REFINED ASYMPTOTIC EXPANSIONS FOR NONLOCAL DIFFUSION EQUATIONS

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Abstract. We study the asymptotic behavior for solutions to nonlocal diffusion models of the form $u_t = J * u - u$ in the whole $\mathbb{R}^d$ with an initial condition $u(x,0) = u_0(x)$. Under suitable hypotheses on J (involving its Fourier transform) and u_0, it is proved an expansion of the form

$$\Big\|u(u) - \sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!}\Big(\int u_0(x)x^\alpha\,dx\Big)\partial^\alpha K_t\Big\|_{L^q(\mathbb{R}^d)} \leqslant Ct^{-A},$$

where K_t is the regular part of the fundamental solution and the exponent A depends on J, q, k and the dimension d.

Moreover, we can obtain bounds for the difference between the terms in this expansion and the corresponding ones for the expansion of the evolution given by fractional powers of the Laplacian, $v_t(x,t) = -(-\Delta)^{\frac{s}{2}} v(x,t)$.

1. Introduction

In this paper we study the asymptotic behavior as $t \to \infty$ of solutions to the nonlocal evolution equation

$$(1.1)\qquad \begin{cases} u_t(x,t) = J * u - u(x,t), & t > 0,\ x \in \mathbb{R}^d,\\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases}$$

where $J : \mathbb{R}^d \to \mathbb{R}$ verifies $\int_{\mathbb{R}^d} J(x)dx = 1$.

Equations like (1.1) and variations of it, have been recently widely used to model diffusion processes, for example, in biology, dislocations dynamics, etc. See, for example, [3], [4], [6], [7], [11], [12], [8], [14] and [15]. As stated in [11], if $u(x,t)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J * u)(x,t) = \int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving

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location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1).

Equation (1.1), is called *nonlocal diffusion equation* since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $J * u$. When J is nonnegative and compactly supported, this equation shares many properties with the classical heat equation, $u_t = cu_{xx}$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed, see [11]. However, there is no regularizing effect in general. For instance, if J is rapidly decaying (or compactly supported) the singularity of the source solution, that is a solution of (1.1) with initial condition a delta measure, $u_0 = \delta_0$, remains with an exponential decay. In fact, this fundamental solution can be decomposed as $w(x,t) = e^{-t}\delta_0 + K_t(x)$ where $K_t(x)$ is smooth, see Lemma 2.1. In this way we see that there is no regularizing effect since the solution u of (1.1) can be written as $u(t) = w(t) * u_0 = e^{-t}u_0 + K_t * u_0$ with K_t smooth, which means that $u(\cdot,t)$ is as regular as u_0 is.

For the heat equation a precise asymptotic expansion in terms of the fundamental solution and its derivatives was found in [9]. In fact, if G_t denotes the fundamental solution of the heat equation, namely, $G_t(x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)}$, under adequate assumptions on the initial condition, we have,

$$\Big\| u(x,t) - \sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!}\Big(\int_{\mathbb{R}^d} u_0(x)x^\alpha\Big)\partial^\alpha G_t\Big\|_{L^q(\mathbb{R}^d)} \leqslant Ct^{-A} \tag{1.2}$$

with $A = (\frac{d}{2})(\frac{(k+1)}{d} + (1-\frac{1}{q}))$. As pointed out by the authors in [9], the same asymptotic expansion can be done in a more general setting, dealing with the equations $u_t = -(-\Delta)^{\frac{s}{2}}u$, $s > 0$.

Now we need to introduce some notation. We will say that $f \sim g$ as $\xi \sim 0$ if $|f(\xi) - g(\xi)| = o(g(\xi))$ when $\xi \to 0$ and $f \lesssim g$ if there exists a constant c independent of the relevant quantities such that $f \leqslant cg$. In the sequel we denote by $\widehat{J}$ the Fourier transform of J.

Our main objective here is to study if an expansion analogous to (1.2) holds for the non-local problem (1.1). Concerning the first term, in [5] it is proved that if J verifies $\widehat{J}(\xi) - 1 \sim -|\xi|^s$ as $\xi \sim 0$, then the asymptotic behavior of the solution to (1.1), $u(x,t)$, is given by

$$\lim_{t\to+\infty} t^{\frac{d}{s}} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = -(-\Delta)^{\frac{s}{2}} v(x,t)$ with initial condition $v(x,0) = u_0(x)$. As a consequence, the decay rate is given by $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \leqslant C\, t^{-\frac{d}{s}}$ and the asymptotic profile is as follows,

$$\lim_{t\to+\infty} \left\| t^{\frac{d}{s}} u(yt^{\frac{1}{s}}, t) - \Big(\int_{\mathbb{R}^d} u_0 \Big) G^s(y) \right\|_{L^\infty(\mathbb{R}^d)} = 0,$$

where $G^s(y)$ satisfies $\widehat{G^s}(\xi) = e^{-|\xi|^s}$.

Here we find a complete expansion for $u(x,t)$, a solution to (1.1), in terms of the derivatives of the regular part of the fundamental solution, K_t. As we have mentioned, the fundamental solution $w(x,t)$ of problem (1.1) satisfies

$$w(x,t) = e^{-t}\delta_0(x) + K_t(x),$$

where the function K_t (the regular part of the fundamental solution) is given by

$$\widehat{K_t}(\xi) = e^{-t}(e^{t\widehat{J}(\xi)} - 1).$$

In contrast with the previous analysis done in [5] where the long time behavior is studied in the $L^\infty(\mathbb{R}^d)$-norm, here we also consider $L^q(\mathbb{R}^d)$ norms. We focus in the case $2 \leqslant q \leqslant \infty$ where we use Hausdorff-Young's inequality and Plancherel's identity as main tools. The case $1 \leq q < 2$ will be treated elsewhere.

Theorem 1.1. *Let be s and m positive such that*

$$\widehat{J}(\xi) - 1 \sim -|\xi|^s, \qquad \xi \sim 0 \tag{1.3}$$

and

$$|\widehat{J}(\xi)| \lesssim \frac{1}{|\xi|^m}, \qquad |\xi| \to \infty. \tag{1.4}$$

Then for any $2 \leqslant q \leqslant \infty$ and $k+1 < m-d$ there exists a constant $C = C(q,k)\||x|^{k+1}u_0\|_{L^1(\mathbb{R}^d)}$ such that

$$\left\| u(x,t) - \sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!} \Big(\int_{\mathbb{R}^d} u_0(x) x^\alpha \Big) \partial^\alpha K_t \right\|_{L^q(\mathbb{R}^d)} \leqslant C t^{-A} \tag{1.5}$$

for all $u_0 \in L^1(\mathbb{R}^d, 1+|x|^{k+1})$. Here $A = \frac{(k+1)}{s} + \frac{d}{s}(1-\frac{1}{q})$.

Remark 1.1. The condition $k+1 < m-d$ guarantees that all the partial derivatives $\partial^\alpha K_t$ of order $|\alpha| = k+1$ make sense. In addition if $\widehat{J}$ decays at infinity faster than any polynomial,

$$\forall\, m > 0,\ \exists\, c(m) \text{ such that } |\widehat{J}(\xi)| \leqslant \frac{c(m)}{|\xi|^m}, \quad |\xi| \to \infty, \tag{1.6}$$

then the expansion (1.5) holds for all k.

Note that, when J has an expansion of the form $\widehat{J}(\xi) - 1 \sim -|\xi|^2$ as $\xi \sim 0$ (this happens for example if J is compactly supported), then the decay rate in $L^\infty(\mathbb{R}^d)$ of the solutions to the non-local problem (1.1) and the heat equation coincide (in both cases they decay as $t^{-\frac{d}{2}}$). Moreover, the first order term also coincide (in both cases it is a Gaussian). See [5] and Theorem 1.1.

Our next aim is to study if the higher order terms of the asymptotic expansion that we have found in Theorem 1.1 have some relation with the corresponding ones for the heat equation. Our next results say that the difference between them is of lower order. Again we deal with $2 \leqslant q \leqslant \infty$.

Theorem 1.2. *Let J as in Theorem 1.1 and assume in addition that there exists $r > 0$ such that*

$$\widehat{J}(\xi) - (1 - |\xi|^s) \sim B|\xi|^{s+r}, \qquad \xi \sim 0, \tag{1.7}$$

for some real number B. Then for any $2 \leqslant q \leqslant \infty$ and $|\alpha| \leqslant m - d$ there exists a positive constant $C = C(q, d, s, r)$ such that the following holds

$$\|\partial^\alpha K_t - \partial^\alpha G_t^s\|_{L^q(\mathbb{R}^d)} \leqslant C t^{-\frac{d}{s}(1-\frac{1}{q})} t^{-\frac{|\alpha|+r}{s}}, \tag{1.8}$$

where G_t^s is defined by its Fourier transform $\widehat{G_t^s}(\xi) = \exp(-t|\xi|^s)$.

Note that these results do not imply that the asymptotic expansion $\sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0(x) x^\alpha\right) \partial^\alpha K_t$ coincides with the expansion that holds for the equation $u_t = -(-\Delta)^{\frac{s}{2}} u$: $\sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0(x) x^\alpha\right) \partial^\alpha G_t^s$. They only say that the corresponding terms agree up to a better order. When J is compactly supported or rapidly decaying at infinity, then $s = 2$ and we obtain an expansion analogous to the one that holds for the heat equation.

Finally, we present a result that gives the first two terms in the asymptotic expansion with very weak assumptions on J.

Theorem 1.3. *Let $u_0 \in L^1(\mathbb{R}^d)$ with $\widehat{u_0} \in L^1(\mathbb{R}^d)$ and $s < l$ be two positive numbers such that $\widehat{J}(\xi) - (1 - |\xi|^s) \sim B|\xi|^l$, when $\xi \sim 0$, for some real number B.*

Then for any $2 \leqslant q \leqslant \infty$

$$\lim_{t\to\infty} t^{\frac{d}{s}(1-\frac{1}{q}) + \frac{l-s}{s}} \|u(t) - v(t) - Bt[(-\Delta)^{\frac{l}{2}} v](t)\|_{L^q(\mathbb{R}^d)} \to 0, \tag{1.9}$$

where v is the solution to $v_t = -(-\Delta)^{\frac{s}{2}} v$ with $v(x, 0) = u_0(x)$.

Moreover

(1.10)
$$\lim_{t\to\infty}\left\| t^{\frac{d}{s}+\frac{l}{s}-1}\left(u(yt^{\frac{1}{s}},t)-v(yt^{\frac{1}{s}},t)\right)-Bh(y)\left(\int_{\mathbb{R}^d}u_0\right)\right\|_{L^\infty(\mathbb{R}^d)}=0,$$

where h is given by $\widehat{h}(\xi)=e^{-|\xi|^s}|\xi|^l$.

Let us point out that the asymptotic expansion given by (1.5) involves K_t (and its derivatives) which is not explicit. On the other hand, the two-term asymptotic expansion (1.9) involves G_t^s, a well known explicit kernel (v is just the convolution of G_t^s and u_0). However, our ideas and methods allow us to find only two terms in the latter expansion.

2. Proofs of the results

2.1. **Preliminaries.** First, let us obtain a representation of the solution using Fourier variables. A proof of existence and uniqueness of solutions using the Fourier transform (see [13]) is given in [5]. We repeat the main arguments here for the sake of completeness.

Theorem 2.1. *Let $u_0\in L^1(\mathbb{R}^d)$ such that $\widehat{u_0}\in L^1(\mathbb{R}^d)$. There exists a unique solution $u\in C^0([0,\infty);L^1(\mathbb{R}^d))$ of (1.1), and it is given by*
$$\widehat{u}(\xi,t)=e^{(\widehat{J}(\xi)-1)t}\widehat{u_0}(\xi).$$

Proof. We have
$$u_t(x,t)=J*u-u(x,t)=\int_{\mathbb{R}^d}J(x-y)u(y,t)\,dy-u(x,t).$$
Applying the Fourier transform we obtain $\widehat{u}_t(\xi,t)=\widehat{u}(\xi,t)(\widehat{J}(\xi)-1)$. Hence, $\widehat{u}(\xi,t)=e^{(\widehat{J}(\xi)-1)t}\widehat{u_0}(\xi)$. Since $\widehat{u_0}\in L^1(\mathbb{R}^d)$ and $e^{(\widehat{J}(\xi)-1)t}$ is continuous and bounded, the result follows by taking the inverse of the Fourier transform. □

Now we prove a lemma concerning the fundamental solution of (1.1).

Lemma 2.1. *Let $J\in\mathcal{S}(\mathbb{R}^d)$, the space of rapidly decreasing functions. The fundamental solution of (1.1), that is the solution of (1.1) with initial condition $u_0=\delta_0$, can be decomposed as*

(2.11)
$$w(x,t)=e^{-t}\delta_0(x)+K_t(x),$$

where the function K_t is smooth and given by
$$\widehat{K_t}(\xi)=e^{-t}(e^{t\widehat{J}(\xi)}-1).$$

Moreover, if u is a solution of (1.1) *it can be written as*

$$u(x,t) = (w * u_0)(x,t) = \int_{\mathbb{R}^d} w(x-z,t)u_0(z)\,dz.$$

Proof. By the previous result we have $\widehat{w}_t(\xi,t) = \widehat{w}(\xi,t)(\widehat{J}(\xi)-1)$. Hence, as the initial datum verifies $\widehat{u_0} = \widehat{\delta_0} = 1$, we get

$$\widehat{w}(\xi,t) = e^{(\widehat{J}(\xi)-1)t} = e^{-t} + e^{-t}(e^{\widehat{J}(\xi)t}-1).$$

The first part of the lemma follows applying the inverse Fourier transform in $\mathcal{S}(\mathbb{R}^d)$.

To finish the proof we just observe that $w * u_0$ is a solution of (1.1) (just use Fubini's theorem) with $(w * u_0)(x,0) = u_0(x)$. $\square$

Remark 2.1. The above proof together with the fact that $\widehat{J}(\xi) \to 0$ (since $J \in L^1(\mathbb{R}^d)$) shows that if $\widehat{J} \in L^1(\mathbb{R}^d)$ then the same decomposition (2.11) holds and the result also applies.

To prove our result we need some estimates on the kernel K_t.

2.2. **Estimates on K_t.** In this subsection we obtain the long time behavior of the kernel K_t and its derivatives.

The behavior of $L^q(\mathbb{R}^d)$-norms with $2 \leqslant q \leqslant \infty$ follows by Hausdorff-Young's inequality in the case $q = \infty$ and Plancherel's identity for $q = 2$.

Lemma 2.2. *Let $2 \leqslant q \leqslant \infty$ and J satisfying* (1.3) *and* (1.4). *Then for all indexes α such that $|\alpha| < m-d$ there exists a constant $c(q,\alpha)$ such that*

$$\|\partial^\alpha K_t\|_{L^q(\mathbb{R}^d)} \leqslant c(q,\alpha)\, t^{-\frac{d}{s}(1-\frac{1}{q})-\frac{|\alpha|}{s}}$$

holds for sufficiently large t. Moreover, if J satisfies (1.6) *then the same result holds with no restriction on α.*

Proof of Lemma 2.2. We consider the cases $q = 2$ and $q = \infty$. The other cases follow by interpolation. We denote by *e.s.* the exponentially small terms.

First, let us consider the case $q = \infty$. Using the definition of K_t, $\widehat{K_t}(\xi) = e^{-t}(e^{t\widehat{J}(\xi)}-1)$, we get, for any $x \in \mathbb{R}^d$,

$$|\partial^\alpha K_t(x)| \leqslant e^{-t}\int_{\mathbb{R}^d} |\xi|^{|\alpha|}|e^{t\widehat{J}(\xi)}-1|\,d\xi.$$

Using that $|e^y - 1| \leqslant 2|y|$ for $|y|$ small, say $|y| \leqslant c_0$, we obtain that

$$|e^{t\widehat{J}(\xi)}-1| \leqslant 2t|\widehat{J}(\xi)| \leqslant \frac{2t}{|\xi|^m}$$

for all $|\xi| \geq h(t) = (c_0 t)^{\frac{1}{m}}$. Then

$$e^{-t} \int_{|\xi| \geq h(t)} |\xi|^{|\alpha|} |e^{t\widehat{J}(\xi)} - 1| \, d\xi \lesssim te^{-t} \int_{|\xi| \geq h(t)} \frac{|\xi|^{|\alpha|}}{|\xi|^m} \, d\xi \leqslant te^{-t} c(m - |\alpha|)$$

provided that $|\alpha| < m - d$.

Is easy to see that if (1.6) holds no restriction on the indexes α has to be assumed.

It remains to estimate

$$e^{-t} \int_{|\xi| \leqslant h(t)} |\xi|^{|\alpha|} |e^{t\widehat{J}(\xi)} - 1| d\xi.$$

We observe that the term $e^{-t} \int_{|\xi| \leqslant h(t)} |\xi|^{|\alpha|} d\xi$ is exponentially small, so we concentrate on

$$I(t) = e^{-t} \int_{|\xi| \leqslant h(t)} |e^{t\widehat{J}(\xi)}| |\xi|^{|\alpha|} d\xi.$$

Now, let us choose $R > 0$ such that

$$|\widehat{J}(\xi)| \leqslant 1 - \frac{|\xi|^s}{2} \text{ for all } |\xi| \leqslant R. \tag{2.12}$$

Once R is fixed, there exists $\delta > 0$ with

$$|\widehat{J}(\xi)| \leqslant 1 - \delta \text{ for all } |\xi| \geq R. \tag{2.13}$$

Then

$$\begin{aligned} |I(t)| &\leqslant e^{-t} \int_{|\xi| \leqslant R} |e^{t\widehat{J}(\xi)}| |\xi|^{|\alpha|} d\xi + e^{-t} \int_{R \leqslant |\xi| \leqslant h(t)} |e^{t\widehat{J}(\xi)}| |\xi|^{|\alpha|} d\xi \\ &\lesssim \int_{|\xi| \leqslant R} e^{t(|\widehat{J}(\xi)| - 1)} |\xi|^{|\alpha|} d\xi + e^{-t\delta} \int_{R \leqslant |\xi| \leqslant h(t)} |\xi|^{|\alpha|} d\xi \\ &\lesssim \int_{|\xi| \leqslant R} e^{-\frac{t|\xi|^s}{2}} |\xi|^{|\alpha|} + e.s. \\ &= t^{-\frac{|\alpha|}{s} - \frac{d}{s}} \int_{|\eta| \leqslant Rt^{\frac{1}{s}}} e^{-\frac{|\eta|^s}{2}} |\eta|^{|\alpha|} + e.s. \lesssim t^{-\frac{|\alpha|}{s} - \frac{d}{s}}. \end{aligned}$$

Now, for $q = 2$, by Plancherel's identity we have

$$\|\partial^\alpha K_t\|^2_{L^2(\mathbb{R}^d)} \leq e^{-2t} \int_{\mathbb{R}^d} |e^{t\widehat{J}(\xi)} - 1|^2 |\xi|^{2|\alpha|} d\xi.$$

Putting out the exponentially small terms, it remains to estimate

$$\int_{|\xi| \leqslant R} |e^{t(\widehat{J}(\xi) - 1)}|^2 |\xi|^{2|\alpha|} d\xi,$$

where R is given by (2.12). The behavior of $\widehat{J}$ near zero gives

$$\int_{|\xi|\leqslant R} |e^{t(\widehat{J}(\xi)-1)}|^2|\xi|^{2|\alpha|}d\xi \lesssim \int_{|\xi|\leqslant R} e^{-t|\xi|^s}|\xi|^{2|\alpha|}d\xi \lesssim t^{-\frac{d}{s}-\frac{2|\alpha|}{s}},$$

which finishes the proof. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Following [9] we obtain that the initial condition $u_0 \in L^1(\mathbb{R}^d, 1+|x|^{k+1})$ has the following decomposition

$$u_0 = \sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\int u_0 x^\alpha\, dx\right) D^\alpha \delta_0 + \sum_{|\alpha|=k+1} D^\alpha F_\alpha$$

where

$$\|F_\alpha\|_{L^1(\mathbb{R}^d)} \leqslant \|u_0\|_{L^1(\mathbb{R}^d,\,|x|^{k+1})}$$

for all multi-indexes α with $|\alpha| = k+1$.

In view of (2.11) the solution u of (1.1) satisfies

$$u(x,t) = e^{-t}u_0(x) + (K_t * u_0)(x).$$

The first term being exponentially small it suffices to analyze the long time behavior of $K_t * u_0$. Using the above decomposition and Lemma 2.2 we get

$$\begin{aligned}
\Big\|K_t * u_0 - \sum_{|\alpha|\leqslant k} \frac{(-1)^{|\alpha|}}{\alpha!}\Big(\int u_0(x)x^\alpha dx\Big)\partial^\alpha K_t\Big\|_{L^q(\mathbb{R}^d)} &\leqslant \\
&\leqslant \sum_{|\alpha|=k+1} \|\partial^\alpha K_t * F_\alpha\|_{L^q(\mathbb{R}^d)} \\
&\leqslant \sum_{|\alpha|=k+1} \|\partial^\alpha K_t\|_{L^q(\mathbb{R}^d)} \|F_\alpha\|_{L^1(\mathbb{R}^d)} \\
&\lesssim t^{-\frac{d}{s}(1-\frac{1}{q})} t^{-\frac{(k+1)}{s}} \|u_0\|_{L^1(\mathbb{R}^d,\,|x|^{k+1})}.
\end{aligned}$$

This ends the proof. □

2.3. **Asymptotics for the higher order terms.** In this subsection we prove Theorem 1.2.

Proof of Theorem 1.2. Recall that we have defined G_t^s by its Fourier transform $\widehat{G_t^s} = \exp(-t|\xi|^s)$.

We consider the case $q = \infty$, the case $q = 2$ can be handled similarly and the rest of the cases, $2 < q < \infty$, follow again by interpolation.

Writing each of the two terms in Fourier variables we obtain

$$\|\partial^\alpha K_t - \partial^\alpha G_t^s\|_{L^\infty(\mathbb{R}^d)} \leqslant \int_{\mathbb{R}^d} |\xi|^{|\alpha|}\left|e^{-t}(e^{t\widehat{J}(\xi)} - 1) - e^{-t|\xi|^s}\right|d\xi.$$

Let us choose a positive R such that

$$\left|\widehat{J}(\xi) - 1 + |\xi|^s\right| \leqslant C|\xi|^{r+s}, \qquad \text{for } |\xi| \leqslant R,$$

satisfying (2.13) for some $\delta > 0$. For $|\xi| \geq R$ all the terms are exponentially small as $t \to \infty$. Thus the behavior of the difference $\partial^\alpha K_t - \partial^\alpha G_t$ is given by the following integral:

$$I(t) = \int_{|\xi|\leqslant R} |\xi|^{|\alpha|}\left|e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s}\right| d\xi.$$

In view of the elementary inequality $|e^y - 1| \leqslant c(R)|y|$ for all $|y| \leqslant R$ we obtain that

$$\begin{aligned} I(t) &= \int_{|\xi|\leqslant R} |\xi|^{|\alpha|} e^{-t|\xi|^s}\left|e^{t(\widehat{J}(\xi)-1+|\xi|^s)} - 1\right| d\xi \\ &\lesssim \int_{|\xi|\leqslant R} |\xi|^{|\alpha|} e^{-t|\xi|^s}\left|t(\widehat{J}(\xi) - 1 + |\xi|^s)\right| d\xi \\ &\lesssim t\int_{|\xi|\leqslant R} |\xi|^{|\alpha|} e^{-t|\xi|^s}|\xi|^{s+r} d\xi \\ &\lesssim t^{-\frac{d}{s}-\frac{r}{s}-\frac{|\alpha|}{s}}. \end{aligned}$$

This finishes the proof. □

2.4. **A different approach.** In this final subsection we obtain the first two terms in the asymptotic expansion of the solution under less restrictive hypotheses on J.

Proof of Theorem 1.3. The method that we use here is just to estimate the difference $\|u(t) - v(t) - Bt(-\Delta)^{\frac{l}{2}} v(t)\|_{L^q(\mathbb{R}^d)}$ using Fourier variables.

As before, it is enough to consider the cases $q = 2$ and $q = \infty$. We analyze the case $q = \infty$, the case $q = 2$ follows in the same manner by applying Plancherel's identity.

By Hausdorff-Young's inequality we get

$$\begin{aligned} &\|u(t) - v(t) - tB(-\Delta)^{\frac{l}{2}} v(t)\|_{L^\infty(\mathbb{R}^d)} \\ &\leqslant \int_{\mathbb{R}^d} \left|\widehat{u}(t,\xi) - \widehat{v}(t,\xi) - tB\widehat{(-\Delta)^{\frac{l}{2}} v}(t,\xi)\right| d\xi \\ &= \int_{\mathbb{R}^d} \left|e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s}(1 + tB|\xi|^l)\right| |\widehat{u_0}(\xi)|\, d\xi. \end{aligned}$$

As before, let us choose $R > 0$ such that

$$|\widehat{J}(\xi)| \leqslant 1 - \frac{|\xi|^s}{2}, \quad |\xi| \leqslant R.$$

Then there exists $\delta > 0$ such that

$$|\widehat{J}(\xi)| \leqslant 1 - \delta, \quad |\xi| \geq R.$$

Hence

$$\int_{|\xi|\geq R} |e^{t(\widehat{J}(\xi)-1)}||\widehat{u_0}(\xi)|d\xi \leqslant e^{-\delta t}\|\widehat{u_0}\|_{L^1(\mathbb{R}^d)}$$

and

$$\int_{t^{-\frac{1}{l}}\leqslant|\xi|\leqslant R} |e^{t(\widehat{J}(\xi)-1)}||\widehat{u_0}(\xi)|d\xi \leqslant \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^d)}\int_{t^{-\frac{1}{l}}\leqslant|\xi|\leqslant R} e^{-t|\xi|^s/2}$$

$$\lesssim t^{-\frac{d}{s}}\int_{|\xi|\geq t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\xi|^s/2}d\xi \lesssim t^{-\frac{d}{s}}e^{-t^{1-\frac{s}{l}}/4}.$$

Also

$$\int_{|\xi|\geq t^{-\frac{1}{l}}} e^{-t|\xi|^s}(1+tB|\xi|^l)|\widehat{u_0}(\xi)|d\xi \lesssim \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^d)}\int_{|\xi|\geq t^{-\frac{1}{l}}} e^{-t|\xi|^s}t|\xi|^l\, d\xi$$

$$\lesssim t^{1-\frac{d}{s}-\frac{l}{s}}\int_{|\eta|\geq t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\eta|^s}|\eta|^l\, d\xi$$

$$\lesssim t^{1-\frac{d}{s}-\frac{l}{s}}e^{-t^{1-\frac{s}{l}}/2}\int_{|\eta|\geq t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\eta|^s/2}|\eta|^l\, d\xi.$$

Therefore, we have to analyze

$$I(t)=\int_{|\xi|\leqslant t^{-\frac{1}{l}}} |e^{t(\widehat{J}(\xi)-1)}-e^{-t|\xi|^s}(1+tB|\xi|^l)||\widehat{u_0}(\xi)|\, d\xi.$$

We write $\widehat{J}(\xi)=1-|\xi|^s+B|\xi|^l+|\xi|^l f(\xi)$ where $f(\xi)\to 0$ as $|\xi|\to 0$. Thus

$$I(t)\leqslant I_1(t)+I_2(t)$$

where

$$I_1(t)=\int_{|\xi|\leqslant t^{-\frac{1}{l}}} e^{-t|\xi|^s}|e^{Bt|\xi|^l+t|\xi|^l f(\xi)}-(1+Bt|\xi|^l+t|\xi|^l f(\xi))||\widehat{u_0}(\xi)|\, d\xi$$

and

$$I_2(t)=\int_{|\xi|\leqslant t^{-\frac{1}{l}}} e^{-t|\xi|^s}t|\xi|^l f(\xi)|\widehat{u_0}(\xi)|\, d\xi.$$

For I_1 we have

$$\begin{aligned} I_1(t) &\leqslant \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^d)}\int_{|\xi|\leqslant t^{-\frac{1}{l}}} e^{-t|\xi|^s}(t|\xi|^l+t|\xi|^l|f(\xi)|)^2\, d\xi \\ &\lesssim \int_{|\xi|\leqslant t^{-\frac{1}{l}}} e^{-t|\xi|^s}t^2|\xi|^{2l}\, d\xi \lesssim t^{-\frac{d}{s}+2-\frac{2l}{s}} \end{aligned}$$

and then

$$t^{\frac{d}{s}+\frac{l}{s}-1}I_1(t)\lesssim t^{1-\frac{l}{s}}\to 0,\quad t\to\infty.$$

It remains to prove that

$$t^{\frac{d}{s}+\frac{l}{s}-1}I_2(t)\to 0,\quad t\to\infty.$$

Making a change of variable we obtain

$$t^{\frac{d}{s}-1+\frac{l}{s}} I_2(t) \leqslant \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^d)} \int_{|\xi|\leqslant t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\xi|^s} |\xi|^l f(\xi t^{-\frac{1}{s}})\, d\xi.$$

The integrand is dominated by $\|f\|_{L^\infty(\mathbb{R}^d)} |\xi|^l \exp(-|\xi|^s)$, which belongs to $L^1(\mathbb{R}^d)$. Hence, as $f(\xi/t^{\frac{1}{s}}) \to 0$ when $t \to \infty$, this shows that

$$t^{\frac{d}{s}+\frac{l}{s}-1} I_2(t) \to 0,$$

and finishes the proof of (1.9).

Thanks to (1.9), the proof of (1.10) is reduced to show that

$$\lim_{t\to\infty} \left\| t^{\frac{d}{s}+\frac{l}{s}} [(-\Delta)^{\frac{l}{2}} v](y t^{\frac{1}{s}}, t) - h(y) \left(\int_{\mathbb{R}^d} u_0 \right) \right\|_{L^\infty(\mathbb{R}^d)} = 0.$$

For any $y \in \mathbb{R}^d$ by making a change of variables we obtain

$$I(y,t) = t^{\frac{d}{s}+\frac{l}{s}} [(-\Delta)^{\frac{l}{2}} v](y t^{\frac{1}{s}}, t) = \int_{\mathbb{R}^d} e^{-|\xi|^s} |\xi|^l e^{iy\xi} \widehat{u_0}(\xi/t^{\frac{1}{s}}).$$

Thus, using the dominated convergence theorem we obtain

$$\left\| I(y,t) - h(y) \int_{\mathbb{R}^d} u_0 \right\|_{L^\infty(\mathbb{R}^d)} \leqslant \int_{\mathbb{R}^d} e^{-|\xi|^s} |\xi|^l |\widehat{u_0}(\xi/t^{\frac{1}{s}}) - \widehat{u_0}(0)|\, d\xi \to 0$$

as $t \to \infty$. $\square$

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