

ASYMPTOTIC BEHAVIOUR FOR A NONLOCAL DIFFUSION EQUATION ON A LATTICE

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ABSTRACT. In this paper we study the asymptotic behaviour as $t \rightarrow \infty$ of solutions to a nonlocal diffusion problem on a lattice, namely, $u'_n(t) = \sum_{j \in \mathbb{Z}^d} J_{n-j} u_j(t) - u_n(t)$ with $t \geq 0$ and $n \in \mathbb{Z}^d$. We assume that J is nonnegative and verifies $\sum_{n \in \mathbb{Z}^d} J_n = 1$. We find that solutions decay to zero as $t \rightarrow \infty$ and prove an optimal decay rate using, as our main tool, the discrete Fourier transform.

1. INTRODUCTION

In this paper our main concern is the study of the asymptotic behaviour of the following nonlocal equation on a lattice

$$(1.1) \quad \begin{cases} u'_n(t) = (J * u)_n(t) - u_n(t), & t \geq 0, n \in \mathbb{Z}^d, \\ u_n(0) = \varphi_n, & n \in \mathbb{Z}^d, \end{cases}$$

where by $(J * u)$ we denote the discrete convolution,

$$(J * u)_n = \sum_{j \in \mathbb{Z}^d} J_{n-j} u_j.$$

Through the paper we assume that the kernel J is nonnegative and satisfies,

$$(1.2) \quad \sum_{n \in \mathbb{Z}^d} J_n = 1.$$

Equation (1.1), is called *nonlocal diffusion equation*. Continuous analogous to (1.1), like $u_t(x, t) = J * u(x, t) - u(x, t)$, have been recently widely used to model diffusion processes, see, for example, [2], [3], [5], [6], [8], [9], [10], [16] and [17]. In particular, let us mention that these equations are also used in models of neuronal activity, see [7], [11], [13] and [14]. Also there is a discrete counterpart for nonlocal models, see [1], [3] and references therein. In all these models the asymptotic behaviour of the solution (see [4]) is relevant, both from its pure mathematical and its applied point of view. Concerning (1.1), as stated in [9] (see also [3]), if $u_i(t)$ is thought of as the density of a single population at the point i at time t , and J_{i-j} is thought of as the probability distribution of jumping from location i to location j , then $(J * u)(t)$ is the rate at which individuals are arriving to position i from all other places and $-u_i(t)$ is the rate at which they are leaving location i to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1).

To study the asymptotic behaviour of solutions to (1.1) let us introduce the discrete Laplacian given by

$$(\Delta_d u)_n = \sum_{k=1}^d (u_{n+e_k} - 2u_n + u_{n-e_k}),$$

where $\{e_k\}_{k=1}^d$ is the canonical basis on \mathbb{R}^d . Note that this is a *local* diffusion operator.

Our first result says that the asymptotic behaviour as $t \rightarrow \infty$ of solutions to (1.1) is the same as the one for the evolution equation associated to a fractional power of the discrete Laplacian.

Theorem 1.1. *Let u be a solution of equation (1.1) with $\varphi \in l^1(\mathbb{Z}^d)$. If there exist positive constants α and A such that*

$$\widehat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \quad \text{as } \xi \rightarrow 0,$$

then the asymptotic behaviour of $u(t)$ is given by

$$\lim_{t \rightarrow \infty} t^{d/\alpha} \|u(t) - v(t)\|_{l^\infty(\mathbb{Z}^d)} = 0,$$

where v is solution of $v' = -A(-\Delta_d)^{\alpha/2}v$ with initial datum $v_n(0) = \varphi_n$, $n \in \mathbb{Z}^d$.

In view of this result, we analyze the asymptotic profile of the solutions to $v' = -A(-\Delta_d)^{\alpha/2}v$.

Theorem 1.2. *Let us consider $\varphi \in l^1(\mathbb{Z}^d)$. Then the solution to*

$$\begin{cases} v'(t) = -A(-\Delta_d)^{\alpha/2}v, & t > 0, \\ v(0) = \varphi, \end{cases}$$

satisfies

$$\lim_{t \rightarrow \infty} \sup_{j \in \mathbb{Z}^d} \left| t^{d/\alpha} v([jt^{1/\alpha}], t) - \left(\sum_{n \in \mathbb{Z}^d} \varphi_n \right) G^A(j) \right| = 0,$$

where G^A is defined by

$$G^A(x) = \int_{\mathbb{R}^d} e^{ix\xi} e^{-A|\xi|^\alpha} d\xi,$$

and $[\cdot]$ is the floor function.

2. PROOFS OF THE RESULTS

In our analysis, we make use of the semidiscrete Fourier transform (SDFT) (we refer to [12] and [15] for the main properties of the SDFT). For any $v \in l^2(\mathbb{Z}^d)$ we define its SDFT by:

$$\widehat{v}(\xi) = \sum_{j \in \mathbb{Z}^d} e^{-i\xi \cdot j} v_j, \quad \xi \in [-\pi, \pi]^d.$$

In view of property (1.2), \widehat{J} belongs to $L^\infty([-\pi, \pi])$ and $\widehat{J}(0) = 1$.

Proof of Theorem 1.1. Applying the SDFT to the solutions of equation (1.1) we get

$$\widehat{u}'(t, \xi) = \widehat{J}(\xi) \widehat{u}(t, \xi) - \widehat{u}(t, \xi), \quad \xi \in [-\pi, \pi]^d, t > 0.$$

Solving this ODE we find that

$$(2.1) \quad \widehat{u}(t, \xi) = e^{t(\widehat{J}(\xi)-1)} \widehat{\varphi}(\xi), \quad \xi \in [-\pi, \pi]^d, t > 0.$$

In the same way, v , the solution to $v' = -A(-\Delta_d)^{\alpha/2}v$ satisfies

$$(2.2) \quad \widehat{v}(t, \xi) = e^{-At|\xi|^\alpha} \widehat{\varphi}(\xi), \quad \xi \in [-\pi, \pi]^d, t > 0,$$

where

$$p(\xi) = \left(4 \sum_{k=1}^d \sin^2\left(\frac{\xi_k}{2}\right)\right)^{1/2}.$$

Using the Fourier representation of u and v given by (2.1) and (2.2) we find that

$$\begin{aligned} \|u(t) - v(t)\|_{l^\infty(\mathbb{Z}^d)} &\leq \int_{[-\pi, \pi]^d} |\widehat{u}(\xi, t) - \widehat{v}(\xi, t)| d\xi \\ &= \int_{[-\pi, \pi]^d} |\exp(t(\widehat{J}(\xi) - 1)) - \exp(-At p^\alpha(\xi))| |\widehat{\varphi}(\xi)| d\xi. \end{aligned}$$

By our hypothesis there exists a positive $R < \pi$ such that

$$|\widehat{J}(\xi)| \leq 1 - \frac{|\xi|^\alpha}{2}, \quad |\xi| \leq R.$$

Once R has been fixed, there exists $\delta > 0$ such that

$$|\widehat{J}(\xi)| \leq 1 - \delta \quad \text{for all } \xi \in \Omega_R = \{\xi \in [-\pi, \pi]^d, |\xi| > R\}.$$

Hence, it is easy to see that

$$\begin{aligned} \int_{\xi \in \Omega_R} |e^{t(\widehat{J}(\xi)-1)} - e^{-At p^\alpha(\xi)}| |\widehat{\varphi}(\xi)| d\xi &\leq \|\widehat{\varphi}\|_{L^\infty([-\pi, \pi]^d)} \int_{\xi \in \Omega_R} (e^{t(|\widehat{J}(\xi)|-1)} + e^{-At p^\alpha(\xi)}) d\xi \\ &\leq \|\widehat{\varphi}\|_{L^\infty([-\pi, \pi]^d)} \int_{\xi \in \Omega_R} (e^{-t\delta} + \exp(-At \inf_{\xi \in \Omega_R} p^\alpha(\xi))) d\xi. \end{aligned}$$

Tacking into account that the right hand side in the last inequality is exponentially small, it remains to analyze the term

$$I(t) = \int_{|\xi| \leq R} |\widehat{u}(\xi, t) - \widehat{v}(\xi, t)| d\xi.$$

Let us choose a function $r(t) \rightarrow 0$ such that $r(t)t^{1/\alpha} \rightarrow \infty$ as $t \rightarrow \infty$. The remaining term $I(t)$ satisfies:

$$I(t) = \int_{|\xi| \leq R} |e^{t(\widehat{J}(\xi)-1)} - e^{-At p^\alpha(\xi)}| |\widehat{\varphi}(\xi)| d\xi \leq I_1(t) + I_2(t)$$

where

$$I_1(t) = \int_{|\xi| \leq r(t)} |e^{t(\widehat{J}(\xi)-1)} - e^{-At p^\alpha(\xi)}| |\widehat{\varphi}(\xi)| d\xi$$

and

$$I_2(t) = \int_{r(t) \leq |\xi| \leq R} |e^{t(\widehat{J}(\xi)-1)} - e^{-At p^\alpha(\xi)}| |\widehat{\varphi}(\xi)| d\xi.$$

Using that, for some positive constant c , the following holds

$$c|\xi| \leq p(\xi) \leq |\xi| \quad \text{for all } \xi \in [-\pi, \pi]^d,$$

the term $I_2(t)$ can be estimated as follows:

$$\begin{aligned}
t^{d/\alpha} I_2(t) &\leq t^{d/\alpha} \|\widehat{\varphi}\|_{L^\infty([- \pi, \pi]^d)} \int_{r(t) \leq |\xi| \leq R} (e^{-At p^\alpha(\xi)} + e^{t(|\widehat{J}(\xi)|-1)}) d\xi \\
&\leq t^{d/\alpha} \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{r(t) \leq |\xi| \leq R} (e^{-At p^\alpha(\xi)} + e^{-t|\xi|^\alpha/2}) d\xi \\
&\leq t^{d/\alpha} \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{r(t) \leq |\xi| \leq R} e^{-Bt|\xi|^\alpha} d\xi \\
&= \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{t^{1/\alpha} r(t) \leq |\xi| \leq t^{1/\alpha}} e^{-B|\xi|^\alpha} d\xi \\
&\leq \|\varphi\|_{L^1(\mathbb{Z}^d)} t^{d/\alpha} e^{-Btr^\alpha(t)} \rightarrow 0.
\end{aligned}$$

To estimate $I_1(t)$ we first observe that there exists a function $h(\xi)$ with $h(\xi) \rightarrow 0$ as $|\xi| \rightarrow 0$ and such that

$$|\widehat{J}(\xi) - 1 - A|\xi|^\alpha| \leq |\xi| h(\xi)$$

for all ξ in a sufficiently small ball centered at the origin. Thus for all such ξ

$$|\widehat{J}(\xi) - 1 - Ap^\alpha(\xi)| \leq |\xi|^\alpha h(\xi) + ||\xi|^\alpha - p^\alpha(\xi)| \lesssim |\xi|^\alpha h(\xi) + |\xi|^{3\alpha}.$$

In view of this property we get

$$\begin{aligned}
I_1(t) &\leq t^{d/\alpha} \|\widehat{\varphi}\|_{L^\infty([- \pi, \pi]^d)} \int_{|\xi| \leq r(t)} e^{-At p^\alpha(\xi)} |e^{t(\widehat{J}(\xi)-1-At p^\alpha(\xi))} - 1| d\xi \\
&\leq t^{d/\alpha} \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{|\xi| \leq r(t)} e^{-At p^\alpha(\xi)} t |\xi|^\alpha (h(\xi) + |\xi|^{3\alpha}) d\xi \\
&\leq t^{d/\alpha} \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{|\xi| \leq r(t)} e^{-Bt|\xi|^\alpha} (t|\xi|^\alpha h(\xi) + t|\xi|^{4\alpha}) d\xi.
\end{aligned}$$

The last term in the right hand side verifies

$$t^{d/\alpha} \int_{|\xi| \leq r(t)} e^{-Bt|\xi|^\alpha} t |\xi|^{4\alpha} d\xi \leq t^{-3} \int_{|\eta| \leq r(t)t^{1/\alpha}} e^{-B|\eta|^\alpha} |\eta|^{4\alpha} d\eta \rightarrow 0.$$

Hence we have to analyze the first one. In this case, by the same change of variables, we get

$$t^{d/\alpha} \int_{|\xi| \leq r(t)} e^{-Bt|\xi|^\alpha} t |\xi|^\alpha h(\xi) = \int_{|\eta| < r(t)t^{1/\alpha}} |\eta|^\alpha e^{-Bt|\eta|^\alpha} h(\eta t^{-1/\alpha}).$$

Applying Lebesgue convergence theorem we obtain that also this term converges to zero as $t \rightarrow \infty$. This ends the proof. \square

Now we prove our second result, Theorem 1.2, that describes the asymptotic profile of solutions to $v' = -A(-\Delta_d)^{\alpha/2} v$.

Proof of Theorem 1.2. Using the Fourier representation of v we have

$$v(j, t) = \int_{[- \pi, \pi]^d} e^{-At p^\alpha(\xi)} e^{ij\xi} \widehat{\varphi}(\xi) d\xi, \quad j \in \mathbb{Z}^d, t > 0.$$

Thus

$$\begin{aligned} t^{d/\alpha} v([jt^{1/\alpha}], t) &= t^{d/\alpha} \int_{[-\pi, \pi]^d} e^{-At p^\alpha(\xi)} e^{i[jt^{1/\alpha}] \xi} \widehat{\varphi}(\xi) d\xi \\ &= \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-At p^\alpha(\xi t^{-1/\alpha})} \exp\left(i\xi \frac{[jt^{1/\alpha}]}{t^{1/\alpha}}\right) \widehat{\varphi}(\xi t^{-1/\alpha}) d\xi \end{aligned}$$

and

$$G^A(j) = \int_{\mathbb{R}^d} e^{-A|\xi|^\alpha} e^{i\xi j} d\xi.$$

Denoting

$$I(j, t) = t^{d/\alpha} v([jt^{1/\alpha}], t) - \widehat{\varphi}(0) G^A(j)$$

we obtain

$$\begin{aligned} |I(j, t)| &\leq \left| \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-At p^\alpha(\xi t^{-1/\alpha})} e^{ij\xi} - \widehat{\varphi}(0) \int_{\mathbb{R}^d} e^{-A|\xi|^\alpha} e^{i\xi j} d\xi \right| \\ &\quad + \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-At p^\alpha(\xi t^{-1/\alpha})} |e^{ij\xi} - e^{i\xi [jt^{1/\alpha}] t^{-1/\alpha}}| |\widehat{\varphi}(\xi t^{-1/\alpha})| d\xi \\ &= I_1(j, t) + I_2(j, t). \end{aligned}$$

Therefore we have to get bounds for $I_1(j, t)$ and $I_2(j, t)$.

Step I. Estimates for $I_2(t)$. For $I_2(t)$ we have the rough estimate

$$\begin{aligned} |I_2(j, t)| &\leq \|\widehat{\varphi}\|_{L^\infty([- \pi, \pi]^d)} \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-At p^\alpha(\xi t^{-1/\alpha})} \left| \sin\left(\frac{jt^{1/\alpha}\xi - [jt^{1/\alpha}]\xi}{2t^{1/\alpha}}\right) \right| d\xi \\ &\leq \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-At p^\alpha(\xi t^{-1/\alpha})} \left| \frac{jt^{1/\alpha}\xi - [jt^{1/\alpha}]\xi}{2t^{1/\alpha}} \right| d\xi \\ &\leq \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-At p^\alpha(\xi t^{-1/\alpha})} \frac{|\xi|}{t^{1/\alpha}} d\xi \\ &\lesssim t^{-1/\alpha} \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-c(\alpha)|\xi|^\alpha} |\xi| d\xi \rightarrow 0. \end{aligned}$$

Step II. Estimates for $I_1(t)$. Observe that I_1 satisfies:

$$\begin{aligned} I_1(j, t) &\leq \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} |e^{-At p^\alpha(\xi t^{-1/\alpha})} - e^{-A|\xi|^\alpha}| |\widehat{\varphi}(\xi t^{-1/\alpha})| d\xi \\ &\quad + \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-A|\xi|^\alpha} |\widehat{\varphi}(\xi t^{-1/\alpha}) - \widehat{\varphi}(0)| d\xi \\ &\quad + |\widehat{\varphi}(0)| \int_{\xi \notin [-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-A|\xi|^\alpha} d\xi \\ &= I_3(t) + I_4(t) + I_5(t). \end{aligned}$$

In the case of the last integral, easily follows that $|\xi| \gtrsim t^{1/\alpha}$. Thus

$$I_5(t) \lesssim \int_{|\xi| \gtrsim t^{1/\alpha}} e^{-A|\xi|^\alpha} d\xi \rightarrow 0.$$

For I_4 we have the following estimate:

$$I_4(t) = \int_{\mathbb{R}^d} e^{-A|\xi|^\alpha} |\widehat{\varphi}(\xi t^{-1/\alpha}) - \widehat{\varphi}(0)| \chi_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} d\xi$$

and the Lebesgue dominated convergence theorem guarantees that $I_4(t) \rightarrow 0$ as $t \rightarrow \infty$.

Using that $p(\xi)$ satisfies $c|\xi| \leq p(\xi) \leq |\xi|$ for some positive c and the mean value theorem we get:

$$\begin{aligned} I_3(t) &\leq \|\widehat{\varphi}\|_{L^\infty([-\pi, \pi]^d)} \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-At p^\alpha(\xi t^{-1/\alpha})} |t p^\alpha(\xi t^{-1/\alpha}) - |\xi|^\alpha| d\xi \\ &\lesssim \|\varphi\|_{L^1(\mathbb{Z}^d)} \int_{[-\pi t^{1/\alpha}, \pi t^{1/\alpha}]^d} e^{-c|\xi|^\alpha} |t p^\alpha(\xi t^{-1/\alpha}) - |\xi|^\alpha| d\xi. \end{aligned}$$

Applying again the dominated convergence theorem we obtain that $I_3(t) \rightarrow 0$ as $t \rightarrow \infty$.

The proof is now complete. \square

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