LIMITS AS $p(x) \to \infty$ OF $p(x)$-HARMONIC FUNCTIONS WITH NON-HOMOGENEOUS NEUMANN BOUNDARY CONDITIONS

M. PEREZ-LLANOS AND J. D. ROSSI

Abstract. In this paper we study the limit as $p(x) \to \infty$ of solutions to $-\Delta_{p(x)} u = 0$ in a domain Ω, with non-homogeneous Neumann boundary conditions, $|\nabla u|^{p(x)} \frac{\partial u}{\partial \eta} = g(x)$. Our approach consists on considering sequences of variable exponents converging uniformly to $+\infty$ and then determining the equation satisfied by a limit of the corresponding solutions.

To Jean Pierre Gossez, with our best wishes in his 65th birthday

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. Our goal is to study the limit, as the exponent $p(x) \to \infty$, of solutions to the following problem

$$\begin{cases} -\Delta_{p(x)} u(x) = 0, & x \in \Omega \subset \mathbb{R}^N, \\ |\nabla u|^{p(x)} \frac{\partial u}{\partial \eta}(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where $\Delta_{p(x)} u(x) := \operatorname{div}\left(|\nabla u(x)|^{p(x)-2} \nabla u(x)\right)$ is the $p(x)$-Laplacian operator with a variable exponent $p(x)$ and the boundary datum g is assumed to be continuous and verifies the compatibility assumption

$$\int_{\partial\Omega} g = 0, \tag{1.2}$$

otherwise there is no solution to (1.1). To obtain uniqueness of the solution we impose the additional condition

$$\int_{\Omega} u = 0. \tag{1.3}$$

When p is constant in Ω, the limit of $p-$harmonic functions as $p \to \infty$ has been extensively studied in the literature (see [5] and the survey [3]) and leads naturally to the infinity Laplacian given by $\Delta_\infty u = \left(D^2 u \, \nabla u\right) \cdot \nabla u$. Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) are related to the optimal Lipschitz extension problem (see [2] and the survey paper [3]) and

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find applications in optimal transportation, image processing and tug-of-war games (see, e.g., [10], [13], [6], [21], [22] and the references therein). Also limits of the eigenvalue problem related to the p-laplacian has been exhaustively studied, see [7], [15], [16], [23].

On the other hand, problems related to PDEs involving variable exponents are used in elasticity and electrorheological fluids. Meanwhile, the underlying functional analytical tools have been extensively developed (cf. [17] and [9]) and new applications to image processing have kept the subject as the focus of an intensive research activity. Although a natural extension of the theory, the problem addressed here is a continuation of recent papers [19] (where the case of a variable exponent that equals infinity in a subdomain of Ω is considered) and [20] (where the Dirichlet case was treated). Closely related to this work is [18], where the authors prove existence and uniqueness (via a comparison principle), as well as the validity of a Harnack inequality, for solutions of our limit PDE in Ω. Concerning the limit as $p \to \infty$ for the Neumann problem we mention [13] where the limit as $p \to \infty$ without dependence on $x \in \Omega$ is studied.

The approach in this paper is based on considering sequences $p_n(x)$ of variable exponents converging uniformly to $+\infty$, analyzing how the corresponding solutions of the problem converge and identifying the equation satisfied by the limit. Before introducing our main result, let us state the assumptions on the sequence $p_n(x)$ that will be assumed from now on: $p_n(x)$ is a sequence of C^1 functions in Ω such that

$$p_n(x) \to +\infty, \quad \text{uniformly in } \Omega, \tag{1.4}$$

hence we may assume that,

$$p_n(x) \geq \alpha > N, \quad \text{for all } x \in \Omega, \tag{1.5}$$

in addition we impose

$$\nabla \ln p_n(x) \longrightarrow \xi(x) \in C(\Omega), \quad \text{uniformly in } \Omega, \tag{1.6}$$

$$\frac{p_n}{n}(x) \to q(x) > 0,\ q \in C(\Omega), \quad \text{uniformly in } \Omega, \tag{1.7}$$

and

$$\limsup_{n\to\infty} \frac{p_n^+}{p_n^-} \leq k; \tag{1.8}$$

where

$$p_n^- = \min_{x\in\overline{\Omega}} p_n(x), \qquad p_n^+ = \max_{x\in\overline{\Omega}} p_n(x). \tag{1.9}$$

The following is the main result of this paper. We prove, under the above assumptions, that the limit (along subsequences) of solutions of (1.1) with $p(x) = p_n(x)$ exists and is a viscosity solution of a limit PDE with the $\infty-$Laplacian and an extra term in which the vector field $\xi(x) = \lim_n \nabla \ln p_n(x)$ appears, together with a boundary condition involving the normal derivative and the function $q(x) = \lim_n \frac{p_n}{n}(x)$, in which only the sign of the datum g is relevant.

Theorem 1.1. *Let u_n be the solution of* (1.1) *normalized according to* (1.3) *with $p(x) = p_n(x)$ satisfying* (1.4)–(1.8)*. Then, along a subsequence,*

$$u_n \longrightarrow u_\infty, \quad \text{uniformly in } \overline{\Omega}, \tag{1.10}$$

where u_∞ is a solution of the problem

$$\begin{cases} -\Delta_\infty u - |\nabla u|^2 \ln |\nabla u| \, \langle \xi, \nabla u \rangle = 0, & \text{in } \Omega, \\ B(x, u, \nabla u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.11}$$

in the viscosity sense. Here

$$B(x, u, \nabla u) \equiv \begin{cases} \min\{|\nabla u|^q - 1 \ , \ \frac{\partial u}{\partial \eta}\} & \text{if } g > 0, \\ \max\{1 - |\nabla u|^q \ , \ \frac{\partial u}{\partial \eta}\} & \text{if } g < 0, \\ H(|\nabla u|^q) \frac{\partial u}{\partial \eta} & \text{if } g = 0, \end{cases}$$

with $H(a)$ given by

$$H(a) = \begin{cases} 1 & \text{if } a > 1, \\ 0 & \text{if } 0 \le a \le 1. \end{cases}$$

Moreover, the limit u_∞ belong to $W^{1,\infty}(\Omega)$ and verifies

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le 1, \tag{1.12}$$

and is a maximizer of the following variational problem

$$\max_K \int_{\partial\Omega} g v \, dS, \qquad K = \Big\{ v \in W^{1,\infty}(\Omega), \ \int_\Omega v = 0, \ |\nabla v| \le 1 \Big\}. \tag{1.13}$$

Remark 1.2. *Notice that we are taking $G(0) = 0$ for $G(s) = s^2 \ln(s)$, hence the term $|\nabla u|^2 \ln |\nabla u|$ in* (1.11) *makes sense when evaluated at a test function with vanishing gradient.*

Remark 1.3. *Note that hypothesis* (1.7) *can be replaced by $p_n(x)/a_n \to q(x)$ for a given sequence $a_n \to \infty$ as $n \to \infty$. The corresponding statements can be rewritten in terms of a_n (instead of n) but we prefer to simplify the notation.*

Remark 1.4. *In the limit problem we note the dependence in x of the sequence p_n. In fact, two limits play a role here, $\nabla \ln p_n(x) \to \xi(x)$ and $\frac{p_n}{n}(x) \to q(x)$.*

Remark 1.5. *The maximization problem* (1.13) *is also obtained by applying the Kantorovich optimality principle to a mass transfer problem for the measures $\mu^+ = g^+ \mathcal{H}^{N-1} \llcorner \partial\Omega$ and $\mu^- = g^- \mathcal{H}^{N-1} \llcorner \partial\Omega$ that are concentrated on $\partial\Omega$. The mass transfer compatibility condition $\mu^+(\partial\Omega) = \mu^-(\partial\Omega)$ holds since g fulfils the compatibility condition* (1.2)*. See* [1] *and* [10].

Let us end the introduction presenting some examples of sequences $p_n(x)$ that fulfill the required conditions.

(1) $p_n(x) = n$; we have $\xi = 0$, $q = 1$ and $k = 1$.
(2) $p_n(x) = p(x) + n$; we get $\xi = 0$, $q = 1$ and $k = 1$.
(3) $p_n(x) = np(x)$; this is a model case. We obtain a nontrivial vector field $\xi(x) = \nabla(\ln(p(x)))$, a nontrivial scalar $q(x) = p(x)$ and $k = \frac{\max_{x\in\overline{\Omega}} p}{\min_{x\in\overline{\Omega}} p}$.
(4) $p_n(x) = n^a p(x/n)$ [scaling in x]; in this case, we have

$$\nabla(\ln p_n(x)) = \frac{\nabla p}{p}(x/n)\frac{1}{n} \to 0$$

and so $\xi = 0$. Moreover, we have also $k = 1$. However,

$$\frac{p_n(x)}{n} = n^{a-1}p(x/n)$$

that does not converge to any nontrivial $q(x)$, unless $a = 1$ in which case $q(x) \equiv p(0)$. The conclusion also hold for $p_n(x) = n^a + p(x/n)$, we have $\xi = 0$ and $k = 1$.
(5) $p_n(x) = n^a p(nx)$; we get

$$\nabla(\ln p_n(x)) = \frac{n\nabla p}{p}(nx),$$

which does not have a limit as $n \to \infty$. The same happens with $p_n(x) = n + p(nx)$, for which

$$\nabla(\ln p_n(x)) = \frac{n\nabla p(nx)}{n + p(nx)},$$

that does not have a uniform limit (although it is bounded).
(6) We can modify the previous example to get a nontrivial limit. Assume that $r = r(\theta)$ is a function of the angular variable and that $0 \notin \Omega$; then consider $p_n(x) = n + r(nx)$ to obtain

$$\nabla(\ln p_n(x)) = \frac{n\nabla r(nx)}{n + r(nx)} \to \nabla r(\theta).$$

Concerning q we obtain

$$\frac{p_n(x)}{n} = 1 + \frac{r(nx)}{n} \to 1.$$

In this case we get $k = 1$.
(7) Finally, we can combine examples (3) and (6). Let $p_n(x) = np(x) + r(nx)$, with Ω as in (6). We get

$$\nabla(\ln p_n(x)) = \frac{n\nabla p(x) + n\nabla r(nx)}{np(x) + r(nx)} \to \frac{\nabla p(x) + \nabla r(\theta)}{p(x)},$$

and

$$\frac{p_n(x)}{n} = p(x) + \frac{r(nx)}{n} \to p(x).$$

In this case $k = \frac{\max_{x\in\overline{\Omega}} p}{\min_{x\in\overline{\Omega}} p}$.

The rest of the paper is organized as follows: in Section 2 we collect some properties of the approximate problems and prove that there is a uniform limit (along subsequences) that is a maximizer in (1.13) and in Section 3 we deal with the limit PDE.

2. Analysis of problem (1.1)

First of all, let us give some brief introduction to variable exponent Sobolev and Lebesgue spaces, and some of their main properties, that we will use in the sequel. See [9], [11], [12], [17] and the survey [14] for more details. The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ is defined as follows

$$L^{p(x)}(\Omega) = \left\{ u \ \text{ such that } \int_\Omega |u(x)|^{p(x)} < +\infty \right\},$$

and is endowed with the norm

$$|u|_{p(x)} = \inf \left\{ \tau > 0 \ \text{ such that } \int_\Omega \left| \frac{u(x)}{\tau} \right|^{p(x)} \leq 1 \right\}.$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is given by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) \ \text{ such that } |\nabla u| \in L^{p(x)}(\Omega) \right\},$$

with the norm

$$\|u\| = \inf \left\{ \tau > 0 \ \text{ such that } \int_\Omega \left| \frac{\nabla u(x)}{\tau} \right|^{p(x)} + \left| \frac{u(x)}{\tau} \right|^{p(x)} \leq 1 \right\}.$$

The following result holds.

Proposition 2.1.

i) *The spaces* $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)} \right)$ *and* $\left(W^{1,p(x)}(\Omega), \ \|\cdot\| \right)$ *are separable, reflexive and uniformly convex Banach spaces.*

ii) *Hölder inequality holds, namely*

$$\int_\Omega |uv| \leq 2|u|_{p(x)}|v|_{q(x)}, \quad \forall u \in L^{p(x)}(\Omega), \, \forall v \in L^{q(x)}(\Omega),$$

where $\frac{1}{p(x)} + \frac{1}{q(x)} = 1$.

iii) *When* $p(x) \geq \alpha > N$ *the embedding from* $W^{1,p(x)}(\Omega)$ *to* $C^\beta(\overline{\Omega})$ *is compact and continuous. In particular,* $W^{1,p(x)}(\Omega) \hookrightarrow C(\overline{\Omega})$.

iv) *There exists a constant* $C > 0$ *such that*

$$|u|_{p(x)} \leq C|\nabla u|_{p(x)},$$

for every $u \in W^{1,p(x)}(\Omega)$ *such that* $\int_\Omega u = 0$. *Therefore,* $|\nabla u|_{p(x)}$ *and* $\|u\|$ *are equivalent norms on* $W^{1,p(x)}(\Omega) \cap \{\int_\Omega u = 0\}$.

Let us introduce now the definition of a weak solution to (1.1). From now on we assume that we deal with a sequence $p_n(x)$ verifying (1.4)–(1.8), but we drop the subscript n when we can simplify the notation.

Definition 2.2. *We say that $u \in W^{1,p(x)}(\Omega)$ is a weak solution to problem* (1.1) *if*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \nabla v = \int_{\partial\Omega} gv, \qquad \forall v \in W^{1,p(x)}(\Omega).$$

We have the following existence result.

Lemma 2.3. *There exists a unique weak solution u to* (1.1)*, which is the unique minimizer of the functional*

$$L(u) = \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)} - \int_{\partial\Omega} gu \tag{2.1}$$

in the set

$$S = \left\{ u \in W^{1,p(\cdot)}(\Omega) \ : \ \int_\Omega u = 0 \right\}. \tag{2.2}$$

Proof. Functions in the variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ are necessarily continuous thanks to the assumption $p_n(x) \geq \alpha > N$. Indeed, the continuous embedding in

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow W^{1,\alpha}(\Omega) \subset C\left(\overline{\Omega}\right) \tag{2.3}$$

follows from [17, Theorem 2.8].

It is standard to show that the functional attains a minimum in S since for every r such that $1 \leq r < \alpha(N-1)/(N-\alpha)$, the embedding $S \hookrightarrow L^r(\partial\Omega)$ is compact.

It is also standard to show that the minimizer of L in S is the unique weak solution of (1.1). $\square$

Let us now recall the definition of viscosity solution (cf. [8]) for a problem like (1.1) or (1.11). Assume we are given continuous functions

$$F : \overline{\Omega} \times \mathbb{R}^N \times \mathbb{S}^{N\times N} \to \mathbb{R},$$

and

$$B : \partial\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}.$$

Following [4] let us recall the definition of viscosity solution taking into account general boundary conditions.

Definition 2.4. *Consider the boundary value problem*

$$\begin{cases} F(x, \nabla u, D^2 u) = 0 & \text{in } \Omega, \\ B(x, u, \nabla u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.4}$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\max\{B(x_0, \phi(x_0), \nabla\phi(x_0)),\ F(x_0, \nabla\phi(x_0), D^2\phi(x_0))\} \geq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

(2) *An upper semi-continuous function u is a viscosity subsolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \psi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\min\{B(x_0, \psi(x_0), \nabla\psi(x_0)),\ F(x_0, \nabla\psi(x_0), D^2\psi(x_0))\} \leq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, \nabla\psi(x_0), D^2\psi(x_0)) \leq 0.$$

(3) *Finally, u is a viscosity solution if it is a super and a subsolution.*

In the sequel, we will use the notation as in the definition: ϕ will always stand for a test function touching the graph of u from below and ψ for a test function touching the graph of u from above.

Proposition 2.5. *Let u be a continuous weak solution of* (1.1). *Then u is a viscosity solution of* (1.1) *in the sense of Definition 2.4.*

Proof. Let $x_0 \in \Omega$ and a let ϕ be a test function such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0. We want to show that

$$\begin{aligned} -\Delta_{p(x_0)}\phi(x_0) \quad = \quad & -|\nabla\phi(x_0)|^{p(x_0)-2}\Delta\phi(x_0) \\ & -(p(x_0) - 2)|\nabla\phi(x_0)|^{p(x_0)-4}\Delta_\infty\phi(x_0) \\ & -|\nabla\phi(x_0)|^{p(x_0)-2}\ln(|\nabla\phi|)(x_0)\,\langle\nabla\phi(x_0), \nabla p(x_0)\rangle \\ \geq \quad & 0. \end{aligned}$$

Assume, *ad contrarium*, that this is not the case; then there exists a radius $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$\begin{aligned} -\Delta_{p(x)}\phi(x) \quad = \quad & -|\nabla\phi(x)|^{p(x)-2}\Delta\phi(x) \\ & -(p(x) - 2)|\nabla\phi(x)|^{p(x)-4}\Delta_\infty\phi(x) \\ & -|\nabla\phi(x)|^{p(x)-2}\ln(|\nabla\phi|)(x)\langle\nabla\phi(x), \nabla p(x)\rangle \\ < \quad & 0, \end{aligned}$$

for every $x \in B(x_0, r)$. Set

$$m = \inf_{|x-x_0|=r} (u - \phi)(x)$$

and let $\Phi(x) = \phi(x) + m/2$. This function Φ verifies $\Phi(x_0) > u(x_0)$ and

$$-\Delta_{p(x)}\Phi = -\mathrm{div}(|\nabla\Phi|^{p(x)-2}\nabla\Phi) < 0 \qquad \text{in } B(x_0, r). \tag{2.5}$$

Multiplying (2.5) by $(\Phi - u)^+$, which vanishes on the boundary of $B(x_0, r)$, we get

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u) < 0.$$

On the other hand, taking $(\Phi - u)^+$, extended by zero outside $B(x_0, r)$, as test function in the weak formulation of (1.1), we obtain

$$\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi-u) = 0.$$

Upon subtraction and using a well know inequality, we conclude

$$\begin{aligned} 0 &> \int_{B(x_0,r)\cap\{\Phi>u\}} \Big(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla(\Phi-u) \\ &\geq c\int_{B(x_0,r)\cap\{\Phi>u\}} |\nabla\Phi-\nabla u|^{p(x)}, \end{aligned}$$

a contradiction.

If $x_0 \in \partial\Omega$ we want to prove

$$\max\left\{|\nabla\phi(x_0)|^{p(x_0)-2}\frac{\partial\phi}{\partial\eta}(x_0) - g(x_0), -\Delta_{p(x_0)}\phi(x_0)\right\} \geq 0.$$

Assume that this is not the case. We proceed as before and we obtain

$$\int_{\{\Phi>u\}} |\nabla\Phi|^{p-2}\nabla\Phi\nabla(\Phi-u) < \int_{\partial\Omega\cap\{\Phi>u\}} g(\Phi-u),$$

and

$$\int_{\{\Phi>u\}} |\nabla u|^{p-2}\nabla u\nabla(\Phi-u) \geq \int_{\partial\Omega\cap\{\Phi>u\}} g(\Phi-u).$$

From where we can reach again again a contradiction.

This proves that u is a viscosity supersolution. The proof that u is a viscosity subsolution runs as above and we omit the details. □

Remark 2.6. *If B is monotone in the variable $\frac{\partial u}{\partial \nu}$ (this is indeed the case for solutions to* (1.1)*) Definition 2.4 takes a simpler form, see* [4]*. More precisely, if u is a supersolution and $\phi \in C^2(\overline{\Omega})$ is such that $u - \phi$ has a strict minimum at x_0 with $u(x_0) = \phi(x_0)$, then*

(1) if $x_0 \in \Omega$, then

$$\begin{aligned} -\Delta_{p(x)}\phi(x) = \;& -|\nabla\phi(x)|^{p(x)-2}\Delta\phi(x) \\ & -(p(x)-2)|\nabla\phi(x)|^{p(x)-4}\Delta_\infty\phi(x) \\ & -|\nabla\phi(x)|^{p(x)-2}\ln(|\nabla\phi|)(x)\langle\nabla\phi(x),\nabla p(x)\rangle \geq 0 \end{aligned}$$

and

(2) if $x_0 \in \partial\Omega$, then

$$|\nabla\phi(x_0)|^{p(x_0)-2}\frac{\partial\phi}{\partial\eta}(x_0) \;\geq g(x_0).$$

Theorem 2.7. *There exists a subsequence $\{u_{p_{n_i}}\}$ of solutions that converge to some nontrivial function u_∞ in $C^\beta(\Omega)$, for some $0 < \beta < 1$. Moreover, the limit u_∞ belongs to $W^{1,\infty}(\Omega)$, verifies*

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le 1, \tag{2.6}$$

and is a maximizer of the following problem

$$\max_K \int_{\partial\Omega} gv, \qquad K = \left\{ v \in W^{1,\infty}(\Omega),\ \int_\Omega v = 0,\ |\nabla v| \le 1 \right\}. \tag{2.7}$$

Proof. If we consider the trivial function in the variational problem verified by u_{p_n} we get

$$\int_\Omega \frac{1}{p_n(x)} |\nabla u_{p_n}|^{p_n(x)} - \int_{\partial\Omega} g u_{p_n} \le 0.$$

Then,

$$\begin{aligned} \int_\Omega \frac{1}{p_n(x)} |\nabla u_{p_n}|^{p_n(x)} &\le \int_{\partial\Omega} g u_{p_n} \\ &\le \|g\|_{L^1(\partial\Omega)} \|u_{p_n}\|_{L^\infty(\partial\Omega)} \\ &\le C(\Omega, g) \|\nabla u_{p_n}\|_{L^q(\Omega)}, \end{aligned}$$

where $p_n(x) \ge q > N$. Now we claim that

$$\|\nabla u_{p_n}\|_{L^q(\Omega)} \le C(\Omega, g) |\nabla u_{p_n}|_{p_n(x)}. \tag{2.8}$$

Indeed, if we apply Hölder inequality for variable exponent Sobolev spaces, see Proposition 2.1, we get

$$\|\nabla u_{p_n}\|^q_{L^q(\Omega)} \le 2 |1|_{a_n'(x)} ||\nabla u_{p_n}|^q|_{a_n(x)} \le 2 \max\{1, \mu(\Omega)\} |\nabla u_{p_n}|^q_{p_n(x)}, \tag{2.9}$$

where $q a_n(x) = p_n(x)$ and $\frac{1}{a_n(x)} + \frac{1}{a_n'(x)} = 1$. Hence, from the above estimate (2.8) straight follows. Summing up we have shown that

$$\int_\Omega \frac{1}{p_n(x)} |\nabla u_{p_n}|^{p_n(x)} \le C(\Omega, g) |\nabla u_{p_n}|_{p_n(x)}. \tag{2.10}$$

Next, we take τ_0 such that

$$\frac{1}{2} \le \int_\Omega \left| \frac{\nabla u_{p_n}}{\tau_0} \right|^{p_n(x)} \le 1. \tag{2.11}$$

Taking into account (2.10) and (2.11) we deduce that

$$\frac{\min\{\tau_0^{p_n^+}, \tau_0^{p_n^-}\}}{2p_n^+} \le \int_\Omega \frac{1}{p_n(x)} |\nabla u_{p_n}|^{p_n(x)} \le C(\Omega, g)\tau_0, \tag{2.12}$$

with p_n^+, p_n^- defined in (1.9). Now we claim that

$$|\nabla u_{p_n}|_{p_n(x)} \le C(n), \quad \text{with } C(n) \to 1, \quad \text{as } n \to \infty. \tag{2.13}$$

If $|\nabla u_{p_n}|_{p_n(x)} \leq 1$, then (2.13) is immediate. Then let us assume that $|\nabla u_{p_n}|_{p_n(x)} > 1$ and let $\tau_0 > 1$ such that (2.11) holds. Note that, from (1.8), we get

$$\limsup_{n\to\infty} \frac{\log(p_n^+)}{p_n^- - 1} = 0. \tag{2.14}$$

Therefore, by (2.12) and (2.14) we obtain that

$$\tau_0 \leq (C(f,\Omega,q)p_n^+)^{\frac{1}{p_n^- -1}} \to 1, \quad \text{as } n \to \infty,$$

and then (2.13) follows. By Proposition 2.1 it follows that u_{p_n} is uniformly bounded in $W^{1,p_n}(\Omega)$. Since $p_n \geq \alpha > N$ we have that $W^{1,p_n}(\Omega)$ embeds compactly into $C^\beta(\Omega)$, for some $0 < \beta < 1$. Then, from (2.13) we get for a subsequence $\{u_{p_{n_i}(x)}\}$ such that $u_{p_{n_i}(x)} \rightharpoonup u_\infty$, weakly in $W^{1,q}(\Omega)$ and $u_{p_{n_i}(x)} \to u_\infty$, strongly in $C^\beta(\Omega)$. Moreover, by the lower semicontinuity of the norm, we have that

$$|\nabla u_\infty|_{L^q(\Omega)} \leq \liminf_{n\to\infty} |\nabla u_{p_n}|_{L^q(\Omega)}.$$

Passing to the limit as $q \to \infty$ using (2.9) and (2.13) in the previous estimate we obtain (2.6).

It just remains to see that u_∞ maximizes (2.7), (thus u_∞ is nontrivial when $g \not\equiv 0$). Note that for n fixed we have that

$$\int_\Omega \frac{1}{p_n(x)} |\nabla u_{p_n}|^{p_n(x)} - \int_{\partial\Omega} g u_{p_n} \leq \int_\Omega \frac{1}{p_n(x)} - \int_{\partial\Omega} g v,$$

for any $v \in K$. Neglecting the first positive term on the left hand side and rearranging we obtain

$$\int_{\partial\Omega} g v \leq \int_{\partial\Omega} g u_{p_n} + \int_\Omega \frac{1}{p_n(x)}.$$

Now, passing to the limit as $n \to \infty$ in the previous expression we get

$$\int_{\partial\Omega} g v \leq \int_{\partial\Omega} g u_\infty,$$

for any function $v \in K$, thus (2.7) holds. □

3. Passing to the limit in the viscosity sense

From the results introduced in the previous section we know that, extracting a subsequence if necessary,

$$u_n \longrightarrow u_\infty, \qquad \text{uniformly in } \Omega,$$

for a certain continuous function u_∞.

To prove that u_∞ is a viscosity supersolution of (1.11), let ϕ be such that $u - \phi$ has a strict local minimum at $x_0 \in \Omega$, with $\phi(x_0) = u(x_0)$. We want to prove that

$$-\Delta_\infty \phi(x_0) - |\nabla\phi(x_0)|^2 \ln|\nabla\phi(x_0)| \langle \xi(x_0), \nabla\phi(x_0)\rangle \geq 0. \tag{3.1}$$

Since $u_n \to u$ uniformly, there is a sequence $(x_n)_n$ such that $x_n \to x_0$ and $u_n - \phi$ has a local minimum at x_n. As u_n is a viscosity solution of (1.1) (cf. Proposition 2.5), we have

$$\begin{aligned} - \quad & \frac{|\nabla\phi(x_n)|^2\Delta\phi(x_n)}{p_n(x_n)-2} - \Delta_\infty\phi(x_n) \\ & -|\nabla\phi(x_n)|^2 \ln|\nabla\phi(x_n)| \left\langle \nabla\phi(x_n), \frac{\nabla p_n(x)}{p_n(x_n)-2} \right\rangle \geq 0. \end{aligned}$$

Using the fact that $x_n \to x_0$ and the assumptions (1.4) and (1.6), we obtain the following convergences

$$\begin{gathered} \frac{|\nabla\phi(x_n)|^2\Delta\phi(x_n)}{p_n(x_n)-2} \longrightarrow 0, \\ \Delta_\infty\phi(x_n) \longrightarrow \Delta_\infty\phi(x_0), \\ |\nabla\phi(x_n)|^2\ln(|\nabla\phi(x_n)|) \longrightarrow |\nabla\phi(x_0)|^2\ln(|\nabla\phi(x_0)|), \end{gathered}$$

and

$$\left\langle \nabla\phi(x_n), \frac{\nabla p_n(x)}{p_n(x_n)-2} \right\rangle \longrightarrow \langle \nabla\phi(x_0), \xi(x_0)\rangle .$$

Hence (3.1) follows. This proves that u is a viscosity supersolution; the fact that it is also a viscosity subsolution follows analogously.

Let us check the boundary condition. There are six cases to be considered. Assume that $u_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$. Using the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty\phi(x_0) - |\nabla\phi(x_0)|^2 \ln|\nabla\phi(x_0)|\langle\xi(x_0), \nabla\phi(x_0)\rangle \geq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla\phi|^{i(\frac{p_i}{i}(x_i)-\frac{2}{i})}(x_i)\frac{\partial\phi}{\partial\eta}(x_i) = |\nabla\phi|^{p_i(x_i)-2}(x_i)\frac{\partial\phi}{\partial\eta}(x_i) \geq g(x_i).$$

Since $g(x_0) > 0$, we have $\nabla\phi(x_0) \neq 0$, and we obtain, using that $\frac{p_i}{i}(x_i) \to q(x_0)$,

$$|\nabla\phi|^{q(x_0)}(x_0) \geq 1.$$

Moreover, we also have

$$\frac{\partial\phi}{\partial\eta}(x_0) \geq 0.$$

Hence, if $u_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$, we get

$$\begin{aligned} \max\Big\{ \min\Big\{ -1 + |\nabla\phi|^{q(x_0)}(x_0), \frac{\partial\phi}{\partial\eta}(x_0)\Big\} \ , \\ -\Delta_\infty\phi(x_0) - |\nabla\phi(x_0)|^2\ln|\nabla\phi(x_0)|\langle\xi(x_0), \nabla\phi(x_0)\rangle\Big\} \geq 0. \end{aligned} \tag{3.2}$$

Next assume that $u_\infty - \psi$ has a strict maximum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$. Using the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i} - \psi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \psi(x_0) - |\nabla \psi(x_0)|^2 \ln |\nabla \psi(x_0)| \langle \xi(x_0), \nabla \psi(x_0)\rangle \le 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla \psi|^{i(\frac{p_i}{i}(x_i) - \frac{2}{i})}(x_i)\frac{\partial \psi}{\partial \eta}(x_i) = |\nabla \psi|^{p_i - 2}(x_i)\frac{\partial \psi}{\partial \eta}(x_i) \le g(x_i).$$

If $1 < |\nabla \psi(x_0)|^{q(x_0)}$ we obtain

$$\frac{\partial \psi}{\partial \eta}(x_0) \le 0.$$

Hence, the following inequality holds

$$\begin{aligned} \min\Big\{ \min\Big\{ -1 + |\nabla\psi|^{q(x_0)}(x_0), \frac{\partial \psi}{\partial \eta}(x_0)\Big\}\ , \\ -\Delta_\infty \psi(x_0) - |\nabla \psi(x_0)|^2 \ln |\nabla \psi(x_0)| \langle \xi(x_0), \nabla \psi(x_0)\rangle \Big\} \le 0. \end{aligned} \tag{3.3}$$

For the following case assume that $u_\infty - \psi$ has a strict maximum at x_0 with $g(x_0) < 0$. Using the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i} - \psi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \psi(x_0) - |\nabla \psi(x_0)|^2 \ln |\nabla \psi(x_0)| \langle \xi(x_0), \nabla \psi(x_0)\rangle \Big\} \le 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla \psi|^{i(\frac{p_i}{i}(x_i) - \frac{2}{i})}(x_i)\frac{\partial \psi}{\partial \eta}(x_i) = |\nabla \psi|^{p_i - 2}(x_i)\frac{\partial \psi}{\partial \eta}(x_i) \le g(x_i).$$

As $g(x_0) < 0$, $\nabla \psi(x_0) \neq 0$ and we obtain, using that $\frac{p_i}{i}(x_i) \to q(x_0)$,

$$|\nabla \psi|^{q(x_0)}(x_0) \ge 1,$$

and

$$\frac{\partial \psi}{\partial \eta}(x_0) \le 0.$$

Hence, the following inequality holds

$$\begin{aligned} \min\Big\{ \max\Big\{ 1 - |\nabla\psi|^{q(x_0)}(x_0), \frac{\partial \psi}{\partial \eta}(x_0)\Big\}\ , \\ -\Delta_\infty \psi(x_0) - |\nabla \psi(x_0)|^2 \ln |\nabla \psi(x_0)| \langle \xi(x_0), \nabla \psi(x_0)\rangle \Big\} \le 0. \end{aligned} \tag{3.4}$$

Now assume that $u_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) < 0$. Using the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) - |\nabla \phi(x_0)|^2 \ln |\nabla \phi(x_0)| \langle \xi(x_0), \nabla \phi(x_0)\rangle \ge 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla \phi|^{i(\frac{p_i}{i}(x_i)-\frac{2}{i})}(x_i)\frac{\partial \phi}{\partial \eta}(x_i) = |\nabla \phi|^{p_i(x_i)-2}(x_i)\frac{\partial \phi}{\partial \eta}(x_i) \geq g(x_i).$$

If $1 < |\nabla \phi|^{q(x_0)}(x_0)$ we obtain

$$\frac{\partial \phi}{\partial \eta}(x_0) \geq 0.$$

Hence, the following inequality holds.

$$\begin{aligned} \max\Big\{ & \max\Big\{ -1 + |\nabla \phi|^{q(x_0)}(x_0), \frac{\partial \phi}{\partial \eta}(x_0)\Big\} \ , \\ & -\Delta_\infty \phi(x_0) - |\nabla \phi(x_0)|^2 \ln|\nabla \phi(x_0)|\langle \xi(x_0), \nabla \phi(x_0)\rangle \Big\} \geq 0. \end{aligned} \tag{3.5}$$

For the next case assume that $u_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) = 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) - |\nabla \phi(x_0)|^2 \ln|\nabla \phi(x_0)|\langle \xi(x_0), \nabla \phi(x_0)\rangle \Big\} \geq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla \phi|^{i(\frac{p_i}{i}(x_i)-\frac{2}{i})}(x_i)\frac{\partial \phi}{\partial \eta}(x_i) = |\nabla \phi|^{p_i(x_i)-2}(x_i)\frac{\partial \phi}{\partial \eta}(x_i) \geq g(x_i).$$

If $\nabla \phi(x_0) = 0$, then we have

$$\frac{\partial \phi}{\partial \eta}(x_0) = 0.$$

If $|\nabla \phi(x_0)|^{q(x_0)} > 1$ then, as before, we obtain

$$\frac{\partial \phi}{\partial \eta}(x_0) \geq 0.$$

Therefore, the following inequality holds

$$\begin{aligned} \max\Big\{ & H(|\nabla \phi|^{q(x_0)}(x_0))\frac{\partial \phi}{\partial \eta}(x_0)\Big\} \ , \\ & -\Delta_\infty \phi(x_0) - |\nabla \phi(x_0)|^2 \ln|\nabla \phi(x_0)|\langle \xi(x_0), \nabla \phi(x_0)\rangle \Big\} \geq 0. \end{aligned} \tag{3.6}$$

Finally, assume that $u_\infty - \psi$ has a strict maximum at x_0 with $g(x_0) = 0$ Using the uniform convergence of u_{p_i} to u_∞ we obtain that $u_{p_i} - \psi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \psi(x_0) - |\nabla \psi(x_0)|^2 \ln|\nabla \psi(x_0)|\langle \xi(x_0), \nabla \psi(x_0)\rangle \Big\} \leq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla \psi|^{i(\frac{p_i}{i}(x_i)-\frac{2}{i})}(x_i)\frac{\partial \psi}{\partial \eta}(x_i) = |\nabla \psi|^{p_i(x_i)-2}(x_i)\frac{\partial \psi}{\partial \eta}(x_i) \leq g(x_i).$$

If $\nabla\psi(x_0) = 0$, then we have

$$\frac{\partial\psi}{\partial\eta}(x_0) = 0.$$

If $|\nabla\psi(x_0)|^{q(x_0)} > 1$ we obtain

$$\frac{\partial\psi}{\partial\eta}(x_0) \le 0.$$

Hence, the following inequality holds

$$\begin{aligned} \min\Big\{ H(|\nabla\psi|^{q(x_0)}(x_0))\frac{\partial\psi}{\partial\eta}(x_0)\Big\} \ , \\ -\Delta_\infty\psi(x_0) - |\nabla\psi(x_0)|^2 \ln|\nabla\psi(x_0)| \langle\xi(x_0), \nabla\psi(x_0)\rangle \Big\} \le 0. \end{aligned} \tag{3.7}$$

This ends the proof. □

3.1. **Examples.** In $1-d$ we find that the limit can be easily computed and, surprisingly, does not depend on the sequence $p_n(x) \to \infty$.

Assume that $\Omega = (-1, 1)$ and that $g(1) = -g(-1) > 0$. We get as the limit variational problem

$$\max_v g(1)(v(1) - v(-1)), \qquad \text{with} \int_{-1}^{1} v = 0, \ |v'| \le 1.$$

It is immediate that the unique solution to this problem is

$$u_\infty(x) = x.$$

Note that $u_\infty(x) = x$ is also a solution to the limit ODE that in this case reads as

$$\begin{cases} u''(x) + \ln|u'(x)| \, \langle\xi(x), u'(x)\rangle = 0, & x \in (-1, 1), \\ \min\big\{|u'(1)|^{q(1)} - 1 \ , \ u'(1)\big\} = 0, \\ \max\big\{1 - |u'(-1)|^{q(-1)} \ , \ -u'(-1)\big\} = 0. \end{cases}$$

This example can be easily generalized to the case where Ω is an annulus, $\Omega = \{r_1 < |x| < r_2\}$ and the function g is a positive constant g_1 on $|x| = r_1$ and a negative constant g_2 on $|x| = r_2$ with the constraint

$$\int_{\partial\Omega} g = \int_{|x|=r_1} g + \int_{|x|=r_2} g = 0.$$

The solutions u_n of (1.1) in the annulus converge uniformly as $n \to \infty$ to a cone

$$u_\infty(x) = C - |x|$$

that is the unique maximizer in (1.13).

Remark 3.1. *Note that in general there is non uniqueness of solutions the limit PDE,* (1.11)*, even when* $\xi = 0$ *and* $q = 1$*, see* [13] *for a counterexample.*

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Mayte Pérez-Llanos
Departamento de Matemática
Instituto Superior Técnico
Av. Rovisco Pais 1049-001, Lisboa,
PORTUGAL.
mayte@math.ist.utl.pt

Julio D. Rossi
Departamento de Análisis Matemático,
Universidad de Alicante,
Ap. correos 99, 03080 Alicante, SPAIN.
On leave from
Departamento de Matemática, FCEyN UBA,
Ciudad Universitaria, Pab 1, (1428),
Buenos Aires, ARGENTINA.

E-mail address: jrossi@dm.uba.ar