The limit as $p \to \infty$ in free boundary problems with fractional p-Laplacians

by

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Abstract

We study the p-fractional optimal design problem under volume constraint taking special care of the case when p is large, obtaining in the limit a free boundary problem modeled by the Hölder Infinity Laplacian operator. A necessary and sufficient condition is imposed in order to obtain the uniqueness of solutions to the limiting problem, and, under such condition, we find precisely the optimal configuration for the limit problem. We also prove the optimal regularity (locally $C^{0,s}$) for any limiting solution. Finally, we establish some geometric properties for solutions and their free boundaries.

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1 Introduction

In the modern mathematical theory of optimization an *Optimal Design Problem* under a volume constraint can be described as follows: Let $\Omega \subset \mathbb{R}^N$ be a smooth and bounded domain and $0 < \alpha < \mathcal{L}^N(\Omega)$ a fixed quantity. For example, one can think about the quantity of insulating material/substance to be used in the best insulation configuration for a "body" with prescribed volume. The problem is to find a best configuration $\mathscr{O} \subset \Omega$ such that minimizes a cost functional associated to a quantitative process (a mapping u), under the prescription of the maximum volume to be used, in others words,

$$\min \left\{ \mathfrak{J}_\alpha[u_\Xi] \;\middle|\; u_\Xi : \Omega \to \mathbb{R}_+,\ \Xi \subset \Omega \text{ such that } 0 < \mathcal{L}^N(\Xi) \leq \alpha \right\},$$

where in several situations of applied mathematics the variational functional $\mathfrak{J}_\alpha[u_\Xi]$ has an integral representation, whose involved functions are linked to the competing configuration Ξ via a prescribed PDE. Notice that some examples of such minimization problems come from the calculus of variations and optimal control theory: In elliptic PDEs (eigenvalue problems with geometric constraints, shape optimization problems with constrained perimeter or volume), optimal design of semiconductor devices and problems in structural optimization, just to mention a few.

Concerning optimization problems with volume constraints the pioneering work is the paper [1], where the authors minimize $\mathfrak{J}_\alpha[v_\Xi] = \int_\Omega |\nabla v_\Xi|^2 dx$ with prescribed volume of the set $\Xi = \{u = 0\}$. By considering the minimization problem

$$\min \left\{ \int_\Omega \Delta u \, dx \;\middle|\; 0 \leq u \in H^1(\Omega),\ \Delta u = 0 \text{ in } \{u > 0\} \cap \Omega,\ u = g \text{ on } \partial\Omega \text{ and } \mathcal{L}^N(\{u > 0\} \cap \Omega) \leq \alpha \right\},$$

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in [2] it is studied the shape optimization problem in heat conduction (u is the temperature in Ω) with non-constant prescribed temperature distribution g. The nonlinear counterpart for such optimal design problems, under non-constant temperature distribution, have been developed in [23] and [24]. On the other hand, optimal design problems governed by degenerate/singular quasi-linear operators just have appeared recently in the literature. Independently, [7] and [18] treated the p-Laplacian case, whose cost functional is $\mathfrak{J}_\alpha[v_\Xi] = \int_\Omega |\nabla v_\Xi|^p dx$ with prescribed volume of set $\Xi = \{u > 0\}$ (cf. [25] for other considerations about nonlinear problems in rough inhomogeneous media governed by degenerate elliptic operators $\mathfrak{L}[u] = \mathrm{div}(\mathfrak{A}(x, \nabla u))$ of p-Laplacian type). The previous overview summarizes the mathematical journey in the local setting.

Recently, the study of optimal design problems driven by fractional diffusion operators was successfully developed. The starting point of this research has been the following minimization problem

$$\min\left\{\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dxdy \mid u\in W^{s,2}\left(\mathbb{R}^N\right), u=\varphi \text{ in } \mathfrak{D}\subset\mathbb{R}^N \text{ and } \mathcal{L}^N(\{u>0\}\cap\mathfrak{D}^c)=\alpha\right\}. \tag{1.1}$$

Recall that, minimizers to (1.1) satisfy

$$(-\Delta)^s u(x) := C_{N,s}.\text{P.V.}\int_{\mathbb{R}^N}\frac{u(x)-u(y)}{|x-y|^{N+2s}}dy = 0 \quad \text{in} \quad \{u>0\}\cap\mathfrak{D}^c, \tag{1.2}$$

where $(-\Delta)^s$ is the well-known *Fractional Laplacian* operator, $s \in (0,1)$, P.V. means the Cauchy principal value and $C_{N,s}$ is a normalizing constant.

In [26] the authors investigated the variational formulation of problem (1.2) and strongly rely on the extension formula by Caffarelli-Silvestre (based on the Dirichlet-to-Neumann operator) given in [11]. Moreover, inspired by devices and results that comes from [9] they were able to obtain optimal regularity of minimizers ($C^{0,s}$ regularity estimates), the s-Hölder growth away from the free boundary and the positive density of $\{u > 0\}$ and $\{u = 0\}$ along the free boundary. Particularly, this implies that blow-up limits have non-trivial free boundaries and that free boundaries can not form cusps (cf. [9] for a survey to Alt-Caffarelli's theory for one-phase problems in the non-local setting).

Those previous studies are the starting point for the present work in the setting of fractional diffusion operators with p-Laplacian structure. Thus, let us consider the minimizing problem for the p-fractional energy with a positive data g prescribed outside Ω and a restriction on the maximum volume of the support of the involved functions inside Ω. From a mathematical point of view we consider the optimization problem:

$$\mathfrak{L}_p^s[\alpha] = \min\left\{[v]_{W^{s,p}(\mathbb{R}^N)} \mid v\in W^{s,p}\left(\mathbb{R}^N\right), v=g \text{ in } \mathbb{R}^N\setminus\Omega \text{ and } \mathcal{L}^N(\{v>0\}\cap\Omega)\le\alpha\right\}. \tag{$\mathfrak{P}_p^s$}$$

Physically speaking, taking into account long-range interactions a model for the problem becomes more accurate when governed by nonlocal operator such as the p-fractional Laplacian.

Existence of a minimizer follows easily by the direct method in calculus of variations. Moreover, recall that any minimizer u_p is a solution to the following Dirichlet problem driven by fractional p-Laplacian operator

$$\begin{cases} -(-\Delta_{\mathbb{R}^N})_p^s u_p(x) &= 0 & \text{in} & \{u_p>0\}\cap\Omega \\ u_p(x) &= g(x) & \text{on} & \mathbb{R}^N\setminus\Omega, \end{cases} \tag{1.3}$$

where

$$(-\Delta_{\mathbb{R}^N})_p^s u_p(x) := C_{N,s,p}.\text{P.V.}\int_{\mathbb{R}^N}\frac{|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))}{|x-y|^{N+ps}}dy.$$

In the present paper, we are interested in the asymptotic behavior, as $p\to\infty$, of optimal shapes to problem ($\mathfrak{P}_p^s$). The limiting configurations for $p\to\infty$ have been inspired by the work of the first author in the local setting, see [20] for details. Analytical and geometric features of a limiting free boundary reveals asymptotic information on the optimal design problem ($\mathfrak{P}_p^s$). Motivated by formal considerations, we are led to consider the following limiting configuration:

$$(\mathfrak{P}_\infty^s)\quad \mathfrak{L}_\infty^s[\alpha] = \min\left\{ \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|v(y)-v(x)|}{|x-y|^s} \;\Big|\; v \in W^{s,\infty}\left(\mathbb{R}^N\right),\, v = g \text{ in } \mathbb{R}^N\setminus\Omega \text{ and } \mathcal{L}^N(\{v>0\}\cap\Omega)\leq \alpha \right\}.$$

We prove here that any sequence of minimizers u_p to $\mathfrak{P}_p^s$ converges, up to a subsequence, to a solution u_∞ of the limiting problem $\mathfrak{P}_\infty^s$. Furthermore, we find the associated equation that u_∞ verifies in its positivity region, $\Omega_\infty^+ := \{u_\infty > 0\}\cap\Omega$, that is,

$$-\mathcal{L}_\infty^s[u_\infty](x) := -\left(\sup_{y\in\mathbb{R}^N} \frac{u_\infty(y)-u_\infty(x)}{|x-y|^s} + \inf_{y\in\mathbb{R}^N} \frac{u_\infty(y)-u_\infty(x)}{|x-y|^s} \right) = 0 \quad \text{in} \quad \Omega_\infty^+,$$

where $\mathcal{L}_\infty^s[\cdot]$ is the *Hölder Infinity Laplacian operator*. For this reason, we have the fact that u_∞ is an extremal for the nonlocal Hölder extension problem. We will show that u_∞ is a minimizer for the Hölder norm within its positivity region. This means, it minimizes the Hölder quotient in every sub-domain of Ω_∞^+ when testing against functions with the same boundary data (cf. [12] and [15]). Therefore, u_∞ is a Hölder-infinity harmonic function in its positivity region. These information are present in the first theorem in this paper.

Theorem 1.1. *Let u_p be a minimizer to ($\mathfrak{P}_p^s$). Then, up to a subsequence,*

$$u_p \to u_\infty \quad as \quad p\to\infty,$$

uniformly in Ω and weakly in $W^{s,q}(\Omega)$ for all $1<q<\infty$, where u_∞ minimizes ($\mathfrak{P}_\infty^s$). Furthermore, the extremal values also converge

$$\mathfrak{L}_p^s[\alpha] \to \mathfrak{L}_\infty^s[\alpha] \quad as \quad p\to\infty.$$

Finally, the limit u_∞ fulfils

$$-\mathcal{L}_\infty^s[u_\infty](x) = 0 \quad in \quad \{u_\infty>0\}\cap\Omega,$$

in the viscosity sense.

We also studied uniqueness of the solution to the limit problem (note that when we have uniqueness of the limit we have convergence of u_p not only along subsequences). Here the key is a geometric compatibility condition on the data,

$$\textbf{(Comp. Assump.)}\qquad \alpha \leq \mathcal{L}^N\left(\bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{[g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)}}\right)^{\frac{1}{s}}}(y)\cap\Omega \right).$$

We observe that when g is constant, then (**Comp. Assump.**) is satisfied. This condition (**Comp. Assump.**) turns out to be necessary and sufficient to obtain uniqueness of solutions to the limit problem.

Theorem 1.2. *Let v_∞ be given by*

$$v_\infty(x) = \sup_{\mathbb{R}^N\setminus\Omega}\left(g(y) - \mathfrak{H}^\sharp|x-y|^s\right)_+.$$

1. *Assume that* (**Comp. Assump.**) *holds. Let $\mathfrak{H}^\sharp$ be the unique positive number such that*
$$\Omega^\sharp := \bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(y)\cap\Omega \quad \textit{fulfils} \quad \mathcal{L}^N\left(\Omega^\sharp\right)=\alpha.$$
Then v_∞ is the unique minimizer for $(\mathfrak{P}^s_\infty)$.

2. *On the other hand, assume that* (**Comp. Assump.**) *does not hold. Then there exists infinitely many minimizers for $(\mathfrak{P}^s_\infty)$. Moreover, v_∞ is the least (or minimal) solution, in the following sense $v_\infty(x)\le u_\infty(x)$ in Ω for any other minimizer u_∞ to $(\mathfrak{P}^s_\infty)$ and verifies*
$$\{v_\infty>0\}\cap\Omega = \bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\frac{g(y)}{\mathfrak{H}^\sharp}}(y)\cap\Omega \quad \textit{fulfils} \quad \mathcal{L}^N(\{v_\infty>0\}\cap\Omega)<\alpha.$$

Now, we state some properties of the limits of u_p. In the next result we don't assume (**Comp. Assump.**).

Theorem 1.3. *Let v_∞ a uniform limit as $p\to\infty$ of u_p. Then the following properties hold:*

1. $C^{0,s}$ **regularity for minimizers.** *v_∞ is uniformly s-Hölder continuous in Ω.*

2. **Strong non-degeneracy for minimizers.** *v_∞ is strongly non-degenerate of order s, i.e., there exists a constant $\mathfrak{c}=\mathfrak{c}(N,s)>0$ such that for any fixed point $x_0\in\overline{\{v_\infty>0\}}\cap\Omega$ there holds*
$$\sup_{B_r(x_0)} v_\infty(x)\ge \mathfrak{c}r^s.$$

3. **Uniform lower positive density.** *Let $x_0\in\{v_\infty>0\}\cap\Omega$ be an interior point. There exists a constant $\mathfrak{c}_0=\mathfrak{c}_0(N,s)>0$ such that for every $r\ll 1$ there holds*
$$\mathcal{L}^N(\{v_\infty>0\}\cap B_r(x_0))\ge \mathfrak{c}_0 r^N.$$

4. **Harnack inequality for minimizers in a touching ball.** *Let $x_0\in\{v_\infty>0\}\cap\Omega$ be an interior point and $\mathfrak{d}:=\mathrm{dist}(x_0,\partial\{v_\infty>0\})$. Then,*
$$\sup_{B_{\tau\mathfrak{d}}(x_0)} v_\infty(x)\le \mathfrak{C}\inf_{B_{\tau\mathfrak{d}}(x_0)} v_\infty(x)$$
for a universal constant $\mathfrak{C}>0$ and for any $0<\tau<1$.

5. **Uniform non-degeneracy for minimizers** *v_∞ grows with an s-rate away from the free boundary, i.e., for a constant $\mathfrak{c}=\mathfrak{c}(N,s)>$ there holds*
$$v_\infty(x)\ge \mathfrak{c}\,\mathrm{dist}(x,\partial\{v_\infty>0\})^s \quad \forall\, x\in\{v_\infty>0\}\cap\Omega.$$

Let us also mention that we have convergence of the positivity sets (in the sense that the measure of the symmetric difference goes to zero as $p\to\infty$) and also convergence of the null sets. For the proof of this result we use some of the regularity properties obtained in Theorem 1.3.

Theorem 1.4. *Let v_p be a sequence of minimizers to $(\mathfrak{P}^s_p)$. If for some subsequence, denoted by v_p yet, $v_p\to v_\infty$ uniformly in $\overline{\Omega}$ and weakly in $W^{s,q}(\Omega)$ for all $1<q<\infty$, being v_∞ a solution to $(\mathfrak{P}^s_\infty)$, then*
$$\lim_{p\to\infty}\mathcal{L}^N(\{v_p>0\}\triangle\{v_\infty>0\})=0.$$
Moreover, the null sets verify
$$\overline{int(\{u_\infty=0\})}\subset\liminf_{p\to\infty}\{u_p=0\}\subset\limsup_{p\to\infty}\{u_p=0\}\subset\{u_\infty=0\}.$$

With the same ideas used here one can also deal with the following two optimal design problems:

$$(\mathfrak{A}_p^s)\qquad \widehat{\mathfrak{L}_p^s}[\alpha] = \min_{\substack{u=g \text{ in } \mathbb{R}^N\setminus\Omega \\ \mathcal{L}^N(\{v>0\}\cap\Omega)\leq\alpha}} \mathfrak{F}_p^s[v]\left(\mathbb{R}^N,\Omega\right),$$

with

$$\mathfrak{F}_p^s[v]\left(\mathbb{R}^N,\Omega\right) := \left(\int_\Omega\int_\Omega \frac{|v(y)-v(x)|^p}{|y-x|^{N+sp}}dxdy + \int_{\mathbb{R}^N\setminus\Omega}\int_\Omega \frac{|g(y)-v(x)|^p}{|y-x|^{N+sp}}dxdy\right)^{\frac{1}{p}}.$$

and

$$(\mathfrak{B}_p^s)\qquad \widetilde{\mathfrak{L}_p^s}[\alpha] = \min_{\substack{u=g \text{ in } D \\ \mathcal{L}^N(\{v>0\}\cap\Omega)\leq\alpha}} \mathfrak{G}_p^s[v]\left(\Omega\cup D,\Omega\right),$$

with

$$\mathfrak{G}_p^s[v]\left(\Omega\cup D,\Omega\right) := \left(\int_\Omega\int_\Omega \frac{|v(y)-v(x)|^p}{|y-x|^{N+sp}}dxdy + \int_D\int_\Omega \frac{|g(y)-v(x)|^p}{|y-x|^{N+sp}}dxdy\right)^{\frac{1}{p}}.$$

Note that in $(\mathfrak{A}_p^s)$ and in $(\mathfrak{B}_p^s)$ we do not need to ask that the exterior datum g is in $W^{s,p}(\mathbb{R}^N)$. In fact, it is enough that there is an extension $u_g : \Omega \to \mathbb{R}$ such that the two integrals that define $\mathfrak{F}_p^s$ and $\mathfrak{G}_p^s$ are finite. We will briefly comment on the limit as $p\to\infty$ for these minimizations problems in the last section of this paper.

Let us end this introduction with a brief survey on recent references concerning limits as $p\to\infty$ in different $p-$Laplacian type problems. It has been well established (cf. [5]) that for a non-negative function g, the corresponding weak solutions (local problem) for the p-Laplacian

$$\left\{\begin{array}{rcll} -\Delta_p\, u_p(x) & = & 0 & \text{in} \quad \Omega \\ u_p(x) & = & g(x) & \text{on} \quad \partial\Omega \end{array}\right. \tag{1.4}$$

converge for a limiting function u_∞, which fulfils in the viscosity sense the following problem

$$\left\{\begin{array}{rcll} -\Delta_\infty\, u_\infty(x) & = & 0 & \text{in} \quad \Omega \\ u_\infty(x) & = & g(x) & \text{on} \quad \partial\Omega, \end{array}\right.$$

where $\Delta_\infty\, v(x) := Dv^T\cdot DvD^2v$ is the well known *Infinity-Laplacian operator*, which is associated to AMLE, *Absolutely Minimizing Lipschitz Extension*, a concept was introduced by G. Aronsson in the end of sixties. (cf. [3]).

The nonlocal counterpart of the problem (1.4), related to the fractional p-Laplacian operator is given of the following form

$$\left\{\begin{array}{rcll} -\left(-\Delta_\Omega\right)_p^s u_p(x) & = & 0 & \text{in} \quad \Omega \\ u_p(x) & = & g(x) & \text{on} \quad \partial\Omega \end{array}\right.$$

and has been studied by Chambolle *et al* in [12]. They proved that the limiting problem as $p\to\infty$ is given by

$$\left\{\begin{array}{rcll} -\mathcal{L}_\infty^s[u_\infty](x) & = & 0 & \text{in} \quad \Omega \\ u_\infty(x) & = & g(x) & \text{on} \quad \partial\Omega, \end{array}\right.$$

Recall that if $g\in C^{0,s}(\partial\Omega)$ then u_∞ is said to be the *Optimal Hölder extension* to $\overline{\Omega}$ of the Hölder boundary data g, in the following sense:

$$\sup_{\substack{x,y\in\Omega\\ x\neq y}} \frac{|u_\infty(y)-u_\infty(x)|}{|x-y|^s} \leq \sup_{\substack{x,y\in\partial\Omega\\ x\neq y}} \frac{|g(y)-g(x)|}{|x-y|^s}.$$

For this reason, the operator $\mathcal{L}_\infty^s$ is called the *Hölder Infinity Laplacian*. Moreover, in [6] it is obtained the best Hölder extension of a function g defined in $\mathbb{R}^N \setminus \Omega$. Such a extension is related to solubility of the Dirichlet problem driven by the *Infinity Fractional Laplacian*. Notice that such an operator arises from a nonlocal and non-variational approach (more precisely from Tug-of-War game theory), different from the variational treatment, see [12]. Furthermore, recently the Hölder Infinity Laplacian have also appeared in [15], where it is studied the behaviour of solutions as $p \to \infty$ of the following Dirichlet problem

$$\begin{cases} (-\Delta)_p^s u_p(x) & = & f(x,u) & \text{in} & \Omega \\ u_p(x) & = & g(x) & \text{on} & \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.5}$$

Regarding free boundary problems, the strategy of passing the limit as $p \to \infty$ in p-variational problems in order to obtain a non-variational limiting configuration (a problem governed by the Infinity-Laplacian operator) has been successful in many contexts of the current literature: In Bernoulli type problems [17], optimal design problems [20], obstacle type problems [21] (See also [19] for a free boundary problem in the context of Tug-of-War games and [22] for a limiting free boundary problem in the two-phases setting). Furthermore, such approach allows us to use several technical features of the corresponding p-sequential problems to their limiting points, via uniform convergence. Optimal regularity estimates, weak geometric and measure-theoretic properties are some of these obtained features, just to mention a few. Finally, we highlight that in this article, such a strategy will also play a key role in our approach in order to study some properties of minimizers of the limiting minimizing problem $\mathfrak{P}_\infty^s$.

Finally, for similar free boundary problems in the local case, that is, when we consider the p-energy $\int_\Omega |\nabla u|^p$ instead of p-fractional energy $\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^p}{|y-x|^{N+sp}} dxdy$ we refer to [20]. Remark that in the nonlocal case treated here some subtle differences appear. For instance, the singular kernel $\mathfrak{K}(x,y) = |x-y|^{-(N+ps)}$ yields a significant obstacle when one looks for a bound for the family $(u_p)_{p\nearrow\infty}$ in the $W^{s,q}$ topology for all $1<q<\infty$. On the other hand, we will follow ideas from [20] when we obtain the necessary condition that is used to pass to the limit as $p \to \infty$ in order to obtain the uniqueness and characterization of limiting profile. In contrast with the local case, there is no uniform in p estimates for the free boundary. Hence the properties of the free boundary of the limit problem and the convergence as $p \to \infty$ of the free boundaries in the non-local case are based in completely different arguments.

The paper is organized as follows: in Section 2 we collect some preliminary results that will be used latter; in Section 3 we show how to pass to the limit as $p \to \infty$ and deal with the uniqueness of solutions to the limit problem (Theorem 1.2), we also include some examples in which the limit solution can be computed explicitly; in Section 4 we collect the proofs of the properties of the limit problem stated in Theorem 1.3 and we include here the proof of the convergence of the positivity and null sets, Theorem 1.4; finally, in Section 5 we include some remarks on possible extensions of our results.

2 Preliminaries

From now on we establish the functional framework for our problem. For $0<s<1$ and $1<p<\infty$ the fractional Sobolev spaces $W^{s,p}(\mathbb{R}^N)$ is defined as

$$W^{s,p}(\mathbb{R}^N) := \left\{ u \in L^p(\mathbb{R}^N) : \frac{|u(y)-u(x)|}{|y-x|^{\frac{N}{p}+s}} \in L^p(\mathbb{R}^N \times \mathbb{R}^N) \right\},$$

whose corresponding norm is given by

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left[\|u\|_{L^p(\mathbb{R}^N)}^p + [u]_{W^{s,p}(\mathbb{R}^N)}^p \right]^{\frac{1}{p}},$$

where

$$[u]_{W^{s,p}(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^p}{|y-x|^{N+sp}}dxdy\right)^{\frac{1}{p}}$$

is the so-called *Gagliardo semi-norm*. Furthermore, the fractional Sobolev space $W^{s,\infty}(\mathbb{R}^N)$ is defined as follows

$$W^{s,\infty}(\mathbb{R}^N) := \left\{u \in L^\infty(\mathbb{R}^N) : \frac{u(y)-u(x)}{|y-x|^s} \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)\right\},$$

endorsed with the norm

$$\|u\|_{W^{s,\infty}(\mathbb{R}^N)} := \|u\|_{L^\infty(\mathbb{R}^N)} + \left\|\frac{u(y)-u(x)}{|y-x|^s}\right\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)}.$$

For the sake of simplicity in notation, it is convenient to designate $s = \gamma - \frac{N}{p}$. Consequently, γ must satisfy that $\gamma p > N$ and $\gamma - \frac{N}{p} < 1$ in order to ensure that $s \in (0,1)$.

Finally, for $\Omega \subset \mathbb{R}^N$ a smooth domain we define

$$W^{s,p}_g(\Omega) := \left\{u \in W^{s,p}(\mathbb{R}^N) : \ s = \gamma - \frac{N}{p} \text{ and } u = g \text{ on } \mathbb{R}^N \setminus \Omega\right\}.$$

Recall that $W^{s,p}(\mathbb{R}^N)$ is a Banach space, interpolated between $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$ (cf. [13]). Moreover, in order to recover $L^p(\mathbb{R}^N)$ and $W^{1,p}(\mathbb{R}^N)$ as $s \to 0$ and $s \to 1$ respectively, we must consider the norm (cf. [8]):

$$\|u\|_{W^{s,p}(\mathbb{R}^N)} := \left[\|u\|^p_{L^p(\mathbb{R}^N)} + s(1-s)[u]^p_{W^{s,p}(\mathbb{R}^N)}\right]^{\frac{1}{p}}.$$

For a complete study about Fractional Sobolev spaces (without the use of interpolation theory) we recommend the survey [13].

In the following we specify the notions of solutions which we will use throughout this article. For a fixed value of $1 < p < \infty$ we consider weak solutions. On the other hand, in the limiting setting, as $p \to \infty$, we will use the concept of viscosity solutions.

Definition 2.1 (Weak solution). $u \in W^{s,p}_g(\Omega)$ is said a weak subsolution (resp. supersolution) to (1.3) provided

- ✓ $u \le g$ in $\mathbb{R}^N \setminus \Omega$;
- ✓ For all $0 \le \varphi \in W^{s,p}_0(\Omega)$ holds

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^{p-2}(u(y)-u(x))}{|y-x|^{N+sp}}(\varphi(y)-\varphi(x))dxdy \le 0 \ \text{(resp. } \ge 0)$$

Finally, we say that u is a weak solution to (1.3) if it is simultaneously u is a weak supersolution and weak subsolution.

Definition 2.2 (Viscosity solution). A upper (resp. lower) semi-continuous function u such that $u \le g$ in $\mathbb{R}^N \setminus \Omega$ is said a viscosity subsolution (resp. supersolution) to (1.3) if whenever $x_0 \in \Omega$ and $\varphi \in C^1_0(\mathbb{R}^N)$ such that

- ✓ $u(x_0) = \varphi(x_0)$;
- ✓ $u(x) \le \varphi(x)$ for $x \ne x_0$ then

$$-\int_{\mathbb{R}^N} \frac{|\varphi(y)-\varphi(x_0)|^{p-2}(\varphi(y)-\varphi(x_0))}{|y-x|^{N+sp}}dy \le 0 \ \text{(resp. } \ge 0)$$

Finally, a continuous function u is a viscosity solution to (1.3) if it is simultaneously u is a viscosity supersolution and a viscosity subsolution.

Concerning general theory of viscosity solutions to integro-differential equations with singular kernels we refer the reader to Barles-Imbert's survey, [4].

Recall that the fractional Sobolev space $W^{s,p}(\mathbb{R}^N)$ embeds, for sufficiently large exponent p, into the Hölder-continuous functions (cf. [13]). Such a result will play an important role in order to pass to the limit in our problem.

Theorem 2.3 (**Hölder embedding**). *Let $0<s<1$, $sp>N$ and $\gamma = s-\frac{N}{p}$. Then, for any $u \in W^{s,p}(\mathbb{R}^N)$ there exists a positive constant $\mathfrak{c}=\mathfrak{c}(N,p,s)$ such that*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^N)} \leq \mathfrak{c}.\|u\|_{W^{s,p}(\mathbb{R}^N)}.$$

The next result plays a key role in order to deduce the limiting operator as $p\to\infty$ in our optimization problem.

Lemma 2.4 ([15, Lemma 6.1]). *Let φ be a test function and $x_p \to x$ as $p\to\infty$. Then*

$$\mathfrak{A}_p(\varphi(x_p)) \to (\mathcal{L}^s_\infty)^+[\varphi](x_0) \quad \textit{and} \quad \mathfrak{B}_p(\varphi(x_p)) \to -(\mathcal{L}^s_\infty)^-[\varphi](x_0),$$

where

$$\mathfrak{A}_p^{p-1}(\varphi(x_p)) = \int_{\mathbb{R}^N} \frac{|\varphi(y)-\varphi(x_p)|^{p-2}(\varphi(y)-\varphi(x_p))_+}{|y-x_p|^{\gamma p}}dy,$$

$$\mathfrak{B}_p^{p-1}(\varphi(x_p)) = \int_{\mathbb{R}^N} \frac{|\varphi(y)-\varphi(x_p)|^{p-2}(\varphi(y)-\varphi(x_p))_-}{|y-x_p|^{\gamma p}}dy,$$

$$(\mathcal{L}^s_\infty)^+[\varphi](x_0) := \sup_{y\in\mathbb{R}^N} \frac{u_\infty(y)-u_\infty(x_0)}{|x_0-y|^s} \qquad \textit{and} \qquad (\mathcal{L}^s_\infty)^-[\varphi](x_0) := \inf_{y\in\mathbb{R}^N} \frac{u_\infty(y)-u_\infty(x_0)}{|x_0-y|^s}.$$

The next result ensures that continuous weak solutions to (1.5) are also viscosity solutions.

Lemma 2.5 ([15, Lemma 3.9] and [16]). *Let $f(x,u)$ be a continuous function such that $f(x,\cdot)$ is nondecreasing. Let $u\in W^{s,p}_g(\Omega)$ be a weak solution to* (1.5) *and $\gamma p>N$. If u is continuous then it is a viscosity solution.*

Remark 2.6. Notice that if $\gamma p > 2N$ then we can remove the continuity assumption in Lemma 2.5, because Theorem 2.3 says that under this hypothesis u is a continuous function.

3 Main results. Proof of Theorems 1.1 and 1.2

Before proving our main result, let us present the notion of *Optimal s-Hölder extension* which plays an important role in our studies of optimal design problems in fractional diffusion.

Definition 3.1. We say that $v_g \in W^{s,\infty}(\mathbb{R}^N)$ is an optimal s-Hölder extension to $g:\mathbb{R}^N\setminus\Omega\to\mathbb{R}$ provided

- ✓ $g\in C^{0,s}(\mathbb{R}^N\setminus\Omega)$;
- ✓ $v_g = g$ in $\mathbb{R}^N\setminus\Omega$;
- ✓ $[v_g]_{C^{0,s}(\mathbb{R}^N)} \leq [g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)}$, where the previous s-Hölder semi-norm is defined as follows

$$[\omega]_{C^{0,s}(\mathscr{O})} := \sup_{\substack{x,y\in\mathscr{O}\\ x\neq y}} \frac{|\omega(y)-\omega(x)|}{|x-y|^s}$$

for any $\omega\in C^0(\mathscr{O})$ with $\mathscr{O}\subset\mathbb{R}^N$.

Now, we can proceed with the proof of Theorem 1.1.

Proof of Theorem 1.1. Let us consider an s-Hölder extension of g, which we will denominate by v_g, among all functions in the class

$$\mathscr{K}_\infty^s = \left\{\varphi \in W^{s,\infty}(\mathbb{R}^N) \;\middle|\; \varphi = g, \text{ in } \mathbb{R}^N \setminus \Omega,\ \mathcal{L}^N\left(\{\varphi > 0\} \cap \Omega\right) \leq \alpha\right\}. \tag{3.1}$$

Since u_p is a minimizer to $(\mathfrak{P}_p^s)$, then

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_p(y)-u_p(x)|^p}{|y-x|^{\gamma p}}dxdy \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v(y)-v(x)|^p}{|y-x|^{\gamma p}}dxdy$$

for all test functions v in the class

$$\mathscr{K}_p^s := \left\{\varphi \in W^{s,p}(\mathbb{R}^N) \;\middle|\; \varphi = g, \text{ in } \mathbb{R}^N \setminus \Omega,\ \mathcal{L}^N\left(\{\varphi > 0\} \cap \Omega\right) \leq \alpha\right\}.$$

Note that v_g competes in the minimization problem $(\mathfrak{P}_p^s)$. Consequently, by using v_g as a test function in problem $(\mathfrak{P}_p^s)$ we obtain, the following

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_p(y)-u_p(x)|^p}{|y-x|^{\gamma p}}dxdy \leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_g(y)-v_g(x)|^p}{|y-x|^{\gamma p}}dxdy.$$

Therefore,

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u_p(y)-u_p(x)|^p}{|y-x|^{\gamma p}}dxdy\right)^{\frac{1}{p}} \leq \left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_g(y)-v_g(x)|^p}{|y-x|^{\gamma p}}dxdy\right)^{\frac{1}{p}}. \tag{3.2}$$

Furthermore, it holds that

$$\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_g(y)-v_g(x)|^p}{|y-x|^{\gamma p}}dxdy\right)^{\frac{1}{p}} \to [v_g]_{C^{0,s}(\mathbb{R}^N)} \qquad \text{as} \quad p \to \infty. \tag{3.3}$$

Now our aim is to show that $u_p \to u_\infty$ in the $W^{s,q}$ weak topology, for all $1 < q < \infty$ and $s = \gamma - \frac{N}{q}$. To this end, fix $1 < q < \infty$ such that $p > q \gg 1$. Now, let us define

$$\mathfrak{S} := \sup_{x,y \in \mathbb{R}^N} [u_p(y) - u_p(x)].$$

Such a quantity is well defined, because according to Hölder embedding, Theorem 2.3, we obtain for p large enough that

$$\begin{aligned}
\|u_p\|_{C^{0,\sigma(n,s)}(\mathbb{R}^N)} \quad &\leq \quad \mathfrak{C}(N,s)[1+\mathfrak{c}^p]^{\frac{1}{p}}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|v_g(y)-v_g(x)|^p}{|y-x|^{\gamma p}}dxdy\right)^{\frac{1}{p}} \\
&\to \quad \mathfrak{c}(N,s) \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|v_g(y)-v_g(x)|}{|y-x|^s} \qquad \text{as} \quad p \to \infty.
\end{aligned}$$

Hence, Arzelá-Áscoli Theorem, as well as the fact that $u_p = g$ in $\mathbb{R}^N \setminus \Omega$, ensures us that, up to a subsequence,

$$\lim_{p\to\infty} u_p(z) = u_\infty(z) \quad \text{uniformly in } \mathbb{R}^N.$$

Moreover, as $u_p = g$ in $\mathbb{R}^N \setminus \Omega$, the limit fulfills

$$u_\infty \in C^0(\mathbb{R}^N), \text{ with } \|u_\infty\|_{L^\infty(\mathbb{R}^N)} \leq \mathfrak{c}(N,s) \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|v_g(y)-v_g(x)|}{|y-x|^s} \quad \text{and} \quad u_\infty = g \text{ in } \mathbb{R}^N \setminus \Omega.$$

Now, we consider the set

$$\mathbb{V}_\gamma := \left\{ y \in \mathbb{R}^N : \operatorname{dist}(y,\Omega)^\gamma \leq \mathfrak{S} \right\}.$$

Since

$$\sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u_\infty(y)-u_\infty(x)|}{|y-x|^s} = \max\left\{ \sup_{\substack{x\in\Omega,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u_\infty(y)-u_\infty(x)|}{|y-x|^s}, \sup_{\substack{x,y\in\mathbb{R}^N\setminus\Omega \\ x\neq y}} \frac{|g(y)-g(x)|}{|y-x|^s} \right\},$$

by virtue of (3.2) and (3.3), we just need to analyze the first supremum. In addiction, by Fatou's Lemma, we have

$$\begin{aligned}
\sup_{\substack{x\in\Omega,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u_\infty(y)-u_\infty(x)|}{|y-x|^s} &= \lim_{q\to\infty} \left(\int_\Omega \int_{\mathbb{R}^N} \frac{|u_\infty(y)-u_\infty(x)|^q}{|y-x|^{\gamma q}} dxdy \right)^{\frac{1}{q}} \\
&\leq \lim_{q\to\infty} \liminf_{p\to\infty} \left(\int_\Omega \int_{\mathbb{R}^N} \frac{|u_p(y)-u_p(x)|^q}{|y-x|^{\gamma q}} dxdy \right)^{\frac{1}{q}}.
\end{aligned}$$

Thus, we can re-write the fractional p-energy functional in the following way

$$\int_\Omega \int_{\mathbb{R}^N} \frac{|u_p(y)-u_p(x)|^q}{|y-x|^{\gamma q}} dxdy = \int_\Omega \int_{\mathbb{V}_\gamma} \frac{|u_p(y)-u_p(x)|^q}{|y-x|^{\gamma q}} dxdy + \int_\Omega \int_{\mathbb{R}^N\setminus\mathbb{V}_\gamma} \frac{|u_p(y)-u_p(x)|^q}{|y-x|^{\gamma q}} dxdy.$$

Applying Hölder inequality for the first integral we get

$$\begin{aligned}
\int_\Omega \int_{\mathbb{V}_\gamma} \frac{|u_p(y)-u_p(x)|^q}{|y-x|^{\gamma q}} dxdy &\leq \left(\int_\Omega \int_{\mathbb{V}_\gamma} \frac{|u_p(y)-u_p(x)|^p}{|y-x|^{\gamma p}} dxdy \right)^{\frac{q}{p}} \left(\mathcal{L}^N(\Omega\times\mathbb{V}_\gamma)\right)^{\frac{p-q}{p}} \\
&\leq \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|v_g(y)-v_g(x)|^p}{|y-x|^{\gamma p}} dxdy \right)^{\frac{q}{p}} \left(\mathcal{L}^N(\Omega\times\mathbb{V}_\gamma)\right)^{1-\frac{q}{p}}.
\end{aligned}$$

Now, analyzing the second integral we obtain

$$\begin{aligned}
\int_\Omega \int_{\mathbb{R}^N\setminus\mathbb{V}_\gamma} \frac{|u_p(y)-u_p(x)|^q}{|y-x|^{\gamma q}} dxdy &\leq \mathfrak{S}^q \int_\Omega \int_{|x-y|^\gamma > \mathfrak{S}} \frac{1}{|y-x|^{\gamma q}} dxdy \\
&= \mathfrak{S}^q \mathcal{L}^N(\Omega)\mathcal{L}^{N-1}(\mathbb{S}^{N-1}) \frac{\mathfrak{S}^{\frac{N-\gamma q}{\gamma}}}{\gamma q - N} \\
&= \frac{\mathfrak{S}^{\frac{N}{\gamma}} \mathcal{L}^N(\Omega)\mathcal{L}^{N-1}(\mathbb{S}^{N-1})}{\gamma q - N}.
\end{aligned}$$

Therefore, the sequence $(u_p)_{p>0}$ is uniformly bounded in the $W^{s,q}$-topology, and its weak limit as $p\to\infty$, verifies

$$\left(\int_\Omega \int_{\mathbb{R}^N} \frac{|u_\infty(y)-u_\infty(x)|^q}{|y-x|^{\gamma q}} dxdy \right)^{\frac{1}{q}} \leq \left[[v_g]^q_{C^{0,s}(\mathbb{R}^N)} .\mathcal{L}^N(\Omega\times\mathbb{V}_\gamma) + \frac{\mathfrak{S}^{\frac{N}{\gamma}} \mathcal{L}^N(\Omega)\mathcal{L}^{N-1}(\mathbb{S}^{N-1})}{\gamma q - N} \right]^{\frac{1}{q}}.$$

Finally, taking $q\to\infty$ and performing a standard diagonal argument, we obtain a subsequence, which we will be still labelled as u_p, that converges weakly in every $W^{s,q}(\mathbb{R}^N)$, for $1<q<\infty$ and $s=\gamma-\frac{n}{q}$, to a limit $u_\infty \in W^{s,\infty}(\mathbb{R}^N)$ such that

$$\sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|u_\infty(y)-u_\infty(x)|}{|x-y|^s} \leq \sup_{\substack{x,y\in\mathbb{R}^N \\ x\neq y}} \frac{|v(y)-v(x)|}{|y-x|^s},$$

for all functions v that belong to the set $\mathscr{K}^s_\infty$ given in (3.1).

Let us now estimate the Lebesgue measure of $\{u_\infty > 0\} \cap \Omega$. Fix an $\varepsilon > 0$. Thanks to the uniform convergence, for p large enough, there holds

$$\{u_\infty(x) > \varepsilon\} \cap \Omega \subset \{u_p(x) > 0\} \cap \Omega.$$

Hence

$$\mathcal{L}^N(\{u_\infty > \varepsilon\} \cap \Omega) \leq \mathcal{L}^N(\{u_p > 0\} \cap \Omega) \leq \alpha,$$

and we conclude that

$$\mathcal{L}^N(\{u_\infty > 0\} \cap \Omega) = \lim_{\varepsilon \to 0} \mathcal{L}^N(\{u_\infty > \varepsilon\} \cap \Omega) \leq \alpha.$$

Therefore, we have proved that u_∞ is an extremal for the limit problem ($\mathfrak{P}^s_\infty$).

We proved that there exists a subsequence of solutions u_{p_k} to ($\mathfrak{P}^s_p$) such that $u_{p_k} \to u_\infty$ uniformly as $p_k \to \infty$. It remains to prove that u_∞ verifies

$$-\mathcal{L}^s_\infty[u_\infty](x) = 0 \quad \text{in} \quad \{u_\infty > 0\} \cap \Omega$$

in the viscosity sense. To prove this fact, we argue as follows: let $x_0 \in \{u_\infty > 0\} \cap \Omega$ and $x_p \to x_0$ be a sequence of minima for the positive functions $u_p - \varphi$ and such that

$$\psi(x) = u_\infty(x) - \varphi(x) > 0$$

achieves a strictly minimum at x_0 (for a test function φ). According to Lemma 2.5, for p sufficiently large, we have

$$-\left[\mathfrak{A}^{p-1}_p(\varphi(x_p)) - \mathfrak{B}^{p-1}_p(\varphi(x_p))\right] \geq 0$$

point-wisely (because $u_p(x_p) > 0$ and u_p is a viscosity solution to (1.5) in $\{u_p > 0\} \cap \Omega$ for $f \equiv 0$). Therefore,

$$\mathfrak{B}_p(\varphi(x_p)) \geq \mathfrak{A}_p(\varphi(x_p))$$

and by using Lemma 2.4 we obtain after passing the limit as $p \to \infty$ the following

$$-(\mathcal{L}^s_\infty)^+[\varphi](x_0) \geq (\mathcal{L}^s_\infty)^-[\varphi](x_0) \quad \Rightarrow \quad -\mathcal{L}^s_\infty[\varphi](x_0) \geq 0.$$

The last one says us that u_∞ is a viscosity supersolution for the Hölder Infinity Laplacian. Similarly we can prove that u_∞ is a viscosity supersolution. □

Theorem 3.2 (**Characterization of minimizers and their optimal sets).** *Assume that* (**Comp. Assump.**) *holds, and let $\mathfrak{H}^\sharp$ be the unique positive number such that*

$$\Omega^\sharp := \bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(y) \cap \Omega \quad \textit{fulfils} \quad \mathcal{L}^N\left(\Omega^\sharp\right) = \alpha. \tag{3.4}$$

Then v_∞ given as

$$\begin{cases} -\mathcal{L}^s_\infty[v_\infty] &=& 0 & \text{in} & \Omega^\sharp \\ v_\infty(x) &=& g(x) & \text{on} & \mathbb{R}^N \setminus \Omega \\ v_\infty(x) &=& 0 & \text{on} & \partial\Omega^\sharp \cap \Omega \end{cases}$$

is the unique minimizer for ($\mathfrak{P}^s_\infty$). Therefore, if v_p minimizes ($\mathfrak{P}^s_p$), then $v_p \to v_\infty$ as $p \to \infty$ uniformly in Ω and weakly in $W^{s,q}(\Omega)$ for all $1 < q < \infty$ and also the extremal values converge $\mathfrak{L}^p \to \mathfrak{L}^\infty = \mathfrak{H}^\sharp$ as $p \to \infty$. Furthermore, v_∞ is given explicitly by the formula,

$$v_\infty(x) = \sup_{\mathbb{R}^N \setminus \Omega} \left(g(y) - \mathfrak{H}^\sharp |x-y|^s\right)_+.$$

Proof. First of all, notice that due to assumption **Comp. Assump.**

$$[g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)} \le \mathfrak{H}^\sharp.$$

Now, Theorem 1.1 assures the existence of at least one minimizer v_∞ for problem $(\mathfrak{P}_\infty^s)$. Next, for each point $z \in \partial\{v_\infty > 0\} \cap \Omega$ we have due to s-Hölder regularity of v_∞ that

$$g(y) = v_\infty(y) - v_\infty(z) \le [v_\infty]_{C^{0,s}(\mathbb{R}^N)}|y-z|^s \quad \forall\, y \in \mathbb{R}^N\setminus\Omega.$$

Consequently,

$$\bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{[v_\infty]_{C^{0,s}(\mathbb{R}^N)}}\right)^{\frac{1}{s}}}(y)\cap\Omega \subset \{v_\infty>0\}\cap\Omega. \tag{3.5}$$

In particular, this means that

$$\mathcal{L}^N\left(\bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{\mathfrak{L}^\sharp}\right)^{\frac{1}{s}}}(y)\cap\Omega\right) = \alpha \ge \mathcal{L}^N\left(\bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{[v_\infty]_{C^{0,s}(\mathbb{R}^N)}}\right)^{\frac{1}{s}}}(y)\cap\Omega\right). \tag{3.6}$$

Moreover,

$$[v_\infty]_{C^{0,s}(\mathbb{R}^N)} \ge \mathfrak{H}^\sharp. \tag{3.7}$$

Now, letting

$$\Omega^\sharp := \bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(y)\cap\Omega$$

we observe that $\hat{v}_\infty$ given by

$$\begin{cases} -\mathcal{L}_\infty^s[\hat{v}_\infty] &=& 0 & \text{in} & \Omega^\sharp \\ \hat{v}_\infty &=& g & \text{on} & \mathbb{R}^N\setminus\Omega \\ \hat{v}_\infty &=& 0 & \text{on} & \partial\Omega^\sharp\cap\Omega \end{cases}$$

is a competitor function for the minimization problem $(\mathfrak{P}_\infty^s)$. Hence,

$$[v_\infty]_{C^{0,s}(\mathbb{R}^N)} \le [\hat{v}_\infty]_{C^{0,s}(\mathbb{R}^N)}.$$

Now, consider the following barrier function $\Theta_g : \mathbb{R}^N \to \mathbb{R}_+$ given by

$$\Theta_g(x) := \sup_{y\in\mathbb{R}^N\setminus\Omega}\left(g(y) - \mathfrak{H}^\sharp|x-y|^s\right)_+,$$

where we have extended g in an s-Hölder way in Ω. We affirm that

$$[\Theta_g]_{C^{0,s}(\mathbb{R}^N)} = \mathfrak{H}^\sharp.$$

First of all, let us show that $[\Theta_g]_{C^{0,s}(\mathbb{R}^N)} \le \mathfrak{H}^\sharp$. Without loss of generality select $x_1, x_2 \in \mathbb{R}^N$ and assume that

$$0 < \Theta_g(x_1) < \Theta_g(x_2).$$

Now, let $\hat{x}_1, \hat{x}_2 \in \mathbb{R}^N\setminus\Omega$ such that

$$\Theta_g(x_i) = g(\hat{x}_i) - \mathfrak{H}^\sharp|x_i - \hat{x}_i|^s \quad \text{for} \quad i = 1, 2.$$

Notice that from the definition of Θ_g it follows that

$$\Theta_g(x_1) \geq g(\hat{x}_2) - \mathfrak{H}^\sharp |x_1 - \hat{x}_2|^s.$$

For this fact and by using that $|\cdot|^s$ is a distance function we get that

$$\Theta_g(x_2) - \Theta_g(x_1) \leq \mathfrak{H}^\sharp (|x_1 - \hat{x}_2|^s - |x_2 - \hat{x}_2|^s) \leq \mathfrak{H}^\sharp |x_1 - x_2|^s.$$

Hence, this implies that $[\Theta_g]_{C^{0,s}(\mathbb{R}^N)} \leq \mathfrak{H}^\sharp$.

Now, let us verify the reverse inequality. Given $\hat{x} \in \Omega$, there exists $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^N \setminus \Omega$ such that

$$\Theta_g(\hat{x}) = \lim_{k\to\infty} g(x_k) - \mathfrak{H}^\sharp |\hat{x} - x_k|^s.$$

Consequently,

$$\limsup_{k\to\infty} \frac{\Theta_g(x_k) - \Theta_g(\hat{x})}{|\hat{x} - x_k|} \geq \lim_{k\to\infty} \frac{g(x_k) - \Theta_g(\hat{x})}{|\hat{x} - x_k|} = \mathfrak{H}^\sharp,$$

which assures that $[\Theta_g]_{C^{0,s}(\mathbb{R}^N)} \geq \mathfrak{H}^\sharp$.

In what follows, we will check that Θ_g satisfies the boundary conditions. From its definition it is immediate that

$$\Theta_g = 0 \quad \text{on} \quad \partial\Omega^\sharp.$$

Moreover, we claim that

$$g(x) = \sup_{y\in\mathbb{R}^N\setminus\Omega} \left(g(y) - \mathfrak{H}^\sharp |x-y|^s \right)_+ . \tag{3.8}$$

In fact, arguing by contradiction, we assume that (3.8) is not satisfied. This would imply there exist points $z, w \in \mathbb{R}^N \setminus \Omega$ such that

$$\mathfrak{H}^\sharp |z - w| < g(z) - g(w),$$

which implies that

$$\mathfrak{H}^\sharp < [g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)}.$$

However, this contradicts the assumption (**Comp. Assump.**).

Next, we will check that Θ_g has the same contour conditions as v_∞. Once we verified it, we know that

$$[v_\infty]_{C^{0,s}(\mathbb{R}^N)} \leq [\Theta_g]_{C^{0,s}(\mathbb{R}^N)} \leq \mathfrak{H}^\sharp. \tag{3.9}$$

Our next aim is to show that

$$v_\infty(x) = \sup_{y\in\mathbb{R}^N\setminus\Omega} \left(g(y) - \mathfrak{H}^\sharp |x-y|^s \right)_+$$

Indeed, assuming that this does not hold, then there exists $\hat{x}$ such that

✓ $v_\infty(\hat{x}) < \Theta_g(\hat{x})$

By considering quotients involving $\hat{x}$ and points in $\mathbb{R}^N \setminus \Omega$, we conclude that

$$[v_\infty]_{C^{0,s}(\mathbb{R}^N)} > \mathfrak{H}^\sharp \geq [\Theta_g]_{C^{0,s}(\mathbb{R}^N)},$$

which is clearly a contradiction, because Θ_g competes with v_∞ in the limit optimization problem.

Therefore, both functions have the same positivity set, because

$$\mathcal{L}^N(\{v_\infty > 0\} \cap \Omega) = \alpha = \mathcal{L}^N(\{\Theta_g > 0\} \cap \Omega)$$

and one set is included into the other.

✓ Or $v_\infty(\hat{x}) > \Theta_g(\hat{x})$.

In this case, by comparing quotients defining the s-Hölder constant with $\hat{x}$ and points on the boundary of the positivity set, we obtain

$$[v_\infty]_{C^{0,s}(\mathbb{R}^N)} > \mathfrak{H}^\sharp \geq [\Theta_g]_{C^{0,s}(\mathbb{R}^N)},$$

which contradicts the fact that v_∞ is optimal for the limit optimization problem.

Finally, we conclude the proof by combining (3.5), (3.6), (3.7), (3.9) and the fact that v_∞ and Θ_g are Hölder infinity harmonic in $\Omega^\sharp$ with the same boundary data. □

Example 3.3. Let us explore the relationship between the α-volume of the optimal set $\Omega^\sharp := \{u_\infty > 0\} \cap \Omega$ and the corresponding constants $0 < s < 1$ and $\mathfrak{H}^\sharp$.

1. Consider $0 < r_0 < r < \mathfrak{R}$, $\Omega = B_{r_0}(0) \subset \mathbb{R}^2$ and $g(x) = \chi_{\overline{B_{\mathfrak{R}}(0)} \setminus B_r(0)}$. Thus is not to hard to see that

$$\{u_\infty > 0\} \cap B_r(0) = \bigcup_{y \in \partial B_r(0)} \left[B_{\left(\frac{1}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(y) \cap B_{r_0}(0) \right].$$

Consequently,

$$\alpha = \mathcal{L}^2\left(\Omega^\sharp\right) = \mathcal{L}^2\left(B_{r_0}(0)\right) - \mathcal{L}^2\left(B_{r - \left(\frac{1}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(0)\right) = \pi\left[r_0^2 - r^2 + 2r\left(\frac{1}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}} - \left(\frac{1}{\mathfrak{H}^\sharp}\right)^{\frac{2}{s}}\right]$$

Finally, $\mathfrak{H}^\sharp = \mathfrak{H}^\sharp(r_0, r, s, \alpha) = \dfrac{1}{\left(r - \sqrt{\frac{\pi r_0^2 - \alpha}{\pi}}\right)^s}$.

2. Now, let us take $0 < r < R$, the domain $\Omega = B_r(0) \subset \mathbb{R}^2$ and the function $g : \mathbb{R}^2 \setminus B_r \to \mathbb{R}^+$ given by $g(y) = (R^2 - |y|^2)_+^s$. Then, it holds that

$$\{u_\infty > 0\} \cap B_r(0) = \bigcup_{y \in \partial B_r(0)} \left[B_{\left(\frac{(R^2 - r^2)^s}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(y) \cap B_r(0) \right].$$

Moreover,

$$\begin{aligned}\alpha = \mathcal{L}^2\left(\{u_\infty > 0\} \cap B_r(0)\right) &= \mathcal{L}^2\left(B_r(0)\right) - \mathcal{L}^2\left(B_{r - \left(\frac{(R^2 - r^2)^s}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(0)\right) \\ &= \pi\left[2r\left(\frac{(R^2 - r^2)^s}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}} - \left(\frac{(R^2 - r^2)^s}{\mathfrak{H}^\sharp}\right)^{\frac{2}{s}}\right].\end{aligned}$$

Therefore, $\mathfrak{H}^\sharp = \mathfrak{H}^\sharp(R, r, s, \alpha) = \left(\dfrac{R^2 - r^2}{r - \sqrt{\frac{\pi r^2 - \alpha}{\pi}}}\right)^s$.

Remark 3.4. Concerning the optimal set $\Omega^\sharp$ an interesting question appears: what should be the (topological, geometrical or analytical) condition under g or Ω in order to the centers of the balls in (3.4) belong to $\partial\Omega$, i.e.,

$$\Omega^\sharp = \bigcup_{y \in \partial\Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^\sharp}\right)^{\frac{1}{s}}}(y) \cap \Omega.$$

To answer this question we argue as follows: Fix $z \in \mathbb{R}^N \setminus \Omega$ and let $y \in \partial\Omega$ such that $|z-y| = \mathrm{dist}(z, \partial\Omega)$. If

$$\text{(\textbf{Geom. Assump.})} \qquad \left| g(z)^{\frac{1}{s}} - g(y)^{\frac{1}{s}} \right| \leq (\mathfrak{H}^{\sharp})^{\frac{1}{s}}.|z-y|$$

then

$$\Omega^{\sharp} = \bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^{\sharp}}\right)^{\frac{1}{s}}}(y) \cap \Omega \subset \bigcup_{y \in \partial\Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^{\sharp}}\right)^{\frac{1}{s}}}(y) \cap \Omega \subset \Omega^{\sharp}.$$

Example 3.5. Let us present some examples in order to explore how the previous geometric condition (**Geom. Assump.**) behave.

1. If $\Omega = B_r(0) \subset \mathbb{R}^2$ and $g(y) = e^{-s|y|^2}$ then $\mathfrak{H}^{\sharp} = \mathfrak{H}^{\sharp}(r,s,\alpha) = \left(\frac{e^{-r^2}}{r - \sqrt{\frac{\pi r^2 - \alpha}{\pi}}} \right)^s$.

2. If $\Omega = B_r(0) \subset \mathbb{R}^2$ and $g(y) = |y|^s$ then $\mathfrak{H}^{\sharp} = \mathfrak{H}^{\sharp}(r,s,\alpha) = \left(\frac{r}{r - \sqrt{\frac{\pi r^2 - \alpha}{\pi}}} \right)^s$.

3. If $\Omega = B_r(0) \subset \mathbb{R}^3$ and $g : \mathbb{R}^3 \setminus B_r(0) \to \mathbb{R}_+$ is a radial s-Hölder function fulfilling (**Geom. Assump.**) then
$$\mathfrak{H}^{\sharp} = \mathfrak{H}^{\sharp}(r,s,\alpha) = \frac{g(r)}{\left(r - \sqrt[3]{\frac{4\pi r^3 - 3\alpha}{4\pi}} \right)^s}.$$
Particularly, if $g(y) = \mathfrak{c}_0.\chi_{\{\mathbb{R}^3 \setminus B_r(0)\}}$ for some $\mathfrak{c}_0 > 0$ then $\mathfrak{H}^{\sharp} = \frac{\mathfrak{c}_0}{\left(r - \sqrt[3]{\frac{4\pi r^3 - 3\alpha}{4\pi}} \right)^s}$.

Next we will show that the assumption (**Comp. Assump.**) is a necessary and sufficient condition in order to obtain uniqueness to ($\mathfrak{P}_{\infty}^s$). In fact, if (**Comp. Assump.**) is not satisfied, we can find multiple solutions for ($\mathfrak{P}_{\infty}^s$). In spite of this multiplicity result, we are able to prove the existence of a minimal solution.

Theorem 3.6. *Let us assume that* (**Comp. Assump.**) *does not hold. Then there exists infinitely many minimizers for* ($\mathfrak{P}_{\infty}^s$). *Moreover,*

$$v_{\infty}(x) = \sup_{y \in \mathbb{R}^N \setminus \Omega} \left(g(y) - \mathfrak{H}^{\sharp}|x-y|^s \right)_+$$

is a minimizer such that

$$\{v_{\infty} > 0\} \cap \Omega = \bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\frac{g(y)}{\mathfrak{H}^{\sharp}}}(y) \cap \Omega \quad \textit{fulfils} \quad \mathcal{L}^N(\{v_{\infty} > 0\} \cap \Omega) < \alpha.$$

Finally, v_{∞} is the least (or minimal) solution, in the following sense

$$v_{\infty}(x) \leq u_{\infty}(x) \quad in \quad \Omega$$

for any other minimizer u_{∞} to ($\mathfrak{P}_{\infty}^s$).

Proof. Let $\mathfrak{H}^{\sharp} > 0$ the unique constant such that

$$\Omega^{\sharp} := \bigcup_{y \in \mathbb{R}^N \setminus \Omega} B_{\left(\frac{g(y)}{\mathfrak{H}^{\sharp}}\right)^{\frac{1}{s}}}(y) \cap \Omega \quad \text{and} \quad \mathcal{L}^N(\Omega^{\sharp}) = \alpha.$$

Since (**Comp. Assump.**) is not satisfied, this means that

$$[g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)} > \mathfrak{H}^{\sharp}.$$

Now, define

$$\Xi := \bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{[g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)}}\right)^{\frac{1}{s}}}(y)\cap\Omega$$

and notice that $\mathcal{L}^N(\Xi)<\alpha$. As previously proved

$$\Theta_g(x) := \sup_{y\in\mathbb{R}^N\setminus\Omega}\left(g(y)-\mathfrak{L}^\sharp|x-y|^s\right)_+,$$

is an extremal for the limiting optimization problem with measure $\mathcal{L}^N(\Xi)$.

Next, let $\hat{v}_\infty$ be an extremal function for the limiting problem with N-dimensional Lebesgue measure α. Thus, since $\hat{v}_\infty = g$ on $\mathbb{R}^N\setminus\Omega$ we have

$$[\hat{v}]_{C^{0,s}(\mathbb{R}^N)} \geq [g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)} \geq [\Theta_g]_{C^{0,s}(\mathbb{R}^N)}. \tag{3.10}$$

Remind that Θ_g competes in the limiting optimization problem with N-dimensional Lebesgue measure α. For this reason,

$$[\hat{v}]_{C^{0,s}(\mathbb{R}^N)} \leq [\Theta_g]_{C^{0,s}(\mathbb{R}^N)}.$$

Therefore, it holds the triple equality in (3.10) and consequently Θ_g maximizes the limiting optimization problem. Furthermore, for $x\in\Xi$ we obtain that

$$\Theta_g(x)\leq\hat{v}_\infty(x),$$

otherwise the s-Hölder semi-norm of $\hat{v}_\infty$ is greater than one for Θ_g. For the sake of contradiction let us assume that there exists $\hat{x}\in\Xi$ such that

$$\hat{v}_\infty(\hat{x}) < \Theta_g(\hat{x}) = \sup_{y\in\mathbb{R}^N\setminus\Omega}\left(g(y)-[g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)}|\hat{x}-y|^s\right)_+.$$

From this fact and using that $\hat{v}_\infty = g$ on $\mathbb{R}^N\setminus\Omega$ we obtain that

$$\sup_{\mathbb{R}^N\setminus\Omega}\frac{\hat{v}_\infty(y)-\hat{v}_\infty(x_0)}{|y-x_0|^s} > [g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)},$$

which implies

$$[\Theta_g]_{C^{0,s}(\mathbb{R}^N)} = [g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)} \leq [\hat{v}_\infty]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)} \leq [\hat{v}_\infty]_{C^{0,s}(\mathbb{R}^N)},$$

yielding a contradiction with the optimality of $\hat{v}_\infty$.

Therefore, Θ_g is the *minimal/extremal* for the limiting optimization problem. Moreover, for any extremal $\hat{v}_\infty$ the following inclusion there holds for its support

$$\Xi := \bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{[g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)}}\right)^{\frac{1}{s}}}(y)\cap\Omega \subset \{\hat{v}_\infty>0\}\cap\Omega.$$

Hereafter, for $0<\sigma<1$ (small enough) consider the σ-neighborhood of Ξ,

$$\Xi_\sigma := \left(\bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{[g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)}}\right)^{\frac{1}{s}}}(y)\cap\Omega\right)+B_\sigma(0) \quad \text{such that} \quad \mathcal{L}^N(\Xi_\sigma)<\alpha.$$

Next, consider u_∞ the viscosity solution to the following boundary value problem

$$\begin{cases} -\mathcal{L}_\infty^s[u_\infty] &= 0 \quad \text{in} \quad \Xi_\sigma \\ u_\infty &= g \quad \text{on} \quad \mathbb{R}^N \setminus \Omega \\ u_\infty &= 0 \quad \text{on} \quad \partial \Xi_\sigma. \end{cases}$$

Since $\Xi \subset \Xi_\sigma$ we claim that

$$[u_\infty]_{C^{0,s}(\mathbb{R}^N)} = [g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}.$$

In order to prove this statement, let us define on the Ξ_σ the auxiliary boundary value function

$$\mathfrak{G}(x) := \begin{cases} g(x) & \text{in} \quad \mathbb{R}^N \setminus \Omega \\ 0 & \text{in} \quad \partial \Xi_\sigma \cap \Omega. \end{cases}$$

Notice that such an exterior datum $\mathfrak{G}$ is an s-Hölder function with corresponding s-Hölder semi-norm given by

$$[\mathfrak{G}]_{C^{0,s}(\partial \Xi_\sigma)} = \sup_{\substack{x,y \in \partial \Xi_\sigma \\ x \neq y}} \frac{|\mathfrak{G}(x) - \mathfrak{G}(y)|}{|x-y|^s}.$$

In the following we will estimate the constant $[\mathfrak{G}]_{C^{0,s}(\partial \Xi_\sigma)}$. To this end, we must consider three cases:

✓ If $x, y \in \partial \Xi_\sigma \cap \Omega$. In this case,

$$\frac{|\mathfrak{G}(x) - \mathfrak{G}(y)|}{|x-y|^s} = 0 < [g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}.$$

✓ If $x, y \in \mathbb{R}^N \setminus \Omega$. Immediately we obtain

$$\frac{|\mathfrak{G}(x) - \mathfrak{G}(y)|}{|x-y|^s} = 0 \leq [g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}.$$

✓ If $x \in \mathbb{R}^N \setminus \Omega$ and $y \in \partial \Xi_\sigma \cap \Omega$ In this last case we have

$$\frac{|\mathfrak{G}(x) - \mathfrak{G}(y)|}{|x-y|^s} = \frac{|f(x)|}{|x-y|^s} < [g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)},$$

where we have been using that $\Xi \subset \Omega_\sigma$, and consequently the distance $|x-y|^s$ is much bigger than $\frac{g(x)}{[g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}}$, which is due to the fact that for $y \in \partial \Xi$ and any $x \in \mathbb{R}^N \setminus \Omega$, we obtain

$$g(x) - [g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)} |x-y|^s \leq 0 \quad \Leftrightarrow \quad |x-y|^s \geq \frac{g(x)}{[g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}}.$$

Therefore, we conclude that

$$[\mathfrak{G}]_{C^{0,s}(\partial \Xi_\sigma)} = [g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}.$$

Moreover, as u_∞ has the same s-Hölder semi-norm that $\mathfrak{G}$, we obtain that

$$[u_\infty]_{C^{0,s}(\mathbb{R}^N)} = [g]_{C^{0,s}(\mathbb{R}^N \setminus \Omega)}.$$

In other words, u_∞ is also an extremal function for the limiting optimization problem, which is positive on $\partial \Xi \subset \text{int}(\Xi_\sigma)$. Finally, we conclude that $u_\infty \neq \Theta_g$, as well as the fact that there is no monotonicity with respect to the measure in the limiting optimization problem. □

An immediate consequence of our previous Theorem 3.6 is the following convergence result:

Corollary 3.7. *If*

$$\hat{\alpha} := \mathcal{L}^N\left(\bigcup_{y\in\mathbb{R}^N\setminus\Omega} B_{\left(\frac{g(y)}{[g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)}}\right)^{\frac{1}{s}}}(y)\cap\Omega\right) < \alpha,$$

and in ($\mathfrak{P}_p^s$), u_p *is an extremal for* $\mathfrak{L}_p^s[\alpha]$ *and* v_p *is an extremal for* $\mathfrak{L}_p^s[\hat{\alpha}]$, *then*

$$\lim_{p\to\infty}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u_p(y)-u_p(x)|^p}{|y-x|^{N+sp}}dxdy\right)^{\frac{1}{p}} = \lim_{p\to\infty}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|v_p(y)-v_p(x)|^p}{|y-x|^{N+sp}}dxdy\right)^{\frac{1}{p}}.$$

Furthermore,

$$u_p \rightrightarrows u_\infty \quad \text{and} \quad v_p \rightrightarrows \Theta_g \quad \text{in} \quad \mathbb{R}^N$$

with

$$[u_\infty]_{C^{0,s}(\mathbb{R}^N)} = [g]_{C^{0,s}(\mathbb{R}^N\setminus\Omega)} = [\Theta_g]_{C^{0,s}(\mathbb{R}^N)} \quad \text{and} \quad \Theta_g(x) \le u_\infty(x).$$

4 Main results. Proof of Theorems 1.3 and 1.4

In this section we study some quantitative regularity and geometric measure properties for the limiting free boundary, namely $\partial\{v_\infty > 0\}\cap\Omega$, as well as convergence issues of the corresponding free boundaries $\partial\{v_p > 0\}\cap\Omega$.

Theorem 4.1 (**$C^{0,s}$ regularity for minimizers**). *Let* v_p *be minimizer to* ($\mathfrak{P}_p^s$) *and assume that for a subsequence (denoted as* v_p *yet)* $v_p \to v_\infty$ *uniformly in* $\overline{\Omega}$ *and weakly in* $W^{s,q}(\Omega)$ *for every* $1 < q < \infty$. *Then* v_∞ *is uniformly s-Hölder continuous in* Ω.

Proof. Revisiting the proof of Theorem 1.1, we conclude that any limit point u_∞ of minimizers u_p to ($\mathfrak{P}_p^s$) converging uniformly in $\overline{\Omega}$ and weakly in $W^{s,q}(\Omega)$ for every $1 < q < \infty$ fulfils:

- ✓ $u_\infty \in W^{s,\infty}(\Omega)$
- ✓ $u_\infty \in C^0(\mathbb{R}^N)$ with

$$\|u_\infty\|_{L^\infty(\mathbb{R}^N)} \le \mathfrak{c}(N,s).\sup_{\substack{x,y\in\mathbb{R}^N\setminus\Omega\\ x\neq y}}\frac{|g(y)-g(x)|}{|y-x|^s} \quad \text{and} \quad u_\infty = g \text{ in } \mathbb{R}^N\setminus\Omega.$$

- ✓ $u_\infty \in C^{0,s}(\mathbb{R}^N)$ with

$$\sup_{\substack{x,y\in\mathbb{R}^N\\ x\neq y}}\frac{|u_\infty(y)-u_\infty(x)|}{|x-y|^s} \le \sup_{\substack{x,y\in\mathbb{R}^N\setminus\Omega\\ x\neq y}}\frac{|g(y)-g(x)|}{|y-x|^s}.$$

□

Theorem 4.2 (**Strong non-degeneracy for minimizers**). *Let* v_∞ *a uniform limit as* $p\to\infty$ *to* ($\mathfrak{P}_\infty^s$). *Then* v_∞ *is strongly non-degenerate of order s, i.e., there is a constant* $\mathfrak{c} = \mathfrak{c}(N,s) >$ *such that for any fixed point* $x_0 \in \overline{\{v_\infty > 0\}}\cap\Omega$ *there holds*

$$\sup_{B_r(x_0)} v_\infty(x) \ge \mathfrak{c}\, r^s. \tag{4.1}$$

Proof. First of all, notice that by continuity, it suffices to prove (4.1) for points within the set $\{v_\infty > 0\} \cap \Omega$. Next, fix $x_0 \in \{v_\infty > 0\}$ and define the scaled function

$$v_r(x) := \frac{v_\infty(x_0 + rx)}{r^s} \quad \text{in} \quad B_1(0)$$

and the auxiliary barrier function

$$\Phi(x) := \mathfrak{c}|x|^s,$$

for a constant $\mathfrak{c} = \mathfrak{c}(N, s) > 0$ to be chosen *a posteriori*. Thus,

$$\mathcal{L}_\infty^s[\Phi](x) \leq 0 \leq \mathcal{L}_\infty^s[v_r](x) \qquad \text{in} \quad B_1$$

Now, if $v_r \leq \Phi$ in whole $\mathbb{R}^N \setminus B_1(0)$, then the Comparison Principle would imply that $v_r \leq \Phi$ in $B_1(0)$. However, this contradicts the fact that $v_r(0) > 0$. Therefore, there exists a point $z \in \mathbb{R}^N \setminus B_1(0)$ such that

$$v_r(z) > \Phi(z) = \mathfrak{c}|z|^s \quad \Rightarrow \quad v_\infty(x_0 + rz) > \mathfrak{c}r^s|z|^s$$

Now, using the s-Hölder continuity for minimizers we obtain

$$\mathfrak{c}r^s|z|^s - v_\infty(x_0 + rx) \leq v_\infty(x_0 + rz) - v_\infty(x_0 + rx) \leq \hat{\mathfrak{c}}r^s|x - z|^s \leq 2^s\hat{\mathfrak{c}}r^s|z|^s.$$

Finally,

$$\sup_{B_r(x_0)} v_\infty(y) \geq v_\infty(x_0 + rx) \geq r^s|z|^s(\mathfrak{c} - 2^s\hat{\mathfrak{c}}) \geq \mathfrak{c}^\sharp r^s,$$

provided we choose $\mathfrak{c}^\sharp := \mathfrak{c} - 2^s\hat{\mathfrak{c}} > 0$. □

Once we have established the asymptotic behaviour for the limiting free boundary problem, it becomes possible to obtain some weak geometric and measure theoretic properties for the free boundaries.

The next result says that the positivity set of a limiting function has uniform positive density along the free boundary, which inhibits the development of cusps pointing inwards to the vanishing region.

Corollary 4.3 (Uniform Lower Positive Density). *Let $x_0 \in \{v_\infty > 0\} \cap \Omega$ be an interior point. If v_∞ is a minimizer to $(\mathfrak{P}_\infty^s)$ in Ω, then there exists a constant $\mathfrak{c}_0 = \mathfrak{c}_0(N, s) > 0$ such that for every $r \ll 1$ it holds*

$$\mathcal{L}^N(\{v_\infty > 0\} \cap B_r(x_0)) \geq \mathfrak{c}_0 r^N.$$

Proof. From the Strong Non-degeneracy, Theorem 4.2, we have that there exists $z \in B_r(x_0)$ such that

$$v_\infty(z) \geq \mathfrak{c}(N, s)r^s > 0.$$

Furthermore, due to s-Hölder regularity, Theorem 4.1, for $y \in B_{\varsigma r}(z)$ we get,

$$v_\infty(y) - \mathfrak{C}(N, s)(\varsigma r)^s \geq v_\infty(z).$$

Hence, by the previous estimate, it is possible to choose $0 < \varsigma \ll 1$ (small enough) such that

$$y \in B_r(x_0) \cap B_{\varsigma r}(z) \quad \text{and} \quad v_\infty(y) > 0.$$

Therefore, we conclude that there exists a portion of $B_r(x_0)$ with volume comparable to r^N within $\{v_\infty > 0\} \cap \Omega$, i.e.,

$$\mathcal{L}^N\left(B_r(x_0) \cap \{v_\infty > 0\}\right) \geq \mathcal{L}^N\left(B_r(x_0) \cap B_{\varsigma r}(z)\right) = \mathfrak{c}_0(N, s)\mathcal{L}^N(B_r(x_0)).$$

□

Definition 4.4. A set $\mathfrak{S} \subset \mathbb{R}^N$ is said porous with porosity $\delta > 0$, if $\exists\, \mathfrak{R} > 0$ such that

$$\forall\, x \in \mathfrak{S},\ \forall\, r \in (0, \mathfrak{R}),\ \exists\, y \in \mathbb{R}^N \text{ such that } B_{\delta r}(y) \subset B_r(x) \setminus \mathfrak{S}.$$

A porous set of porosity δ has Hausdorff dimension not exceeding $N - \mathfrak{c}\delta^N$, where $\mathfrak{c} > 0$ is a dimensional constant. Particularly, a porous set has Lebesgue measure zero.

As a consequence of the optimal growth rate, Theorem 4.1 and non-degeneracy property, Theorem 4.2 we will obtain porosity for the free boundary.

Corollary 4.5. *Let v_∞ be a minimizer to $(\mathfrak{P}^s_\infty)$. Then the free boundary $\partial\{v_\infty > 0\} \cap \Omega$ is a porous set.*

Proof. Let $\mathfrak{R} > 0$ and $x_0 \in \Omega$ be such that $\overline{B_{4\mathfrak{R}}(x_0)} \subset \Omega$. We will prove that the set $\partial\{v_\infty > 0\} \cap B_{\mathfrak{R}}(x_0)$ is $\frac{\delta}{2}$-porous, for a universal constant $0 < \delta \leq 1$. To this end, let $x \in \partial\{v_\infty > 0\} \cap B_{\mathfrak{R}}(x_0)$. For each $r \in (0, \mathfrak{R})$ we have $\overline{B_r(x)} \subset B_{2\mathfrak{R}}(x_0) \subset \Omega$. Now, let $y \in \partial B_r(x)$ such that $v_\infty(y) = \sup\limits_{\partial B_r(x)} v_\infty$. By Non-degeneracy

$$v_\infty(y) \geq \mathfrak{c} r^s, \tag{4.2}$$

where $\mathfrak{c} > 0$ is a universal constant. On the other hand, near the free boundary

$$v_\infty(y) \leq \mathfrak{C}\mathfrak{d}(y)^s, \tag{4.3}$$

where $\mathfrak{C} > 0$ is a universal constant and $\mathfrak{d}(y) := \operatorname{dist}(y, \partial\{v_\infty > 0\} \cap \overline{B_{2\mathfrak{R}}(x_0)})$. Now, from (4.2) and (4.3) we get

$$\mathfrak{d}(y) \geq \delta r \tag{4.4}$$

for a universal positive constant $0 < \delta \leq 1$. Now, let $\hat{y} \in [x, y]$ be such that $|y - \hat{y}| = \frac{\delta r}{2}$. Then, there holds

$$B_{\frac{\delta}{2} r}(\hat{y}) \subset B_{\delta r}(y) \cap B_r(x). \tag{4.5}$$

Indeed, for each $z \in B_{\frac{\delta}{2} r}(\hat{y})$

$$|z - y| \leq |z - \hat{y}| + |y - \hat{y}| < \frac{\delta r}{2} + \frac{\delta r}{2} = \delta r,$$

and

$$|z - x| \leq |z - \hat{y}| + \big(|x - y| - |\hat{y} - y|\big) \leq \frac{\delta r}{2} + \left(r - \frac{\delta r}{2}\right) = r,$$

and hence (4.5) follows. Finally, since by (4.4) $B_{\delta r}(y) \subset B_{\mathfrak{d}(y)}(y) \subset \{v_\infty > 0\}$, we have

$$B_{\delta r}(y) \cap B_r(x) \subset \{v_\infty > 0\} \cap \Omega,$$

which together with (4.5) yields

$$B_{\frac{\delta}{2} r}(\hat{y}) \subset B_{\delta r}(y) \cap B_r(x) \subset B_r(x) \setminus \partial\{v_\infty > 0\} \subset B_r(x) \setminus \partial\{v_\infty > 0\} \cap B_R(x_0).$$

Therefore, the free boundary is a $\frac{\delta}{2}$-porous set. □

Theorem 4.6 (**Harnack inequality for minimizers in a touching ball).** *Let v_∞ be a solution of $(\mathfrak{P}^s_\infty)$, $x_0 \in \{v_\infty > 0\} \cap \Omega$ an interior point and $\mathfrak{d} := \operatorname{dist}(x_0, \partial\{v_\infty > 0\})$. Then,*

$$\sup_{B_{\tau\mathfrak{d}}(x_0)} v_\infty(x) \leq \mathfrak{C} \inf_{B_{\tau\mathfrak{d}}(x_0)} v_\infty(x)$$

for a universal constant $\mathfrak{C} > 0$ and for any $0 < \tau < 1$.

Proof. Let $z_1, z_2 \in \{v_\infty > 0\} \cap \Omega$ be points such that

$$\inf_{B_{\tau\mathfrak{d}}(x_0)} v_\infty(x) = v_\infty(z_1) \quad \text{and} \quad \sup_{B_{\tau\mathfrak{d}}(x_0)} v_\infty(x) = v_\infty(z_2).$$

Since $\operatorname{dist}(z_1, \partial\{v_\infty > 0\}) \geq \tau\mathfrak{d}$, by Theorem 4.2

$$v_\infty(z_1) \geq \mathfrak{C}_1 (\tau\mathfrak{d})^s. \tag{4.6}$$

Moreover, by Theorem 4.1

$$v_\infty(z_2) \leq \mathfrak{C}_2 [(\tau\mathfrak{d})^s + v_\infty(x_0)]. \tag{4.7}$$

Now, by choosing $y \in \partial\{v_\infty > 0\}$ such that $\mathfrak{d} = |x_0 - y|$, we get as consequence from Theorem 4.1

$$v_\infty(x_0) \leq \sup_{B_\mathfrak{d}(y)} v_\infty(x) \leq \mathfrak{C}_2.\mathfrak{d}^s. \tag{4.8}$$

Combining (4.6), (4.7) and (4.8), we conclude

$$\sup_{B_{\tau\mathfrak{d}}(x_0)} v_\infty(x) \leq \mathfrak{C}(N, s, \tau). \inf_{B_{\tau\mathfrak{d}}(x_0)} v_\infty(x).$$

□

Theorem 4.7 (Uniform non-degeneracy for minimizers). *Let v_p be minimizer to $(\mathfrak{P}_p^s)$ and v_∞ a uniform limit as $p \to \infty$ to $(\mathfrak{P}_\infty^s)$. Then v_∞ has an s-grows rate away from the free boundary, i.e., for a constant $\mathfrak{c} = \mathfrak{c}(N, s) >$ there holds*

$$v_\infty(x) \geq \mathfrak{c} \operatorname{dist}(x, \partial\{v_\infty > 0\})^s \quad \forall\, x \in \{v_\infty > 0\} \cap \Omega.$$

Proof. Let $x_0 \in \{v_\infty > 0\} \cap \Omega$ and $\hat{x}_0 \in \partial\{v_\infty > 0\} \cap \Omega$ such that

$$\operatorname{dist}(x_0, \partial\{v_\infty > 0\} \cap \Omega) = |x_0 - \hat{x}_0| := \mathfrak{R}.$$

Now, let us define

$$\Theta(x) := \begin{cases} \mathfrak{c}.(\mathfrak{R}^s - |x - x_0|^s) & \text{in} \quad B_\mathfrak{R}(x_0) \setminus B_{\frac{\mathfrak{R}}{2}}(x_0) \\ \kappa & \text{in} \quad B_{\frac{\mathfrak{R}}{2}}(x_0) \\ 0 & \text{on} \quad \mathbb{R}^N \setminus B_\mathfrak{R}(x_0) \end{cases},$$

where we have chosen $\kappa > 0$ such that

$$\kappa \leq \inf_{B_{\frac{\mathfrak{R}}{2}}(x_0)} v_\infty(z).$$

Then, it is easy to check that

$$\begin{array}{ccccccc} \mathcal{L}_\infty^s[v_r](x) & \leq & 0 & \leq & \mathcal{L}_\infty^s[\Theta](x) & \text{in} & B_\mathfrak{R}(x_0) \setminus B_{\frac{\mathfrak{R}}{2}(x_0)} \\ & & \Theta & \leq & v_\infty & \text{on} & \mathbb{R}^N \setminus \left(B_\mathfrak{R}(x_0) \setminus B_{\frac{\mathfrak{R}}{2}}(x_0)\right). \end{array}$$

Therefore, using the Comparison Principle, we get

$$v_\infty(z) \geq \Theta(z) \quad \text{in} \quad B_\mathfrak{R}(x_0) \setminus B_{\frac{\mathfrak{R}}{2}}(x_0). \tag{4.9}$$

Now, for $z \in \partial B_{\frac{3\mathfrak{R}}{4}}(x_0)$, applying the Harnack inequality (Theorem 4.6), we obtain the following inequalities

$$v_\infty(z) \leq \sup_{\overline{B_{\frac{3\mathfrak{R}}{4}}(x_0)}} v_\infty(t) \leq \mathfrak{C} \inf_{\overline{B_{\frac{3\mathfrak{R}}{4}}(x_0)}} v_\infty(t) \leq \mathfrak{C} v_\infty(x_0). \tag{4.10}$$

Finally, by combining (4.9) and (4.10) we obtain

$$v_\infty(x_0) \geq \mathfrak{C}^{-1} v_\infty(z) \geq \mathfrak{C}^{-1}\Theta(z) = \mathfrak{C}^{-1}\mathfrak{c}.\mathfrak{R}^s = \mathfrak{C}_\sharp(N,s)\,\mathrm{dist}^s(x_0, \partial\{v_\infty > 0\} \cap \Omega).$$

□

Theorem 4.8 (**Convergence of the positivity sets).** *Let v_p be a sequence of minimizers to ($\mathfrak{P}_p^s$). If for some subsequence, denoted by v_p yet, $v_p \to v_\infty$ uniformly in $\overline{\Omega}$ and weakly in $W^{s,q}(\Omega)$ for all $1 < q < \infty$, being v_∞ a solution to ($\mathfrak{P}_\infty^s$), then*

$$\lim_{p\to\infty} \mathcal{L}^N(\{v_p > 0\} \triangle \{v_\infty > 0\}) = 0.$$

Proof. First of all, given $\varepsilon > 0$ there exists p large enough such that

$$\{v_\infty > \varepsilon\} \cap \Omega \subset \{v_p > 0\} \cap \Omega.$$

Moreover, since

$$\mathcal{L}^N(\{v_\infty > 0\} \cap \Omega) = \alpha,$$

then for such a $\varepsilon > 0$ there exists $0 < \delta < 2\alpha$ such that

$$\alpha - \frac{\delta}{2} \leq \mathcal{L}^N(\{v_\infty > \varepsilon\} \cap \Omega) \leq \alpha.$$

Now, by defining

$$\Omega_\infty^\varepsilon := \{v_\infty > \varepsilon\} \cap \Omega,$$

then $\Omega_\infty^\varepsilon$ is increasing as $\varepsilon \searrow 0$. Moreover,

$$\lim_{\varepsilon\to 0+} \mathcal{L}^N(\Omega_\infty^\varepsilon) = \mathcal{L}^N(\{v_\infty > 0\} \cap \Omega) = \alpha.$$

On the other hand,

$$\alpha - \frac{\delta}{2} \leq \mathcal{L}^N(\{v_\infty > \varepsilon\} \cap \Omega) \leq \mathcal{L}^N(\{v_p > 0\} \cap \Omega) \leq \alpha,$$

which implies that

$$\mathcal{L}^N(\{v_p > 0\} \setminus \{v_\infty > \varepsilon\} \cap \Omega) \leq \frac{\delta}{2}.$$

Finally,

$$\begin{array}{rcl} \mathcal{L}^N(\{v_p > 0\} \triangle \{v_\infty > 0\}) & \leq & \mathcal{L}^N(\{v_p > 0\} \setminus \{v_\infty > 0\}) + \mathcal{L}^N(\{v_\infty > 0\} \setminus \{v_p > 0\}) \\ & \leq & \mathcal{L}^N(\{v_p > 0\} \setminus \{v_\infty > 0\}) + \mathcal{L}^N(\{v_\infty > 0\} \setminus \{v_\infty > \varepsilon\}) \\ & \leq & \frac{\delta}{2} + \frac{\delta}{2} = \delta. \end{array}$$

Therefore,

$$\lim_{p\to\infty} \mathcal{L}^N(\{v_p > 0\} \triangle \{v_\infty > 0\}) = 0$$

and the theorem is proved. □

Remark 4.9 (**Convergence of the free boundaries**). The previous theorem gives the convergence in the sense of symmetric difference of sets. However, by assuming the strong non-degeneracy for family of p-minimizers v_p, i.e., there exists a universal modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ such that

$$\sup_{B_r(x_0)} v_p(x) \geq \mathfrak{c}\,\omega(r),$$

then

$$\partial\{v_p > 0\} \to \partial\{v_\infty > 0\} \quad \text{as} \quad p \to \infty,$$

in the sense of the Hausdorff distance.

In order to show a short proof of this convergence let us introduce the following notation: for $\delta > 0$ we will denote the δ-neighborhood of a set $\mathfrak{S} \subset \mathbb{R}^N$ as follows

$$\mathcal{N}_\delta(\mathfrak{S}) := \left\{x \in \mathbb{R}^N : \text{dist}(x, \mathfrak{S}) < \delta\right\}.$$

Now, we must show that, given $0 < \delta \ll 1$ and $p = p(\delta)$ large enough, one obtains

$$\partial\{v_p > 0\} \subset \mathcal{N}_\delta(\partial\{v_\infty > 0\}) \quad \text{and} \quad \partial\{v_\infty > 0\} \subset \mathcal{N}_\delta(\partial\{v_p > 0\}).$$

Let us prove the first inclusion. Suppose for sake of contradiction that such an inclusion does not hold. Thus, should exist a point $x_0 \in \partial\{v_p > 0\} \cap \left(\mathbb{R}^N \setminus \mathcal{N}_\delta(\partial\{v_\infty > 0\})\right)$. The last sentence implies in particular that

$$\text{dist}(x_0, \partial\{v_\infty > 0\}) \geq \delta.$$

Now, if $x_0 \in \{v_\infty > 0\}$ then by uniform non-degeneracy, Theorem 4.7 we get

$$v_\infty(x_0) \geq \mathfrak{c}\,\text{dist}(x_0, \partial\{v_\infty > 0\})^s \geq \mathfrak{c}\delta^s.$$

On the other hand, due to uniform convergence we must have for p large enough

$$v_p(x_0) \geq \frac{\mathfrak{c}\delta^{\mathfrak{s}}}{7} > 0.$$

However, this contradicts the assumption that $x_0 \in \partial\{v_p > 0\}$. Therefore, $v_\infty(x_0) = 0$ and consequently $v_\infty \equiv 0$ in $B_\delta(x_0)$, which contradicts the strong non-degeneracy property for the sequence of p-minimizers.

The second inclusion can be proved similarly.

In what follows, we analyze the behaviour of the coincidence sets for the p-variational problem and its corresponding limiting problem. We recall the following notion of limits of sets

$$\liminf_{p\to\infty} \mathrm{U}_p := \bigcap_{p=1}^{\infty} \bigcup_{k\geq p} \mathrm{U}_k \quad \text{and} \quad \limsup_{p\to\infty} \mathrm{U}_p := \bigcup_{p=1}^{\infty} \bigcap_{k\geq p} \mathrm{U}_k$$

Theorem 4.10. *Let $\mathrm{U}_p := \{u_p = 0\}$ be the null sets of the nonlocal p-variational problems and $\mathrm{U}_\infty := \{u_\infty = 0\}$ be the corresponding null set of the limiting problem. Then,*

$$\overline{int(\mathrm{U}_\infty)} \subset \liminf_{p\to\infty} \mathrm{U}_p \subset \limsup_{p\to\infty} \mathrm{U}_p \subset \mathrm{U}_\infty.$$

Proof. Given $0 < \varepsilon \ll 1$ (small enough), consider $\mathscr{V}_\varepsilon$ an ε-neighbourhood of U_∞. Thus, $\Omega \setminus \mathscr{V}_\varepsilon \subset \{u_\infty > 0\}$ been it a closed set. By using the continuity of limiting u_∞, there exists a $0 < \delta = \delta(\varepsilon)$ such that

$$u_\infty(x) > \delta \quad \forall\, x \in \Omega \setminus \mathscr{V}_\varepsilon.$$

Moreover, by the uniform convergence (up a subsequence $u_p \to u_\infty$) we obtain that for p large enough

$$u_p(x) > \delta \quad \forall\, x \in \Omega \setminus \mathscr{V}_\varepsilon.$$

Therefore,

$$\Omega \setminus \mathscr{V}_\varepsilon \subset \{u_p > 0\} \quad \Rightarrow \quad \mathrm{U}_p \subset \mathscr{V}_\varepsilon \quad \text{for every } p \gg 1.$$

This implies that

$$\limsup_{p\to\infty} \mathrm{U}_p \subset \mathscr{V}_\varepsilon,$$

for any ε-neighbourhood of U_∞ of of U_∞. Particularly, we obtain that

$$\limsup_{p\to\infty} \mathrm{U}_p \subset \mathrm{U}_\infty$$

since U_∞ is a compact set.

Let $x_0 \in \mathrm{int}(\mathrm{U}_\infty)$. We claim that there exists a $\hat{p} = \hat{p}(x_0)$ such that

$$u_k(x_0) = 0 \quad \forall\, k \geq \hat{p}.$$

If we suppose the opposite, i.e.,

$$u_{p_j}(x_0) > 0$$

for some subsequence $p_j \to \infty$, then

$$-(-\Delta)^s_{p_j} u_{p_j}(x_0) = 0.$$

Passing to the limit we conclude that

$$-\mathcal{L}^s_\infty[u_\infty](x_0) = 0,$$

which implies $x_0 \in \overline{\{u_\infty > 0\}} \cap \Omega$, a contradiction with $x_0 \in \mathrm{int}(\mathrm{U}_\infty)$. This proves our claim. Consequently,

$$x_0 \in \bigcup_{k\geq\hat{p}} \mathrm{U}_k \quad \Rightarrow \quad \mathrm{int}(\mathrm{U}_\infty) \subset \liminf_{p\to\infty} \mathrm{U}_p.$$

Finally, we conclude that

$$\overline{\mathrm{int}(\mathrm{U}_\infty)} \subset \liminf_{p\to\infty} \mathrm{U}_p$$

where we have used that $\liminf_{p\to\infty} \mathrm{U}_p$ is a closed set. □

Definition 4.11. The *reduced free boundary* $\mathfrak{F}^{\Omega}_{\mathrm{red}}[u_\infty]$ is the set of points x_0 at which the following condition holds: given the half ball $B^+_r(x_0) := \{(x-x_0)\cdot\eta \geq 0\} \cap B_r(x_0)$ we get

$$\lim_{r\to 0} \frac{\mathcal{L}^N(B^+_r(x_0) \triangle \Omega^+[u_\infty])}{\mathcal{L}^N(B_r(x_0))} = 0. \tag{4.11}$$

Such a definition says us (cf. [14]) that the vector measure $\nabla\chi_\Omega(B_r(x_0))$ has a density at the point, i.e., there exists $\eta(x_0)$ (with $|\eta(x_0)| = 1$) such that fulfils the following

$$\lim_{r\to 0} \frac{\nabla\chi_\Omega(B_r(x_0))}{|\nabla\chi_\Omega(B_r(x_0))|} = \eta(x_0).$$

Recall that from the uniform positive density of $\Omega^+[u_\infty]$ (Corollary 4.3) we have, as $r \to 0$, at the free boundary point x_0 the following

$$B_r(x_0) \cap \mathfrak{F}^{\Omega}_{\mathrm{red}}[u_\infty] \subset \{|(x-x_0)\cdot\eta(x_0)| \leq o(r)\}.$$

In fact, if we suppose that $u_\infty(x) = 0$ for $(x-x_0)\cdot\eta(x_0) \geq \varepsilon r$, there exists $\mathfrak{c}_0 > 0$ such that $\mathcal{L}^N(B_{\varepsilon r}(x) \cap \Omega_0[u_\infty]) \geq \mathfrak{c}_0 \varepsilon r^N$, implying

$$\liminf_{r\to 0} \frac{\mathcal{L}^N(B^+_r(x_0) \triangle \Omega^+[u_\infty])}{\mathcal{L}^N(B_r(x_0))} \geq \mathfrak{c}_0\varepsilon,$$

which yields a contradiction with (4.11).

Next, we will show that free boundary points at which we have a tangent ball from inside are regular points. To this end, let us introduce the following definition:

Definition 4.12. A free boundary point $y \in \mathfrak{F}_\Omega[u] := \partial\{u > 0\} \cap \Omega$ is said to have a *tangent ball from inside* if there exists a ball $\mathscr{B} \subset \Omega^+[u] := \{u > 0\} \cap \Omega$ such that $y \in \mathscr{B} \cap \Omega^+[u]$. Finally, we say that a free boundary point $y \in \mathfrak{F}_\Omega[u]$ is regular if $\mathfrak{F}_\Omega[u]$ has a tangent hyperplane at y.

Theorem 4.13. *A free boundary point $y \in \mathfrak{F}_\Omega[u_\infty]$ which has a tangent ball from inside is regular.*

Proof. The proof is similar to the one in [10, Lemma 11.17], thus we will only write the modifications for the reader's convenience. Let us suppose that $B_1(y_1)$ is tangent to $\mathfrak{F}_\Omega[u_\infty]$ at y. Now, consider the following function

$$\Phi(x) = 1 - |x - y_1|^s.$$

As before, from non-degeneracy, some multiple, say $\mathfrak{c}_0\Phi$, is a lower barrier of u_∞ in $B_1(y_1)$. Now, let $\mathfrak{c}_r > 0$ be the supremum of all $\mathfrak{c}$'s such that

$$u(x) \geq \mathfrak{c}\Phi(x) \quad \text{in} \quad B_r(y_1).$$

Notice that such values $\mathfrak{c}_r$ increase with r. For this reason, by optimal regularity, converges to some constant $\mathfrak{c}_\infty$ as $r \to 0$. Finally, according to [10], this implies the following asymptotic behaviour near free boundary

$$u_\infty(x) = \mathfrak{c}_\infty.((x-y)\cdot\eta(y))^s + o\left(((x-y)\cdot\eta(y))^s\right),$$

where $\eta(y) = y_1 - y$. Therefore, the plane orthogonal to $\eta(y)$ is tangent to $\mathfrak{F}_\Omega[u_\infty]$ and, we conclude that y is a regular point. □

5 Generalizations and comments

In this final section we will present some remarks and extensions for our previous results.

First of all, we highlight that our approach can be applied for weak solutions of possibly degenerate/singular non-local equations of the form

$$(-\Delta_{\mathbb{R}^N})^s_{\mathfrak{K}} u(x) := C_{N,p,s}.\text{P.V.}\int_{\mathbb{R}^N} |u(x)-u(y)|^{p-2}(u(x)-u(y))\mathfrak{K}(x,y)dy,$$

where $\mathfrak{K}: \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a general singular kernel fulfilling the following properties: There exist constants $\Lambda \geq \lambda > 0$ and $\mathfrak{M}, \varsigma > 0$ such that

- ✓ [**Symmetry**] $\mathfrak{K}(x,y) = \mathfrak{K}(y,x)$ for all $x,y \in \mathbb{R}^N$;
- ✓ [**Growth condition**] $\lambda \leq \mathfrak{K}(x,y).|x-y|^{N+ps} \leq \Lambda$ for $x,y \in \mathbb{R}^N$, $x \neq y$;
- ✓ [**Integrability at infinity**] $0 \leq \mathfrak{K}(x,y) \leq \frac{\mathfrak{M}}{|x-y|^{N+\varsigma}}$ for $x \in B_2$ and $y \in \mathbb{R}^N \setminus B_{\frac{1}{4}}$.
- ✓ [**Translation invariance**] $\mathfrak{K}(x+z, y+z) = \mathfrak{K}(x,y)$ for all $x,y,z \in \mathbb{R}^N$, $x \neq y$.
- ✓ [**Continuity**] The map $x \mapsto \mathfrak{K}(x,y)$ is continuous in $\mathbb{R}^N \setminus \{y\}$.

Clearly this previous class of operators have as prototype the fractional p-Laplacian operator provided $\mathfrak{K}(x,y) = |x-y|^{-(N+ps)}$.

Moreover, any minimizer for

$$\mathfrak{I}^p_{\mathfrak{K}}[u](\mathbb{R}^N) := \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |u(x)-u(y)|^p \mathfrak{K}(x,y)dydx$$

satisfies in the weak sense

$$(-\Delta_{\mathbb{R}^N})^s_{\mathfrak{K}} u(x) = 0 \quad \text{in} \quad \mathbb{R}^N.$$

Although $\mathfrak{K}$ is a general kernel, we obtain that any minimizer sequence $(u_p)_{p>0}$, still converges to a Hölder infinity harmonic function u_∞ as previously.

Another interesting issue which we want to stress is that we can recover, under suitable assumptions, the corresponding "local counterpart" taking the limit as $s \to 1^-$. More precisely, by studying the minimization problem with the corrected p-fractional energy

$$\hat{\mathfrak{J}}_p^s[u](\mathbb{R}^N) := \mathscr{K}(N,p)[u]_{W^{s,p}(\Omega)},$$

where the above normalization constant is given explicitly by

$$\mathscr{K}(N,p) = \frac{p\Gamma(\frac{N+p}{2})}{2\pi^{\frac{N-1}{2}}\Gamma(\frac{p+1}{2})}.$$

Recall that it is proved that for any smooth bounded domain $\Omega \subset \mathbb{R}^N$, $u \in W^{1,p}$ with $1 < p < \infty$ there holds

$$\lim_{s\to 1^-} \mathscr{K}(N,p)[u]_{W^{s,p}(\Omega)} = \|\nabla u\|_{L^p(\Omega)}$$

(cf. [8] for more derails). Therefore, if $u_p \rightrightarrows u_\infty$ in $\mathbb{R}^N$ then

$$\lim_{p\to\infty}\left[\lim_{s\to 1^-} \hat{\mathfrak{J}}_p^s[u_p](\mathbb{R}^N)\right] = \mathrm{Lip}[u_\infty].$$

Moreover, the limit satisfies in the viscosity sense

$$\begin{cases} -\Delta_\infty u_\infty(x) &= 0 \quad \text{in} \quad \{u_\infty > 0\} \cap \Omega \\ u_\infty(x) &= g(x) \quad \text{on} \quad \mathbb{R}^N \setminus \Omega. \end{cases}$$

On the other hand, notice that the reverse double limit, namely first take $p \to \infty$ and then $s \to 1^-$, does not yield an infinity-harmonic function in the limit, because we get that the limit verifies

$$\mathcal{L}_\infty^1[u_\infty](x) := \sup_{y\in\mathbb{R}^N\setminus\Omega} \frac{u_\infty(y)-u_\infty(x)}{|x-y|} + \inf_{y\in\mathbb{R}^N\setminus\Omega} \frac{u_\infty(y)-u_\infty(x)}{|x-y|}, \tag{5.1}$$

an equation that does not coincide with the infinity-Laplacian operator. In fact, the Aronsson's function

$$u(x_1,\cdots,x_N) := \mathfrak{a}_1.|x_1|^{\frac{4}{3}} + \cdots + \mathfrak{a}_N.|x_N|^{\frac{4}{3}} \quad \left(\sum_{i=1}^N \mathfrak{a}_i = 0\right)$$

is an infinity harmonic function, however it does not satisfies (5.1) in

$$\Omega = \left\{x \in \mathbb{R}^N : \max_{1\le i\le N} |x_i| \le 1\right\}.$$

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