Incomplete quenching in a system of heat equations coupled at the boundary

Raúl Ferreira
Departamento de Matemáticas, U. Complutense de Madrid, 28040 Madrid, Spain.
e-mail: raul_ferreira@ucm.es

Arturo de Pablo
Departamento de Matemáticas, U. Carlos III de Madrid, 28911 Leganés, Spain.
e-mail: arturop@math.uc3m.es

Mayte Pérez-LLanos
Departamento de Matemáticas, U. Carlos III de Madrid, 28911 Leganés, Spain.
e-mail: mtperez@math.uc3m.es

Julio D. Rossi
Departamento de Matemática, FCEyN UBA, 1428 Buenos Aires, Argentine.
e-mail: jrossi@dm.uba.ar

Abstract

In this paper we find a possible continuation for quenching solutions to a system of heat equations coupled at the boundary condition. This system exhibits simultaneous and non-simultaneous quenching. For non-simultaneous quenching our continuation is a solution of a parabolic problem with Neumann boundary conditions. We also give some results for simultaneous quenching and present some numerical experiments that suggest that the approximations are not uniformly bounded in this case.

2000 AMS Subject Classification: 35B40, 35B60, 35K50.

Keywords and phrases: quenching, non-simultaneous, incomplete.

1 Introduction and main results

Our main concern in this paper is to look for a possible continuation after quenching of solutions to a system of heat equations coupled at the boundary.

We consider the following parabolic system: two heat equations,

$$\begin{cases} u_t = u_{xx} \\ v_t = v_{xx} \end{cases} \qquad 0 < x < 1\,, 0 < t < T, \tag{1.1}$$

coupled through a non linear flux at $x = 0$,

$$\begin{cases} u_x(0,t) = v^{-p}(0,t) \\ v_x(0,t) = u^{-q}(0,t) \end{cases} \qquad 0 < t < T, \tag{1.2}$$

and zero flux at $x = 1$,

$$\begin{cases} u_x(1,t) = 0 \\ v_x(1,t) = 0 \end{cases} \qquad 0 < t < T. \tag{1.3}$$

As initial condition we take

$$\begin{cases} u(x,0) = u_0(x) \\ v(x,0) = v_0(x) \end{cases} \qquad 0 < x < 1. \tag{1.4}$$

being u_0, v_0 positive, smooth and satisfying the compatibility conditions with the boundary data. We also assume that $u_0', v_0' \geq 0$ and $u_0'', v_0'' < 0$. By classical theory, local existence for the solutions up to some time $t = T$ (maximal existence time) is easily deduced. Moreover, solutions are decreasing in time and increasing in space.

In [FPQR] the authors study this problem and find that, due to the absorption generated by the boundary condition at $x = 0$, the solutions decrease to zero at this point. If they vanish in finite time $t = T_0$, the boundary condition (1.2) blows up and the solution, being classical up to $t = T$, no longer exists (as a classical solution) for greater times, thus the maximal existence time of a classical solution is $T = T_0$.

This phenomenon of existence of a finite time $t = T$ at which some term of the problem ceases to make sense is known as quenching (T denotes the quenching time). It was studied for the first time in [K]. Since then, the phenomenon of quenching for different problems has been the issue of intensive study in recent years, see for example, [C, CK, DX, FPQR, FL, KN, L, L2, L3, PQR], and the references therein.

Some questions related to this situation naturally arise. For instance: how rapidly the solutions tend to zero, the quenching rate, see [CK, L]; the extinction set for the solutions and their behavior near these points, the quenching set and the quenching profile, see [L, L2]; the possibility of extending the solution in some weak sense after quenching, see [FG].

Dealing with a system of equations it is also interesting to guess whether the components of the solution quench (reach zero) at the same time (*simultaneous quenching*) or if some component quenches at time T while the other components remain bounded away from zero (*non-simultaneous quenching*), [FPQR, PQR]. In the case of non-simultaneous quenching, the flux at the boundary of the quenching variable remains bounded. Nevertheless, its time derivative blows up. In fact, both time derivatives blow up, see [FPQR]. Hence quenching is always simultaneous in the sense of [K]. Simultaneous vs. non-simultaneous phenomenon has been also analyzed in the case of blow-up, see for instance [PQR, QR1, QR2, ST].

We enclose in the following theorem the results obtained in [FPQR] concerning simultaneous vs. non-simultaneous quenching for our problem (1.1)–(1.4).

Theorem ([FPQR])
i) If $p, q \geq 1$ the quenching is always simultaneous, while if $p < 1$, we can find initial conditions giving rise to non-simultaneous quenching, i.e., such that u quenches and v remains bounded below (analogous result when $q < 1$).
ii) If $q < 1$ and $p \geq p_0 = (1+q)/(1-q) > 1$, simultaneous quenching is not possible. Nevertheless, if $0 \leq p, q \leq 1$ simultaneous quenching occurs for some initial conditions.

The restriction $p \geq p_0$ instead of $p \geq 1$ in *ii*) seems to be technical. We will not require such a condition in the present work.

Another important issue, as we have mentioned, is to see if it is possible in some sense to continue the solution (u, v) beyond $t = T$. This question was raised first for blow-up problems, see [BC, GV1, GV2, L, QRV], etc. In [FG] it was answered for a quenching problem for the heat equation with a nonlinear boundary condition. There the authors find that the continuation is a solution to the heat equation with a Dirichlet boundary condition, $u(0, t) = 0$, replacing the nonlinear flux at $x = 0$ and as initial condition at $t = T$ the final profile of the original solution, $u(x, T)$.

The purpose of this work is to find, if possible, a natural continuation for problem (1.1)–(1.4) for times beyond T. To this end we approximate the involved powers by bounded functions and then try to pass to the limit in the approximations. Let

$$f_n(s) = \begin{cases} s^{-q}, & \text{if } s > 1/n, \\ n^{q+1}s, & \text{if } 0 < s \leq 1/n, \\ 0 & \text{if } s < 0, \end{cases} \tag{1.5}$$

and

$$g_n(s) = \begin{cases} s^{-p}, & \text{if } s > 1/n, \\ n^{p+1}s, & \text{if } 0 < s \leq 1/n, \\ 0 & \text{if } s < 0, \end{cases} \tag{1.6}$$

and let (u_n, v_n) be the solution to problem (1.1) – (1.4) with f_n and g_n as boundary data, i.e.,

$$(u_n)_x(0, t) = g_n(v_n(0, t)), \qquad (v_n)_x(0, t) = f_n(u_n(0, t)).$$

Since f_n and g_n are bounded functions the solution (u_n, v_n) is defined for all $t > 0$. A natural attempt to obtain a continuation of (u, v) after quenching is to pass to the limit as $n \to \infty$ in (u_n, v_n).

Our first result assures that it is possible to take this limit and that it indeed gives a continuation of (u, v) after T when blow-up is non-simultaneous. Moreover, we identify the PDE system verified by the continuation after quenching.

Theorem 1.1 *Assume that u quenches while v remains bounded from below. Then the approximations (u_n, v_n) have a finite limit*

$$(\overline{u}, \overline{v}) = \lim_{n\to\infty} (u_n, v_n), \qquad \text{for all} \quad 0 \leq x \leq 1, \quad t > 0, \tag{1.7}$$

which is an extension of (u, v), that is, for every $t < T$ it holds that $(\overline{u}, \overline{v}) \equiv (u, v)$.

Moreover, for every $t > T$, $(\overline{u}, \overline{v})$ is the solution to the system

$$\begin{cases} \overline{u}_t = \overline{u}_{xx}, & \overline{v}_t = \overline{v}_{xx}, & 0 < x < 1, \quad t > T, \\ \overline{u}_x(0, t) = (\overline{v})^{-p}(0, t), & \overline{v}_x(0, t) = 0, & t > T, \\ \overline{u}_x(1, t) = 0, & \overline{v}_x(1, t) = 0, & t > T, \\ \overline{u}(x, T) = u(x, T), & \overline{v}(x, T) = v(x, T), & 0 \leq x \leq 1. \end{cases} \tag{1.8}$$

This result provides us with a natural continuation $(\overline{u}, \overline{v})$ of (u, v) after quenching. Note that the v variable does not quench and continues as a solution of the heat equation with zero boundary flux, while the quenching variable u continues with boundary flux given by v^{-p}. Therefore the system becomes partially decoupled.

For a single equation, see [FG], it happens that the continuation verifies a Dirichlet boundary condition at $x = 0$. However, in Theorem 1.1 the boundary conditions verified by the continuation are of Neumann type. This says that a possible continuation for systems may strongly differ from a possible continuation for a single equation.

In the simultaneous quenching case the situation becomes more involved.

As we have mentioned, when $p = q$ and $u_0 = v_0$, the system reduces to a single equation, and the continuation verifies a Dirichlet problem after T. We can show that this type of continuation is not generic.

In the general case, we can only prove that u_n, v_n are bounded in compact intervals of time when p and q are less than one. However, passing to the limit in the system seems delicate, since we cannot find a priori estimates uniformly in n that ensure that the fluxes $f_n(u_n(0,t))$ and $g_n(v_n(0,t))$ converge to some limits.

The situation can be even worse for p or q greater than one. We conjecture in this case that the sequence (u_n, v_n) is not bounded below near T (and therefore we cannot take the limit). Numerical experiments support this conjecture, see Section 4.

Remark 1.1 *All the results in this article are also valid if we replace the flux at the boundary of the regularized problems, f_n and g_n, by smooth approximating functions, $\overline{f}_n$ and $\overline{g}_n$, respectively, such that $f_n \le C\overline{f}_n$ and $g_n \le C\overline{g}_n$, for some $C > 0$.*

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1.1 that deals with the non-simultaneous quenching case; in Section 3 we present some partial results concerning the simultaneous case and finally in Section 4 we perform some numerical experiments that illustrate our results.

2 Non-simultaneous quenching.

Recall that we are considering the approximating problems (P_n) where we have replaced the involved powers by continuous and bounded functions, that is,

$$(P_n) \quad \begin{cases} (u_n)_t = (u_n)_{xx}, , & (v_n)_t = (v_n)_{xx}, & 0 < x < 1, \quad t > 0, \\ (u_n)_x(0,t) = g_n(v_n(0,t)), & (v_n)_x(0,t) = f_n(u_n(0,t)), & t > 0, \\ (u_n)_x(1,t) = 0, & (v_n)_x(1,t) = 0, & t > 0, \\ u_n(x,0) = u_0(x), & v_n(x,0) = v_0(x), & 0 \le x \le 1, \end{cases}$$

where f_n and g_n are given by (1.5) and (1.6), respectively.

Solutions to this problem satisfy the following lemma.

Lemma 2.1 *There exists a unique global in time solution to (P_n), such that*

$$u_n, v_n \in C^{2,1}((0,1) \times [0,\tau]),$$

for every $\tau > 0$, verifying:

i) (u_n, v_n) is uniformly bounded from above;

ii) $(u_n, v_n) \ge (u, v)$, for $(x,t) \in [0,1] \times [0,T)$.

Proof. To prove *i)* we only note that both functions are subsolutions of problem

$$\begin{cases} w_t = w_{xx}, & 0 < x < 1,\ t > 0, \\ w_x(0,t) = w_x(1,t) = 0, & t > 0, \\ w(x,0) = \max(\|v_0\|_\infty, \|u_0\|_\infty), & 0 \le x \le 1. \end{cases}$$

In order to prove *ii)*, let us denote $\psi = u - u_n$ and $\omega = v - v_n$, with (u_n, v_n) solution to (P_n) with initial condition $(u_0 - \varepsilon, v_0 - \varepsilon)$ for some $\varepsilon > 0$. So $\psi(x,0), \omega(x,0) < 0$.

Let us suppose that there exists a first time t_0 and some point $x_0 \in [0,1]$ such that $\psi(x_0,t_0) = 0$ and $\omega(x,t_0) \le 0$ (the opposite situation is similar). By the Strong Maximum Principle $x_0 \in \{0,1\}$. This cannot happen at $x_0 = 1$, since $\psi_x(1,t) = 0$ and it contradicts Hopf's Lemma. Thus, $x_0 = 0$ and from Hopf's Lemma it follows that $\psi_x(0,t_0) < 0$.

But, on the other hand

$$\psi_x(0,t) = v^{-p}(0,t) - g_n(u_n(0,t)) \ge v^{-p} - (v_n)^{-p} \ge -p|\xi|^{-p-1}\omega(0,t) \ge 0$$

and we arrive to a contradiction. Finally, taking $\varepsilon \to 0$ we obtain de desired result. $\square$

The estimate proved in the last lemma allows us to consider the limit in (1.7), at least for $0 < t < T$. This limit coincides with the solution (u,v) for $t < T$ as we show now.

Lemma 2.2 *Let $(\overline{u}, \overline{v})$ be the function defined in* (1.7). *Then, for every $t < T$ it holds that $(\overline{u}, \overline{v}) \equiv (u,v)$.*

Proof. For any fixed $t_0 < T$ there exists a constant $c > 0$ such that $u(x,t),\ v(x,t) \ge c$, for every $0 \le x \le 1$ and $t \le t_0$. If we take n_0 verifying $1/n_0 < c$, then for every $n \ge n_0$ (u,v) solves problem (P_n) in $(0,t_0] \times (0,1)$ and by uniqueness of the solution we conclude $(u_n,v_n) = (u,v)$ in $[0,t_0] \times [0,1]$ for $n \ge n_0$. Therefore

$$(\overline{u}, \overline{v}) = \lim_{n\to\infty} (u_n, v_n) = (u,v), \tag{2.1}$$

for every $t \in [0,t_0]$ and every $x \in [0,1]$. The arbitrariness of t_0 gives that (2.1) holds for any $t < T$. $\square$

Note that (u_n,v_n) verifies $(u_n)_t, (v_n)_t \le 0$ and $(u_n)_{xx}, (v_n)_{xx} \le 0$ for times smaller than τ_n, the first time where one of the components reaches $1/n$.

Let us suppose, from now on, that u quenches while v does not, i.e., $v(0,t) \ge c > 0$ for all $0 \le t \le T$. Note that, by the results of [FPQR] this fact implies $q < 1$.

We want to show that the possible extension, $(\overline{u}, \overline{v})$, of the solution for $t > T$ is the unique pair of functions satisfying the system (1.8). To this end we prove that, for n large enough (u_n,v_n) is a solution to the same system (1.8). We remark that, since v does not quench g_n remains being the power $(v_n)^{-q}(0,t)$ for all $t > 0$ while the function f_n in (P_n) turns to be $n^{p+1}\, u_n(0,t)$.

The next lemma will play a crucial role in our arguments. It says that for n large enough, in the approximating problems the u_n variable reaches zero, while v_n stays positive and bounded away from zero uniformly in n.

Lemma 2.3 *For each n sufficiently large, there exists a time T_n, such that $u_n(0,t) \le 0$, for all $t \ge T_n$. Moreover, at that time $c \le v_n(0,T_n) \le C$ for some constants $c, C > 0$ independent of n.*

Proof. Since the quenching is non-simultaneous there exists a time $\tau_n < T$ (with $\tau_n \to T$) such that $u(0,\tau_n) = 1/n$ and $v(0,\tau_n) = c_n \geq c$. Notice that we have $(u_n, v_n) = (u, v)$ for $t \in (0, \tau_n)$. Then, at time $t = \tau_n$ the functions u_n and v_n are increasing and concave. Therefore,

$$\begin{aligned} & c \leq v_n(x,\tau_n) \leq v_n(0,\tau_n) + n^q x \leq C + n^q x\,, \\ & \frac{1}{n} \leq u_n(x,\tau_n) \leq \frac{1}{n} + (v_n)^{-p}(0,\tau_n)\,x \leq \frac{1}{n} + Cx\,. \end{aligned} \tag{2.2}$$

Now, we estimate the time $\widehat{\tau}_n$ at which v_n reaches the level $c/2$ (if there is such a time). Denote by $s(x,t) = v_n(x, t+\tau_n)$. Since $s(x,0) = v(x,\tau_n) \geq c$ and $s_x(0,t) = f_n(u_n(0,t)) \leq n^q$, we have that s is supersolution to the problem

$$\begin{cases} h_t = h_{xx}, & 0 < x < 1,\ 0 < t < \infty, \\ h_x(0,t) = n^q, & 0 \leq t < \infty, \\ h_x(1,t) = 0, & 0 \leq t < \infty, \\ h(x,0) = c, & 0 \leq x \leq 1. \end{cases}$$

It is easy to see that the function h is decreasing in time and then, it is concave and increasing. Moreover, integrating the equation we have that

$$\frac{d}{dt}\int_0^1 h(x,t)\,dx = -n^q\,.$$

Therefore, $h(0,t)$ vanishes in finite time. Let us denote by τ_0 a time such that $h(0,\tau_0) = c/2$. We wish now to estimate τ_0, that is, a lower bound for $\widehat{\tau}_n$. Rescaling h as follows we take off the dependence on n in the boundary condition. Let

$$\psi(y,\tau) = h(y/n^q, \tau/n^{2q}).$$

which satisfies the problem

$$\begin{cases} \psi_\tau = \psi_{yy}, & 0 < y < n^q,\ 0 < \tau < \infty, \\ \psi_y(0,\tau) = 1, & 0 \leq \tau < \infty, \\ \psi_y(n^q,\tau) = 0, & 0 \leq \tau < \infty, \\ \psi(y,0) = c, & 0 \leq y \leq n^q. \end{cases}$$

Then, there exists a time τ_1 at which $\psi(0,\tau_1) = c/2$. We have also that $\psi(0,\tau_1) = h(0,\tau_1/n^{2q})$, thus,

$$\widehat{\tau}_n \geq \tau_0 = \tau_1/n^{2q}.$$

We claim that, for n large enough, there exists a time $\bar{\tau}_n \leq \widehat{\tau}_n$ such that $u(0,\bar{\tau}_n) = 0$. We observe that for $t \in (0,\widehat{\tau}_n)$ the function u_n verifies that $(u_n)_x(0,t) = v_n^{-p}(0,t)$. Therefore, denoting $r(x,t) = u_n(x,t+\tau_n)$, we have that r is subsolution to the linear problem

$$\begin{cases} r_t = r_{xx}, & 0 < x < 1,\ 0 < t < \widehat{\tau}_n, \\ r_x(0,t) = C^{-p}, & 0 < t \leq \widehat{\tau}_n \\ r_x(1,t) = 0, & 0 < t \leq \widehat{\tau}_n, \\ r(x,0) = u(x,\tau_n), & 0 \leq x \leq 1, \end{cases} \tag{2.3}$$

for a constant $C > 0$ such that $v_n(0,t) \le C$. Integrating the equation in (2.3) we obtain that

$$\frac{d}{dt}\int_0^1 r(x,t)\,dx = -C^{-p},$$

which implies that there exists a time $\overline{\tau}_0$ such that $r(0,\overline{\tau}_0) = 0$. Moreover $r(0,t) < 0$ from this time on. In order to estimate $\overline{\tau}_0$, which is an upper bound for $\overline{\tau}_n$, we rescale r as follows

$$\omega(y,\tau) = n\, r(y/n, \tau/n^2)\,.$$

The problem satisfied by ω is

$$\begin{cases} \omega_\tau = \omega_{yy}, & 0 < y < n,\ 0 < \tau < \infty, \\ \omega_y(0,\tau) = C^{-p}, & 0 \le \tau < \infty, \\ \omega_y(n,\tau) = 0, & 0 \le \tau < \infty, \\ \omega(y,0) = n\,u(y/n, \tau_n), & 0 \le y \le n. \end{cases}$$

Using (2.2) to estimate the initial value $\omega(y,0) = n\,u(y/n,\tau_n) \le Cy + 1$, it is easy to see that there exists a time $\overline{\tau}_1$ (bounded independently of n), such that $\omega(0,\overline{\tau}_1) = 0$.

Observe that $0 = \omega(0,\overline{\tau}_1) = n\,r(0,\overline{\tau}_1/n^2)$. Thus

$$\overline{\tau}_n \le \overline{\tau}_0 = \overline{\tau}_1/n^2 \le C/n^2.$$

Finally, from our bounds on $\overline{\tau}_n$ and $\widehat{\tau}_n$, using precisely that $q < 1$, for n large enough it holds that

$$\overline{\tau}_n \le \widehat{\tau}_n.$$

This fact means that at the time T_n at which u_n vanishes, v_n remains positive. Note that for times greater than T_n, v_n is a solution to the heat equation with homogeneous Neumann boundary conditions. The proof is now complete. $\square$

To finish this section we have to prove that we can pass to the limit as in (1.7) and that $(\overline{u},\overline{v})$ is indeed a solution to (1.8).

Proof.[End of the proof of Theorem 1.1] From the previous lemma we obtain that the sequence v_n are uniformly bounded away zero uniformly in n. On the other hand, by Lemma 2.1 the sequence v_n are uniformly bounded from above. Thus, we have that

$$C_1 < v_n(x,t) < C_2.$$

Also from Lemma 2.1, we obtain that $u_n(x,t) < C_3$. To obtain a lower bound, we note that, for n large, u_n is supersolution of

$$\begin{cases} w_t = w_{xx} & 0 < x < 1,\ t > 0, \\ w_x(0,t) = C_1^{-p}, & t > 0, \\ w_x(1,t) = 0, & t > 0, \\ w(x,0) = u_0(x). \end{cases}$$

Therefore, (u_n, v_n) are uniformly bounded in compact sets. So, taking a subsequence if necessary, we have that there exists the limit $(u_n, v_n) \to (\overline{u},\overline{v})$.

Our next aim is to identify the PDE system verified by this limit after T.

Now we just observe that (u_n, v_n) is a solution to

$$\begin{cases} (u_n)_t = (u_n)_{xx},, & (v_n)_t = (v_n)_{xx}, & 0 < x < 1, \quad t > T_n, \\ (u_n)_x(0,t) = (v_n)^{-p}(0,t), & (v_n)_x(0,t) = 0, & t > T_n, \\ (u_n)_x(1,t) = 0, & (v_n)_x(1,t) = 0, & t > T_n, \end{cases} \tag{2.4}$$

where T_n is the first time at which $u_n(x, T_n) = 0$. From the estimates obtained in the previous lemma, we have that $\tau_n < T_n < \tau_n + C/n^2$. Thus, it holds that

$$\lim_{n\to\infty} T_n = T.$$

First, we want to pass to the limit in the v-variable. To this end we write down the integral version of the problem for v_n,

$$-\int_{T_n}^t \int_0^1 v_n \varphi_t + \int_0^1 v_n(t)\varphi(t) - \int_0^1 v_n(T_n)\varphi(T_n) = \int_{T_n}^t \int_0^1 v_n \varphi_{xx} - \int_{T_n}^t v_n \varphi_x \Big|_0^1. \tag{2.5}$$

It is easy to see that we can pass to the limit in all the terms of the above identity, the only tricky point is to show that

$$v_n(x, T_n) \to v(x, T). \tag{2.6}$$

In order to prove this we just consider the Green function of the Neumann problem, G. Therefore, for times $\tau_n \le t \le T_n$ we can write

$$v_n(x,t) = \int_0^1 G(x-y,t) v(y,\tau_n)\, dy + \int_{\tau_n}^t G(x, t-s)(v_n)_x(0,s)\, ds. \tag{2.7}$$

We have that the first integral goes uniformly to $v(x, T)$ while the second one is bounded by Cn^{q-1} (we are using here that $T_n - \tau_n \le Cn^{-2}$ and that $(v_n)_x = f_n(u_n) \le Cn^{q+1}$). Since $q < 1$ this last term goes to zero. This completes the proof for the v-component.

For the u-component passing to the limit is even easier since v_n are uniformly bounded below away from zero. Thus we can pass to the limit in the weak form of the problem (analogous to (2.5)) beginning at $t = 0$. $\square$

3 Simultaneous quenching

In this section we collect some results concerning the simultaneous case. Hence, let us suppose that u and v quench at the same time T.

Now we state a lemma that shows that, under certain conditions on the exponents and the initial conditions, we can compare u_n and v_n.

Lemma 3.1 *i) Let $q \le p$ and $u_0(x) < v_0(x)$ with $\|v_0\|_\infty \le 1$. Then $u_n(x,t) \le v_n(x,t)$*

ii) Let $q \ge p$ and $u_0(x) > v_0(x)$ with $u_0(0) \le 1$. Then $u_n(x,t) \ge v_n(x,t)$

Proof. To prove *i)*, let us denote $\psi = u_n - v_n$. So $\psi(x, 0) < 0$. Assume that there exists a first time t_0 and some point $x_0 \in [0, 1]$ such that $\psi(x_0, t_0) = 0$. By the Strong Maximum Principle

$x_0 \in \{0,1\}$. This cannot happen at $x_0 = 1$, since $\psi_x(1,t) = 0$ and it contradicts Hopf's Lemma. Thus $x_0 = 0$ and from Hopf's Lemma it follows that $\psi_x(0,t_0) < 0$.

In order to get a contradiction we consider 3 different cases:

1) If no truncation takes place, then $\psi_x(0,t_0) = v_n^{-p}(0,t_0)(1 - v_n^{p-q}(0,t_0)) \geq 0$; in this case $v_n(0,t_0) \leq 1$ (since the initial data are both bounded by one).

2) if only one truncation takes place, then $u_n(0,t_0) \leq 1/n < v_n(0,t_0)$;

3) if both truncations take place, then $\psi_x(0,t_0) = n^{p+1} v_n(0,t_0)(1 - n^{q-p}) \geq 0$.

Interchanging the roles of p and q, u_n and v_n, we obtain the second statement. □

We define for n fixed the sets

$$\begin{aligned}
A_n &= \{(u_0,v_0) \mid \exists t_n \text{ such that} : u_n(0,t_n),\ v_n(0,t_n) \leq 0\}, \\
B_n &= \{(u_0,v_0) \mid \exists t_n \text{ such that} : u_n(0,t_n) \leq 0, \text{ and } v_n(0,t) > 0,\ \forall t\}, \\
\widehat{B}_n &= \{(u_0,v_0) \mid \exists t_n \text{ such that} : v_n(0,t_n) \leq 0, \text{ and } u_n(0,t) > 0,\ \forall t\}, \\
C_n &= \{(u_0,v_0) \mid u_n(0,t) > 0,\ v_n(0,t) > 0, \forall t\}.
\end{aligned} \tag{3.1}$$

Notice that from Lemma 3.1 we deduce that $B_n \cup \widehat{B}_n$ are nonempty. In the next lemma we study the sets A_n and C_n.

Lemma 3.2 *For a fixed n, let us consider the sets defined above. It holds that*
i) A_n is empty.
ii) The conditions ensuring that the initial data belong to C_n are, in general, quite difficult to be fulfilled. Hence, C_n is a nongeneric set.

Proof. We start by proving i). Let us argue by contradiction and let $(u_0,v_0) \in A_n$. Then, there must exist a first time t^* such that $u_n(0,t^*) = 0$, and $v_n(0,t^*) \leq 0$, (the opposite possibility $v_n(0,t^*) = 0$, and $u_n(0,t^*) \leq 0$ can be regarded analogously). However, by Hopf's Lemma $(u_n)_x(0,t^*) > 0$, which is a contradiction.

To prove ii) and complete the proof let us take first $p = q$ and suppose that $(u_0,v_0) \in C_n$. We define $z = u_n + v_n$ and $\omega = u_n - v_n$. Denote by t_n the first time at which $u_n(0,t_n) = 1/n$ and $v_n(0,t_n) \leq 1/n$ (the reverse situation is analogous). Performing the change of variables $y = n^{p+1}x$, $\tau = (n^{p+1})^2 t$, we have that z and ω verify the following linear problems

$$\begin{cases}
z_\tau = z_{yy}, & \omega_\tau = \omega_{yy}, & 0 < y < n^{p+1},\ \tau_n < \tau < \infty, \\
z_y(0,\tau) = z(0,\tau), & \omega_y(0,\tau) = -\omega(0,\tau), & \tau_n \leq \tau < \infty, \\
z_y(n^{p+1},\tau) = 0, & \omega_y(n^{p+1},\tau) = 0, & \tau_n \leq \tau < \infty,
\end{cases} \tag{3.2}$$

with $\tau_n = (n^{p+1})^2 t_n$. We can expand the solution z as

$$z(y,\tau) = \sum_{k=1}^{\infty} c_k e^{-\lambda_k \tau} \varphi_k(y)\,,$$

with λ_k solving the equation $\tan(\sqrt{\lambda_k} n^{p+1}) = 1/\sqrt{\lambda_k}$, with $k \geq 1$. Consequently, all the eigenvalues are positive, $\lambda_k > 0$, and then $z(y,\tau) \to 0$ as $\tau \to \infty$. Since both $u_n(0,t)$ and $v_n(0,t)$ are positive, this convergence implies that

$$u_n(0,t) \to 0 \quad \text{and} \quad v_n(0,t) \to 0 \quad \text{as } t \to \infty. \tag{3.3}$$

On the other hand, expanding the solution w as

$$\omega(y,\tau)=\sum_k d_k e^{-\alpha_k\tau}\phi_k(y), \tag{3.4}$$

we obtain that the first eigenvalue is negative and the rest of them are positive. More precisely, the eigenvalues α_k are given by

$$\frac{\sqrt{|\alpha_1|}+1}{\sqrt{|\alpha_1|}-1}e^{2n^{p+1}\sqrt{|\alpha_1|}}=1, \qquad \tan(\sqrt{\alpha_k}n^{p+1})=-1/\sqrt{\alpha_k}.$$

This fact implies that $u_n(x,t)-v_n(x,t)\to 0$ as $t\to\infty$, just in the case that the coefficient d_1 corresponding to the eigenvalue α_1 is equal to zero. This coefficient is determined for each n by the initial datum $\omega(0,\tau_n)=u_n(0,\tau_n)-v_n(0,\tau_n)$.

As example of such a solution whose both components remain positive, we take the corresponding solution to the initial condition $u_0=v_0$, (recall that we are considering $p=q$). This implies $u_n=v_n$ for all $t>0$ and all n. Thus, they are positive, [FG].

However, generally it holds that $d_1\neq 0$ and then $u_n(x,t)-v_n(x,t)$ is unbounded. But this is a contradiction with (3.3). Therefore, u_n and v_n cannot be both positive for all times and we conclude that $(u_0,v_0)\in B_n\cup\widehat{B}_n$.

Notice that $(u_0,v_0)\in C_n$ if $d_1=0$, which implies that C_n is a closed set.

For the general case $p\neq q$ it is always possible to find positive constants a,b,c,d such that, the new functions $z=au_n+bv_n$ and $\omega=cu_n-dv_n$, satisfy the boundary conditions at $x=0$

$$z_x(0,t)=k_1z(0,t), \qquad \omega_x(0,t)=-k_2\omega(0,t),$$

for some k_1,k_2 determined by the relations

$$\begin{aligned} bn^{q+1}&=k_1a, \qquad & cn^{p+1}&=k_2d,\\ an^{p+1}&=k_1b, & dn^{q+1}&=k_2c. \end{aligned}$$

As before, changing variables, we get that z and ω solve problems similar to (3.2), and the previous conclusion follows also for $p\neq q$. □

Let us conclude by summing up the results obtained through this section up to this point. As before, denote by t_n the first time at which both truncations, f_n and g_n, take part. By Lemma 3.2, if (u_0,v_0) is such that the initial datum $\omega(0,\tau_n)$ given in (3.2) with $\tau_n=n^{p+1}t_n$, makes the coefficient in (3.4), $d_1=d_1(n)=0$, for every n, then $u_n(0,t),v_n(0,t)$, remain positive for every n and t.

Now, we consider the case in which one of the components changes its sign. We study the case $(u_0,v_0)\in B_n$, the other case is analogous. Let T_n be the first time for which $u_n(0,T_n)=0$.

Lemma 3.3 *If $(u_0,v_0)\in B_n$ for every n, and $p<1$ then both components u_n and v_n are bounded below.*

Proof. We begin by observing that $0\le v_n(x,t)\le C$. Now, consider the explicit solution of the heat equation

$$w(x,t)=a\left(1-\cos(\pi x)e^{-\pi^2(t-T_n)}\right).$$

Taking a small enough we have that $v_n(x,T_n) > w(x,T_n)$ for every n.

Therefore, by comparison with the problem with homogeneous Neumann boundary conditions, we have that $w(x,t) < v_n(x,t)$ for $T_n < t < t_*$ fixed.

On the other hand, integrating the equation verified by u_n we obtain

$$\begin{aligned} \int_0^1 u_n(x,t)\,dx - \int_0^1 u_n(x,T_n)\,dx &= -\int_{T_n}^t f_n(v(0,s))\,ds \geq -\int_{T_n}^t v_n^{-p}(0,s)\,ds \\ &\geq -\int_{T_n}^t w^{-p}(0,s)\,ds \geq -C. \end{aligned}$$

We have used the fact that $p < 1$.

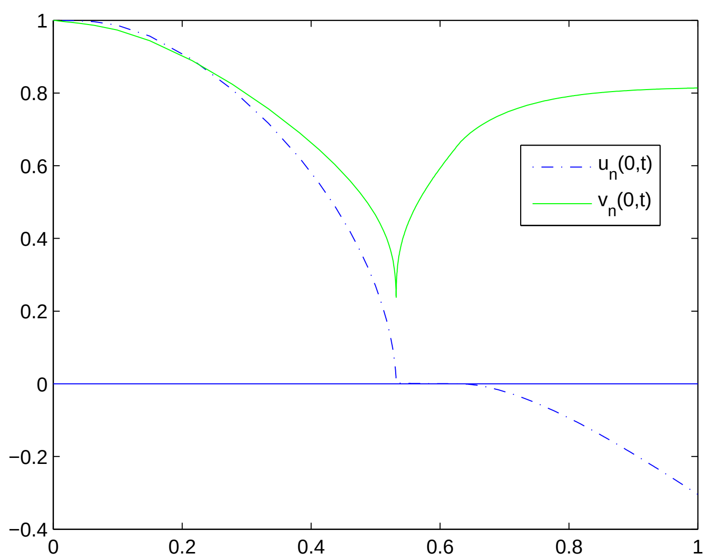
Consequently, using the variation of constants formula associated with the heat semigroup we deduce that $u_n(0,t) \geq -C$ with C independent of n. $\Box$

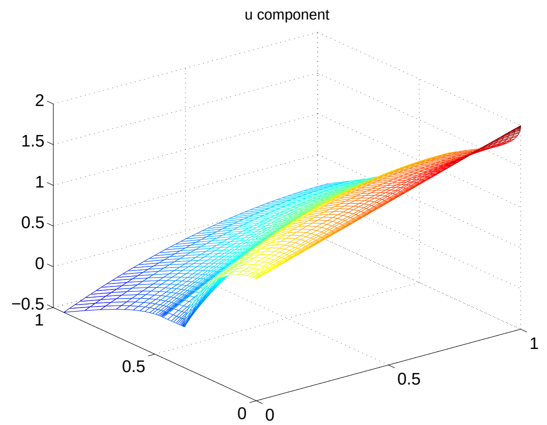
Remark 3.1 *We conjecture that for $p > 1$, u_n is not bounded below in general. Numerical experiments support this conjecture, as shown in the next section.*

4 Numerical experiments

In this section we perform some numerical experiments that illustrate our results. We use finite elements with mass lumping (that, as is well known, coincides with a classical finite differences method in one space dimension). Taking a uniform space discretization of the interval $[0,1]$ of size h we get an ODE system that can be integrated with some adaptive solver. For similar analysis for blow-up problem we refer to [BQR], [FGR] and the survey [BB].

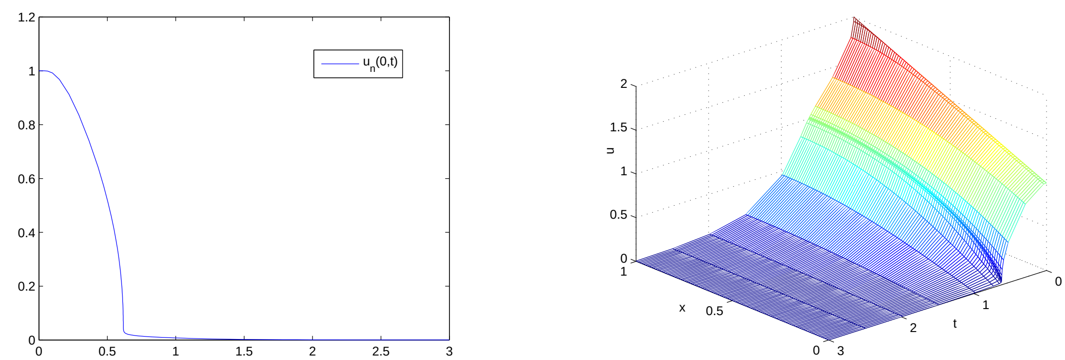
First, we take $p = 1$, $q = 1/3$ and initial conditions $u_0 = 1 + x$ and $v_0 = 1 + x - x^2$. We obtain the following pictures for the approximate problem with $n = 100$.





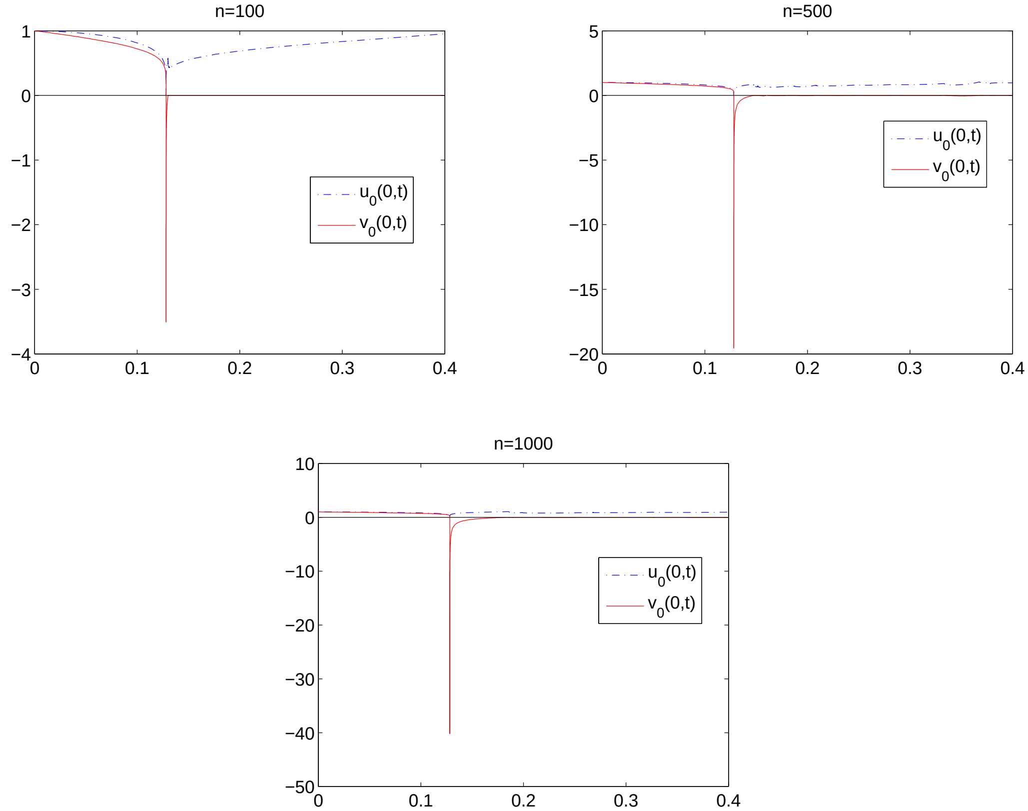
We can observe that the v component converges to the mean value of $v(x,T)$ as $t \to \infty$ (it is a solution to the heat equation with homogeneous Neumann boundary data), while the u component goes to $-\infty$ (it is a solution to the heat equation with boundary flux $(v_n)^{-p}(0,t)$). Also it can be observed that the time derivative of both components at $x = 0$ becomes very large at times $t \approx T$.

Next, we take $p = q = 1/3$ and $u_0 = v_0 = 1 + x$. In this case we have $u(x,t) = v(x,t)$ and therefore simultaneous quenching with continuation given by a solution to the Dirichlet problem. We remark again that this case is not generic.



These pictures illustrate the Dirichlet condition taken by the limit after T.

Finally, we take $p = 2$, $q = 2$ and $u_0(x) = x + 1$ and $v_0(x) = x + 1 - x^2$. We obtain the following results for different values of n, which suggest that solutions are not uniformly bounded from below in this case. Indeed it can be observed that $min u_n \to -\infty$ as n increases.



Acknowledgements

RF, AdP and MPLl are partially supported by DGICYT grant MTM2005-08760-C02-01 and 02 (Spain) and JDR by ANPCyT PICT 5009, UBA X066 and CONICET (Argentina).

References

[BC] Baras, P. and Cohen, L.. *Complete blow-up after T_{max} for the solution of a semilinear heat equation.* J. Funct. Anal. **71** (1987), 142–174.

[BB] Bandle, C. and Brunner, H. *Blowup in diffusion equations: a survey.* J. Comput. Appl. Math. **97** (1998), 3–22.

[BQR] Brandle, C., Quiros, F. and Rossi, J. D. *An adaptive numerical method to handle blow-up in a parabolic system.* Numerische Mathematik. Vol. 102(1) (2005), 39–59.

[C] Chan, C. Y. *Recent advances in quenching phenomena.* Proc. Dynam. Systems. Appl. **2** (1996), 107–113.

[CK] Chan, C. Y. and Kwong, M. K. *Quenching phenomena for singular nonlinear parabolic equations.* Nonlinear Analysis, TMA **12** (1988), 1377–1383.

[DX] Deng, K. and Xu, M. *Quenching for a nonlinear diffusion equation with a singular boundary condition.* Z. Angew. Math. Phys. **50** (1999), 574–584.

[FGR] Ferreira, R., Groisman, P. and Rossi, J. D. *Numerical blow-up for a nonlinear problem with a nonlinear boundary condition.* Math. Mod. Meth. Appl. Sci. **12(4)** (2002), 461–484.

[FPQR] Ferreira, R., de Pablo, A., Quiros, F. and Rossi, J. D. *Non-simultaneous quenching in a system of heat equations coupled at the boundary.* Z. Angew. Math. Phys. **57** (2006), no. 4, 586–594.

[FG] Fila, M., and Guo, J. S.*Complete blow-up and incomplete quenching for the heat equation with a nonlinear boundary condition. Nonlinear Anal.* **48** (2002), no. 7, 995–1002.

[FL] Fila, M. and Levine, H. A. *Quenching on the boundary.* Nonlinear Anal. **21** (1993), 795–802.

[GV1] Galaktionov, V. A. and Vázquez, J. L. *Necessary and sufficient conditions for complete blow-up and extinction for one-dimensional quasilinear heat equations.* Arch. Rational Mech. Anal. **129** (1995), 225–244.

[GV2] Galaktionov, V. A. and Vázquez, J. L. *Continuation of blowup solutions of nonlinear heat equations in several space dimensions.* Comm. Pure Appl. Math. **50** (1997), 1–67.

[K] Kawarada, H. *On solutions of initial-boundary problem for* $u_t = u_{xx} + 1/(1-u)$. Publ. Res. Inst. Math. Sci. **10** (1974/75), no. 3, 729–736.

[KN] Ke, L. and Ning, S. *Quenching for degenerate parabolic equations.* Nonlinear Anal. **34** (1998), 1123–1135.

[L] Levine, H. A. *Quenching and beyond: a survey of recent results.* GAKUTO Internat. Ser. Math. Sci. Appl. **2** (1993) , Nonlinear mathematical problems in industry II , Gakkōtosho, Tokyo, 501–512.

[L2] Levine, H. A. *The phenomenon of quenching: a survey.* In "Trends in the Theory and Practice of Nonlinear Analysis", (V. Lakshmikantham, ed.), Elsevier Science Publ., North Holland, 1985, pp. 275–286.

[L3] Levine, H. A. *The quenching of solutions of nonlinear parabolic and hyperbolic equations with nonlinear boundary conditions.* SIAM J. Math. Anal. **14** (1983), 1139–1153.

[PQR] de Pablo, A., Quiros, F. and Rossi, J. D. *Nonsimultaneous quenching.* Appl. Math. Lett. **15** (2002), no. 3, 265–269.

[QR1] Quiros, F. and Rossi, J. D. *Non-simultaneous blow-up in a semilinear parabolic system.* Z. Angew. Math. Phys. **52** (2001), no. 2, 342–346.

[QR2] Quiros, F. and Rossi, J. D. *Non-simultaneous blow-up in a nonlinear parabolic system.* Adv. Nonlinear Stud. **3** (2003), no. 3, 397–418.

[QRV] Quiros, F., Rossi, J. D. and Vazquez, J. L. *Thermal avalanche for blow-up solutions of semilinear heat equations.* Comm. Pure Appl. Math. **LVII**, (2004), 59–98.

[ST] Souplet, P and Tayachi, S. *Optimal condition for non-simultaneous blow-up in a reaction-diffusion system.* J. Math. Soc. Japan **56** (2004), no. 2, 571–584.