A NONLOCAL CONVECTION-DIFFUSION EQUATION

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ABSTRACT. In this paper we study a nonlocal equation that takes into account convective and diffusive effects, $u_t = J * u - u + G * (f(u)) - f(u)$ in $\mathbb{R}^d$, with J radially symmetric and G not necessarily symmetric.

First, we prove existence, uniqueness and continuous dependence with respect to the initial condition of solutions. This problem is the nonlocal analogous to the usual local convection-diffusion equation $u_t = \Delta u + b \cdot \nabla(f(u))$. In fact, we prove that solutions of the nonlocal equation converge to the solution of the usual convection-diffusion equation when we rescale the convolution kernels J and G appropriately.

Finally we study the asymptotic behaviour of solutions as $t \to \infty$ when $f(u) = |u|^{q-1}u$ with $q > 1$. We find the decay rate and the first order term in the asymptotic regime.

1. Introduction

In this paper we analyze a nonlocal equation that takes into account convective and diffusive effects. We deal with the nonlocal evolution equation

$$(1.1) \qquad \begin{cases} u_t(t,x) = (J * u - u)(t,x) + (G * (f(u)) - f(u))(t,x), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases}$$

Let us state first our basic assumptions. The functions J and G are nonnegatives and verify $\int_{\mathbb{R}^d} J(x)dx = \int_{\mathbb{R}^d} G(x)dx = 1$. Moreover, we consider J smooth, $J \in \mathcal{S}(\mathbb{R}^d)$, the space of rapidly decreasing functions, and radially symmetric and G smooth, $G \in \mathcal{S}(\mathbb{R}^d)$, but not necessarily symmetric. To obtain a diffusion operator similar to the Laplacian we impose in addition that J verifies

$$\frac{1}{2}\partial^2_{\xi_i\xi_i}\widehat{J}(0) = \frac{1}{2}\int_{supp(J)} J(z)z_i^2\,dz = 1.$$

This implies that

$$\widehat{J}(\xi) - 1 + \xi^2 \sim |\xi|^3, \qquad \text{for } \xi \text{ close to } 0.$$

Here $\widehat{J}$ is the Fourier transform of J and the notation $A \sim B$ means that there exist constants C_1 and C_2 such that $C_1 A \leq B \leq C_2 A$. We can consider more general kernels J with expansions in Fourier variables of the form $\widehat{J}(\xi) - 1 + A\,\xi^2 \sim |\xi|^3$. Since the results (and the proofs) are almost the same, we do not include the details for this more general case, but we comment on how the results are modified by the appearance of A.

The nonlinearity f will de assumed nondecreasing with $f(0) = 0$ and locally Lipschitz continuous (a typical example that we will consider below is $f(u) = |u|^{q-1}u$ with $q > 1$).

Equations like $w_t = J * w - w$ and variations of it, have been recently widely used to model diffusion processes, for example, in biology, dislocations dynamics, etc. See, for example,

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[2], [4], [6], [7], [10], [13], [14], [20] and [21]. As stated in [13], if $w(t,x)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J*w)(t,x) = \int_{\mathbb{R}^N} J(y-x)w(t,y)\,dy$ is the rate at which individuals are arriving to position x from all other places and $-w(t,x) = -\int_{\mathbb{R}^N} J(y-x)w(t,x)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density w satisfies the equation $w_t = J*w - w$.

In our case, see the equation in (1.1), we have a diffusion operator $J*u-u$ and a nonlinear convective part given by $G*(f(u)) - f(u)$. Concerning this last term, if G is not symmetric then individuals have greater probability of jumping in one direction than in others, provoking a convective effect.

We will call equation (1.1), a *nonlocal convection-diffusion equation*. It is nonlocal since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$ and its derivatives at that point (t,x), but on all the values of u in a fixed spatial neighborhood of x through the convolution terms $J*u$ and $G*(f(u))$ (this neighborhood depends on the supports of J and G).

First, we prove existence, uniqueness and well-possedness of a solution with an initial condition $u(0,x) = u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Theorem 1.1. *For any $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique global solution*

$$u \in C([0,\infty); L^1(\mathbb{R}^d)) \cap L^\infty([0,\infty); \mathbb{R}^d).$$

If u and v are solutions of (1.1) corresponding to initial data $u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ respectively, then the following contraction property

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}$$

holds for any $t \ge 0$. In addition,

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

We have to emphasize that a lack of regularizing effect occurs. This has been already observed in [5] for the linear problem $w_t = J*w - w$. In [12], the authors prove that the solutions to the local convection-diffusion problem, $u_t = \Delta u + b\cdot\nabla f(u)$, satisfy an estimate of the form $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le C(\|u_0\|_{L^1(\mathbb{R}^d)})\, t^{-d/2}$ for any initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In our nonlocal model, we cannot prove such type of inequality independently of the $L^\infty(\mathbb{R}^d)$-norm of the initial data. Moreover, in the one-dimensional case with a suitable bound on the nonlinearity that appears in the convective part, f, we can prove that such an inequality does not hold in general, see Section 3. In addition, the $L^1(\mathbb{R}^d) - L^\infty(\mathbb{R}^d)$ regularizing effect is not available for the linear equation, $w_t = J*w - w$, see Section 2.

When J is nonnegative and compactly supported, the equation $w_t = J*w - w$ shares many properties with the classical heat equation, $w_t = \Delta w$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed, see [13]. However, there is no regularizing effect in general. Moreover, in [8] and [9] nonlocal Neumann boundary conditions where taken into account. It is proved there that solutions of the nonlocal problems converge to solutions of the heat equation with Neumann boundary conditions when a rescaling parameter goes to zero.

Concerning (1.1) we can obtain a solution to a standard convection-diffusion equation

$$v_t(t,x) = \Delta v(t,x) + b \cdot \nabla f(v)(t,x), \qquad t>0,\ x\in\mathbb{R}^d, \tag{1.2}$$

as the limit of solutions to (1.1) when a scaling parameter goes to zero. In fact, let us consider

$$J_\varepsilon(s) = \frac{1}{\varepsilon^d} J\left(\frac{s}{\varepsilon}\right), \qquad G_\varepsilon(s) = \frac{1}{\varepsilon^d} G\left(\frac{s}{\varepsilon}\right),$$

and the solution $u_\varepsilon(t,x)$ to our convection-diffusion problem rescaled adequately,

$$\begin{cases} (u_\varepsilon)_t(t,x) &= \dfrac{1}{\varepsilon^2}\displaystyle\int_{\mathbb{R}^d} J_\varepsilon(x-y)(u_\varepsilon(t,y)-u_\varepsilon(t,x))\,dy \\ & \qquad + \dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^d} G_\varepsilon(x-y)(f(u_\varepsilon(t,y))-f(u_\varepsilon(t,x)))\,dy, \\ u_\varepsilon(x,0) &= u_0(x). \end{cases} \tag{1.3}$$

Remark that the scaling is different for the diffusive part of the equation $J*u-u$ and for the convective part $G*f(u)-f(u)$. The same different scaling properties can be observed for the local terms Δu and $b\cdot\nabla f(u)$.

Theorem 1.2. *With the above notations, for any $T>0$, we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]} \|u_\varepsilon - v\|_{L^2(\mathbb{R}^d)} = 0,$$

where $v(t,x)$ is the unique solution to the local convection-diffusion problem (1.2) *with initial condition $v(x,0)=u_0(x)\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and $b=(b_1,...,b_d)$ given by*

$$b_j = \int_{\mathbb{R}^d} x_j\, G(x)\,dx, \qquad j=1,...,d.$$

This result justifies the use of the name "nonlocal convection-diffusion problem" when we refer to (1.1).

From our hypotheses on J and G it follows that they verify $|\widehat{G}(\xi)-1-ib\cdot\xi|\le C|\xi|^2$ and $|\widehat{J}(\xi)-1+\xi^2|\le C|\xi|^3$ for every $\xi\in\mathbb{R}^d$. These bounds are exactly what we are using in the proof of this convergence result.

Remark that when G is symmetric then $b=0$ and we obtain the heat equation in the limit. Of course the most interesting case is when $b\neq 0$ (this happens when G is not symmetric). Also we note that the conclusion of the theorem holds for other $L^p(\mathbb{R}^d)$-norms besides $L^2(\mathbb{R}^d)$, however the proof is more involved.

We can consider kernels J such that

$$A = \frac{1}{2}\int_{supp(J)} J(z) z_i^2\,dz \neq 1.$$

This gives the expansion $\widehat{J}(\xi)-1+A\xi^2 \sim |\xi|^3$, for ξ close to 0. In this case we will arrive to a convection-diffusion equation with a multiple of the Laplacian as the diffusion operator, $v_t = A\Delta v + b\cdot\nabla f(v)$.

Next, we want to study the asymptotic behaviour as $t\to\infty$ of solutions to (1.1). To this end we first analyze the decay of solutions taking into account only the diffusive part (the linear part) of the equation. These solutions have a similar decay rate as the one that holds for the heat equation, see [5] and [15] where the Fourier transform play a key role. Using similar techniques we can prove the following result that deals with this asymptotic decay rate.

Theorem 1.3. *Let $p \in [1,\infty]$. For any $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the solution $w(t,x)$ of the linear problem*

$$\begin{cases} w_t(t,x) = (J * w - w)(t,x), & t>0,\, x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{1.4}$$

satisfies the decay estimate

$$\|w(t)\|_{L^p(\mathbb{R}^d)} \le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\, \langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})}.$$

Throughout this paper we will use the notation $A \le \langle t\rangle^{-\alpha}$ to denote $A \le (1+t)^{-\alpha}$.

Now we are ready to face the study of the asymptotic behaviour of the complete problem (1.1). To this end we have to impose some grow condition on f. Therefore, in the sequel we restrict ourselves to nonlinearities f that are pure powers

$$f(u) = |u|^{q-1}u \tag{1.5}$$

with $q > 1$.

The analysis is more involved than the one performed for the linear part and we cannot use here the Fourier transform directly (of course, by the presence of the nonlinear term). Our strategy is to write a variation of constants formula for the solution and then prove estimates that say that the nonlinear part decay faster than the linear one. For the local convection diffusion equation this analysis was performed by Escobedo and Zuazua in [12]. However, in the previously mentioned reference energy estimates were used together with Sobolev inequalities to obtain decay bounds. These Sobolev inequalities are not available for the nonlocal model, since the linear part does not have any regularizing effect, see Remark 5.4 in Section 5. Therefore, we have to avoid the use of energy estimates and tackle the problem using a variant of the Fourier splitting method proposed by Schonbek to deal with local problems, see [17], [18] and [19].

We state our result concerning the asymptotic behaviour (decay rate) of the complete nonlocal model as follows:

Theorem 1.4. *Let f satisfies* (1.5) *with $q > 1$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for every $p \in [1,\infty)$ the solution u of equation* (1.1) *verifies*

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\, \langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})}. \tag{1.6}$$

Finally, we look at the first order term in the asymptotic expansion of the solution. For $q > (d+1)/d$, we find that this leading order term is the same as the one that appears in the linear local heat equation. This is due to the fact that the nonlinear convection is of higher order and that the radially symmetric diffusion leads to gaussian kernels in the asymptotic regime, see [5] and [15].

Theorem 1.5. *Let f satisfies* (1.5) *with $q > (d+1)/d$ and let the initial condition u_0 belongs to $L^1(\mathbb{R}^d, 1+|x|) \cap L^\infty(\mathbb{R}^d)$. For any $p \in [2,\infty)$ the following holds*

$$t^{-\frac{d}{2}(1-\frac{1}{p})}\|u(t) - MH(t)\|_{L^p(\mathbb{R}^d)} \le C(J,G,p,d)\, \alpha_q(t),$$

where

$$M = \int_{\mathbb{R}^d} u_0(x)\, dx,$$

$H(t)$ is the Gaussian,

$$H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}},$$

and

$$\alpha_q(t) = \begin{cases} \langle t \rangle^{-\frac{1}{2}} & \text{if} \quad q \geq (d+2)/d, \\ \langle t \rangle^{\frac{1-d(q-1)}{2}} & \text{if} \quad (d+1)/d < q < (d+2)/d. \end{cases}$$

Remark that we prove a weak nonlinear behaviour, in fact the decay rate and the first order term in the expansion are the same that appear in the linear model $w_t = J * w - w$, see [15].

As before, recall that our hypotheses on J imply that $\widehat{J}(\xi) - (1 - |\xi|^2) \sim B|\xi|^3$, for ξ close to 0. This is the key property of J used in the proof of Theorem 1.5. We note that when we have an expansion of the form $\widehat{J}(\xi) - (1 - A|\xi|^2) \sim B|\xi|^3$, for $\xi \sim 0$, we get as first order term a Gaussian profile of the form $H_A(t) = H(At)$.

Also note that $q = (d+1)/d$ is a critical exponent for the local convection-diffusion problem, $v_t = \Delta v + b \cdot \nabla(v^q)$, see [12]. When q is supercritical, $q > (d+1)/d$, for the local equation it also holds an asymptotic simplification to the heat semigroup as $t \to \infty$.

The first order term in the asymptotic behaviour for critical or subcritical exponents $1 < q \leq (d+1)/d$ is left open. One of the main difficulties that one has to face here is the absence of a self-similar profile due to the inhomogeneous behaviour of the convolution kernels.

The rest of the paper is organized as follows: in Section 2 we deal with the estimates for the linear semigroup that will be used to prove existence and uniqueness of solutions as well as for the proof of the asymptotic behaviour. In Section 3 we prove existence and uniqueness of solutions, Theorem 1.1. In Section 4 we show the convergence to the local convection-diffusion equation, Theorem 1.2 and finally in Sections 5 and 6 we deal with the asymptotic behaviour, we find the decay rate and the first order term in the asymptotic expansion, Theorems 1.4 and 1.5.

2. The linear semigroup

In this section we analyze the asymptotic behavior of the solutions of the equation

$$\begin{cases} w_t(t,x) = (J * w - w)(t,x), & t > 0, \ x \in \mathbb{R}^d, \\ w(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{2.1}$$

As we have mentioned in the introduction, when J is nonnegative and compactly supported, this equation shares many properties with the classical heat equation, $w_t = \Delta w$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed, see [13]. However, there is no regularizing effect in general. In fact, the singularity of the source solution, that is a solution to (2.1) with initial condition a delta measure, $u_0 = \delta_0$, remains with an exponential decay. In fact, this fundamental solution can be decomposed as $S(t,x) = e^{-t}\delta_0 + K_t(x)$ where $K_t(x)$ is smooth, see Lemma 2.1. In this way we see that there is no regularizing effect since the solution w of (2.1) can be written as $w(t) = S(t) * u_0 = e^{-t}u_0 + K_t * u_0$ with K_t smooth, which means that $w(\cdot,t)$ is as regular as u_0 is. This fact makes the analysis of (2.1) more involved.

Lemma 2.1. *The fundamental solution of* (2.1)*, that is the solution of* (2.1) *with initial condition* $u_0 = \delta_0$*, can be decomposed as*

$$S(t,x) = e^{-t}\delta_0(x) + K_t(x), \tag{2.2}$$

with $K_t(x) = K(t,x)$ *smooth. Moreover, if* u *is the solution of* (2.1) *it can be written as*

$$w(t,x) = (S * u_0)(t,x) = \int_{\mathbb{R}} S(t, x-y)u_0(y)\,dy.$$

Proof. Applying the Fourier transform to (2.1) we obtain that

$$\widehat{w}_t(\xi,t) = \widehat{w}(\xi,t)(\widehat{J}(\xi) - 1).$$

Hence, as the initial datum verifies $\widehat{u_0} = \widehat{\delta_0} = 1$,

$$\widehat{w}(\xi,t) = e^{(\widehat{J}(\xi)-1)t} = e^{-t} + e^{-t}(e^{\widehat{J}(\xi)t} - 1).$$

The first part of the lemma follows applying the inverse Fourier transform.

To finish the proof we just observe that $S * u_0$ is a solution of (2.1) (just use Fubini's theorem) with $(S * u_0)(0,x) = u_0(x)$. □

In the following we will give estimates on the regular part of the fundamental solution K_t defined by:

$$K_t(x) = \int_{\mathbb{R}^d} (e^{t(\widehat{J}(\xi)-1)} - e^{-t})\, e^{ix\cdot\xi}\, d\xi, \tag{2.3}$$

that is, in the Fourier space,

$$\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)} - e^{-t}.$$

The behavior of $L^p(\mathbb{R}^d)$-norms of K_t will be obtained by analyzing the cases $p=\infty$ and $p=1$. The case $p=\infty$ follows by Hausdorff-Young's inequality. The case $p=1$ follows by using the fact that the $L^1(\mathbb{R}^d)$-norm of the solutions to (2.1) does not increase.

The analysis of the behaviour of the gradient ∇K_t is more involved. The behavior of $L^p(\mathbb{R}^d)$-norms with $2 \le p \le \infty$ follows by Hausdorff-Young's inequality in the case $p=\infty$ and Plancherel's identity for $p=2$. However, the case $1 \le p < 2$ is more tricky. In order to evaluate the $L^1(\mathbb{R}^d)$-norm of ∇K_t we will use the Carlson inequality (see for instance [3])

$$\|\varphi\|_{L^1(\mathbb{R}^d)} \le C\,\|\varphi\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{2m}} \||x|^m\varphi\|_{L^2(\mathbb{R})}^{\frac{d}{2m}}, \tag{2.4}$$

which holds for $m > d/2$. The use of the above inequality with $\varphi = \nabla K_t$ imposes that $|x|^m\nabla K_t$ belongs to $L^2(\mathbb{R}^d)$. To guarantee this property and to obtain the decay rate for the $L^2(\mathbb{R}^d)$-norm of $|x|^m\nabla K_t$ we will use in Lemma 2.3 that $J \in \mathcal{S}(\mathbb{R}^d)$.

The following lemma gives us the decay rate of the $L^p(\mathbb{R}^d)$-norms of the kernel K_t for $1 \le p \le \infty$.

Lemma 2.2. *Let* J *be such that* $\widehat{J}(\xi) \in L^1(\mathbb{R}^d)$, $\partial_\xi \widehat{J}(\xi) \in L^2(\mathbb{R}^d)$ *and*

$$\widehat{J}(\xi) - 1 + \xi^2 \sim |\xi|^3, \qquad \partial_\xi \widehat{J}(\xi) \sim -\xi \qquad \text{as } \xi \sim 0.$$

For any $p \ge 1$ *there exists a positive constant* $c(p,J)$ *such that* K_t*, defined in* (2.3)*, satisfies:*

$$\|K_t\|_{L^p(\mathbb{R}^d)} \le c(p,J)\,\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})} \tag{2.5}$$

for any $t > 0$.

Remark 2.1. *In fact, when $p = \infty$, a stronger inequality can be proven,*

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \le Cte^{-\delta t}\|\widehat{J}\|_{L^1(\mathbb{R}^d)} + C\langle t\rangle^{-d/2},$$

for some positive $\delta = \delta(J)$.

Moreover, for $p = 1$ we have,

$$\|K_t\|_{L^1(\mathbb{R}^d)} \le 2$$

and for any $p \in [1, \infty]$

$$\|S(t)\|_{L^p(\mathbb{R}^d)-L^p(\mathbb{R}^d)} \le 3.$$

Proof of Lemma 2.2. We analyze the cases $p = \infty$ and $p = 1$, the others can be easily obtained applying Hölder's inequality.

Case $p = \infty$**.** Using Hausdorff-Young's inequality we obtain that

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi.$$

Observe that the symmetry of J guarantees that $\widehat{J}$ is a real number. Let us choose $R > 0$ such that

$$|\widehat{J}(\xi)| \le 1 - \frac{|\xi|^2}{2} \text{ for all } |\xi| \le R. \tag{2.6}$$

Once R is fixed, there exists $\delta = \delta(J)$, $0 < \delta < 1$, with

$$|\widehat{J}(\xi)| \le 1 - \delta \text{ for all } |\xi| \ge R. \tag{2.7}$$

Using that for any real numbers a and b the following inequality holds:

$$|e^a - e^b| \le |a - b| \max\{e^a, e^b\}$$

we obtain for any $|\xi| \ge R$,

$$|e^{t(\widehat{J}(\xi)-1)} - e^{-t}| \le t|\widehat{J}(\xi)| \max\{e^{-t}, e^{t(\widehat{J}(\xi)-1)}\} \le te^{-\delta t}|\widehat{J}(\xi)|. \tag{2.8}$$

Then the following integral decays exponentially,

$$\int_{|\xi|\ge R} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi \le e^{-\delta t} t \int_{|\xi|\ge R} |\widehat{J}(\xi)|d\xi.$$

Using that this term is exponentially small, it remains to prove that

$$I(t) = \int_{|\xi|\le R} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi \le C\langle t\rangle^{-d/2}. \tag{2.9}$$

To handle this case we use the following estimates:

$$|I(t)| \le \int_{|\xi|\le R} e^{t(\widehat{J}(\xi)-1)}d\xi + e^{-t}C(R) \le \int_{|\xi|\le R} d\xi + e^{-t}C(R) \le C(R)$$

and

$$\begin{aligned} |I(t)| &\le \int_{|\xi|\le R} e^{t(\widehat{J}(\xi)-1)}d\xi + e^{-t}C(R) \le \int_{|\xi|\le R} e^{-\frac{t|\xi|^2}{2}} + e^{-t}C(R) \\ &= t^{-d/2} \int_{|\eta|\le Rt^{1/2}} e^{-\frac{|\eta|^2}{2}} + e^{-t}C(R) \le Ct^{-d/2}. \end{aligned}$$

The last two estimates prove (2.9) and this finishes the analysis of this case.

Case $p = 1$. First we prove that the $L^1(\mathbb{R}^d)$-norm of the solutions to equation (1.4) does not increase. Multiplying equation (1.4) by $\operatorname{sgn}(w(t,x))$ and integrating in space variable we obtain,

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|w(t,x)|\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)w(t,y)\operatorname{sgn}(w(t,x))\,dy\,ds - \int_{\mathbb{R}^d}|w(t,x)|\,dx\\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)|w(t,y)|\,dx\,dy - \int_{\mathbb{R}^d}|w(t,x)|\,dx \leq 0,\end{aligned}$$

which shows that the $L^1(\mathbb{R}^d)$-norm does not increase. Hence, for any $u_0 \in L^1(\mathbb{R}^d)$, the following holds:

$$\int_{\mathbb{R}^d}|e^{-t}u_0(x) + (K_t * u_0)(x)|\,dx \leq \int_{\mathbb{R}^d}|u_0(x)|\,dx,$$

and as a consequence,

$$\int_{\mathbb{R}^d}|(K_t * u_0)(x)|\,dx \leq 2\int_{\mathbb{R}^d}|u_0(x)|\,dx.$$

Choosing $(u_0)_n \in L^1(\mathbb{R}^d)$ such that $(u_0)_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ we obtain in the limit that

$$\int_{\mathbb{R}^d}|K_t(x)|dx \leq 2.$$

This ends the proof of the L^1-case and finishes the proof. $\square$

The following lemma will play a key role when analyzing the decay of the complete problem (1.1). In the sequel we will denote by $L^1(\mathbb{R}^d, a(x))$ the following space:

$$L^1(\mathbb{R}^d, a(x)) = \left\{\varphi : \int_{\mathbb{R}^d} a(x)|\varphi(x)|dx < \infty\right\}.$$

Lemma 2.3. *Let $p \geq 1$ and $J \in \mathcal{S}(\mathbb{R}^d)$. There exists a positive constant $c(p,J)$ such that*

$$\|K_t * \varphi - K_t\|_{L^p(\mathbb{R}^d)} \leq c(p)\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}\|\varphi\|_{L^1(\mathbb{R}^d,|x|)}$$

holds for all $\varphi \in L^1(\mathbb{R}^d, 1+|x|)$.

Proof. Explicit computations shows that

$$\begin{aligned}(K_t * \varphi - K_t)(x) &= \int_{\mathbb{R}^d}K_t(x-y)\varphi(y)dy - \int_{\mathbb{R}^d}K_t(x)\,dx\\ &= \int_{\mathbb{R}^d}\varphi(y)(K_t(x-y) - K_t(x))\,dy\\ &= \int_{\mathbb{R}^d}\varphi(y)\int_0^1 \nabla K_t(x-sy)\cdot(-y)\,ds\,dy. \qquad (2.10)\end{aligned}$$

We will analyze the cases $p = 1$ and $p = \infty$, the others cases follow by interpolation.

For $p = \infty$ we have,

$$\|K_t * \varphi - K_t\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla K_t\|_{L^\infty(\mathbb{R}^d)}\int_{\mathbb{R}^d}|y||\varphi(y)|\,dy. \qquad (2.11)$$

In the case $p=1$, by using (2.10) the following holds:

$$\begin{aligned}\int_{\mathbb{R}^d} |(K_t*\varphi - K_t)(x)|\,dx &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y||\varphi(y)| \int_0^1 |\nabla K_t(x-sy)|\,ds\,dy\,dx\\ &= \int_{\mathbb{R}^d} |y||\varphi(y)| \int_0^1 \int_{\mathbb{R}^d} |\nabla K_t(x-sy)|\,dx\,ds\,dy\\ &= \int_{\mathbb{R}^d} |y||\varphi(y)|\,dy \int_{\mathbb{R}^d} |\nabla K_t(x)|\,dx. \qquad (2.12)\end{aligned}$$

In view of (2.11) and (2.12) it is sufficient to prove that

$$\|\nabla K_t\|_{L^\infty(\mathbb{R}^d)} \le C\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}}$$

and

$$\|\nabla K_t\|_{L^1(\mathbb{R}^d)} \le C\langle t\rangle^{-\frac{1}{2}}.$$

In the first case, with R and δ as in (2.6) and (2.7), by Hausdorff-Young's inequality and (2.8) we obtain:

$$\begin{aligned}\|\nabla K_t\|_{L^\infty(\mathbb{R}^d)} &\le \int_{\mathbb{R}^d} |\xi||e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi\\ &= \int_{|\xi|\le R} |\xi||e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi + \int_{|\xi|\ge R} |\xi||e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi\\ &\le \int_{|\xi|\le R} |\xi|e^{-t|\xi|^2/2}d\xi + e^{-t}\int_{|\xi|\le R}|\xi|d\xi + t\int_{|\xi|\ge R}|\xi||\widehat{J}(\xi)|e^{-\delta t}d\xi\\ &\le C(R)\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}} + C(R)e^{-t} + C(J)\,t\,e^{-\delta t}\\ &\le C(J)\,\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}},\end{aligned}$$

provided that $|\xi|\widehat{J}(\xi)$ belongs to $L^1(\mathbb{R}^d)$.

In the second case it is enough to prove that the $L^1(\mathbb{R}^d)$-norm of $\partial_{x_1}K_t$ is controlled by $\langle t\rangle^{-1/2}$. In this case Carlson's inequality gives us

$$\|\partial_{x_1}K_t\|_{L^1(\mathbb{R}^d)} \le C\,\|\partial_{x_1}K_t\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{2m}} \||x|^m\partial_{x_1}K_t\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2m}},$$

for any $m > d/2$.

Now our aim is to prove that, for any $t>0$, we have

$$\|\partial_{x_1}K_t\|_{L^2(\mathbb{R}^d)} \le C(J)\langle t\rangle^{-\frac{d}{4}-\frac{1}{2}} \qquad (2.13)$$

and

$$\||x|^m\partial_{x_1}K_t\|_{L^2(\mathbb{R}^d)} \le C(J)\langle t\rangle^{\frac{m-1}{2}-\frac{d}{4}}. \qquad (2.14)$$

By Plancherel's identity, estimate (2.8) and using that $|\xi|\widehat{J}(\xi)$ belongs to $L^2(\mathbb{R}^d)$ we obtain

$$\begin{aligned}
\|\partial_{x_1} K_t\|^2_{L^2(\mathbb{R}^d)} &= \int_{\mathbb{R}^d} |\xi_1|^2 |e^{t(\widehat{J}(\xi)-1)} - e^{-t}|^2 \, d\xi \\
&\leq 2\int_{|\xi|\leq R} |\xi_1|^2 e^{-t|\xi|^2} d\xi + e^{-2t} \int_{|\xi|\leq R} |\xi_1|^2 d\xi + \int_{|\xi|\geq R} |\xi_1|^2 e^{-2\delta t} t^2 |\widehat{J}(\xi)|^2 d\xi \\
&\leq C(R)\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}} + C(R)e^{-2t} + C(J)e^{-2\delta t}t^2 \\
&\leq C(J)\,\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}}.
\end{aligned}$$

This shows (2.13).

To prove (2.14), observe that

$$\||x|^m \partial_{x_1} K_t\|^2_{L^2(\mathbb{R}^d)} \leq c(d) \int_{\mathbb{R}^d} (x_1^{2m} + \cdots + x_d^{2m}) |\partial_{x_1} K_t(x)|^2 dx.$$

Thus, by symmetry it is sufficient to prove that

$$\int_{\mathbb{R}^d} |\partial^m_{\xi_1}(\xi_1 \widehat{K}_t(\xi))|^2 d\xi \leq C(J)\,\langle t\rangle^{m-1-\frac{d}{2}}$$

and

$$\int_{\mathbb{R}^d} |\partial^m_{\xi_2}(\xi_1 \widehat{K}_t(\xi))|^2 d\xi \leq C(J)\,\langle t\rangle^{m-1-\frac{d}{2}}.$$

Observe that

$$|\partial^m_{\xi_1}(\xi_1 \widehat{K}_t(\xi))| = |\xi_1 \partial^m_{\xi_1} \widehat{K}_t(\xi) + m\partial^{m-1}_{\xi_1} \widehat{K}_t(\xi)| \leq |\xi||\partial^m_{\xi_1}\widehat{K}_t(\xi)| + m|\partial^{m-1}_{\xi_1}\widehat{K}_t(\xi)|$$

and

$$|\partial^m_{\xi_2}(\xi_1 \widehat{K}_t)| \leq |\xi||\partial^m_{\xi_2}\widehat{K}_t(\xi)|.$$

Hence we just have to prove that

$$\int_{\mathbb{R}^d} |\xi|^{2r} |\partial^n_{\xi_1} \widehat{K}_t(\xi)|^2 d\xi \leq C(J)\,\langle t\rangle^{n-r-\frac{d}{2}}, \qquad (r,n) \in \{(0,m-1),\,(1,m)\}.$$

Choosing $m = [d/2]+1$ (the notation $[\cdot]$ stands for the floor function) the above inequality has to hold for $n = [d/2], [d/2]+1$.

First we recall the following elementary identity

$$\partial^n_{\xi_1}(e^g) = e^g \sum_{i_1+2i_2+\ldots+ni_n=n} a_{i_1,\ldots,i_n} (\partial^1_{\xi_1} g)^{i_1} (\partial^2_{\xi_1} g)^{i_2} \ldots (\partial^n_{\xi_1} g)^{i_n},$$

where $a_{i_1,\ldots,i_n}$ are universal constants independent of g. Tacking into account that

$$\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)} - e^{-t}$$

we obtain

$$\partial^n_{\xi_1}\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)} \sum_{i_1+2i_2+\ldots+ni_n=n} a_{i_1,\ldots,i_n} t^{i_1+\cdots+i_n} \prod_{j=1}^n [\partial^j_{\xi_1}\widehat{J}(\xi)]^{i_j}$$

and hence

$$|\partial^n_{\xi_1}\widehat{K_t}(\xi)|^2 \leq C\, e^{2t(\widehat{J}(\xi)-1)} \sum_{i_1+2i_2+\ldots+ni_n=n} t^{2(i_1+\cdots+i_n)} \prod_{j=1}^n [\partial^j_{\xi_1}\widehat{J}(\xi)]^{2i_j}.$$

Using that all the partial derivatives of $\widehat{J}$ decay faster than any polinomial in $|\xi|$, as $|\xi| \to \infty$, we obtain that

$$\int_{|\xi|>R} |\xi|^{2r} |\partial^n_{\xi_1} \widehat{K_t}(\xi)|^2 d\xi \leq C(J)\, e^{-2\delta t} \langle t \rangle^{2n}$$

where R and δ are chosen as in (2.6) and (2.7). Tacking into account that $\widehat{J}(\xi)$ is smooth (since $J \in \mathcal{S}(\mathbb{R}^d)$) we obtain that for all $|\xi| \leq R$ the following hold:

$$|\partial_{\xi_1} \widehat{J}(\xi)| \leq C\, |\xi|$$

and

$$|\partial^j_{\xi_1} \widehat{J}(\xi)| \leq C, \qquad j = 2, \ldots, n.$$

Then for all $|\xi| \leq R$ we have

$$|\partial^n_{\xi_1} \widehat{K_t}(\xi)|^2 \leq C\, e^{-t|\xi|^2} \sum_{i_1+2i_2+\ldots+ni_n=n} t^{2(i_1+\cdots+i_n)} |\xi|^{2i_1}.$$

Finally, using that for any $l \geq 0$

$$\int_{|\xi|\leq R} e^{-t|\xi|^2} |\xi|^l d\xi \leq C(R) \langle t \rangle^{-\frac{d}{2}-\frac{l}{2}},$$

we obtain

$$\int_{|\xi|\leq R} |\xi|^{2r} |\partial^n_{\xi_1} K_t(\xi)|^2 d\xi \leq C(R) \langle t \rangle^{-\frac{d}{2}} \sum_{i_1+2i_2+\cdots+ni_n=n} \langle t \rangle^{2p(i_1,\ldots,i_d)-r}$$

where

$$\begin{aligned} p(i_1, \ldots, i_n) &= (i_1 + \cdots + i_n) - \frac{i_1}{2} \\ &= \frac{i_1}{2} + i_2 + \cdots + i_n \leq \frac{i_1 + 2i_2 + \ldots ni_n}{2} = \frac{n}{2}. \end{aligned}$$

This ends the proof. □

We now prove a decay estimate that takes into account the linear semigroup applied to the convolution with a kernel G.

Lemma 2.4. *Let $1 \leq p \leq r \leq \infty$, $J \in \mathcal{S}(\mathbb{R}^d)$ and $G \in L^1(\mathbb{R}^d, |x|)$. There exists a positive constant $C = C(p, J, G)$ such that the following estimate*

$$\|S(t) * G * \varphi - S(t) * \varphi\|_{L^r(\mathbb{R}^d)} \leq C \langle t \rangle^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} (\|\varphi\|_{L^p(\mathbb{R}^d)} + \|\varphi\|_{L^r(\mathbb{R}^d)}). \tag{2.15}$$

holds for all $\varphi \in L^p(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$.

Remark 2.2. *In fact the following stronger inequality holds:*

$$\|S(t) * G * \varphi - S(t) * \varphi\|_{L^r(\mathbb{R}^d)} \leq C\, \langle t \rangle^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} \|\varphi\|_{L^p(\mathbb{R}^d)} + C\, e^{-t} \|\varphi\|_{L^r(\mathbb{R}^d)}.$$

Proof. We write $S(t)$ as $S(t) = e^{-t}\delta_0 + K_t$ and we get

$$S(t) * G * \varphi - S(t) * \varphi = e^{-t}(G * \varphi - \varphi) + K_t * G * \varphi - K_t * \varphi.$$

The first term in the above right hand side verifies:

$$e^{-t} \|G * \varphi - \varphi\|_{L^r(\mathbb{R}^d)} \leq e^{-t} (\|G\|_{L^1(\mathbb{R}^d)} \|\varphi\|_{L^r(\mathbb{R}^d)} + \|\varphi\|_{L^r(\mathbb{R}^d)}) \leq 2e^{-t} \|\varphi\|_{L^r(\mathbb{R}^d)}.$$

For the second one, by Lemma 2.3 we get that K_t satisfies

$$\|K_t * G - K_t\|_{L^a(\mathbb{R}^d)} \le C(r,J)\|G\|_{L^1(\mathbb{R}^d,|x|)}\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{a})-\frac{1}{2}}$$

for all $t \ge 0$ where a is such that $1/r = 1/a + 1/p - 1$. Then, using Young's inequality we end the proof. $\square$

3. Existence and uniqueness

In this section we use the previous results and estimates on the linear semigroup to prove the existence and uniqueness of the solution to our nonlinear problem (1.1). The proof is based on the variation of constants formula and uses the previous properties of the linear diffusion semigroup.

Proof of Theorem 1.1. Recall that we want prove the global existence of solutions for initial conditions $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Let us consider the following integral equation associated with (1.1):

$$u(t) = S(t) * u_0 + \int_0^t S(t-s) * (G * (f(u)) - f(u))(s)\, ds, \tag{3.1}$$

the functional

$$\Phi[u](t) = S(t) * u_0 + \int_0^t S(t-s) * (G * (f(u)) - f(u))(s)\, ds$$

and the space

$$X(T) = C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T];\, \mathbb{R}^d)$$

endowed with the norm

$$\|u\|_{X(T)} = \sup_{t\in[0,T]} \left(\|u(t)\|_{L^1(\mathbb{R}^d)} + \|u(t)\|_{L^\infty(\mathbb{R}^d)}\right).$$

We will prove that Φ is a contraction in the ball of radius R, B_R, of X_T, if T is small enough.

Step I. Local Existence. Let $M = \max\{\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}\}$ and $p = 1, \infty$. Then, using the results of Lemma 2.2 we obtain,

$$\begin{aligned}
\|\Phi[u](t)\|_{L^p(\mathbb{R}^d)} &\le \|S(t) * u_0\|_{L^p(\mathbb{R}^d)} \\
&\quad + \int_0^t \|S(t-s) * G * (f(u)) - S(t-s) * f(u)\|_{L^p(\mathbb{R}^d)}\, ds \\
&\le (e^{-t} + \|K_t\|_{L^1(\mathbb{R}^d)})\|u_0\|_{L^p(\mathbb{R}^d)} \\
&\quad + \int_0^t 2(e^{-(t-s)} + \|K_{t-s}\|_{L^1(\mathbb{R}^d)})\|f(u)(s)\|_{L^p(\mathbb{R}^d)}\, ds \\
&\le 3\,\|u_0\|_{L^p(\mathbb{R}^d)} + 6\,Tf(R) \le 3M + 6\,Tf(R).
\end{aligned}$$

This implies that

$$\|\Phi[u]\|_{X(T)} \le 6M + 12\,Tf(R).$$

Choosing $R = 12M$ and T such that $12\,Tf(R) < 6M$ we obtain that $\Phi(B_R) \subset B_R$.

Let us choose u and v in B_R. Then for $p = 1, \infty$ the following hold:

$$\begin{aligned}
\|\Phi[u](t) - \Phi[v](t)\|_{L^p(\mathbb{R}^d)} &\leq \int_0^t \|(S(t-s) * G - S(t-s)) * (f(u) - f(v))\|_{L^p(\mathbb{R}^d)}\, ds \\
&\leq 6 \int_0^t \|f(u)(s) - f(v)(s)\|_{L^p(\mathbb{R}^d)}\, ds \\
&\leq C(R) \int_0^t \|u(s) - v(s)\|_{L^p(\mathbb{R}^d)}\, ds \\
&\leq C(R)\, T\, \|u - v\|_{X(T)}.
\end{aligned}$$

Choosing T small we obtain that $\Phi[u]$ is a contraction in B_R and then there exists a unique local solution u of (3.1).

Step II. Global existence. To prove the global well posedness of the solutions we have to guarantee that both $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$-norms of the solutions do not blow up in finite time. We will apply the following lemma to control the $L^\infty(\mathbb{R}^d)$-norm of the solutions.

Lemma 3.1. *Let $\theta \in L^1(\mathbb{R}^d)$ and K be a nonnegative function with mass one. Then for any $\mu \geq 0$ the following hold:*

$$\int_{\theta(x)>\mu} \int_{\mathbb{R}^d} K(x-y)\theta(y)\, dy\, dx \leq \int_{\theta(x)>\mu} \theta(x)\, dx \tag{3.2}$$

and

$$\int_{\theta(x)<-\mu} \int_{\mathbb{R}^d} K(x-y)\theta(y)\, dy\, dx \geq \int_{\theta(x)<-\mu} \theta(x)\, dx. \tag{3.3}$$

Proof of Lemma 3.1. First of all we point out that we only have to prove (3.2). Indeed, once it is proved, then (3.3) follows immediately applying (3.2) to the function $-\theta$.

First, we prove estimate (3.2) for $\mu = 0$ and then we apply this case to prove the general case, $\mu \neq 0$.

For $\mu = 0$ the following inequalities hold:

$$\begin{aligned}
\int_{\theta(x)>0} \int_{\mathbb{R}^d} K(x-y)\theta(y)\, dy\, dx &\leq \int_{\theta(x)>0} \int_{\theta(y)>0} K(x-y)\theta(y)\, dy\, dx \\
&= \int_{\theta(y)>0} \theta(y) \int_{\theta(x)>0} K(x-y)\, dx\, dy \\
&\leq \int_{\theta(y)>0} \theta(y) \int_{\mathbb{R}^d} K(x-y)\, dx\, dy \\
&= \int_{\theta(y)>0} \theta(y)\, dy.
\end{aligned}$$

Now let us analyze the general case $\mu > 0$. In this case the following inequality

$$\int_{\theta(x)>\mu} \theta(x)\, dx \leq \int_{\mathbb{R}^d} |\theta(x)|\, dx$$

shows that the set $\{x \in \mathbb{R}^d : \theta(x) > \mu\}$ has finite measure. Then we obtain

$$\begin{aligned}\int_{\theta(x)>\mu}\int_{\mathbb{R}^d} K(x-y)\theta(y)\,dy\,dx &= \int_{\theta(x)>\mu}\int_{\mathbb{R}^d} K(x-y)(\theta(y)-\mu)\,dy\,dx + \int_{\theta(x)>\mu}\mu\,dx\\ &\le \int_{\theta(x)>\mu}(\theta(x)-\mu)\,dx + \int_{\theta(x)>\mu}\mu\,dx = \int_{\theta(x)>\mu}\theta(x)\,dx.\end{aligned}$$

This completes the proof of (3.2). □

Control of the L^1-norm. As in the previous section, we multiply equation (1.1) by $\operatorname{sgn}(u(t,x))$ and integrate in $\mathbb{R}^d$ to obtain the following estimate

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(t,y)\operatorname{sgn}(u(t,x))\,dy\,dx - \int_{\mathbb{R}^d}|u(t,x)|\,dx\\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(u(t,y))\operatorname{sgn}(u(t,x))\,dy\,dx - \int_{\mathbb{R}^d} f(u(t,x))\operatorname{sgn}(u(t,x))\,dx\\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)|u(t,y)|\,dy\,dx - \int_{\mathbb{R}^d}|u(t,x)|\,dx\\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)|f(u)(t,y)|\,dy\,dx - \int_{\mathbb{R}^d}|f(u)(t,x)|\,dx\\ &= \int_{\mathbb{R}^d}|u(t,y)|\int_{\mathbb{R}^d} J(x-y)\,dx\,dy - \int_{\mathbb{R}^d}|u(t,x)|\,dx\\ &\quad + \int_{\mathbb{R}^d}|f(u)(t,y)|\int_{\mathbb{R}^d} G(x-y)\,dx\,dy - \int_{\mathbb{R}^d}|f(u)(t,x)|\,dx\\ &\le 0,\end{aligned}$$

which shows that the L^1-norm does not increase.

Control of the L^∞-norm. Let us denote $m = \|u_0\|_{L^\infty(\mathbb{R}^d)}$. Multiplying the equation in (1.1) by $\operatorname{sgn}(u-m)^+$ and integrating in the x variable we get,

$$\frac{d}{dt}\int_{\mathbb{R}^d}(u(t,x)-m)^+dx = I_1(t) + I_2(t)$$

where

$$I_1(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(t,y)\operatorname{sgn}(u(t,x)-m)^+\,dy\,dx - \int_{\mathbb{R}^d} u(t,x)\operatorname{sgn}(u(t,x)-m)^+\,dx$$

and

$$\begin{aligned}I_2(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} &G(x-y)f(u)(t,y)\operatorname{sgn}(u(t,x)-m)^+\,dy\,dx\\ &- \int_{\mathbb{R}^d} f(u)(t,x)\operatorname{sgn}(u(t,x)-m)^+\,dx.\end{aligned}$$

We claim that both I_1 and I_2 are negative. Thus $(u(t,x)-m)^+ = 0$ a.e. $x \in \mathbb{R}^d$ and then $u(t,x) \le m$ for all $t > 0$ and a.e. $x \in \mathbb{R}^d$.

In the case of I_1, applying Lemma 3.1 with $K = J$, $\theta = u(t)$ and $\mu = m$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(t,y)\,\mathrm{sgn}(u(t,x)-m)^+\,dy\,dx &= \int_{u(x)>m}\int_{\mathbb{R}^d} J(x-y)u(t,y)\,dy\,dx \\ &\le \int_{u(x)>m} u(t,x)\,dx.\end{aligned}$$

To handle the second one, I_2, we proceed in a similar manner. Applying Lemma 3.1 with

$$\theta(x) = f(u)(t,x) \quad \text{and} \quad \mu = f(m)$$

we obtain

$$\int_{f(u(t,x))>f(m)}\int_{\mathbb{R}^d} G(x-y)f(u)(t,y)\,dy\,dx \le \int_{f(u(t,x))>f(m)} f(u)(t,x)\,dx.$$

Using that f is a nondecreasing function, we rewrite this inequality in an equivalent form te obtain the desired inequality:

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(u)(t,y)\,\mathrm{sgn}(u(t,x)-m)^+\,dy\,dx \\ &\qquad = \int_{u(t,x)\ge m}\int_{\mathbb{R}^d} G(x-y)f(u)(t,y)\,dy\,dx \\ &\qquad = \int_{f(u)(t,x)\ge f(m)}\int_{\mathbb{R}^d} G(x-y)f(u)(t,y)\,dy\,dx \\ &\qquad \le \int_{u(t,x)\ge m} f(u)(t,x)\,dx.\end{aligned}$$

In a similar way, by using inequality (3.3) we get

$$\frac{d}{dt}\int_{\mathbb{R}^d} (u(t,x)+m)^-\,dx \le 0,$$

which implies that $u(t,x) \ge -m$ for all $t > 0$ and a.e. $x \in \mathbb{R}^d$.

We conclude that $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}$.

Step III. Uniqueness and contraction property. Let us consider u and v two solutions corresponding to initial data u_0 and v_0 respectively. We will prove that for any $t > 0$ the following holds:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)-v(t,x)|\,dx \le 0.$$

To this end, we multiply by $\operatorname{sgn}(u(t,x)-v(t,x))$ the equation satisfied by $u-v$ and using the symmetry of J, the positivity of J and G and that their mass equals one we obtain,

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)-v(t,x)|\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(t,y)-v(t,y))\operatorname{sgn}(u(t,x)-v(t,x))\,dx\,dy\\
&\quad-\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}|u(t,x)-v(t,x)|\,dx\\
&\quad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)(f(u)(t,y)-f(v)(t,y))\operatorname{sgn}(u(t,x)-v(t,x))\,dx\,dy\\
&\quad-\int_{\mathbb{R}^d}|f(u)(t,x)-f(v)(t,x)|\,dx\\
&\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)|u(t,y)-v(t,y)|\,dx\,dy-\int_{\mathbb{R}^d}|u(t,x)-v(t,x)|\,dx\\
&\quad+\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)|f(u)(t,y)-f(v)(t,y)|\,dx\,dy-\int_{\mathbb{R}^d}|f(u)(t,x)-f(v)(t,x)|\,dx\\
&=0.
\end{aligned}$$

Thus we get the uniqueness of the solutions and the contraction property

$$\|u(t)-v(t)\|_{L^1(\mathbb{R}^d)}\le\|u_0-v_0\|_{L^1(\mathbb{R}^d)}.$$

This ends the proof of Theorem 1.1. □

Now we prove that, due to the lack of regularizing effect, the $L^\infty(\mathbb{R})$-norm does not get bounded for positive times when we consider initial conditions in $L^1(\mathbb{R})$. This is in contrast to what happens for the local convection-diffusion problem, see [12].

Proposition 3.1. *Let $d=1$ and $|f(u)|\le C|u|^q$ with $1\le q<2$. Then*

$$\sup_{u_0\in L^1(\mathbb{R})}\sup_{t\in[0,1]}\frac{t^{\frac12}\|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}}=\infty.$$

Proof. Assume by contradiction that

$$\sup_{u_0\in L^1(\mathbb{R})}\sup_{t\in[0,1]}\frac{t^{\frac12}\|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}}=M<\infty. \tag{3.4}$$

Using the representation formula (3.1) we get:

$$\|u(1)\|_{L^\infty(\mathbb{R})}\ge\|S(1)*u_0\|_{L^\infty(\mathbb{R})}-\left\|\int_0^1 S(1-s)*(G*(f(u))-f(u))(s)\,ds\right\|_{L^\infty(\mathbb{R})}$$

Using Lemma 2.4 the last term can be bounded as follows:

$$\begin{aligned}
\left\|\int_0^1 S(1-s)*(G*(f(u))-f(u))(s)\,ds\right\|_{L^\infty(\mathbb{R})}&\le\int_0^1\langle 1-s\rangle^{-\frac12}\|f(u(s))\|_{L^\infty(\mathbb{R})}\,ds\\
&\le C\int_0^1\|u(s)\|^q_{L^\infty(\mathbb{R})}ds\le CM^q\|u_0\|^q_{L^1(\mathbb{R})}\int_0^1 s^{-\frac q2}ds\\
&\le CM^q\|u_0\|^q_{L^1(\mathbb{R})},
\end{aligned}$$

provided that $q<2$.

This implies that the $L^\infty(\mathbb{R})$-norm of the solution at time $t=1$ satisfies

$$\begin{aligned}\|u(1)\|_{L^\infty(\mathbb{R})} &\geq \|S(1)*u_0\|_{L^\infty(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}\\ &\geq e^{-1}\|u_0\|_{L^\infty(\mathbb{R})} - \|K_1\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^1(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}\\ &\geq e^{-1}\|u_0\|_{L^\infty(\mathbb{R})} - C\|u_0\|_{L^1(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}.\end{aligned}$$

Choosing now a sequence $u_{0,\varepsilon}$ with $\|u_{0,\varepsilon}\|_{L^1(\mathbb{R})}=1$ and $\|u_{0,\varepsilon}\|_{L^\infty(\mathbb{R})}\to\infty$ we obtain that

$$\|u_{0,\varepsilon}(1)\|_{L^\infty(\mathbb{R})}\to\infty,$$

a contradiction with our assumption (3.4). The proof of the result is now completed. □

4. Convergence to the local problem

In this section we prove the convergence of solutions of the nonlocal problem to solutions of the local convection-diffusion equation when we rescale the kernels and let the scaling parameter go to zero.

As we did in the previous sections we begin with the analysis of the linear part.

Lemma 4.1. *Assume that $u_0\in L^2(\mathbb{R}^d)$. Let w_ε be the solution to*

$$\begin{cases}(w_\varepsilon)_t(t,x) = \dfrac{1}{\varepsilon^2}\displaystyle\int_{\mathbb{R}^d} J_\varepsilon(x-y)(w_\varepsilon(t,y)-w_\varepsilon(t,x))\,dy,\\ w_\varepsilon(0,x)=u_0(x),\end{cases}\tag{4.1}$$

and w the solution to

$$\begin{cases} w_t(t,x)=\Delta w(t,x),\\ w(0,x)=u_0(x).\end{cases}\tag{4.2}$$

Then, for any positive T,

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|w_\varepsilon - w\|_{L^2(\mathbb{R}^d)}=0.$$

Proof. Taking the Fourier transform in (4.1) we get

$$\widehat{w_\varepsilon}(t,\xi)=\frac{1}{\varepsilon^2}\Big(\widehat{J_\varepsilon}(\xi)\widehat{w_\varepsilon}(t,\xi)-\widehat{w_\varepsilon}(t,\xi)\Big).$$

Therefore,

$$\widehat{w_\varepsilon}(t,\xi)=\exp\left(t\frac{\widehat{J_\varepsilon}(\xi)-1}{\varepsilon^2}\right)\widehat{u_0}(\xi).$$

But we have,

$$\widehat{J_\varepsilon}(\xi)=\widehat{J}(\varepsilon\xi).$$

Hence we get

$$\widehat{w_\varepsilon}(t,\xi)=\exp\left(t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}\right)\widehat{u_0}(\xi).$$

By Plancherel's identity, using the well known formula for solutions to (4.2),

$$\widehat{w}(t,\xi)=e^{-t\xi^2}\widehat{u_0}(\xi).$$

we obtain that

$$\|w_\varepsilon(t)-w(t)\|^2_{L^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right|^2|\widehat{u_0}(\xi)|^2\,d\xi$$

With R and δ as in (2.6) and (2.7) we get

$$\begin{aligned}\int_{|\xi|\geq R/\varepsilon}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right|^2|\widehat{u}_0(\xi)|^2d\xi &\leq \int_{|\xi|\geq R/\varepsilon}(e^{-\frac{t\delta}{\varepsilon^2}}+e^{\frac{-tR^2}{\varepsilon^2}})^2|\widehat{u_0}(\xi)|^2d\xi\\ &\leq (e^{-\frac{t\delta}{\varepsilon^2}}+e^{\frac{-tR^2}{\varepsilon^2}})^2\|u_0\|^2_{L^2(\mathbb{R}^d)}\underset{\varepsilon\to 0}{\longrightarrow}0. \end{aligned}\tag{4.3}$$

To treat the integral on the set $\{\xi\in\mathbb{R}^d : |\xi|\leq R/\varepsilon\}$ we use the fact that on this set the following holds:

$$\begin{aligned}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right| &\leq t\left|\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}+\xi^2\right|\max\{e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}},e^{-t\xi^2}\}\\ &\leq t\left|\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}+\xi^2\right|\max\{e^{-\frac{t\xi^2}{2}},e^{-t\xi^2}\}\\ &\leq t\left|\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}+\xi^2\right|e^{-\frac{t\xi^2}{2}}.\end{aligned}\tag{4.4}$$

Thus:

$$\begin{aligned}\int_{|\xi|\leq R/\varepsilon}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right|^2|\widehat{u_0}(\xi)|^2d\xi &\leq \int_{|\xi|\leq R/\varepsilon}e^{-t|\xi|^2}t^2\left|\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}+\xi^2\right|^2|\widehat{u_0}(\xi)|^2\,d\xi\\ &\leq \int_{|\xi|\leq R/\varepsilon}e^{-t\xi^2}t^2|\xi|^4\left|\frac{\widehat{J}(\varepsilon\xi)-1+\varepsilon^2\xi^2}{\varepsilon^2\xi^2}\right|^2|\widehat{u_0}(\xi)|^2\,d\xi.\end{aligned}$$

From $|\widehat{J}(\xi)-1|\leq K|\xi|^2$ for all $\xi\in\mathbb{R}^d$ we get

$$\left|\frac{\widehat{J}(\varepsilon\xi)-1+\varepsilon^2\xi^2}{\varepsilon^2\xi^2}\right|\leq\frac{(K+1)}{\varepsilon^2|\xi|^2}\varepsilon^2|\xi|^2\leq K+1.\tag{4.5}$$

Using this bound and that $e^{-|s|}s^2\leq C$, we get that

$$\sup_{t\in[0,T]}\int_{|\xi|\leq R/\varepsilon}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right|^2|\widehat{u_0}(\xi)|^2d\xi\leq C\int_{\mathbb{R}^d}\left|\frac{\widehat{J}(\varepsilon\xi)-1+\varepsilon^2|\xi|^2}{\varepsilon^2|\xi|^2}\right|^2|\widehat{u}_0(\xi)|^2\mathbf{1}_{\{|\xi|\leq R/\varepsilon\}}d\xi.$$

By inequality (4.5) together with the fact that

$$\lim_{\varepsilon\to 0}\frac{\widehat{J}(\varepsilon\xi)-1+\varepsilon^2|\xi|^2}{\varepsilon^2|\xi|^2}=0$$

and that $\widehat{u}_0\in L^2(\mathbb{R}^d)$, by Lebesgue dominated convergence theorem, we have that

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\int_{|\xi|\leq R/\varepsilon}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right|^2|\widehat{u_0}(\xi)|^2d\xi=0.\tag{4.6}$$

From (4.3) and (4.6) we obtain

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|w_\varepsilon(t)-w(t)\|^2_{L^2(\mathbb{R}^d)}=0,$$

as we wanted to prove. $\square$

Next we prove a lemma that provides us with a uniform (independent of ε) decay for the nonlocal convective part.

Lemma 4.2. *There exists a positive constant $C=C(J,G)$ such that*

$$\Big\|\Big(\frac{S_\varepsilon(t)*G_\varepsilon-S_\varepsilon(t)}{\varepsilon}\Big)*\varphi\Big\|_{L^2(\mathbb{R}^d)}\le C\,t^{-\frac12}\|\varphi\|_{L^2(\mathbb{R}^d)}$$

holds for all $t>0$ and $\varphi\in L^2(\mathbb{R}^d)$, uniformly on $\varepsilon>0$. Here $S_\varepsilon(t)$ is the linear semigroup associated to (4.1).

Proof. Let us denote by $\Phi_\varepsilon(t,x)$ the following quantity:

$$\Phi_\varepsilon(t,x)=\frac{(S_\varepsilon(t)*G_\varepsilon)(x)-S_\varepsilon(t)(x)}{\varepsilon}.$$

Then by the definition of S_ε and G_ε we obtain

$$\begin{aligned}\Phi_\varepsilon(t,x)&=\int_{\mathbb{R}^d}e^{ix\cdot\xi}\exp\Big(\frac{t(\widehat{J}(\varepsilon\xi)-1)}{\varepsilon^2}\Big)\frac{\widehat{G}(\xi\varepsilon)-1}{\varepsilon}d\xi\\&=\varepsilon^{-d-1}\int_{\mathbb{R}^d}e^{i\varepsilon^{-1}x\cdot\xi}\exp\Big(\frac{t(\widehat{J}(\xi)-1)}{\varepsilon^2}\Big)(\widehat{G}(\xi)-1)\,d\xi\\&=\varepsilon^{-d-1}\Phi_1(t\varepsilon^{-2},x\varepsilon^{-1})\end{aligned}.$$

At this point, we observe that for $\varepsilon=1$, Lemma 2.4 gives us

$$\|\Phi_1(t)*\varphi\|_{L^2(\mathbb{R}^d)}\le C(J,G)\langle t\rangle^{-\frac12}\|\varphi\|_{L^2(\mathbb{R}^d)}.$$

Hence

$$\begin{aligned}\|\Phi_\varepsilon(t)*\varphi\|_{L^2(\mathbb{R}^d)}&=\varepsilon^{-d-1}\|\Phi_1(t\varepsilon^{-2},\varepsilon^{-1}\cdot)*\varphi\|_{L^2(\mathbb{R}^d)}=\varepsilon^{-1}\|[\Phi_1(t\varepsilon^{-2})*\varphi(\varepsilon\cdot)](\varepsilon^{-1}\cdot)\|_{L^2(\mathbb{R}^d)}\\&=\varepsilon^{-1+\frac d2}\|\Phi_1(t\varepsilon^{-2}*\varphi(\varepsilon\cdot))\|_{L^2(\mathbb{R}^d)}\le\varepsilon^{-1+\frac d2}(t\varepsilon^{-2})^{-\frac12}\|\varphi(\varepsilon\cdot)\|_{L^2(\mathbb{R}^d)}\\&\le t^{-\frac12}\|\varphi\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

This ends the proof. $\square$

Lemma 4.3. *Let be $T>0$ and $M>0$. Then the following*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\int_0^t\Big\|\Big(\frac{S_\varepsilon(s)*G_\varepsilon-S_\varepsilon(s)}{\varepsilon}-b\cdot\nabla H(s)\Big)*\varphi(s)\Big\|_{L^2(\mathbb{R}^d)}ds=0,$$

holds uniformly for all $\|\varphi\|_{L^\infty([0,T];L^2(\mathbb{R}^d))}\le M$. Here H is the linear heat semigroup given by the Gaussian

$$H(t)=\frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac d2}}$$

and $b=(b_1,...,b_d)$ is given by

$$b_j=\int_{\mathbb{R}^d} x_j\, G(x)\, dx, \qquad j=1,...,d.$$

Proof. Let us denote by $I_\varepsilon(t)$ the following quantity:

$$I_\varepsilon(t)=\int_0^t \Big\| \Big(\frac{S_\varepsilon(s)*G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b\cdot\nabla H(s)\Big) * \varphi(s)\Big\|_{L^2(\mathbb{R}^d)} ds.$$

Choose $\alpha\in(0,1)$. Then

$$I_\varepsilon(t)\le \begin{cases} I_{1,\varepsilon} & \text{if } t\le \varepsilon^\alpha,\\ I_{1,\varepsilon}+I_{2,\varepsilon}(t) & \text{if } t\ge\varepsilon^\alpha,\end{cases}$$

where

$$I_{1,\varepsilon}=\int_0^{\varepsilon^\alpha} \Big\| \Big(\frac{S_\varepsilon(s)*G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b\cdot\nabla H(s)\Big) * \varphi(s)\Big\|_{L^2(\mathbb{R}^d)} ds$$

and

$$I_{2,\varepsilon}(t)=\int_{\varepsilon^\alpha}^t \Big\| \Big(\frac{S_\varepsilon(s)*G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b\cdot\nabla H(s)\Big) * \varphi(s)\Big\|_{L^2(\mathbb{R}^d)} ds.$$

The first term $I_{1,\varepsilon}$ satisfies,

$$\begin{aligned} I_{1,\varepsilon} &\le \int_0^{\varepsilon^\alpha} \Big\| \Big(\frac{S_\varepsilon(s)*G_\varepsilon - S_\varepsilon(s)}{\varepsilon}\Big) * \varphi\Big\|_{L^2(\mathbb{R}^d)} ds + \int_0^{\varepsilon^\alpha} \|b\cdot\nabla H(s)*\varphi\|_{L^2(\mathbb{R}^d)}\, ds \\ &\le C\int_0^{\varepsilon^\alpha} s^{-\frac12}\|\varphi(s)\|_{L^2(\mathbb{R}^d)} ds + C\int_0^{\varepsilon^\alpha} \|\nabla H(s)\|_{L^1(\mathbb{R}^d)}\|\varphi(s)\|_{L^2(\mathbb{R}^d)} ds \\ &\le CM\int_0^{\varepsilon^\alpha} s^{-\frac12}\, ds = 2CM\varepsilon^{\frac{\alpha}{2}}. \end{aligned} \tag{4.7}$$

To bound $I_{2,\varepsilon}(t)$ we observe that, by Plancherel's identity, we get,

$$\begin{aligned} I_{2,\varepsilon}(t) &= \int_{\varepsilon^\alpha}^t \Big\| \Big(e^{s(\widehat{J}(\varepsilon\xi)-1)/\varepsilon^2} \Big(\frac{\widehat{G}(\varepsilon\xi)-1}{\varepsilon}\Big) - i\, b\cdot\xi e^{-s|\xi|^2}\Big)\widehat{\varphi}(s)\Big\|_{L^2_\xi(\mathbb{R}^d)} ds \\ &\le \int_{\varepsilon^\alpha}^t \Big\| \Big(e^{s(\widehat{J}(\varepsilon\xi)-1)/\varepsilon^2} - e^{-s|\xi|^2}\Big) \Big(\frac{\widehat{G}(\varepsilon\xi)-1}{\varepsilon}\Big)\widehat{\varphi}(s)\Big\|_{L^2_\xi(\mathbb{R}^d)} ds \\ &\quad + \int_{\varepsilon^\alpha}^t \Big\| e^{-s|\xi|^2} \Big(\frac{\widehat{G}(\varepsilon\xi)-1}{\varepsilon} - i\, b\cdot\xi\Big)\widehat{\varphi}(s)\Big\|_{L^2_\xi(\mathbb{R}^d)} ds \\ &= \int_{\varepsilon^\alpha}^t R_{1,\varepsilon}(s)\, ds + \int_{\varepsilon^\alpha}^t R_{2,\varepsilon}(s)\, ds. \end{aligned}$$

In the following we obtain upper bounds for $R_{1,\varepsilon}$ and $R_{2,\varepsilon}$. Observe that $R_{1,\varepsilon}$ satisfies:

$$(R_{1,\varepsilon})^2(s)\le 2((R_{3,\varepsilon})^2(s)+(R_{4,\varepsilon})^2(s))$$

where

$$(R_{3,\varepsilon})^2(s) = \int_{|\xi|\leq R/\varepsilon} \left(e^{s(\widehat{J}(\varepsilon\xi)-1)/\varepsilon^2} - e^{-s|\xi|^2}\right)^2 \left|\frac{\widehat{G}(\varepsilon\xi)-1}{\varepsilon}\right|^2 |\widehat{\varphi}(s,\xi)|^2 d\xi$$

and

$$(R_{4,\varepsilon})^2(s) = \int_{|\xi|\geq R/\varepsilon} \left(e^{s(\widehat{J}(\varepsilon\xi)-1)/\varepsilon^2} - e^{-s|\xi|^2}\right)^2 \left|\frac{\widehat{G}(\varepsilon\xi)-1}{\varepsilon}\right|^2 |\widehat{\varphi}(s,\xi)|^2 d\xi.$$

With respect to $R_{3,\varepsilon}$ we proceed as in the proof of Lemma 4.2 by choosing δ and R as in (2.6) and (2.7). Using estimate (4.4) and the fact that $|\widehat{G}(\xi)-1|\leq C|\xi|$ and $|\widehat{J}(\xi)-1+\xi^2|\leq C|\xi|^3$ for every $\xi\in\mathbb{R}^d$ we obtain:

$$\begin{aligned}
(R_{3,\varepsilon})^2(s) &\leq C\int_{|\xi|\leq R/\varepsilon} e^{-s|\xi|^2} s^2 \left|\frac{\widehat{J}(\varepsilon\xi)-1+\xi^2\varepsilon^2}{\varepsilon^2}\right|^2 |\xi|^2|\widehat{\varphi}(s,\xi)|^2 d\xi\\
&\leq C\int_{|\xi|\leq R/\varepsilon} e^{-s|\xi|^2} s^2 \left[\frac{(\varepsilon\xi)^3}{\varepsilon^2}\right]^2 |\xi|^2|\widehat{\varphi}(s,\xi)|^2 d\xi\\
&= C\int_{|\xi|\leq R/\varepsilon} e^{-s|\xi|^2} s^2\varepsilon^2|\xi|^8|\widehat{\varphi}(s,\xi)|^2 d\xi \leq \varepsilon^2 s^{-2}\int_{\mathbb{R}^d} e^{-s|\xi|^2} s^4|\xi|^8|\widehat{\varphi}(s,\xi)|^2 d\xi\\
&\leq C\varepsilon^{2-2\alpha}\int_{\mathbb{R}^d} |\widehat{\varphi}(s,\xi)|^2 d\xi \leq C\varepsilon^{2-2\alpha}M^2.
\end{aligned}$$

In the case of $R_{4,\varepsilon}$, we use that $|\hat{G}(\xi)|\leq 1$ and we proceed as in the proof of (4.3):

$$\begin{aligned}
(R_{4,\varepsilon})^2(s) &\leq \int_{|\xi|\geq R/\varepsilon} (e^{-\frac{s\delta}{\varepsilon^2}} + e^{-\frac{sR^2}{\varepsilon^2}})^2 \varepsilon^{-2}|\widehat{\varphi}(s,\xi)|^2 d\xi\\
&\leq (e^{-\frac{\delta}{\varepsilon^{2-\alpha}}} + e^{-\frac{R^2}{\varepsilon^{2-\alpha}}})^2\varepsilon^{-2}\int_{|\xi|\geq R/\varepsilon} |\widehat{\varphi}(s,\xi)|^2 d\xi\\
&\leq M^2(e^{-\frac{\delta}{\varepsilon^{2-\alpha}}} + e^{-\frac{R^2}{\varepsilon^{2-\alpha}}})^2\varepsilon^{-2}\\
&\leq CM^2\varepsilon^{2-2\alpha}
\end{aligned}$$

for sufficiently small ε.

Then

$$\int_{\varepsilon^\alpha}^t R_{1,\varepsilon}(s)ds \leq CTM\varepsilon^{1-\alpha}. \tag{4.8}$$

The second term can be estimated in a similar way, using that $|\widehat{G}(\xi)-1-ib\cdot\xi|\leq C|\xi|^2$ for every $\xi\in\mathbb{R}^d$, we get

$$\begin{aligned}
(R_{2,\varepsilon})^2(s) &\leq \int_{\mathbb{R}^d} e^{-2s|\xi|^2}\left|\frac{\widehat{G}(\varepsilon\xi)-1-i\,b\cdot\xi\varepsilon}{\varepsilon}\right|^2 |\widehat{\varphi}(s,\xi)|^2 d\xi\\
&\leq C\int_{\mathbb{R}^d} e^{-2s|\xi|^2}\left[\frac{(\xi\varepsilon)^2}{\varepsilon}\right]^2 |\widehat{\varphi}(s,\xi)|^2 d\xi = C\int_{\mathbb{R}^d} e^{-2s|\xi|^2}\varepsilon^2|\xi|^4|\widehat{\varphi}(s,\xi)|^2 d\xi\\
&= C\varepsilon^2 s^{-2}\int_{\mathbb{R}^d} e^{-2s|\xi|^2} s^2|\xi|^4|\widehat{\varphi}(s,\xi)|^2 d\xi \leq C\varepsilon^{2(1-\alpha)}\int_{\mathbb{R}^d}|\widehat{\varphi}(s,\xi)|^2 d\xi\\
&\leq CM^2\varepsilon^{2(1-\alpha)},
\end{aligned}$$

and we conclude that

$$\int_{\varepsilon^\alpha}^t R_{2,\varepsilon}(s)ds \leq CTM\varepsilon^{1-\alpha}. \tag{4.9}$$

Now, by (4.7), (4.8) and (4.9) we obtain that

$$\sup_{t\in[0,T]} I_\varepsilon(t) \leq CM(\varepsilon^{\frac{\alpha}{2}} + \varepsilon^{1-\alpha}) \to 0, \qquad \text{as } \varepsilon \to 0, \tag{4.10}$$

which finishes the proof. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. First we write the two problems in the semigroup formulation,

$$u_\varepsilon(t) = S_\varepsilon(t) * u_0 + \int_0^t \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * f(u_\varepsilon(s))\, ds$$

and

$$v(t) = H(t) * u_0 + \int_0^t b\cdot \nabla H(t-s) * f(v(s))\, ds.$$

Then

$$\sup_{t\in[0,T]} \|u_\varepsilon(t) - v(t)\|_{L^2(\mathbb{R}^d)} \leq \sup_{t\in[0,T]} I_{1,\varepsilon}(t) + \sup_{t\in[0,T]} I_{2,\varepsilon}(t) \tag{4.11}$$

where

$$I_{1,\varepsilon}(t) = \|S_\varepsilon(t) * u_0 - H(t) * u_0\|_{L^2(\mathbb{R}^d)}$$

and

$$I_{2,\varepsilon}(t) = \left\| \int_0^t \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * f(u_\varepsilon(s)) - \int_0^t b\cdot \nabla H(t-s) * f(v(s)) \right\|_{L^2(\mathbb{R}^d)}.$$

In view of Lemma 4.1 we have

$$\sup_{t\in[0,T]} I_{1,\varepsilon}(t) \to 0 \qquad \text{as } \varepsilon \to 0.$$

So it remains to analyze the second term $I_{2,\varepsilon}$. To this end, we split it again

$$I_{2,\varepsilon}(t) \leq I_{3,\varepsilon}(t) + I_{4,\varepsilon}(t)$$

where

$$I_{3,\varepsilon}(t) = \int_0^t \left\| \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * \big(f(u_\varepsilon(s)) - f(v(s))\big) \right\|_{L^2(\mathbb{R}^d)} ds$$

and

$$I_{4,\varepsilon}(t) = \int_0^t \left\| \left(\frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} - b\cdot\nabla H(t-s) \right) * f(v(s)) \right\|_{L^2(\mathbb{R}^d)} ds.$$

Using Young's inequality and that from our hypotheses we have an uniform bound for u_ε and u in terms of $\|u_0\|_{L^1(\mathbb{R}^d)}$, $\|u_0\|_{L^\infty(\mathbb{R}^d)}$ we obtain

$$\begin{aligned} I_{3,\varepsilon}(t) &\le \int_0^t \frac{\|f(u_\varepsilon(s))-f(v(s))\|_{L^2(\mathbb{R}^d)}}{|t-s|^{\frac12}}ds \\ &\le \|f(u_\varepsilon)-f(v)\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))}\int_0^t \frac{ds}{|t-s|^{\frac12}} \qquad (4.12)\\ &\le 2T^{1/2}\|u_\varepsilon - v\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))}C(\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)}). \end{aligned}$$

By Lemma 4.3 we obtain, choosing $\alpha = 2/3$ in (4.10), that

$$\sup_{t\in[0,T]} I_{4,\varepsilon} \le C\varepsilon^{\frac13}\|f(v)\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \le C\varepsilon^{\frac13}C(\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)}). \qquad (4.13)$$

Using (4.11), (4.12) and (4.13) we get:

$$\begin{aligned} \|u_\varepsilon - v\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \le\ & \|I_{1,\varepsilon}\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \\ & + T^{\frac12}C(\|u_0\|_{L^1(\mathbb{R})},\|u_0\|_{L^\infty(\mathbb{R})})\|u_\varepsilon - v\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))}. \end{aligned}$$

Choosing $T = T_0$ sufficiently small, depending on $\|u_0\|_{L^1(\mathbb{R})}$ and $\|u_0\|_{L^\infty(\mathbb{R})}$ we get

$$\|u_\varepsilon - v\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \le \|I_{1,\varepsilon}\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \to 0,$$

as $\varepsilon \to 0$.

Using the same argument in any interval $[\tau, \tau + T_0]$, the stability of the solutions of the equation (1.3) in $L^2(\mathbb{R}^d)$-norm and that for any time $\tau > 0$ it holds that

$$\|u_\varepsilon(\tau)\|_{L^1(\mathbb{R}^d)} + \|u_\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} + \|u_0\|_{L^\infty(\mathbb{R}^d)},$$

we obtain

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|u_\varepsilon - v\|_{L^2(\mathbb{R}^d)} = 0,$$

as we wanted to prove. □

5. Long time behaviour of the solutions

The aim of this section is to obtain the first term in the asymptotic expansion of the solution u to (1.1). The main ingredient for our proofs is the following lemma inspired in the Fourier splitting method introduced by Schonbek, see [17], [18] and [19].

Lemma 5.1. *Let R and δ be such that the function $\widehat{J}$ satisfies:*

$$\widehat{J}(\xi) \le 1 - \frac{|\xi|^2}{2}, \qquad |\xi| \le R \qquad (5.1)$$

and

$$\widehat{J}(\xi) \le 1 - \delta, \qquad |\xi| \ge R. \qquad (5.2)$$

Let us assume that the function $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following differential inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^2\,dx \le c\int_{\mathbb{R}^d}(J*u-u)(t,x)u(t,x)\,dx, \qquad (5.3)$$

for any $t > 0$. Then for any $1 \leq r < \infty$ there exists a constant $a = rd/c\delta$ such that

$$\int_{\mathbb{R}^d} |u(at,x)|^2 dx \leq \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{rd\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}} \int_0^t (s+1)^{rd-\frac{d}{2}-1} \|u(as)\|^2_{L^1(\mathbb{R}^d)} ds \tag{5.4}$$

holds for all positive time t where ω_0 is the volume of the unit ball in $\mathbb{R}^d$. In particular

$$\|u(at)\|_{L^2(\mathbb{R}^d)} \leq \frac{\|u(0)\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{rd}{2}}} + \frac{(2\omega_0)^{\frac{1}{2}}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}} \|u\|_{L^\infty([0,\infty);\, L^1(\mathbb{R}^d))}. \tag{5.5}$$

Remark 5.1. *Condition* (5.1) *can be replaced by $\widehat{J}(\xi) \leq 1 - A|\xi|^2$ for $|\xi| \leq R$ but omitting the constant A in the proof we simplify some formulas.*

Remark 5.2. *The differential inequality* (5.3) *can be written in the following form:*

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^2\, dx \leq -\frac{c}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(t,x)-u(t,y))^2\, dx\, dy.$$

This is the nonlocal version of the energy method used in [12]. *However, in our case, exactly the same inequalities used in* [12] *could not be applied.*

Proof. Let R and δ be as in (5.1) and (5.2). We set $a = rd/c\delta$ and consider the following set:

$$A(t) = \left\{ \xi \in \mathbb{R}^d : |\xi| \leq M(t) = \left(\frac{2rd}{c(t+a)}\right)^{1/2} \right\}.$$

Inequality (5.3) gives us:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(t,x)|^2 dx \leq c\int_{\mathbb{R}^d} (\widehat{J}(\xi)-1)|\widehat{u}(\xi)|^2 d\xi \leq c\int_{A(t)^c} (\widehat{J}(\xi)-1)|\widehat{u}(\xi)|^2 d\xi. \tag{5.6}$$

Using the hypotheses (5.1) and (5.2) on the function $\widehat{J}$ the following inequality holds for all $\xi \in A(t)^c$:

$$c(\widehat{J}(\xi)-1) \leq -\frac{rd}{t+a}, \qquad \text{for every } \xi \in A(t)^c, \tag{5.7}$$

since for any $|\xi| \geq R$

$$c(\widehat{J}(\xi)-1) \leq -c\delta = -\frac{rd}{a} \leq -\frac{rd}{t+a}$$

and

$$c(\widehat{J}(\xi)-1) \leq -\frac{c|\xi|^2}{2} \leq -\frac{c}{2}\frac{2rd}{c(t+a)} = -\frac{rd}{t+a}$$

for all $\xi \in A(t)^c$ with $|\xi| \leq R$.

Introducing (5.7) in (5.6) we obtain

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^2dx &\leq -\frac{rd}{t+a}\int_{A(t)^c}|\widehat{u}(t,\xi)|^2d\xi\\ &\leq -\frac{rd}{t+a}\int_{\mathbb{R}^d}|\widehat{u}(t,\xi)|^2d\xi+\frac{rd}{t+a}\int_{|\xi|\leq M(t)}|\widehat{u}(t,\xi)|^2d\xi\\ &\leq -\frac{rd}{t+a}\int_{\mathbb{R}^d}|u(t,x)|^2dx+\frac{rd}{t+a}M(t)^d\omega_0\|\widehat{u}(t)\|^2_{L^\infty(\mathbb{R}^d)}\\ &\leq -\frac{rd}{t+a}\int_{\mathbb{R}^d}|u(t,x)|^2dx+\frac{rd}{t+a}\left[\frac{2rd}{c(t+a)}\right]^{\frac{d}{2}}\omega_0\|u(t)\|^2_{L^1(\mathbb{R}^d))}.\end{aligned}$$

This implies that

$$\begin{aligned}&\frac{d}{dt}\Big[(t+a)^{rd}\int_{\mathbb{R}^d}|u(t,x)|^2dx\Big]\\ &\qquad=(t+a)^{rd}\left[\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^2dx\right]+rd(t+a)^{rd-1}\int_{\mathbb{R}^d}|u(t,x)|^2dx\\ &\qquad\leq (t+a)^{rd-\frac{d}{2}-1}rd\left(\frac{2rd}{c}\right)^{\frac{d}{2}}\omega_0\|u(t)\|^2_{L^1(\mathbb{R}^d)}.\end{aligned}$$

Integrating on the time variable the last inequality we obtain:

$$(t+a)^{rd}\int_{\mathbb{R}^d}|u(t,x)|^2dx-a^{rd}\int_{\mathbb{R}^d}|u(0,x)|^2dx\leq rd\omega_0\left(\frac{2rd}{c}\right)^{\frac{d}{2}}\int_0^t(s+a)^{rd-\frac{d}{2}-1}\|u(s)\|^2_{L^1(\mathbb{R}^d)}ds$$

and hence

$$\begin{aligned}\int_{\mathbb{R}^d}|u(t,x)|^2\,dx &\leq \frac{a^{rd}}{(t+a)^{rd}}\int_{\mathbb{R}^d}|u(0,x)|^2\,dx\\ &\quad+\frac{rd\omega_0}{(t+a)^{rd}}\left(\frac{2rd}{c}\right)^{\frac{d}{2}}\int_0^t(s+a)^{rd-\frac{d}{2}-1}\|u(s)\|^2_{L^1(\mathbb{R}^d)}\,ds.\end{aligned}$$

Replacing t by ta we get:

$$\begin{aligned}\int_{\mathbb{R}^d}|u(at,x)|^2\,dx &\leq \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{rd\omega_0}{(t+1)^{rd}a^{rd}}\left(\frac{2rd}{c}\right)^{\frac{d}{2}}\int_0^{at}(s+a)^{rd-\frac{d}{2}-1}\|u(s)\|^2_{L^1(\mathbb{R}^d)}\,ds\\ &= \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{rd\omega_0}{(t+1)^{rd}}\left(\frac{2rd}{ca}\right)^{\frac{d}{2}}\int_0^t(s+1)^{rd-\frac{d}{2}-1}\|u(as)\|^2_{L^1(\mathbb{R}^d)}\,ds\\ &= \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{rd\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}}\int_0^t(s+1)^{rd-\frac{d}{2}-1}\|u(as)\|^2_{L^1(\mathbb{R}^d)}\,ds\end{aligned}$$

which proves (5.4).

Estimate (5.5) is obtained as follows:

$$\begin{aligned}\int_{\mathbb{R}^d}|u(at,x)|^2dx &\leq \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{rd\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}}\|u\|^2_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}\int_0^t(s+1)^{rd-\frac{d}{2}-1}ds\\ &\leq \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{2\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{\frac{d}{2}}}\|u\|^2_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}.\end{aligned}$$

This ends the proof. □

Lemma 5.2. *Let $2 \le p < \infty$. For any function $u : \mathbb{R}^d \mapsto \mathbb{R}^d$, $I(u)$ defined by*

$$I(u) = \int_{\mathbb{R}^d} (J * u - u)(x)|u(x)|^{p-1} \operatorname{sgn}(u(x))\, dx$$

satisfies

$$\begin{aligned} I(u) &\le \frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} (J * |u|^{p/2} - |u|^{p/2})(x)|u(x)|^{p/2}\, dx \\ &= -\frac{2(p-1)}{p^2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(|u(y)|^{p/2} - |u(x)|^{p/2})^2\, dx\, dy. \end{aligned}$$

Remark 5.3. *This result is a nonlocal counterpart of the well known identity*

$$\int_{\mathbb{R}^d} \Delta u\, |u|^{p-1} \operatorname{sgn}(u)\, dx = -\frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} |\nabla(|u|^{p/2})|^2\, dx.$$

Proof. Using the symmetry of J, $I(u)$ can be written in the following manner,

$$\begin{aligned} I(u) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(y)-u(x))|u(x)|^{p-1} \operatorname{sgn}(u(x))\, dx\, dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x)-u(y))|u(y)|^{p-1} \operatorname{sgn}(u(y))\, dx\, dy. \end{aligned}$$

Thus

$$I(u) = -\frac{1}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x)-u(y)) \left(|u(x)|^{p-1} \operatorname{sgn}(u(x)) - |u(y)|^{p-1} \operatorname{sgn}(u(y))\right)\, dx\, dy.$$

Using the following inequality,

$$||\alpha|^{p/2} - |\beta|^{p/2}|^2 \le \frac{p^2}{4(p-1)}(\alpha-\beta)(|\alpha|^{p-1} \operatorname{sgn}(\alpha) - |\beta|^{p-1} \operatorname{sgn}(\beta))$$

which holds for all real numbers α and β and for every $2 \le p < \infty$, we obtain that $I(u)$ can be bounded from above as follows:

$$\begin{aligned} I(u) &\le -\frac{4(p-1)}{2p^2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(|u(y)|^{p/2} - |u(x)|^{p/2})^2\, dx\, dy \\ &= -\frac{4(p-1)}{2p^2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(|u(y)|^{p} - 2|u(y)|^{p/2}|u(x)|^{p/2} + |u(x)|^{p})\, dx\, dy \\ &= \frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} (J * |u|^{p/2} - |u|^{p/2})(x)|u(x)|^{p/2}\, dx. \end{aligned}$$

The proof is finished. □

Now we are ready to proceed with the proof of Theorem 1.4.

Proof of Theorem 1.4. Let u be the solution to the nonlocal convection-diffusion problem. Then, by the same arguments that we used to control the $L^1(\mathbb{R}^d)$-norm, we obtain the following:

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^p dx &= p\int_{\mathbb{R}^d}(J*u-u)(t,x)|u(t,x)|^{p-1}\operatorname{sgn}(u(t,x))\,dx\\ &\quad+\int_{\mathbb{R}^d}(G*f(u)-f(u))(t,x)|u(t,x)|^{p-1}\operatorname{sgn}(u(t,x))\,dx\\ &\le p\int_{\mathbb{R}^d}(J*u-u)(t,x)|u(t,x)|^{p-1}\operatorname{sgn}(u(t,x))\,dx.\end{aligned}$$

Using Lemma 5.2 we get that the $L^p(\mathbb{R}^d)$-norm of the solution u satisfies the following differential inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^p\,dx \le \frac{4(p-1)}{p}\int_{\mathbb{R}^d}(J*|u|^{p/2}-|u|^{p/2})(x)|u(x)|^{p/2}\,dx. \tag{5.8}$$

First, let us consider $p=2$. Then

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^2\,dx \le 2\int_{\mathbb{R}^d}(J*|u|-|u|)(t,x)|u(t,x)|\,dx.$$

Applying Lemma 5.1 with $|u|$, $c=2$, $r=1$ and using that $\|u\|_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}\le\|u_0\|_{L^1(\mathbb{R}^d)}$ we obtain

$$\begin{aligned}\|u(td/2\delta)\|_{L^2(\mathbb{R})} &\le \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{d}{2}}}+\frac{(2\omega_0)^{\frac12}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}}\|u\|_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}\\ &\le \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{d}{2}}}+\frac{(2\omega_0)^{\frac12}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}}\|u_0\|_{L^1(\mathbb{R}^d))}\\ &\le \frac{C(J,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})}{(t+1)^{\frac{d}{4}}},\end{aligned}$$

which proves (1.6) in the case $p=2$. Using that the $L^1(\mathbb{R}^d)$-norm of the solutions to (1.1), does not increase, $\|u(t)\|_{L^1(\mathbb{R}^d)}\le\|u_0\|_{L^1(\mathbb{R}^d)}$, by Hölder's inequality we obtain the desired decay rate (1.6) in any $L^p(\mathbb{R}^d)$-norm with $p\in[1,2]$.

In the following, using an inductive argument, we will prove the result for any $r=2^m$, with $m\ge1$ an integer. By Hölder's inequality this will give us the $L^p(\mathbb{R}^d)$-norm decay for any $2<p<\infty$.

Let us choose $r=2^m$ with $m\ge1$ and assume that the following

$$\|u(t)\|_{L^r(\mathbb{R}^d)}\le C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{r})}$$

holds for some positive constant $C=C(J,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})$ and for every positive time t. We want to show an analogous estimate for $p=2r=2^{m+1}$.

We use (5.8) with $p=2r$ to obtain the following differential inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^{2r}dx\le\frac{4(2r-1)}{2r}\int_{\mathbb{R}^d}(J*|u|^r-|u|^r)(t,x)|u(t,x)|^r dx.$$

Applying Lemma (5.1) with $|u|^r$, $c(r)=2(2r-1)/r$ and $a=rd/c(r)\delta$ we get:

$$\begin{aligned}\int_{\mathbb{R}^d}|u(at)|^{2r} &\leq \frac{\|u_0^r\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{d\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}}\int_0^t(s+1)^{rd-\frac{d}{2}-1}\|u^r(as)\|^2_{L^1(\mathbb{R}^d)}ds\\ &\leq \frac{\|u_0\|^{2r}_{L^{2r}(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{C(J)}{(t+1)^{rd}}\int_0^t(s+1)^{rd-\frac{d}{2}-1}\|u(as)\|^{2r}_{L^r(\mathbb{R}^d)}ds\\ &\leq \frac{C(J,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})}{(t+1)^d}\times\\ &\quad\left(1+\int_0^t(s+1)^{rd-\frac{d}{2}-1}(s+1)^{-dr(1-\frac{1}{r})}ds\right)\\ &\leq \frac{C}{(t+1)^{dr}}(1+(t+1)^{\frac{d}{2}})\leq C(t+1)^{\frac{d}{2}-dr}\end{aligned}$$

and then

$$\|u(at)\|_{L^{2r}(\mathbb{R}^d)}\leq C(J,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})(t+1)^{-\frac{d}{2}(1-\frac{1}{2r})},$$

which finishes the proof. □

Let us close this section with a remark concerning the applicability of energy methods to study nonlocal problems.

Remark 5.4. *If we want to use energy estimates to get decay rates (for example in $L^2(\mathbb{R}^d)$), we arrive easily to*

$$\frac{d}{dt}\int_{\mathbb{R}^d}|w(t,x)|^2\,dx=-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)(w(t,x)-w(t,y))^2\,dx\,dy$$

when we deal with a solution of the linear equation $w_t=J*w-w$ and to

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(t,x)|^2\,dx\leq-\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)(u(t,x)-u(t,y))^2\,dx\,dy$$

when we consider the complete convection-diffusion problem. However, we can not go further since an inequality of the form

$$\left(\int_{\mathbb{R}^d}|u(x)|^p\,dx\right)^{\frac{2}{p}}\leq C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}J(x-y)(u(x)-u(y))^2\,dx\,dy$$

is not available for $p>2$.

6. Weakly nonlinear behaviour

In this section we find the leading order term in the asymptotic expansion of the solution to (1.1). We use ideas from [12] showing that the nonlinear term decays faster than the linear part.

We recall a previous result of [15] that extends to nonlocal diffusion problems the result of [11] in the case of the heat equation.

Lemma 6.1. *Let $J\in\mathcal{S}(\mathbb{R}^d)$ such that*

$$\widehat{J}(\xi)-(1-|\xi|^2)\sim B|\xi|^3,\qquad \xi\sim 0,$$

for some constant B. For every $p \in [2,\infty)$, there exists some positive constant $C = C(p,J)$ such that

$$\|S(t) * \varphi - MH(t)\|_{L^p(\mathbb{R}^d)} \leq Ce^{-t}\|\varphi\|_{L^p(\mathbb{R}^d)} + C\|\varphi\|_{L^1(\mathbb{R}^d,|x|)}\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}},\ t>0, \tag{6.1}$$

for every $\varphi \in L^1(\mathbb{R}^d, 1+|x|)$ with $M = \int_{\mathbb{R}} \varphi(x)\,dx$, where

$$H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}},$$

is the gaussian.

Remark 6.1. *We can consider a condition like $\widehat{J}(\xi) - (1 - A|\xi|^2) \sim B|\xi|^3$ for $\xi \sim 0$ and obtain as profile a modified Gaussian $H_A(t) = H(At)$, but we omit the tedious details.*

Remark 6.2. *The case $p \in [1,2)$ is more subtle. The analysis performed in the previous sections to handle the case $p = 1$ can be also extended to cover this case when the dimension d verifies $1 \leq d \leq 3$. Indeed in this case, if J satisfies $\widehat{J}(\xi) \sim 1 - A|\xi|^s$, $\xi \sim 0$, then s has to be grater than $[d/2]+1$ and $s = 2$ to obtain the Gaussian profile.*

Proof. We write $S(t) = e^{-t}\delta_0 + K_t$. Then it is sufficient to prove that

$$\|K_t * \varphi - MK_t\|_{L^p(\mathbb{R}^d)} \leq C\|\varphi\|_{L^1(\mathbb{R}^d,|x|)}\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

and

$$t^{\frac{d}{2}(1-\frac{1}{p})}\|K_t - H(t)\|_{L^p(\mathbb{R}^d)} \leq C\langle t\rangle^{-\frac{1}{2}}.$$

The first estimate follows by Lemma 2.3. The second one uses the hypotheses on $\widehat{J}$. A detailed proof can be found in [15]. □

Now we are ready to prove that the same expansion holds for solutions to the complete problem (1.1) when $q > (d+1)/d$.

Proof of Theorem 1.5. In view of (6.1) it is sufficient to prove that

$$t^{-\frac{d}{2}(1-\frac{1}{p})}\|u(t) - S(t) * u_0\|_{L^p(\mathbb{R}^d)} \leq C\langle t\rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.$$

Using the representation (3.1) we get that

$$\|u(t) - S(t) * u_0\|_{L^p(\mathbb{R}^d)} \leq \int_0^t \|[S(t-s) * G - S(t-s)] * |u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)}\,ds.$$

We now estimate the right hand side term as follows: we will split it in two parts, one in which we integrate on $(0,t/2)$ and another one where we integrate on $(t/2,t)$. Concerning the second term, by Lemma 2.4, Theorem 1.4 we have,

$$\begin{aligned}
&\int_{t/2}^t \|[S(t-s) * G - S(t-s)] * |u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)}ds \\
&\qquad \leq C(J,G)\int_{t/2}^t \langle t-s\rangle^{-\frac{1}{2}}\|u(s)\|^q_{L^{pq}(\mathbb{R}^d)}ds \\
&\qquad \leq C(J,G,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R})})\int_{t/2}^t \langle t-s\rangle^{-\frac{1}{2}}\langle s\rangle^{-\frac{d}{2}(q-\frac{1}{p})}ds \\
&\qquad \leq C\langle t\rangle^{-\frac{d}{2}(q-\frac{1}{p})+\frac{1}{2}} \leq Ct^{-\frac{d}{2}(1-\frac{1}{p})}\langle t\rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.
\end{aligned}$$

To bound the first term we proceed as follows,

$$\begin{aligned}
&\int_0^{t/2} \|[S(t-s)*G - S(t-s)] * |u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)}\, ds \\
&\quad\leq C(p,J,G)\int_0^{t/2} \langle t-s\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac12}(\||u(s)|^q\|_{L^1(\mathbb{R}^d)} + \||u(s)|^q\|_{L^p(\mathbb{R}^d)})\, ds \\
&\quad\leq C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac12}\Big(\int_0^{t/2} \|u(s)\|^q_{L^q(\mathbb{R}^d)} ds + \int_0^{t/2} \|u(s)\|^q_{L^{pq}(\mathbb{R}^d)} ds\Big) \\
&\quad= C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac12}(I_1(t)+I_2(t)).
\end{aligned}$$

By Theorem 1.4, for the first integral, $I_1(t)$, we have the following estimate:

$$I_1(t) \leq \int_0^{t/2} \|u(s)\|^q_{L^q(\mathbb{R}^d)} ds \leq C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}) \int_0^{t/2} \langle s\rangle^{-\frac{d}{2}(q-1)} ds,$$

and an explicit computation of the last integral shows that

$$\langle t\rangle^{-\frac12} \int_0^{t/2} \langle s\rangle^{-\frac{d}{2}(q-1)} ds \leq C\langle t\rangle^{-\frac{d}{2}(q-1)+\frac12}.$$

Arguing in the same manner for I_2 we get

$$\begin{aligned}
\langle t\rangle^{-\frac12} I_2(t) &\leq C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac12}\int_0^{t/2} \langle s\rangle^{-\frac{dq}{2}(1-\frac{1}{pq})} ds \\
&\leq C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac{d}{2}(q-\frac{1}{p})+\frac12} \\
&\leq C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac{d}{2}(q-1)+\frac12}.
\end{aligned}$$

This ends the proof. □

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