A NONLOCAL DIFFUSION PROBLEM THAT APPROXIMATES THE HEAT EQUATION WITH NEUMANN BOUNDARY CONDITIONS

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Abstract. In this paper we discuss a nonlocal approximation to the classical heat equation with Neumann boundary conditions. We consider

$$w_t^\epsilon(x,t) = \frac{1}{\epsilon^{N+2}} \int_\Omega J\Big(\frac{x-y}{\epsilon}\Big)(w^\epsilon(y,t) - w^\epsilon(x,t))dy + \frac{C_1}{\epsilon^N} \int_{\partial\Omega} J\Big(\frac{x-y}{\epsilon}\Big) g(y,t)\, dS_y, \qquad (x,t) \in \overline{\Omega} \times (0,T),$$
$$w(x,0) = u_0(x), \qquad x \in \overline{\Omega},$$

and we show that the corresponding solutions, w^ϵ, converge to the classical solution of the local heat equation $v_t = \Delta v$ with Neumann boundary conditions, $\frac{\partial v}{\partial n}(x,t) = g(x,t)$,and initial condition $v(0) = u_0$, as the parameter ϵ goes to zero. The obtained convergence is in the weak star on L^∞ topology.

1. Introduction

The nonlocal evolution equation

$$(1.1)\qquad \begin{aligned} &u_t(x,t) = \int_\Omega J(x-y)[u(y,t) - u(x,t)]dy, \quad (x,t) \in \Omega \times (0,T), \\ &u(x,0) = u_0(x), \quad x \in \Omega, \end{aligned}$$

see [19], can be seen as similar to the local heat equation with homogeneous Neumann boundary conditions

$$(1.2)\qquad \begin{aligned} &v_t(x,t) = \triangle v(x,t), && (x,t) \in \Omega \times (0,T), \\ &\frac{\partial v}{\partial n}(x,t) = 0, && (x,t) \in \partial\Omega \times (0,T), \\ &v(x,0) = u_0(x), && x \in \Omega. \end{aligned}$$

Here, J is a symmetric continuous nonnegative real function defined on $\mathbb{R}^N$, compactly supported in the unit ball, and such that $\int_{\mathbb{R}^N} J(x)dx = 1$, Ω is a bounded smooth domain in $\mathbb{R}^N$ and u_0 stands for the initial condition. The problem is known as a nonlocal model since the diffusion of the density u at a point x and time t not only depends on $u(x,t)$ locally, but also on all values of u through the convolution like term $\int_\Omega J(x-y)u(y,t)\,dy$. Following [19], the model (1.1) can be interpreted as follows: If $u(x,t)$ is the density of a population at point x and time t, and $J(x-y)$ is thought as the probability distribution of jumping from location y to location x, then the convolution $\int_\Omega J(y-x)u(y,t)\,dy$, is the rate at which the individuals are arriving to location x from all other places $y \in \Omega$ (notice that no individuals may arrive to x coming from outside Ω). In the same way, $-\int_\Omega J(y-x)u(x,t)dy$, is the rate at which individuals are leaving the location x to travel to other sites $y \in \Omega$ (notice that no individual can jump outside Ω). So, in absence of external or internal sources, the density u satisfies the nonlocal equation (1.1). Now we remark that the fact that there is no individuals that enter or leave the domain makes this problem a zero flux diffusion problem and therefore the total mass is preserved, $\int_\Omega u(x,t)\,dx = \int_\Omega u_0(x)\,dx$, as happens with solutions to (1.2). The model (1.1) shares more properties with (1.2) such as: bounded stationary solutions

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are constant, a maximum principle is satisfied and perturbations propagate with infinite speed ([19]). Concerning applications, (1.1) and some variants of it have been used, for instance, in Biology [12], image processing, [21], and particle systems, [9]; see also [4] and [20]. For the mathematical analysis of nonlocal models the list of references is large and we refer to [1], [2], [3], [6], [8], [10], [11], [13], [17], [18], [22], and to the book [5] and references therein.

Moreover, when one rescales the kernel J considering $J(\xi)=\frac{1}{\epsilon^{N+2}}J(\frac{\xi}{\epsilon})$, it was shown in [15] that the corresponding solutions to (1.1) with a fixed initial condition converge to the solution to (1.2) as $\epsilon\to 0$. In addition, concerning the non-homegeneous problem, that is, (1.2) with $\frac{\partial v}{\partial n}(x,t)=g(x,t)$, in [15] it is proved that this problem can be approximated with

$$
\begin{aligned}
&(u^\epsilon)_t(x,t)=\frac{1}{\epsilon^{N+2}}\int_\Omega J\Big(\frac{x-y}{\epsilon}\Big)(u^\epsilon(y,t)-u^\epsilon(x,t))\,dy+\frac{K}{\epsilon^{N+1}}\int_{\mathbb{R}^N\setminus\Omega}J\Big(\frac{x-y}{\epsilon}\Big)g(y,t)\,dy,\\
&u^\epsilon(x,0)=u_0(x),
\end{aligned}
\tag{1.3}
$$

as $\epsilon\to 0$ (with an appropriate choice of the constant K). Notice that in this model there are individuals that enter the domain coming from outside (this is the meaning of the integral in $\mathbb{R}^N\setminus\Omega$). This model has the following disadvantage: when one tries to approximate solutions to (1.2) with $\frac{\partial v}{\partial n}(x,t)=g(x,t)$ the datum $g(x,t)$ is given only for $x\in\partial\Omega$ and therefore one has to extend it outside (to a strip around $\partial\Omega$ inside $\mathbb{R}^N\setminus\Omega$ of width of order ϵ) in order to consider (1.3). An easy way to obtain such extension is to consider $g(\tilde y+n(\tilde y)s)=g(\tilde y)$ where $\tilde y\in\partial\Omega$ and $s\in[0,\delta]$ (with δ small), being n the exterior normal vector to $\partial\Omega$ at $\tilde y$.

To overcome the fact that an extension of the Neumann datum g outside Ω is needed, recently, following [16], the authors of [10] have introduced the nonlocal diffusion model

$$
\begin{aligned}
&u_t(x,t)=\int_\Omega J(x-y)(u(y,t)-u(x,t))dy+\int_{\partial\Omega}G(x-y)g(y,t)\,dS_y, \qquad (x,t)\in\overline{\Omega}\times(0,T),\\
&u(x,0)=u_0(x), \qquad x\in\overline{\Omega},
\end{aligned}
\tag{1.4}
$$

where Ω is a bounded domain, $J:\mathbb{R}^N\mapsto\mathbb{R}$ and $G:\mathbb{R}^N\mapsto\mathbb{R}$ are continuous, nonnegative, radially symmetric functions compactly supported in the unit ball and such that $\int_{\mathbb{R}^N}J(z)dz=1$, $\int_{\mathbb{R}^N}G(z)dz=1$, and $g\in L^\infty_{loc}[(0,\infty);L^1(\partial\Omega)]$.

Remark 1.1. The main advantage of the model given by (1.4) compared with (1.3) is that when one deals with a nonlinear datum of the form $g(y,t)=f(u(y,t))$ it is necessary to use an extension of the solution u from Ω to $\mathbb{R}^N\setminus\Omega$ in the case of (1.3) (notice that such an extension is not trivial since g depends on the solution itself). However, it is not necessary to perform such extension when dealing with (1.6).

For the problem (1.4) in [10] it is proved existence and uniqueness of solutions for $u_0\in L^1(\Omega)$, that a comparison principle is satisfied and it is also studied the asymptotic behavior of the solutions as $t\to\infty$.

Our main goal in this paper is to study the behaviour of solutions to this nonlocal model when the involved kernels are rescaled appropriately. If we consider the new kernels

$$
\begin{aligned}
J(\xi)&=\frac{1}{\epsilon^{N+2}}J(\frac{\xi}{\epsilon}),\\
G(\xi)&=\frac{C_1}{\epsilon^{N}}J(\frac{\xi}{\epsilon}),
\end{aligned}
\tag{1.5}
$$

then we arrive to

$$
\begin{aligned}
&w^\epsilon_t(x,t)=\frac{1}{\epsilon^{N+2}}\int_\Omega J\Big(\frac{x-y}{\epsilon}\Big)(w^\epsilon(y,t)-w^\epsilon(x,t))dy+\frac{C_1}{\epsilon^N}\int_{\partial\Omega}J\Big(\frac{x-y}{\epsilon}\Big)g(y,t)\,dS_y, \quad (x,t)\in\overline{\Omega}\times(0,T),\\
&w(x,0)=u_0(x), \qquad x\in\overline{\Omega}.
\end{aligned}
\tag{1.6}
$$

We have the following existence and uniqueness result:

Theorem 1.1. *For every $u_0\in L^1(\Omega)$ and every $g\in L^\infty_{loc}[(0,\infty);L^1(\partial\Omega)]$ there exists a unique solution $w^\epsilon\in C[[0,\infty);L^1(\Omega)]$ to problem* (1.6).

The proof follow the same lines of Theorem 2.2 in [10] and then we omit the details here.

As we have mentioned, our main objective is to show that the solution of the non-homogeneous Neumann problem for the heat equation (1.2), can be approximated by solutions of (1.6) when the parameter ϵ goes to zero. We have the following theorem:

Theorem 1.2. *Let Ω be a bounded $C^{2+\alpha}$ domain, $g \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{(\mathbb{R}^N \setminus \Omega)} \times [0,T])$, let v the solution to* (1.2) *and assume that $v \in C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times [0,T])$, for some $0<\alpha<1$. Let w^ϵ be the solution to* (1.6). *Then, for each $t \in [0,T]$*

$$w^\epsilon(x,t) \rightharpoonup v(x,t) \qquad \text{as } \epsilon \to 0 \ \ * -\text{weakly in } L^\infty(\Omega).$$

We remark that the obtained convergence is the weak$-*$ topology is the same that was obtained in [15] for the problem (1.3).

2. Proof of Theorem 1.2. Weak$-*$ convergence in L^∞

In this section we give the proof of Theorem 1.2. To this end we use a result proved in [15] for the problem (1.3). More precisely, we will use the following theorem:

Theorem 2.1. *Let Ω be a bounded $C^{2+\alpha}$ domain, $g \in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{(\mathbb{R}^N \setminus \Omega)} \times [0,T])$, $v \in C^{2+\alpha,\frac{\alpha}{2}}(\overline{\Omega} \times [0,T])$ the solution to* (1.2), *for some $0<\alpha<1$. Let u^ϵ be a solution to* (1.3). *Then, there is an adequate constant K such that, for each $t \in [0,T]$,*

$$u^\epsilon(x,t) \rightharpoonup v(x,t) \qquad * -\text{weakly in } L^\infty(\Omega).$$

We fix K in such a way that the conclusion of Theorem 2.1 holds.

Proof of Theorem 1.2. With the aim to prove Theorem 1.2, we just want to obtain that $w^\epsilon(x,t)$ the solution to (1.6) is close to $u^\epsilon(x,t)$ the solution to (1.3). To this end we consider equation verified by the difference

$$u^\epsilon(x,t) - w^\epsilon(x,t)$$

that is,

$$\begin{aligned} &(u^\epsilon(x,t) - w^\epsilon(x,t))_t = \frac{1}{\epsilon^{N+2}} \int_\Omega J\Big(\frac{x-y}{\epsilon}\Big)((u^\epsilon(y,t) - w^\epsilon(y,t)) - (u^\epsilon(x,t) - w^\epsilon(x,t)))dy + A_\epsilon(x,t) \\ &(u^\epsilon(x,0) - w^\epsilon(x,0)) = 0, \end{aligned} \tag{2.1}$$

where, from the fact that w^ϵ, u^ϵ satisfies (1.6) and (1.3) respectively, we get that

$$A_\epsilon(x,t) = \frac{K}{\epsilon^{N+1}} \int_{\mathbb{R}^N \setminus \Omega} J\Big(\frac{x-y}{\epsilon}\Big) g(y)dy - \frac{C_1}{\epsilon^N} \int_{\partial\Omega} J\Big(\frac{x-y}{\epsilon}\Big) g(y) dS_y.$$

After integration in Ω of (2.1), we consider $A_\epsilon(x,t)$ and we decompose it as follows:

$$\int_\Omega A_\epsilon(x,t) = I_1 - I_2, \tag{2.2}$$

with

$$I_1 = \frac{K}{\epsilon^{N+1}} \int_\Omega \int_{\mathbb{R}^N - \setminus \Omega} J\Big(\frac{x-y}{\epsilon}\Big) g(y)dy\, dx,$$

and

$$I_2 = \frac{C_1}{\epsilon^N} \int_\Omega \int_{\partial\Omega} J\Big(\frac{x-y}{\epsilon}\Big) g(y) dS_y\, dx.$$

Let δ such that $2\epsilon < \delta$. We consider the following sets:

$$\Gamma_\delta^+ = \{\tilde{x} \in \Omega \ / \ d(\tilde{x}, \partial\Omega) \le \delta\},$$

$$\Gamma_\delta^- = \{\tilde{x} \in (\mathbb{R}^N \setminus \Omega) \ / d(\tilde{x}, \partial\Omega) \le \delta\},$$

and

$$\Gamma_{[-\delta,\delta]} = \{\tilde{x} \in \mathbb{R}^N \diagup d(\tilde{x}, \partial\Omega) \le \delta\} = \Gamma_\delta^+ \cup \Gamma_\delta^- .$$

Note that

$$I_1 = \frac{K}{\epsilon^{N+1}} \int_{\Gamma_\epsilon^+} \int_{\Gamma_\epsilon^-} J\Big(\frac{x-y}{\epsilon}\Big) g(y)dy\, dx.$$

When $\tilde{y} \in \partial\Omega$ and $s \in [-\delta, \delta]$, we use the change of variables

$$y = \tilde{y} + n(\tilde{y})s$$

being $n(\tilde{y})$ the normal vector at $\tilde{y}$ to obtain

$$I_1 = \frac{1}{\epsilon^N} \int_{\Gamma_\epsilon^+} \Big[\frac{1}{\epsilon} \int_0^\epsilon \int_{\partial\Omega} KJ\Big(\frac{x-(\tilde{y}+n(\tilde{y})s)}{\epsilon}\Big) g(\tilde{y}+n(\tilde{y})s)|\overline{J}(\tilde{y},s)| dS_{\tilde{y}}\, ds \Big] dx, \tag{2.3}$$

where $|\overline{J}(\tilde{y},s)|$ is the Jacobian of the change of variable. In the same way we can transform I_2 and obtain

$$I_2 = \frac{C_1}{\epsilon^N} \int_{\Gamma_\epsilon^+} \int_{\partial\Omega} J\Big(\frac{x-\tilde{y}}{\epsilon}\Big) g(\tilde{y}) dS_{\tilde{y}}\, dx. \tag{2.4}$$

Now we consider the extension of g given by

$$g(\tilde{y} + n(\tilde{y})s) = g(\tilde{y})$$

and from (2.3) and (2.4) we get

$$I_1 - I_2 = \frac{1}{\epsilon^N} \int_{\Gamma_\epsilon^+} \Big[\int_{\partial\Omega} \Big(I_3 \Big) d\tilde{y} \Big] dx, \tag{2.5}$$

being

$$I_3 = \frac{1}{\epsilon} \int_0^\epsilon KJ\Big(\frac{x-\tilde{y}}{\epsilon} - n(\tilde{y})\frac{s}{\epsilon}\Big) g(\tilde{y}+n(\tilde{y})s)|\overline{J}(\tilde{y},s)| ds - C_1 J\Big(\frac{x-\tilde{y}}{\epsilon}\Big) g(\tilde{y}+n(\tilde{y})s).$$

Using the change of variables

$$\tilde{s} = \frac{s}{\epsilon}$$

in the first integral we obtain

$$I_3 = \int_0^1 KJ\Big(\frac{x-\tilde{y}}{\epsilon} - n(\tilde{y})\tilde{s}\Big) g(\tilde{y}+n(\tilde{y})s)|\overline{J}(\tilde{y},\epsilon\tilde{s})| d\tilde{s} - C_1 J\Big(\frac{x-\tilde{y}}{\epsilon}\Big) g(\tilde{y}+n(\tilde{y})s). \tag{2.6}$$

Taking into account (2.2), (2.5) and (2.6), making the change of variables

$$z = \frac{x-\tilde{y}}{\epsilon}$$

and considering that

$$x = \tilde{y} + \epsilon z$$

we obtain

$$\int_\Omega A_\epsilon(x,t) = \int_{\partial\Omega} \Big[\int_{\{z \diagup (\tilde{y}+\epsilon z)\in\Omega\}} \Big(\int_0^1 KJ(z - n(\tilde{y})\tilde{s})|\overline{J}(\tilde{y},\epsilon\tilde{s})| d\tilde{s} - C_1 J(z) \Big) dz \Big] g(\tilde{y}) dS_{\tilde{y}}. \tag{2.7}$$

Choosing C_1 appropriately in such a way that

$$\lim_{\epsilon\to 0} \int_{\{z \diagup (\tilde{y}+\epsilon z)\in\Omega\}} \Big(\int_0^1 KJ(z - n(\tilde{y})\tilde{s})|\overline{J}(\tilde{y},\epsilon\tilde{s})| d\tilde{s} - C_1 J(z) \Big) dz = 0,$$

we conclude

$$\lim_{\epsilon\to 0} \int_\Omega A_\epsilon(x,t) = 0. \tag{2.8}$$

With this estimate we can control

$$u^\epsilon(x,t) - w^\epsilon(x,t).$$

In fact, let z_ϵ be a nonnegative solution of the problem

$$
\begin{aligned}
&z_t^\epsilon(x,t) = \frac{1}{\epsilon^{N+2}}\int_\Omega J\Big(\frac{x-y}{\epsilon}\Big)(z^\epsilon(y,t)-z^\epsilon(x,t))dy + |A_\epsilon(x,t)| \qquad (x,t)\in\overline{\Omega}\times(0,T),\\
&z(x,0)=0, \qquad x\in\overline{\Omega}.
\end{aligned}
\tag{2.9}
$$

Integrating (2.9) in Ω and using the symmetry of J we have

$$\frac{\partial}{\partial t}\int_\Omega z^\epsilon(x,t)dx = \int_\Omega |A_\epsilon(x,t)|dx$$

and therefore

$$\int_\Omega z^\epsilon(x,t)dx = \int_0^T\int_\Omega |A_\epsilon(x,t)|dxdt.$$

From (2.8) we obtain that

$$\int_\Omega z^\epsilon(x,t)dx \to 0 \qquad \text{when } \epsilon\to 0. \tag{2.10}$$

Now, since

$$A_\epsilon(x,t)\le |A_\epsilon(x,t)|,$$

we get that

$$|u^\epsilon(x,t)-w^\epsilon(x,t)|\le z_\epsilon(x,t).$$

Hence, from (2.10), it holds that

$$\int_\Omega |u^\epsilon(x,t)-w^\epsilon(x,t)|dx\to 0, \qquad \text{when } \epsilon\to 0. \tag{2.11}$$

To finish the proof of the theorem, let $\theta\in L^\infty(\Omega)$, then

$$\int_\Omega w^\epsilon\theta\,dx \le \int_\Omega |w^\epsilon(x,t)-u^\epsilon(x,t)||\theta(x)|dx + \int_\Omega u^\epsilon(x,t)\theta(x)\,dx.$$

Taking into account (2.11) and Theorem 2.1 we conclude that

$$\int_\Omega w^\epsilon\theta\,dx \to \int_\Omega v(x,t)\theta\,dx, \ \text{ when } \ \epsilon\to 0,$$

as we wanted to show. ☑

Remark that we obtained $\|u^\epsilon - w^\epsilon\|_{L^\infty(0,T);L^1(\Omega)}\to 0$ in (2.11). This says that solutions to the nonlocal models (1.3) and (1.6) are close for ϵ small in a topology that is stronger than weak$-*$ in L^∞.

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