Non-simultaneous blow-up in a semilinear parabolic system

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Abstract

In this paper we study the possibility of non-simultaneous blow-up for positive solutions of a fully coupled system of two semilinear heat equations, $u_t = \Delta u + u^{p_{11}} v^{p_{12}}$, $v_t = \Delta v + u^{p_{21}} v^{p_{22}}$. Under adequate hypotheses, we prove that if u blows up then v can fail to blow up if and only if $p_{11} > 1$ and $p_{21} < p_{11} - 1$.

1 Introduction.

In this note we study blowing up solutions of the following parabolic system,

$$\begin{cases} u_t = \Delta u + u^{p_{11}} v^{p_{12}} & \text{in } \mathbb{R}^N \times (0,T), \\ v_t = \Delta v + u^{p_{21}} v^{p_{22}} & \text{in } \mathbb{R}^N \times (0,T), \end{cases} \tag{1.1}$$

with initial data

$$\begin{cases} u(x,0) = u_0(x) & \text{in } \mathbb{R}^N, \\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{1.2}$$

Throughout the paper we assume that $p_{ij} \geq 0$ and u_0, v_0 are positive and bounded. There exist solutions (u,v) that blow up in finite time, T, if and only if the exponents p_{ij} verify any of the conditions, $p_{11} > 1$, $p_{22} > 1$ or $(p_{11}-1)(p_{22}-1) - p_{12}p_{21} < 0$, see [1]. At that time T we have,

$$\limsup_{t \nearrow T} \{ \|u(\cdot,t)\|_{L^\infty} + \|v(\cdot,t)\|_{L^\infty} \} = \infty.$$

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However, a priori there is no reason why both functions, u and v, should go to infinity simultaneously at time T. Indeed, as we will show, for certain choices of the parameters p_{ij} there are initial data for which one of the components of the system remains bounded while the other blows up. We denote this phenomenon as *non-simultaneous blow-up*. In this paper we characterize the range of parameters for which non-simultaneous blow-up occurs (for suitable initial data). Our results are contained in the following two theorems.

Theorem 1.1 *If $p_{11} > 1$ and $p_{21} < p_{11} - 1$, then there exist initial data u_0, v_0 such that u blows up while v remains bounded.*

The condition $p_{11} > 1$ guarantees blow-up for u, while the condition $p_{21} < p_{11} - 1$ implies that the coupling between u and v is "weak" enough to have non-simultaneous blow-up.

The possibility of non-simultaneous blow-up for a non-trivial parabolic system was first shown in [7]. In that case the authors consider two porous medium equations coupled through nonlinear boundary conditions. The case we study here is the first example of such a behaviour in a semilinear parabolic system.

Under a hypothesis on the blow-up rate for u we can prove the reciprocal.

Theorem 1.2 *If u blows up at time T at some point x_0 and v remains bounded up to time T, with*

$$u(x,t) \geq C(T-t)^{-\frac{1}{p_{11}-1}} \qquad |x - x_0| \leq K\sqrt{T-t} \tag{1.3}$$

then $p_{11} > 1$ and $p_{21} < p_{11} - 1$.

Remark 1: Both theorems can be extended to the case of a bounded smooth domain Ω with homogeneous Dirichlet boundary data, provided that the blow-up set of u lies in a compact subset of Ω (see [3] for conditions that guarantee this fact). The only difference arises from an extra boundary term in the right hand side of (2.1) of the form

$$\int_0^t \int_{\partial\Omega} \Gamma(x-y, t-\tau) \frac{\partial u}{\partial \eta}(y,\tau)\ d\sigma(y) d\tau.$$

As u remains bounded in a small neighbourhood of $\partial\Omega$, this term is negative and bounded. Hence it does not play any role in the proofs. For simplicity we perform the proofs in $\mathbb{R}^N$ and leave the details in the case of a bounded domain to the reader.

Remark 2: The hypothesis on the blow-up rate, (1.3), is known to hold for p_{11} subcritical, i.e. $1 < p_{11} < (N+2)/(N-2)_+$, both in $\mathbb{R}^N$ and in a bounded convex smooth domain, see [4], [6], and [8]. It also holds without any restriction on the exponent if $u_t \geq 0$ in the case of a bounded convex smooth domain, see [3]. However, it may fail for large p_{11} in high dimensions, [5].

2 Proofs.

THEOREM 1.1. We choose u_0 and v_0 radial and decreasing with $|x|$. Thus u and v are radial and decreasing with $|x|$.

The idea is to fix v_0 and then choose u_0 large enough to guarantee that u blows up alone. Observe that as $p_{11} > 1$ every positive solution with u_0 large has finite time blow-up, T. Moreover, if u_0 is large with v_0 fixed then T becomes small.

We will consider two cases, $p_{22} > 1$ and $p_{22} \le 1$. In the first case, we fix $0 < v_0(x) < 1/4$. We want to use the representation formula obtained from the fundamental solution. Let $\Gamma(x,t)$ be the fundamental solution of the heat equation, namely

$$\Gamma(x,t) = \frac{1}{(4\pi t)^{N/2}} \exp\left(-\frac{|x|^2}{4t}\right).$$

As v is a solution of

$$v_t = \Delta v + u^{p_{21}} v^{p_{22}},$$

we have (see [2])

$$v(x,t) = \int_{\mathbb{R}^N} \Gamma(x-y,t) v_0(y)\ dy \tag{2.1}$$

$$+ \int_0^t \int_{\mathbb{R}^N} \Gamma(x-y,t-\tau) u(y,\tau)^{p_{21}} v(y,\tau)^{p_{22}}\ dy d\tau.$$

We set $V(t) = \sup_{\mathbb{R}^N \times [0,t]} v$. As the initial data are radial and decreasing, u verifies

$$u(x,t) \le \frac{C}{(T-t)^{\frac{1}{p_{11}-1}}},$$

with C independent of u_0, see [4]. Hence, $u^{p_{21}} v^{p_{22}} \le C v^{p_{22}} (T-t)^{-\frac{p_{21}}{p_{11}-1}}$. Using that $V(t)$ is nondecreasing, from (2.1) we obtain,

$$V(t) \le V(0) + C V(t)^{p_{22}} \int_0^t \frac{1}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}}\ d\tau.$$

We choose u_0 large enough in order to get T small. Since $\frac{p_{21}}{p_{11}-1} < 1$, the integral is smaller than $1/2C$ if T is small. Hence,

$$V(t) \le V(0) + \frac{1}{2} V(t)^{p_{22}}. \tag{2.2}$$

We claim that $V(t) < 1$ for all $0 < t < T$. Suppose not and let $0 < t_0 < T$ be the first time such that $V(t_0) = 1$ (i.e $V(t) < 1$ for all $0 < t < t_0$). For $0 \le t \le t_0$ we have $V(t)^{p_{22}} \le V(t)$. Therefore

$$\frac{1}{2} V(t_0) \le V(0) < \frac{1}{4},$$

a contradiction with $V(t_0) = 1$.

Next, assume that $p_{22} \le 1$. We choose $C \ge v_0 \ge 1$. Thus $V(t) \ge 1$. Arguing as before we obtain again (2.2). Now $V(t)^{p_{22}} \le V(t)$. Hence (2.2) produces,

$$\frac{1}{2}V(t) \le V(0) \le C. \qquad \square$$

THEOREM 1.2. The function u is a subsolution of

$$u_t = \Delta u + C u^{p_{11}} \tag{2.3}$$

that has finite time blow-up. Solutions of (2.3) are global in time if $p_{11} \le 1$. To see this we can compare with a flat solution of (2.3), i.e. a solution of $u_t = C u^{p_{11}}$ with initial data $u(x,0) = \|u_0\|_{L^\infty}$. Hence we must have $p_{11} > 1$.

Next we prove that $p_{21} < p_{11} - 1$. Let x_0 be a blow-up point for u. As $v(y,\tau)$ is bounded from below in $|y - x_0| \le C\sqrt{T-\tau}$, from formula (2.1) we get

$$V(t) \ge v(x_0,t) \ge C \int_0^t \int_{\{|y-x_0| \le C\sqrt{T-\tau}\}} u^{p_{21}}(y,\tau)\Gamma(x_0 - y, t-\tau)\, dy d\tau.$$

By assumption, there exists a constant c such that

$$u(y,\tau) \ge \frac{c}{(T-\tau)^{\frac{1}{p_{11}-1}}} \qquad |y - x_0| \le K\sqrt{T-\tau}.$$

With this bound for u we obtain,

$$\begin{aligned} V(t) &\ge C \int_0^t \frac{1}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} \int_{\{|y-x_0| \le K\sqrt{T-\tau}\}} \Gamma(x_0 - y, t-\tau)\, dy d\tau \\ &= C \int_0^t \frac{1}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}} \int_{\{|w| \le K\sqrt{T-\tau}/\sqrt{t-\tau}\}} e^{-|w|^2/4}\, dw\, d\tau \\ &\ge C \int_0^t \frac{1}{(T-\tau)^{\frac{p_{21}}{p_{11}-1}}}\, d\tau. \end{aligned}$$

If $\frac{p_{21}}{p_{11}-1} \ge 1$, the last integral diverges as $t \nearrow T$. Hence, so does v. Therefore, if v remains bounded $p_{21} < p_{11} - 1$. $\square$

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