_{\mathscr{V}_\varepsilon \cap \{u_0 > \varsigma\}} \frac{1}{p}|\nabla u_0|^p dx & \leq & \displaystyle\liminf_{j\to\infty} \int_{\mathscr{V}_\varepsilon \cap \{u_0 > \varsigma\}} \frac{1}{p}|\nabla u_j|^p dx \quad \leq \quad \liminf_{j\to\infty} \int_{\mathscr{V}_\varepsilon \cap \{u_j > \frac{\varsigma}{3}\}} \frac{1}{p}|\nabla u_j|^p dx \\ & & \displaystyle\leq \quad \liminf_{j\to\infty} \int_{\mathscr{V}_\varepsilon \cap \{u_j > 0\}} \frac{1}{p}|\nabla u_j|^p dx \\ & & \displaystyle\leq \quad \liminf_{j\to\infty} \int_{\Omega \cap \{u_j > 0\}} \frac{1}{p}|\nabla u_j|^p dx. \end{array}$$

Letting $\varsigma \to 0$ in the previous inequality we get that

$$\int_{\mathscr{V}_\varepsilon \cap \{u_0 > 0\}} \frac{1}{p}|\nabla u_0|^p dx \leq \liminf_{j\to\infty} \int_{\Omega \cap \{u_j > 0\}} \frac{1}{p}|\nabla u_j|^p dx. \tag{3.7}$$

Furthermore, from L^p bounds on ∇u_0 we obtain that

$$\int_{(\Omega \setminus \mathscr{V}_\varepsilon) \cap \{u_0 > 0\}} \frac{1}{p}|\nabla u_0|^p dx = \mathrm{O}(\varepsilon). \tag{3.8}$$

Finally, combining (3.7), (3.8) and letting $\varepsilon \to 0+$ we conclude that

$$\int_{\Omega \cap \{u_0 > 0\}} \frac{1}{p}|\nabla u_0|^p dx \leq \liminf_{j\to\infty} \int_{\Omega \cap \{u_j > 0\}} \frac{1}{p}|\nabla u_j|^p dx. \tag{3.9}$$

A similar reasoning can be used in order to obtain the complementary estimate, namely

$$\int_{\Omega \cap \{u_0 \leq 0\}} \frac{1}{q}|\nabla u_0|^q dx \leq \liminf_{j\to\infty} \int_{\Omega \cap \{u_j < 0\}} \frac{1}{q}|\nabla u_j|^q dx. \tag{3.10}$$

Hence, $u_0 \in \mathscr{K}_g^{(p,q)}(\Omega)$. Next, assuming $\lambda_+^p(x) > \lambda_-^q(x)$, we have

$$\begin{array}{rcl} \displaystyle\int_\Omega \lambda_-^q(x)\chi_{\{u_0 \leq 0\}} dx & = & \displaystyle\int_{\{u_0 \leq 0\}} \lambda_-^q(x)\chi_{\{u_j > 0\}} dx + \int_{\{u_0 \leq 0\}} \lambda_-^q(x)\chi_{\{u_j \leq 0\}} dx \\ & \leq & \displaystyle\int_{\{u_0 \leq 0\}} \lambda_+^p(x)\chi_{\{u_j > 0\}} dx + \int_\Omega \lambda_-^q(x)\chi_{\{u_j \leq 0\}} dx. \end{array}$$

Then,

$$\int_\Omega \lambda_-^q(x)\chi_{\{u_0 \leq 0\}} dx \leq \liminf_{j\to\infty} \left[\int_{\{u_0 \leq 0\}} \lambda_+^p(x)\chi_{\{u_j > 0\}} dx + \int_\Omega \lambda_-^q(x)\chi_{\{u_j \leq 0\}} dx\right]$$

Furthermore, since $u_j \to u_0$ a.e. in Ω we obtain

$$\int_\Omega \lambda_+^p(x)\chi_{\{u_0 > 0\}} dx = \int_{\{u_0 > 0\}} \lim_{j\to\infty} \left(\lambda_+^p(x)\chi_{\{u_j > 0\}}\right) dx = \lim_{j\to\infty} \int_{\{u_0 > 0\}} \lambda_+^p(x)\chi_{\{u_j > 0\}} dx.$$

In the same way, under the regime $\lambda_+^p(x) \leq \lambda_-^q(x)$ we obtain the estimates

$$\int_\Omega \lambda_+^p(x)\chi_{\{u_0 > 0\}} dx \leq \liminf_{j\to\infty} \left[\int_{\{u_0 > 0\}} \lambda_-^q(x)\chi_{\{u_j \leq 0\}} dx + \int_\Omega \lambda_+^p(x)\chi_{\{u_j > 0\}} dx\right]$$

and

$$\int_\Omega \lambda_-^q(x)\chi_{\{u_0\leq 0\}}dx = \int_{\{u_0\leq 0\}} \lim_{j\to\infty}\left(\lambda_-^q(x)\chi_{\{u_j\leq 0\}}\right)dx = \lim_{j\to\infty}\int_{\{u_0\leq 0\}} \lambda_-^q(x)\chi_{\{u_j\leq 0\}}dx.$$

Therefore, in any case, we get

$$\int_\Omega \mathfrak{F}_0(\lambda_+,\lambda_-)[u_0]dx \leq \liminf_{j\to\infty}\int_\Omega \mathfrak{F}_0(\lambda_+,\lambda_-)[u_j]dx. \tag{3.11}$$

Similarly, we may prove the lower semi-continuity for f, i.e,

$$\int_\Omega f u_0 dx \leq \liminf_{j\to\infty}\int_\Omega f u_j dx. \tag{3.12}$$

Finally, combining (3.9), (3.10), (3.11) and (3.12) we conclude that

$$\mathfrak{J}_{p,q}[u_0] \leq \liminf_{j\to\infty}\mathfrak{J}_{p,q}[u_j] = \mathfrak{J}_0^{(p,q)}.$$

Therefore, the limiting function u_0 is a minimizer to $\mathfrak{J}_{p,q}[\cdot]$ and this finishes the proof of the theorem. □

Example 3.2. We must stress that uniqueness of minimizers of the variational problem does not hold in general. In fact, take $\Omega = B_R \subset \mathbb{R}^N$ and a constant boundary datum $g = \alpha > 0$ on $\partial\Omega$, we have for $u_0 = \alpha$ on $\overline{\Omega}$ and $p = q$

$$\mathfrak{J}_{p,q}[u_0] = \lambda_+^p R^N \omega_N,$$

where ω_N is the volume of the unit ball. Now, let us suppose there exists a unique minimizer v of the functional $\mathfrak{J}_{p,q}[\cdot]$. Then, such a minimizer is radially symmetric, because the operator p-Laplacian is invariant under rotations. For this reason, there exists a constant $a > 0$ such that

$$v(x) := \begin{cases} \mathfrak{c}_1|x|^{\frac{p-N}{p-1}} + \mathfrak{c}_2 & \text{if} \quad a \leq |x| \leq R \\ 0 & \text{if} \quad |x| \leq a, \end{cases}$$

where $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are positive constants satisfying the following relation

$$\begin{cases} \mathfrak{c}_1|R|^{\frac{p-N}{p-1}} + \mathfrak{c}_2 &= \alpha \\ \mathfrak{c}_1|a|^{\frac{p-N}{p-1}} + \mathfrak{c}_2 &= 0, \end{cases}$$

from which we find that

$$\mathfrak{c}_1 = \frac{\alpha}{|R|^{\frac{p-N}{p-1}} - |a|^{\frac{p-N}{p-1}}} \qquad \text{and} \qquad \mathfrak{c}_2 = \frac{-\alpha|a|^{\frac{p-N}{p-1}}}{|R|^{\frac{p-N}{p-1}} - |a|^{\frac{p-N}{p-1}}}.$$

Then, an straightforward calculation shows that

$$\mathfrak{J}_{p,q}[u_0] - \mathfrak{J}_{p,q}[v] = a^N\omega_N(\lambda_-^p - \lambda_+^p) - \frac{N\omega_N\alpha^p}{p}\frac{1}{\left(|R|^{\frac{p-N}{p-1}} - |a|^{\frac{p-N}{p-1}}\right)^{p-1}}\left|\frac{p-N}{p-1}\right|^{p-1}.$$

Finally, if we select carefully the values λ_+ and λ_-, we can make this difference vanish obtaining two different minimizersu_0 and v. For complete details of this computation we refer to [19].

Next, we turn our attention to $L^\infty-$bounds for minimizers.

Theorem 3.3. *Let $p,q>N$ and $f_\pm \in L^r(\Omega)$ with* $\max\left\{\frac{1}{p}+\frac{1}{r},\frac{1}{q}+\frac{1}{r}\right\}\le 1$. *Then, any minimizer u_0 to* (**Min**) *fulfils*

$$\|u\|_{L^\infty(\Omega)} \le C\left(N,p,q,\lambda_+,\lambda_-,\|g\|_{L^\infty(\Omega)},\|f\|_{L^r(\Omega)}\right).$$

Proof. First, let $\mathrm{j}_0 := \left\lceil \sup\limits_{\partial\Omega} g(x)\right\rceil$, i.e., the smallest natural number greater than or equal to $\sup\limits_{\partial\Omega} g(x)$. Next, for each $j\ge \mathrm{j}_0$ we consider the truncation $u_j:\Omega\to\mathbb{R}$ given by

$$u_j(x) := \begin{cases} j\cdot \mathrm{sign}(u(x)) & \text{if} \quad |u|>j \\ u(x) & \text{if} \quad |u|\le j. \end{cases}$$

Moreover, if we call $\mathscr{A}_j := \{|u|>j\}$, then for each $j>\mathrm{j}_0$ we have

$$u(x)=u_j(x) \quad \text{in} \quad \Omega\setminus\mathscr{A}_j \quad \text{and} \quad u_j(x)=j\cdot\mathrm{sign}(u(x)) \quad \text{in} \quad \mathscr{A}_j.$$

From the fact that u is a minimizer we obtain

$$\begin{aligned} \int_{\mathscr{A}_j\cap\{u>0\}} |\nabla u|^p dx &= \int_{\Omega\cap\{u>0\}} |\nabla u|^p - |\nabla u_j|^p dx \\ &\le \int_{\mathscr{A}_j\cap\{u>0\}} f_+(u_j-u)dx \end{aligned} \tag{3.13}$$

Furthermore, notice that

$$\begin{aligned} \int_{\mathscr{A}_j} f_+(u_j-u)dx &= \int_{\mathscr{A}_j\cap\{u>0\}} f_+(j-u)dx + \int_{\mathscr{A}_j\cap\{u\le 0\}} f_+(u-j)dx \\ &\le 2\int_{\mathscr{A}_j} |f_+|(|u|-j)dx. \end{aligned}$$

Now, recall that u_j and u have the same sign. Consequently, it follows that $(|u|-j)^+\in W_0^{1,p}(\Omega)$. Thus, using Hölder and Gagliardo-Nirenberg-Sobolev inequalities we obtain

$$\begin{aligned} \int_{\mathscr{A}_j\cap\{u>0\}} |f_+|(|u|-j)^+dx &\le \|f_+\|_{L^{\frac{p}{p-1}}(\mathscr{A}_j\cap\{u>0\})} \|(|u|-j)^+\|_{L^p(\mathscr{A}_j\cap\{u>0\})} \\ &\le \|f_+\|_{L^{p'}(\mathscr{A}_j\cap\{u>0\})} \mathcal{L}^N(\mathscr{A}_j)^\theta \|\nabla u\|_{L^p(\mathscr{A}_j\cap\{u>0\})}, \end{aligned}$$

where $\theta := 1-\frac{1}{p^*}-\frac{1}{p'}$ and p^* is the critical Sobolev exponent. Now, using Young inequality we get

$$\|f_+\|_{L^{p'}(\mathscr{A}_j\cap\{u>0\})} \mathcal{L}^N(\mathscr{A}_j)^\theta \|\nabla u\|_{L^p(\mathscr{A}_j\cap\{u>0\})} \le \mathfrak{C}\mathcal{L}^N(\mathscr{A}_j)^{\frac{p}{p-1}\theta} + \frac{1}{2}\|\nabla u\|^p_{L^p(\mathscr{A}_j\cap\{u>0\})}. \tag{3.14}$$

Therefore, from (3.13) and (3.14) we obtain

$$\int_{\mathscr{A}_j\cap\{u>0\}} |\nabla u|^p dx \le \mathfrak{C}\mathcal{L}^N(\mathscr{A}_j)^{1-\frac{p}{N}+\frac{p(pp'-N)}{N(p-1)p'}}$$

and (cf. (3.1) and (3.4) changing $\mathfrak{J}_0^{(p,q)}$ by $\mathfrak{J}_{p,q}[g]$)

$$\|u\|_{L^1(\mathscr{A}_{\mathrm{j}_0}\cap\{u>0\})} \le \mathcal{L}^N(\mathscr{A}_{\mathrm{j}_0}\cap\{u>0\})\|u\|_{L^p(\mathscr{A}_{\mathrm{j}_0}\cap\{u>0\})}.$$

Mutatis mutandis a similar estimate holds for the negative part of u. Finally, boundedness of u will follow from general mathematical tools come from elliptic PDE theory (cf. [17, Chapter 2, Lemma 5.2]). □

Remark 3.4. As a byproduct of L^∞ bounds for a minimizer u of the functional $\mathfrak{J}_{p,q}[\cdot]$ we obtain a universal control of u in the $W^{1,p}(\{u>0\}) \cap W^{1,q}(\{u<0\})$ topology. In fact, we get

$$\begin{aligned}
\int_{\Omega\cap\{u>0\}} |\nabla u|^p dx &\leq \mathfrak{J}_{p,q}[g] - \int_{\Omega\cap\{u>0\}} (\lambda_+^p(x) + |f_+||u|) dx \\
&\leq \mathfrak{J}_{p,q}[g] + \mathfrak{C}\left(N, p, \Omega, \|f_+\|_{L^{p'}(\Omega\cap\{u>0\})}\right) \\
&\leq \mathfrak{C}_\sharp\left(N, p, \Omega, \|g\|, \|f_+\|_{L^{p'}}\right),
\end{aligned}$$

with a similar estimate holding in the negativity set of u. Therefore,

$$\max\left\{\|u\|_{W^{1,p}(\{u>0\})}, \|u\|_{W^{1,q}(\{u<0\})}\right\} \leq \mathfrak{C}_\sharp\left(N, p, q, \Omega, \|g\|, \|f_+\|_{L^{p'}}, \|f_-\|_{L^{q'}}\right).$$

We will finish this section by bringing to light the Euler-Lagrange equation related to the functional $\mathfrak{J}_{p,q}[\cdot]$, as well as the free boundary condition (the flux balance through the phase transition) which is satisfied by any minimizer u_0 along the free boundary.

Proposition 3.5. *Let u_0 be a solution to the minimization problem* **Min**. *Then u_0 satisfies in the weak sense*

$$\left\{\begin{array}{rcllc}
\Delta_p u_0 &=& f_+(x) & in & \{u_0>0\}\cap\Omega \\
\Delta_q u_0 &=& f_-(x) & in & \{u_0\leq 0\}^\circ\cap\Omega \\
u_0(x) &=& g(x) & on & \partial\Omega.
\end{array}\right. \tag{3.15}$$

A proof for such a result is rather standard. For this reason, we will omit it.

Next, we will focus our attention at the equation satisfied through the free boundary for minimizers to **Min**. For this purpose, consider $x_0 \in \partial\{u_0>0\}\cup\partial\{u_0<0\}$ a free boundary point, $\mathscr{B}$ a small ball centered at x_0, $\Phi \in C_0^1(B_s(x_0), \mathbb{R}^N)$ a vector field and $\varepsilon = \mathrm{o}(1)$. Thus, we define the quantities

$$\Xi_\varepsilon^+(x_0) := \int_{\mathscr{B}_s(x_0)\cap\partial\{u_0>\varepsilon\}} \left(\frac{p-1}{p}|\nabla u_0(x)|^p - \lambda_+^p(x)\right) \nu_1\cdot\Phi d\mathscr{H}^{N-1}$$

and

$$\Xi_\varepsilon^-(x_0) := \int_{\mathscr{B}_s(x_0)\cap\partial\{u_0<-\delta\}} \left(\frac{q-1}{q}|\nabla u_0(x)|^q - \lambda_-^q(x)\right) \nu_2\cdot\Phi d\mathscr{H}^{N-1}$$

Proposition 3.6. *Let u_0 be a minimizer to* **Min** *with $\mathcal{L}^N(\{u_0=0\}\cap\Omega)=0$, $x_0\in\partial\{u_0>0\}\cup\partial\{u_0<0\}$ a free boundary point and $\mathscr{B}_s(x_0)\subset\Omega$. Then, for any $\Phi\in C_0^1(B_s(x_0),\mathbb{R}^N)$, there holds*

$$\lim_{\varepsilon\searrow 0}\Xi_\varepsilon^+(x_0) + \lim_{\delta\nearrow 0}\Xi_\delta^-(x_0) = 0.$$

The proof of the previous result follows by Hadamard's methods, i.e., domain variation techniques. We will also omit such details of the proof here and refer to [24, Lemma 2.4]. In particular, we must highlight that the balance flux

$$\frac{p-1}{p}(u_\nu^+)^p(x_0) - \frac{q-1}{q}(u_\nu^-)^q(x_0) = \lambda_+^p(x_0) - \lambda_-^q(x_0),$$

holds in the classical sense along C^1 pieces of the free boundary, where $u_\nu^\pm$ are respectively the normal derivatives in the inward direction to $\partial\Omega^\pm[u]$.

4 Further properties for minimizers

In this section we will show that any minimizer u_0 to (**Min**) grows linearly away from the free boundary $\mathfrak{F}^+_{\Omega}[u_{p,q}] := \partial\{u_{p,q} > 0\} \cap \Omega$ (resp. $\mathfrak{F}^-_{\Omega}[u_{p,q}] := \partial\{u_{p,q} < 0\} \cap \Omega$). An essential tool we will use is the non-homogeneous Harnack inequality, which we state below for completeness.

Theorem 4.1 (**Serrin's Harnack inequality, see [25] and [26]).** *Let $0 \le \phi \in W^{1,p}(B_R)$, satisfying*

$$\Delta_p \phi(x) = f(x) \quad in \quad B_R$$

in the weak sense ,with $f \in L^s(B_R)$and $s > \frac{N}{p}$. Then, there exists a constant $\mathfrak{C} = \mathfrak{C}(N,p,s,R-r) > 0$ such that

$$\sup_{B_r} \phi(x) \le \mathfrak{C}\left[\inf_{B_r} \phi(x) + \left(r^{p-\frac{N}{s}} \|f\|_{L^s(B_R)}\right)^{\frac{1}{p-1}}\right].$$

Theorem 4.2. *Let u_0 be a minimizer to* (**Min**)*, with $f_\pm \in L^r(\Omega)$,$r > N$, $\lambda_+ \in L^p(\Omega)$, $\lambda_- \in L^q(\Omega)$, $\Omega' \Subset \Omega$ and $x_0 \in \mathfrak{F}^+_{\Omega'}[u_0]$ (resp. $x_0 \in \mathfrak{F}^-_{\Omega'}[u_0]$). Then, there exists a constant $\mathfrak{c}_\pm > 0$ depending only on N, p, q, $\|\lambda_+\|_{L^p}$, $\|\lambda_-\|_{L^q}$ and $\|f_\pm\|_{L^r}$ such that*

$$\pm u_0(x) \ge \mathfrak{c}_\pm \mathrm{dist}(x, \mathfrak{F}^\pm_{\Omega}[u_0]). \tag{4.1}$$

Proof. We will prove the estimate just in the positive phase, because the other one can be obtained in a similar way. Fix $x_0 \in \mathfrak{F}^+_{\Omega'}[u_0]$. Notice that it suffices to show such an estimate for points close enough to the free boundary, in other words,

$$0 < \mathrm{dist}(x_0, \mathfrak{F}^+_{\Omega'}[u_0]) \ll \iota$$

where ι depends on dimension, p,q, and data of the problem and, it will be choosed a posteriori. Now, define $\mathfrak{d} := \mathrm{dist}(x_0, \mathfrak{F}^+_{\Omega'}[u_0])$ and the scaled function

$$\omega(x) := \frac{u(x_0 + \mathfrak{d}x)}{\mathfrak{d}}.$$

Notice that the thesis of our Theorem is equivalent to establishing that $\omega(0) \ge \mathfrak{c}$ (bounded away from zero) for a universal constant $\mathfrak{c} > 0$. It is easy to check that ω is a minimizer to

$$\mathfrak{J}^{\mathfrak{d}}_p[\phi] := \int_{B_1} \left(\frac{1}{p}|\nabla\phi|^p + \lambda_+(y)^p \chi_{\{\phi>0\}} + \mathfrak{d} f_+(y)\phi\right) dy$$

where $y = x_0 + \mathfrak{d}x$. By our construction $\omega > 0$ in B_1, as well as

$$\Delta_p \omega(y) = \mathfrak{d} f_+(y) \quad \text{in} \quad B_1$$

in the weak sense. Then, by using the Harnack's inequality (Theorem 4.1) we obtain

$$\omega(z) \le \mathfrak{C}(N,p)\left[\omega(0) + \left(\mathfrak{d}^{\frac{r-N}{r}} \|f_+\|_{L^r(\Omega)}\right)^{\frac{1}{p-1}}\right],$$

for any $x \in B_{\frac{4}{5}}$. Next, we will choose a non-negative, smooth radially symmetric cut-off function Θ verifying

$$\begin{cases} 0 \le \Theta \le 1 & \text{in} \quad B_1 \\ \Theta = 0 & \text{in} \quad B_{\frac{1}{7}} \\ \Theta = 1 & \text{in} \quad B_1 \setminus B_{\frac{1}{2}}, \end{cases}$$

as well as we define $\Psi : B_1 \to \mathbb{R}_+$ as the following test function

$$\Phi(x) := \begin{cases} \min\left\{\omega(x), \mathfrak{C}(N,p)\left[\omega(0)+\mathfrak{d}^{\frac{r-N}{r}}\|f_+\|_{L^r(\Omega)}\right].\Theta(x)\right\} & \text{in} \quad B_1 \\ \omega(x) & \text{in} \quad B_1 \setminus B_{\frac{1}{2}}. \end{cases}$$

Now, let us define the following set

$$\Xi := \left\{z \in B_{\frac{1}{2}} \mid \mathfrak{C}(N,p)\left[\omega(0)+\mathfrak{d}^{\frac{r-N}{r}}\|f_+\|_{L^r(\Omega)}\right].\Theta(z) < \omega(z)\right\}.$$

It is easy to verify that $B_{\frac{1}{7}} \subset \Xi \subset B_{\frac{1}{2}}$. From minimality of ω we obtain

$$\begin{aligned} \Pi &:= \int_{\Xi}\left[\lambda_+^p(x_0+\mathfrak{d}x)(1-\chi_{\{\Phi>0\}})+\mathfrak{d}f_+(x_0+\mathfrak{d}x)[\omega(x)-\Phi(x)]\right]dx \\ &\leq \int_{\Xi}(|\nabla\Phi|^p-|\nabla\omega|^p)dx \\ &\leq \left[\mathfrak{C}(N,p)\left(\omega(0)+(\mathfrak{d}^{\frac{r-N}{r}}\|f_+\|_{L^r(\Omega)})^{\frac{1}{p-1}}\right).\|\Theta\|_{L^\infty(B_1)}\right]^p \\ &\leq 2^p\mathfrak{C}(N,p)^p\left[\omega(0)^p+\left(\mathfrak{d}^{\frac{r-N}{r}}\|f_+\|_{L^r(\Omega)}\right)^{\frac{p}{p-1}}\right]. \end{aligned} \tag{4.2}$$

Now, we turn our attention towards a lower bound control for the LHS of (4.2). Thus, we estimate

$$\int_{\Xi}\lambda_+^p(x_0+\mathfrak{d}x)(1-\chi_{\{\Phi>0\}})dx = \int_{\Xi}\lambda_+^p(x_0+\mathfrak{d}x)\chi_{\{\Phi=0\}}dx \geq \|\lambda_+\|^p_{L^p(B_{1/7})} \tag{4.3}$$

Applying the Harnack inequality (Theorem 4.1) and the fact that $\Xi \subset B_{\frac{1}{2}}$ we have that

$$0 \leq \omega-\Phi \leq \omega \leq \mathfrak{C}(N,p)\left[\omega(0)+\left(\mathfrak{d}^{\frac{r-N}{r}}\|f_+\|_{L^r(\Omega)}\right)^{\frac{1}{p-1}}\right] \quad \text{in} \quad \Xi.$$

Thus, we estimate

$$\begin{aligned} \Pi_0 &:= -\int_{\Xi}\mathfrak{d}f_+(x_0+\mathfrak{d}x)[\omega(x)-\Phi(x)]dx \\ &\leq \mathfrak{d}\|\omega-\Phi\|_{L^{\frac{r}{r-1}}(\Xi)}\|f_+(x_0+\mathfrak{d}x)\|_{L^r(\Xi)} \\ &\leq \mathfrak{d}^{\frac{r-N}{r}}\mathfrak{C}(N,p,r,\Xi)\left[\omega(0)+\left(\mathfrak{d}^{\frac{r-N}{r}}\|f_+\|_{L^r(\Omega)}\right)^{\frac{1}{p-1}}\right]\|f_+\|_{L^r(\Omega)} \end{aligned} \tag{4.4}$$

holds. Now, combining (4.2), (4.3) and (4.4) we obtain

$$2^p\mathfrak{C}(N,p)^p\left[\omega(0)^p+\left(\mathfrak{d}^{\frac{r-N}{r}}\|f_+\|_{L^r(\Omega)}\omega(0)\right)\right] \geq \|\lambda_+\|^p_{L^p(B_{1/7})}-\mathfrak{C}(N,p,r,\Xi)\left(\mathfrak{d}^{\frac{r-N}{r}}\|f_+\|_{L^r(\Omega)}\right)^{\frac{p}{p-1}}. \tag{4.5}$$

Therefore, choosing appropriately $0 < \mathfrak{d} \leq \iota(N,p,\lambda_+,\|f_+\|_{L^r(\Omega)}) \ll 1$ we conclude

$$\omega(0) \geq \mathfrak{c}(N,p,\lambda_+,\|f_+\|_{L^r(\Omega)}) > 0$$

as desired. This concludes the proof of the Theorem. □

Remark 4.3. From the proof of Theorem 4.2, according to (4.5), we have the following estimate

$$\omega(0) \geq \frac{1}{2\mathfrak{C}(N,p)}.\|\lambda_+\|_{L^p(B_{1/7})}$$

where $\mathfrak{C}(N,p) > 0$ is the constant from Harnack inequality (Theorem 4.1).

In what follows we are going to iterate the linear growth estimates obtained in the Theorem 4.2 in order to establish a strong non-degeneracy property for minimizers u_0 near a free boundary point. More precisely, we have the following result:

Theorem 4.4. *Let u_0 be a minimizer to* (**Min**)*, with $f_\pm \in L^r(\Omega)$, $r > N$, $\lambda_+ \in L^p(\Omega)$, $\lambda_- \in L^q(\Omega)$, $\Omega' \Subset \Omega$ and $x_0 \in \{u_0 \geq 0\} \cap \Omega'$ (resp. $x_0 \in \{u_0 \leq 0\} \cap \Omega'$). Then, there exist constants $\mathfrak{c}^*_\pm > 0$ depending only on $N, p, q, \|\lambda_+\|_{L^p}$, $\|\lambda_-\|_{L^q}$ and $\|f_\pm\|_{L^r}$ such that*

$$\sup_{B_{r_0}(x_0)\cap\Omega^\pm} \pm u_0(x) \geq \mathfrak{c}^*_\pm r_0 \quad \textit{for any} \quad 0 < r_0 \leq \operatorname{dist}(\partial\Omega', \partial\Omega). \tag{4.6}$$

Proof. First of all, it suffices to show the thesis of the Theorem within the positive phase $(\Omega')^+[u_0]$ due to continuity for minimizers.

Let us begin by establishing the existence of a $\sigma_0 = \sigma_0(N, p, \lambda_+.\|f_+\|_{L^r(\Omega)}, \Omega') > 0$ and the data of the problem, such that if $x \in (\Omega')^+[u_0]$, then there holds

$$\sup_{B_{\mathfrak{d}(x)}} u_0(x) \geq (1+\sigma_0)u_0(x_0), \tag{4.7}$$

where $\mathfrak{d}(x) := \operatorname{dist}(x, \partial(\Omega')^+[u_0])$. In order to check (4.7), we will assume, for sake of contradiction, that such a σ_0 does not exist. Then, we can find sequences $\sigma_j = \mathrm{o}(1)$ and $x_j \in (\Omega')^+[u_0]$ such that

$$\sup_{B_{\mathfrak{d}_j(x_j)}} u_0(x) \geq (1+\sigma_j)u_0(x_j), \tag{4.8}$$

where $\mathfrak{d}_j(x_j) := \operatorname{dist}(x_j, \partial(\Omega')^+[u_0]) = \mathrm{o}(1)$ as $j \to \infty$. Now, we define the normalized sequence $v_j : B_1 \to \mathbb{R}$ given by

$$v_j(y) := \frac{u_0(x_j + \mathfrak{d}_j y)}{u_0(x_j)}.$$

We have that $v_j(0) = 1$ and, from (4.8) we get

$$0 \leq v_j \leq 1 + \sigma_j \quad \text{in} \quad B_1.$$

Moreover, v_j satisfies in B_1 in the weak sense

$$\Delta_p v_j = \frac{\mathfrak{d}_j^p}{u(x_j)^{p-1}} f_+(x_j + \mathfrak{d}_j y). \tag{4.9}$$

Hence, taking into account the linear growth from Theorem 4.2 and estimate (4.9), we obtain

$$\Delta_p v_j \leq \mathfrak{C}\mathfrak{d}_j f_+(x_j + \mathfrak{d}_j y).$$

By Harnack inequality (Theorem 4.1), we deduce that $\{v_j\}_{j\geq 1}$ is an equicontinuous sequence in B_1. Thus, we may assume that $v_j \to v$ locally uniformly in B_1. One more time Harnack inequality revels that for any x such that $|x| \leq r_0 < 1$, there holds

$$0 \leq 1 + \sigma_j - v_j(x) \leq \mathfrak{C}\left[1 + \sigma_j - v_j(0) + \left(\mathfrak{d}_j^{1-\frac{N}{r}} \|f_+\|_{L^r(\Omega)}\right)^{\frac{1}{p-1}}\right] = \mathfrak{C} \cdot \mathrm{o}(1).$$

By letting $j \to \infty$ we conclude that the limiting blow-up profile v is identically 1 in B_1.

Now, we will show that such a conclusion yields a contradiction. To this end, let $y_j \in \partial(\Omega')^+[u_0]$ such that $\mathfrak{d}_j = |x_j - y_j|$. Thus, up to a subsequence, there would hold

$$1 + \mathrm{o}(1) = v_j\left(\frac{y_j - x_j}{\mathfrak{d}_j}\right) = 0$$

which clearly is an absurd for j large enough.

Therefore, we just need to prove that the estimate (4.7) hold. Such a conclusion will follow by Caffarelli's polygonal type of argument. Precisely, we construct a polygonal along which u_0 grows linearly. Starting from $x = x_0$ we find a sequence of points $\{x_k\}_{k\geq 1}$ such that:

- ✓ $u_0(x_k) \geq (1+\sigma_0)^k u_0(x_0)$;
- ✓ $\text{dist}(x_{k-1}, \partial(\Omega')^+[u_0]) = |x_k - x_{k-1}|$;
- ✓ $u_0(x_k) - u_0(x_{k-1}) \geq \mathfrak{c}|x_k - x_{k-1}|$. In particular, we get that

$$u_0(x_k) - u_0(x_0) \geq \mathfrak{c}|x_k - x_0|.$$

Since $u_0(x_k) \to \infty$ as $k \to \infty$ this process must be finite. Then, there exists a last $x_{k_0} \in B_{r_0}(x_0)$ and for such a point, we have $|x_{k_0} - x_0| \geq \mathfrak{c}(N,p)r_0$. Finally, we conclude that

$$\sup_{B_{r_0}(x_0)} u_0(x) \geq u_0(x_{k_0}) \geq u_0(x_0) + \mathfrak{c}(p,N)|x_k - x_0| \geq \mathfrak{c}(p,N)r_0,$$

which finishes the proof. □

5 The limit problem as $p, q \to \infty$

In this last Section we will establish the limit profile as p, q go to infinity for our minimization problem.

Lemma 5.1. *Let $u_{p,q}$ be a minimizer to* (**Min**)*. Then, there exists a constant $\mathfrak{C}_0 = \mathfrak{C}_0(g, \Omega, p, q, \lambda_\pm, f_\pm) > 0$ such that*

$$\max\left\{\|\nabla u_{p,q}\|_{L^p(\Omega)}, \|\nabla u_{p,q}\|_{L^q(\Omega)}\right\} \leq \mathfrak{C}_0.$$

Furthermore, it holds that

$$\lim_{p,q\to\infty} \mathfrak{C}_0 = \max\left\{1, [g]_{C^{0,1}(\overline{\Omega})}, [g]^{\mathscr{Q}}_{C^{0,1}(\overline{\Omega})}, [g]^{\frac{1}{\mathscr{Q}}}_{C^{0,1}(\overline{\Omega})}, \|\lambda_+\|_{L^\infty(\Omega)}, \|\lambda_+\|^{\frac{1}{\mathscr{Q}}}_{L^\infty(\Omega)}, \|\lambda_-\|^{\mathscr{Q}}_{L^\infty(\Omega)}, \|\lambda_-\|_{L^\infty(\Omega)}\right\}.$$

Proof. Let Ψ be a Lipschitz extension of g among functions in the set

$$\mathscr{K}_\infty := \{v \in W^{1,\infty}(\Omega) \mid v = g \text{ on } \partial\Omega\}.$$

Since Ω is bounded, Ψ competes in the minimization problem (**Min**). Thus, by using Ψ as a test function in (**Min**), we obtain

$$\mathfrak{J}_{p,q}[u_{p,q}] \leq \mathfrak{J}_{p,q}[\Psi].$$

On the other hand,

$$\begin{aligned}\mathfrak{J}_{p,q}[\Psi] \leq \mathcal{L}^N(\Omega)\left(\frac{1}{p}\text{Lip}(\Psi)^p + \frac{1}{q}\text{Lip}(\Psi)^q\right) + \|\lambda_+\|^p_{L^p(\Omega)} \\ +\|\lambda_-\|^q_{L^q(\Omega)} + \|f_+\|_{L^{p'}(\Omega)}\|\Psi\|_{L^p(\Omega)} + \|f_+\|_{L^{q'}(\Omega)}\|\Psi\|_{L^q(\Omega)}.\end{aligned}$$

Now, notice that

$$\begin{aligned}\int_{\Omega\cap\{u_{p,q}>0\}} \frac{1}{p}|\nabla u_{p,q}|^p dx &\leq \mathfrak{J}_{p,q}[\Psi] - \int_{\Omega\cap\{u_{p,q}>0\}} \left(\lambda_+^p(x) + |f_+||u|\right) dx \\ &\leq \mathfrak{J}_{p,q}[\Psi].\end{aligned}$$

Similarly, one obtains

$$\int_{\Omega\cap\{u_{p,q}<0\}} \frac{1}{q}|\nabla u_{p,q}|^q dx \le \mathfrak{J}_{p,q}[\Psi].$$

Now, if $q>p$, then

$$\left(\int_\Omega |\nabla u_{p,q}|^p\right)^{\frac{1}{p}} \le \sqrt[p]{p\mathfrak{J}_{p,q}[\Psi]} + \sqrt[q]{p\mathfrak{J}_{p,q}[\Psi]}\mathcal{L}^N(\Omega)^{\frac{q-p}{pq}}.$$

On the other hand, if $p>q$,

$$\left(\int_\Omega |\nabla u_{p,q}|^q\right)^{\frac{1}{q}} \le \sqrt[q]{q\mathfrak{J}_{p,q}[\Psi]} + \sqrt[p]{q\mathfrak{J}_{p,q}[\Psi]}\mathcal{L}^N(\Omega)^{\frac{q-p}{pq}}.$$

Therefore,

$$\max\left\{\|\nabla u_{p,q}\|_{L^p(\Omega)}, \|\nabla u_{p,q}\|_{L^q(\Omega)}\right\} \le \mathfrak{C}(\Psi,\Omega,p,q,\lambda_\pm,f_\pm),$$

where

$$\mathfrak{C}(\Psi,\Omega,p,q,\lambda_\pm,f_\pm) = \max\left\{\sqrt[p]{p\mathfrak{J}_{p,q}[\Psi]} + \sqrt[q]{p\mathfrak{J}_{p,q}[\Psi]}\mathcal{L}^N(\Omega)^{\frac{q-p}{pq}}, \sqrt[q]{q\mathfrak{J}_{p,q}[\Psi]} + \sqrt[p]{q\mathfrak{J}_{p,q}[\Psi]}\mathcal{L}^N(\Omega)^{\frac{q-p}{pq}}\right\}.$$

Therefore, the sequence $u_{p,q}$ is uniformly bounded in $W^{1,p}(\Omega)\cap W^{1,q}(\Omega)$, and its weak limit as $p,q\to\infty$ fulfils

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le \max\left\{1, \mathrm{Lip}[\Psi], \mathrm{Lip}[\Psi]^{\mathcal{Q}}, \mathrm{Lip}[\Psi]^{\frac{1}{\mathcal{Q}}}, \|\lambda_+\|_{L^\infty(\Omega)}, \|\lambda_+\|^{\frac{1}{\mathcal{Q}}}_{L^\infty(\Omega)}, \|\lambda_-\|^{\mathcal{Q}}_{L^\infty(\Omega)}, \|\lambda_-\|_{L^\infty(\Omega)}\right\}.$$

□

As an immediate consequence of previous analysis we are able to prove the following theorem:

Theorem 5.2. *Let $p,q>N$ and $f_\pm\in L^r(\Omega)$ with $\max\left\{\frac{1}{p}+\frac{1}{r},\frac{1}{q}+\frac{1}{r}\right\}\le 1$. Then, for all sequence of solutions $u_{p,q}$ to* (**Min**)*, there exists a subsequence, denoted by $u_{p,q}$ yet, such that $u_{p,q}\to u_\infty$ uniformly in $\overline{\Omega}$. Furthermore, $u_\infty\in W^{1,\infty}_g(\Omega)$ with*

$$[u_\infty]_{C^{0,1}(\overline{\Omega})} := \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{|w(x)-w(y)|}{|x-y|} \le \lim_{p,q\to\infty}\mathfrak{C}(\Psi,\Omega,p,q,\lambda_\pm,f_\pm).$$

Proof. From Lemma 5.1 we have that

$$\max\left\{\|\nabla u_{p,q}\|_{L^p(\Omega)}, \|\nabla u_{p,q}\|_{L^q(\Omega)}\right\} \le \mathfrak{C}_0.$$

Next, fix m, and take $p,q>m$. We have,

$$\left(\int_\Omega |\nabla u_{p,q}|^m\right)^{1/m} \le |\Omega|^{\frac{1}{m}-\frac{1}{p}}\|\nabla u_{p,q}\|_{L^p(\Omega)} \le |\Omega|^{\frac{1}{m}-\frac{1}{p}}\mathfrak{C}_0.$$

Hence, there exists a weak limit in $W^{1,m}(\Omega)$ that we will denote by u_∞. This weak limit has to verify

$$\left(\int_\Omega |\nabla u_\infty|^m\right)^{1/m} \le |\Omega|^{\frac{1}{m}}\lim_{p,q\to\infty}\mathfrak{C}_0.$$

As the above inequality holds for every m, we get that $u_\infty\in W^{1,\infty}(\Omega)$ and moreover, taking the limit $m\to\infty$,

$$|\nabla u_\infty| \le \lim_{p,q\to\infty}\mathfrak{C}_0, \qquad \text{a.e. } x\in\Omega.$$

Therefore, we have

$$[u_\infty]_{C^{0,1}(\overline{\Omega})} := \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{|w(x)-w(y)|}{|x-y|} \leq \lim_{p,q\to\infty} \mathfrak{C}(\Psi,\Omega,p,q,\lambda_\pm,f_\pm).$$

□

We will comment throughout this section how the source term f influences on the limit, it is through its support and sign.

Before starting let us define the following space

$$\mathfrak{Z} := \left\{ w\in C^{0,1}(\overline{\Omega}) \;\middle|\; w=g \text{ in } \partial\Omega \text{ and } [w]_{C^{0,1}(\overline{\Omega})} := \sup_{\substack{x,y\in\overline{\Omega}\\ x\neq y}} \frac{|w(x)-w(y)|}{|x-y|} \leq 1 \right\}.$$

Under such a definition the following Theorem holds (See [6] for similar result in isotropic case).

Theorem 5.3. *Let $f_\pm \in L^r(\Omega)$ with $\max\left\{\frac{1}{p}+\frac{1}{r}, \frac{1}{q}+\frac{1}{r}\right\} \leq 1$, $g\in W^{1,p}(\overline{\Omega})\cap W^{1,q}(\overline{\Omega})$ with $[g]_{C^{0,1}(\overline{\Omega})} \leq 1$, $\|\lambda_\pm\|_{L^\infty(\Omega)} < 1$ and $u_{p,q}$ the corresponding minimizer to* (**Min**)*. Then, u_∞ obtained as a uniform limit of a subsequence of $\{u_{p,q}\}$, fulfils the maximization problem*

$$\max_{v\in\mathfrak{Z}} \left(\int_{\{v>0\}\cap\Omega} f_+ v dx + \int_{\{v<0\}\cap\Omega} f_- v dx \right) = \int_{\{u_\infty>0\}\cap\Omega} f_+ u_\infty dx + \int_{\{u_\infty<0\}\cap\Omega} f_- u_\infty dx. \tag{5.1}$$

Remark 5.4. Under the same conditions of Theorem 5.3 but with $\lambda_+ \equiv 1$ and $\|\lambda_-\|_{L^\infty(\Omega)} < 1$ we get

$$\begin{aligned} &\max_{v\in\mathfrak{Z}} \left(\int_{\{v>0\}\cap\Omega} f_+ v dx + \int_{\{v<0\}\cap\Omega} f_- v dx + \mathcal{L}^N(\{v>0\}) \right) \\ &\quad = \int_{\{u_\infty>0\}\cap\Omega} f_+ u_\infty dx + \int_{\{u_\infty<0\}\cap\Omega} f_- u_\infty dx + \mathcal{L}^N(\{u_\infty>0\}) \end{aligned} \tag{5.2}$$

as the variational limit problem.

Finally, when $\lambda_+ > 1$ the corresponding term in the functional diverges (recall that $(\lambda_+)^p$ appears) and therefore we don't have a limit variational problem in this case.

Theorem 5.5. *Let $f_\pm \in C^0(\overline{\Omega})$ and $g\in W^{1,p}(\overline{\Omega})\cap W^{1,q}(\overline{\Omega})$ such that $[g]_{C^{0,1}(\overline{\Omega})} \leq 1$. Then, $u_\infty \in \mathfrak{Z}$ obtained as uniform limit of a subsequence $\{u_{p,q}\}_{p,q>0}$, fulfils in the viscosity sense*

$$\left\{\begin{array}{rcll} -\Delta_\infty u_\infty &=& 0 & \text{in} \quad (\{u_\infty>0\}\cup\{u_\infty<0\})\cap(\Omega\setminus \operatorname{supp} f_\pm)^\circ \\ |\nabla u_\infty| &=& 1 & \text{in} \quad (\{u_\infty>0\}\cup\{u_\infty<0\})\cap(\Omega\cap\{f_\pm>0\}) \\ -|\nabla u_\infty| &=& -1 & \text{in} \quad (\{u_\infty>0\}\cup\{u_\infty<0\})\cap(\Omega\cap\{f_\pm<0\}) \\ -\Delta_\infty u_\infty &\geq& 0 & \text{in} \quad (\{u_\infty>0\}\cup\{u_\infty<0\})\cap(\Omega\cap\partial\{f_\pm>0\}\setminus\partial\{f_\pm<0\}) \\ -\Delta_\infty u_\infty &\leq& 0 & \text{in} \quad (\{u_\infty>0\}\cup\{u_\infty<0\})\cap(\Omega\cap\partial\{f_\pm<0\}\setminus\partial\{f_\pm>0\}) \\ u_\infty(x) &=& g(x) & \text{on} \quad \partial\Omega. \end{array}\right. \tag{5.3}$$

Proof. First, from the uniform convergence, it holds that $u_\infty = g$ on $\partial\Omega$. Next, we will prove that the limit function u_∞ is an $\infty-$harmonic function outside of support of source term, i.e.,

$$-\Delta_\infty u_\infty(x) = 0 \quad \text{in} \quad (\{u_\infty>0\}\cup\{u_\infty<0\})\cap(\Omega\setminus \operatorname{supp} f_\pm)^\circ.$$

To this end, let $x_0 \in (\{u_\infty > 0\} \cup \{u_\infty < 0\}) \cap (\Omega \setminus \operatorname{supp} f)^\circ$ and $\phi \in C^2(\Omega)$ such that $u_\infty - \phi$ has a strict local maximum (resp. strict local minimum) at x_0. Since, up to subsequence, $u_{p,q} \to u_\infty$ local uniformly, there exists a sequence $x_{p,q} \to x_0$ such that $u_{p,q} - \phi$ has a local maximum (resp. local minimum) at $x_{p,q}$. Moreover, if $u_{p,q}$ is a weak solution (consequently a viscosity solution according to Lemma 2.4) to (1.3) we obtain

$$-\left[|\nabla\phi(x_{p,q})|^{p-2}\Delta\phi(x_{p,q}) + (p-2)|\nabla\phi(x_{p,q})|^{p-4}\Delta_\infty\phi(x_{p,q})\right] \le -f_\pm(x_{p,q}) \quad (\text{resp. } \ge -f_\pm(x_{p,q})).$$

Now, if $|\nabla\phi(x_0)| \neq 0$ we may divide both sides of the above inequality by $(p-2)|\nabla\phi(x_{p,q})|^{p-4}$ (which is different from zero for p (resp. q) large enough). Thus, we obtain that

$$-\Delta_\infty\phi(x_{p,q}) \le \frac{|\nabla\phi(x_{p,q})|^2\Delta\phi(x_{p,q})}{p-2} - \frac{f_\pm(x_{p,q})}{(p-2)|\nabla\phi(x_{p,q})|^{p-4}} \quad (\text{resp. } \ge \cdots),$$

where the RHS tends to zero as $p \to \infty$ (resp. $q \to \infty$), because $f_\pm(x_{p,q}) \to 0$. Therefore,

$$-\Delta_\infty\phi(x_0) \le 0 \quad (\text{resp. } \ge 0),$$

and since such an inequality is immediately satisfied if $|\nabla\phi(x_0)| = 0$ we conclude that u_∞ is a viscosity sub-solution (resp. super-solution) to the desired equation.

Observe that *mutatis mutandis* the previous reasoning also proves that u_∞ fulfils

$$-\Delta_\infty u_\infty \ge 0 \quad \text{in} \quad (\{u_\infty > 0\} \cup \{u_\infty < 0\}) \cap (\Omega \cap \partial\{f_\pm > 0\} \setminus \partial\{f_\pm < 0\})$$

respectively

$$-\Delta_\infty u_\infty \le 0 \quad \text{in} \quad (\{u_\infty > 0\} \cup \{u_\infty < 0\}) \cap (\Omega \cap \partial\{f_\pm < 0\} \setminus \partial\{f_\pm > 0\})$$

in the viscosity sense.

Next, we will prove that u_∞ is a viscosity solution to

$$\max\{-\Delta_\infty u_\infty(x), -|\nabla u_\infty(x)| + 1\} = 0 \quad \text{in} \quad (\{u_\infty > 0\} \cup \{u_\infty < 0\}) \cap (\Omega \cap \{f_\pm > 0\}).$$

First let us prove that u_∞ is a viscosity super-solution. Fix $x_0 \in (\{u_\infty > 0\} \cup \{u_\infty < 0\}) \cap (\Omega \cap \{f_\pm > 0\})$ and let $\phi \in C^2(\Omega)$ be a test function such that $u_\infty(x_0) = \phi(x_0)$ and the inequality $u_\infty(x) > \phi(x)$ holds for all $x \neq x_0$. We want to show that

$$-\Delta_\infty\phi(x_0) \ge 0 \quad \text{or} \quad -|\nabla\phi(x_0)| + 1 \ge 0.$$

Notice that if $|\nabla\phi(x_0)| = 0$ there is nothing to prove. Hence, as a matter of fact, we may assume that

$$-|\nabla\phi(x_0)| + 1 < 0. \tag{5.4}$$

As in the previous case, there exists a sequence $x_{p,q} \to x_0$ such that $u_{p,q} - \phi$ has a local minimum at $x_{p,q}$. Since $u_{p,q}$ is a weak super-solution (consequently a viscosity super-solution according to Lemma 2.4) to (1.3) we get

$$-\left[|\nabla\phi(x_{p,q})|^{p-2}\Delta\phi(x_{p,q}) + (p-2)|\nabla\phi(x_{p,q})|^{p-4}\Delta_\infty\phi(x_{p,q})\right] \ge -f_\pm(x_{p,q}).$$

Now, dividing both sides by $(p-2)|\nabla\phi(x_{p,q})|^{p-4}$ (which is different from zero for p (resp. q) large enough due to (5.4)) we get

$$-\Delta_\infty\phi(x_{p,q}) \ge -\frac{|\nabla\phi(x_{p,q})|^2\Delta\phi(x_{p,q})}{p-2} - \left(\frac{\sqrt[p-4]{f_\pm(x_{p,q})}}{|\nabla\phi(x_{p,q})|}\right)^{p-4}.$$

Passing the limit as $p, q \to \infty$ in the above inequality we conclude that

$$-\Delta_\infty \phi(x_0) \geq 0.$$

That proves that u_∞ is a viscosity super-solution.

Now, we will analyze the other case. To this end, fix $x_0 \in (\{u_\infty > 0\} \cup \{u_\infty < 0\}) \cap (\Omega \cap \{f_\pm > 0\})$ and a test function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and the inequality $u_\infty(x) < \phi(x)$ holds for $x \neq x_0$. We want to prove that

$$-\Delta_\infty \phi(x_0) \leq 0 \quad \text{and} \quad -|\nabla \phi(x_0)| - 1 \leq 0. \tag{5.5}$$

Again, as before, there exists a sequence $x_{p,q} \to x_0$ such that $u_{p,q} - \phi$ has a local maximum at $x_{p,q}$ and since $u_{p,q}$ is a weak sub-solution (resp. viscosity sub-solution) to (1.3), we have that

$$-\frac{|\nabla \phi(x_{p,q})|^2 \Delta \phi(x_{p.q})}{p-2} - \Delta_\infty \phi(x_{p,q}) \leq -\left(\frac{\sqrt[p-4]{f_\pm(x_{p,q})}}{|\nabla \phi(x_{p,q})|}\right)^{p-4} \leq 0.$$

Thus, we obtain $-\Delta_\infty \phi(x_0) \leq 0$ letting $p, q \to \infty$. If $-|\nabla \phi(x_0)| - 1 > 0$, as $p, q \to \infty$, then the right hand side goes to $-\infty$, which clearly yields a contradiction. Therefore (5.5) holds.

The last part of the proof consists in proving that u_∞ is a viscosity solution to

$$\max\{-\Delta_\infty u_\infty(x), -|\nabla u_\infty(x)| + 1\} = 0 \quad \text{in} \quad (\{u_\infty > 0\} \cup \{u_\infty < 0\}) \cap (\Omega \cap \{f_\pm < 0\}).$$

The argument holds like the previous case and for this reason we will omit it here. □

Remark 5.6. It is worth to highlight that combining the information from Lemma 5.1 and Theorem 5.2 we are able to infer that when $f_\pm = 0$ the positive and the negative parts of the solutions to the limit problem are, in fact, an AMLE for its boundary data under the limit free boundary condition ($\infty-$**FBC**). This is due to the fact that they are $\infty-$harmonic functions.

The limiting free boundary condition

In this short part we will deduce (formally) the so-called limiting free boundary condition coming from (**FBC**). Precisely, by supposing that solutions and their corresponding free boundaries are appropriated regular we can proceeding as following: Recall the $(p,q)-$flux balance (**FBC**), that is,

$$\mathscr{G}_{p,q}(u_\nu^+, u_\nu^-, \lambda_+, \lambda_-) = \frac{p-1}{p}(u_\nu^+)^p(x) - \frac{q-1}{q}(u_\nu^-)^q(x) - \lambda_+^p(x) + \lambda_-^q(x) = 0.$$

Now, we rewrite this as follows:

$$\left(\frac{p-1}{p}(u_\nu^+)^p(x) + [\lambda_-^{\frac{q}{p}}]^p(x)\right)^{\frac{1}{p}} = \left(\frac{q-1}{q}[(u_\nu^-)^{\frac{q}{p}}]^p(x) + \lambda_+^p(x)\right)^{\frac{1}{p}}.$$

Hence, using the well-know fact that

$$(\mathfrak{A}^p + \mathfrak{B}^p)^{\frac{1}{p}} \to \max\{\mathfrak{A}, \mathfrak{B}\}, \qquad \text{as } p \to \infty,$$

we obtain as formal limit of the previous identity,

$$\max\left\{u_\nu^+(x),\ \lambda_-^{\mathscr{Q}}(x)\right\} = \max\left\{(u_\nu^-)^{\mathscr{Q}}(x),\ \lambda_+(x)\right\}. \tag{5.6}$$

Remark that this limit procedure in the free boundary condition is only formal since we do not have enough regularity of the normal derivatives $u_\nu^\pm$ (note that they depend on p, q) and the associated free boundaries (in order to have that (**FBC**) holds point wise and that the free boundaries converge uniformly (together with its normal vectors) as $p, q \to \infty$).

Let us point out that in the 1-D case (see the next example) the limit verifies the limit free boundary condition (5.6) point-wise in all cases.

Examples

Finally, let us present some examples in which we are able to compute the limit as $p,q\to\infty$.

Example 5.7. Let us analyze the 1-D minimization problem: Given an interval $(0,L)$, let $\lambda_\pm>0$ be two positive constants, α,β be positive numbers and impose the boundary conditions $u_{\mathfrak{J}}(0)=\alpha$ and $u_{\mathfrak{J}}(L)=-\beta$ (that is, we take $g(0)=\alpha$, $g(L)=-\beta$). Finally, we take $f_\pm\equiv 0$.

The functional to be minimized is given by

$$\mathfrak{J}_{p.q}[v]=\int_{\{v>0\}}\left(\frac{1}{p}|v'|^p+\lambda_+^p\right)dx+\int_{\{v<0\}}\left(\frac{1}{q}|v'|^q+\lambda_-^q\right)dx$$

First of all, we will deal with the case in which there is a dead-core region. In other words, there are points

$$0<x_p^+<x_q^-<L$$

such that

$$u_{\mathfrak{J}}(x)=0 \qquad \forall\, x\in(x_p^+,x_q^-).$$

Thus, the energy is minimized by a function of following form

$$u_{\mathfrak{J}}(x):=\begin{cases} -\dfrac{\alpha}{x_p^+}(x-x_p^+) & \text{if} \quad x\in(0,x_p^+)\\ 0 & \text{if} \quad x\in(x_p^+,x_q^-)\\ -\dfrac{\beta}{L-x_q^-}(x-x_q^-) & \text{if} \quad x\in(x_q^-,L).\end{cases}$$

Moreover, the minimum of the energy is given by

$$\mathfrak{J}_{p.q}[u_{\mathfrak{J}}]=\frac{1}{p}\alpha^p(x_p^+)^{1-p}+\frac{1}{q}\beta^q(L-x_q^-)^{1-q}+\lambda_+^px_p^++\lambda_-^q(L-x_q^-).$$

Notice that $\mathfrak{J}_{p.q}$ achieves a minimum at $u_{\mathfrak{J}}$, thus by minimizing the previous sentence with respect to x_p^+ and x_q^- we obtain

$$x_p^+=\sqrt[p]{\frac{p-1}{p}}\frac{\alpha}{\lambda_+} \quad \text{and} \quad L-x_q^-=\sqrt[q]{\frac{q-1}{q}}\frac{\beta}{\lambda_-}.$$

Recall that we have assumed that $0<x_p^+<x_q^-<L$. Thus, in this case, we conclude that a solution with a dead-core exists if and only if

$$\sqrt[p]{\frac{p-1}{p}}\frac{\alpha}{\lambda_+}+\sqrt[q]{\frac{q-1}{q}}\frac{\beta}{\lambda_-}<L.$$

Moreover, the limits as $p,q\to\infty$ of x_p^+ and $L-x_q^-$ are the following

$$x_\infty^+=\frac{\alpha}{\lambda_+} \quad \text{and} \quad L-x_\infty^-=\frac{\beta}{\lambda_-},$$

and hence the limiting profile is given by

$$u_{\mathfrak{J}_\infty}(x):=\begin{cases} -\lambda_+(x-x_\infty^+) & \text{if} \quad x\in(0,x_\infty^+)\\ 0 & \text{if} \quad x\in(x_\infty^+,x_\infty^-)\\ -\lambda_-(x-x_\infty^-) & \text{if} \quad x\in(x_\infty^-,L).\end{cases}$$

Next, we will assume that there is no dead-core region, in other words, $x_p^+ = x_q^- = x_j$. Hence, such a point must verify the condition

$$\frac{p-1}{p}\left|\frac{\alpha}{x_j}\right|^p - \frac{q-1}{q}\left|\frac{\beta}{L-x_j}\right|^q = \lambda_+^p - \lambda_-^q. \tag{5.7}$$

Now, for such a fixed point we have that $u_{\mathfrak{J}}$ is given by

$$u_{\mathfrak{J}}(x) := \begin{cases} \alpha\left(1-\dfrac{1}{x_j}x\right) & \text{if} \quad x \in (0,x_j) \\ \beta\left[\dfrac{1}{L-x_l}(L-x)-1\right] & \text{if} \quad x \in (x_j,L) \end{cases}$$

Since $(x_j)_{j\in\mathbb{N}}$ is a bounded sequence we have, up to a subsequence, $x_j \to x_\infty$. Now, we divide the analysis of (5.7) in two cases (note that we assumed that $\lambda_+^p \neq \lambda_-^q$).

✓ If $\lambda_+^p > \lambda_-^q$ we have

$$\frac{p-1}{p}\left|\frac{\alpha}{x_j}\right|^p\left(1-\frac{p(q-1)}{q(p-1)}\left|\frac{x_j}{\alpha}\right|^p\left|\frac{\beta}{L-x_j}\right|^q\right) = \lambda_+^p - \lambda_-^q = \lambda_+^p\left[1-\left(\frac{\lambda_-^{\frac{q}{p}}}{\lambda_+}\right)^p\right],$$

which yield in the limit as $p,q\to\infty$

$$\frac{\alpha}{x_\infty} = \lambda_+,$$

provided

$$\frac{x_\infty}{\alpha}\left|\frac{\beta}{L-x_\infty}\right|^{\mathcal{Q}} < 1.$$

This holds if and only if

$$\frac{\alpha}{\lambda_+} + \frac{\beta}{\sqrt[\mathcal{Q}]{\lambda_+}} < L.$$

Therefore, in this case $u_{\mathfrak{J}_\infty}$ (the uniform limit of the $u_{p,q}$) is uniquely determined and is given by

$$u_{\mathfrak{J}_\infty}(x) := \begin{cases} \alpha\left(1-\frac{1}{x_\infty}x\right) & \text{if} \quad x \in (0,x_\infty) \\ \beta\left[\frac{1}{L-x_\infty}(L-x)-1\right] & \text{if} \quad x \in (x_\infty,L). \end{cases} \tag{5.8}$$

Furthermore, in the case

$$\frac{\beta}{\sqrt[\mathcal{Q}]{\lambda_+}} + \frac{\alpha}{\lambda_+} \geq L$$

we obtain from the previous analysis that $u_{\mathfrak{J}_\infty}$ is a Lipschitz function with boundary values α and $-\beta$ and Lipschitz constant less or equal to $\frac{\beta}{\sqrt[\mathcal{Q}]{\lambda_+}} + \frac{\alpha}{\lambda_+}$. Therefore, the only possibility is the strait line given by

$$u_{\mathfrak{J}_\infty}(x) = \alpha - \left(\frac{\alpha+\beta}{L}\right)x. \tag{5.9}$$

Finally, note that in this case we have lost the free boundary condition since the limit does not depends on $\lambda_\pm$.

✓ If $\lambda_+^p < \lambda_-^q$ then re-writing (5.7) as

$$\frac{q-1}{q}\left|\frac{\beta}{L-x_j}\right|^q\left(1-\frac{(p-1)q}{p(q-1)}\left|\frac{L-x_j}{\beta}\right|^q\left|\frac{\alpha}{x_j}\right|^p\right)=\lambda_-^q-\lambda_+^p=\lambda_-^q\left[1-\left(\frac{\lambda_+^{\frac{p}{q}}}{\lambda_-}\right)^q\right],$$

we obtain in the limit as $p,q\to\infty$

$$\frac{\beta}{L-x_\infty}=\lambda_-$$

provided

$$\frac{\alpha}{x_\infty}\left|\frac{L-x_\infty}{\beta}\right|^{\mathcal{Q}}<1.$$

This holds if and only if

$$\frac{\alpha}{\lambda_-^{\mathcal{Q}}}+\frac{\beta}{\lambda_-}<L.$$

One more time we obtain the limit profile (5.8). Similarly to the previous case, if $\frac{\alpha}{\lambda_-^{\mathcal{Q}}}+\frac{\beta}{\lambda_-}\geq L$ we obtain the limit characterization (5.9).

Example 5.8. If $f_-=0>f_+$ in Ω and $g\equiv 0$ in $\partial\Omega$, then the unique (positive) maximizer to (5.1) is given by

$$u_\infty(x):=\operatorname{dist}(x,\partial\Omega).$$

In effect, we have that

$$|\operatorname{dist}(x,\partial\Omega)-\operatorname{dist}(y,\partial\Omega)|\leq|x-y|,$$

in other words, $u_\infty\in\mathfrak{Z}$. Finally, since u_∞ fulfils (5.1), it is suffices to show that $w(x)\leq\operatorname{dist}(x,\partial\Omega)$ for any $w\in\mathfrak{Z}$. In fact, since $w\in\mathfrak{Z}$ we have that

$$\frac{|w(x)|}{|x-y|}\leq 1\quad\forall\ y\in\partial\Omega.$$

Therefore,

$$|w(x)|\leq\inf_{y\in\partial\Omega}|x-y|=\operatorname{dist}(x,\partial\Omega).$$

Furthermore, we must to observe that u_∞ is, in fact, a viscosity solution of the limit problem (1.4). In effect, given $x\in\Omega$ we have

$$|\nabla u_\infty|=1\quad\text{a.e in}\quad\Omega.$$

Finally, u_∞ is a supersolution for $\infty-$Laplacian, we conclude that

$$\max\{-\Delta_\infty u_\infty(x),\,|\nabla u_\infty(x)|-1\}=0\quad\text{in}\quad\Omega$$

in the viscosity sense.

Similarly, if $f_->0=f_+$ in Ω and $g\equiv 0$ in $\partial\Omega$, then the unique maximizer to (5.1) is given by

$$u_\infty(x):=\begin{cases}-\operatorname{dist}(x,\partial\Omega), & x\in\Omega,\\ 0, & \text{otherwise}.\end{cases}$$

and it satisfies in the viscosity sense

$$\max\{-\Delta_\infty u_\infty(x),-|\nabla u_\infty(x)|+1\}=0\quad\text{in}\quad\Omega.$$

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References

[1] H. W. ALT AND L. A. CAFFARELLI. *Existence and regularity for a minimum problem with free boundary*. J. Reine Angew. Math. 325 (1981), 105–144.

[2] H. W. ALT, L. A. CAFFARELLI AND A. FRIEDMAN. *Variational problems with two phases and their free boundaries*. Trans. Amer. Math. Soc. 282 (1984), no. 2, 431–461.

[3] M. AMARAL AND E.V. TEIXEIRA. *Free transmission problems*. Comm. Math. Phys. 337 (2015), no. 3, 1465–1489.

[4] G. ARONSSON. *Extension of functions satisfying Lipschitz conditions*. Ark. Mat. 6, (1967), 551–561.

[5] G. ARONSSON, M.G. CRANDALL AND P. JUUTINEN. *A tour of the theory of absolutely minimizing functions*. Bull. Amer. Math. Soc. 41 (2004), 439–505.

[6] T. BHATTACHARYA, E. DIBENEDETTO AND J. MANFREDI. *Limits as $p \to \infty$ of $\Delta_p u_p = f$ and related extremal problems.* Rend. Sem. Mat. Univ. Politec. Torino 1989, Special Issue, 1568 (1991).

[7] M. BORSUK. *Transmission problems for elliptic second-order equations in non-smooth domains*. In: Frontiers in Mathematics. Birkhäuser/Springer Basel AG, Basel (2010).

[8] L.A. CAFFARELLI, D. JERISON, C.E. KENIG. *Some new monotonicity theorems with applications to free boundary problems*. Ann. Math. 155 (2002) 369–404.

[9] EDITORS: A.V. CHERKAEV AND R. KOHN. *Topics in the Mathematical Modelling of Composite Materials*. Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser (1997).

[10] M.G. CRANDALL, H. ISHII AND P.L. LIONS. *User's guide to viscosity solutions of second order partial differential equations*. Bull. Amer. Math. Soc. 27 (1992), 1–67.

[11] D. DANIELLI AND A. PETROSYAN. *A minimum problem with free boundary for a degenerate quasi-linear operator*. Calc. Var. Partial Differential Equations 23 (1) (2005) 97–124.

[12] J.V. DA SILVA AND J.D. ROSSI. *The limit as $p \to \infty$ in free boundary problems with fractional p-Laplacians*. Submitted.

[13] J.V. DA SILVA, J.D. ROSSI AND A. SALORT. *The limit as $p \to \infty$ in p-dead core type problems: Existence, uniqueness and regularity for solutions*. In preparation.

[14] J.V. DA SILVA AND A. SALORT. *The limiting obstacle problems for fractional p-Laplacian*. In preparation.

[15] R. JENSEN. *Uniqueness of Lipschitz extensions minimizing the sup-norm of the gradient*. Arch. Ration. Mech. Anal. 123 (1993) 51–74.

[16] P. JUUTINEN, P. LINDQVIST AND J.J. MANFREDI. *On the equivalence of viscosity solutions and weak solutions for a quasilinear equation*. SIAM J. Math. Anal. 33 (3), (2001), 699–717.

[17] O.A. LADYZHENSKAYA AND N.N. URAL'TSEVA. *Linear and Quasilinear Elliptic Equations*. Math. Sci. Eng., vol.46, Academic Press, NewYork, 1968.

[18] R. A. LEITÃO, O.S. QUEIROZ AND E.V. TEIXEIRA. *Regularity for degenerate two-phase free boundary problems*. Ann. Inst. H. Poincare Anal. Non Lineaire. 32 (2015), pp. 741–762. -

[19] G. LU, AND P. WANG. *On the uniqueness of a viscosity solution of a two-phase free boundary problem*. J. Funct. Anal. 258 (2010), 2817–2833.

[20] J.J. MANFREDI, A. PETROSYAN AND H. SHAHGHOLIAN. *A free boundary problem for ∞-Laplace equation*. Calc. Var. Partial Differential Equations 14 (2002), 359–384.

[21] P. JUUTINEN; M. PARVIAINEN AND J. D. ROSSI. *Discontinuous gradient constraints and the infinity Laplacian*. International Mathematics Research Notices. Vol. 2016 (8), 2451–2492, (2016).

[22] J.D. ROSSI AND E.V. TEIXEIRA. *A limiting free boundary problem ruled by Aronsson's equation*. Trans. Amer. Math. Soc. 364 (2012), no. 2, 703–719.

[23] J.D. ROSSI, E.V. TEIXEIRA AND J.M. URBANO. *Optimal regularity at the free boundary for the infinity obstacle problem*. Interfaces Free Bound. 7 (2015), 381–398.

[24] J.D. ROSSI AND P. WANG. *The limit as $p \to \infty$ in a two-phase free boundary problem for the p-Laplacian*. Interfaces Free Bound. 18 (2016), 117–137.

[25] J. SERRIN. *A Harnack inequality for nonlinear equations* Bull. Am. Math. Soc. 69 (4) (1963) 481–486.

[26] J. SERRIN. *Local behavior of solutions of quasi-linear equations*. Acta Math. 111 (1964), 247–302.