NONLOCAL EVOLUTION PROBLEMS IN THIN DOMAINS

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Abstract. In this paper we consider parabolic nonlocal problems in thin domains. Fix $\Omega \subset \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ and consider u^ϵ be the solution to $u_t(t,x) = \int_\Omega J_\epsilon(x-y)(u^\epsilon(t,y) - u^\epsilon(t,x))dy + f(t,x)$ with initial condition $u(0,x) = u_0(x)$ and a kernel of the form $J_\epsilon(x) = J(x_1, \epsilon x_2)$ with J non-singular. This corresponds (via a simple change of variables) to the usual nonlocal evolution problem $v_t(t,z) = \frac{1}{\epsilon^{N_2}} \int_{\Omega_\epsilon} J(z-w)(v(w) - v(z))\, dw + f^\epsilon(t,z)$, in the thin domain $\Omega_\epsilon = \{(x_1, \epsilon x_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} : (x_1, x_2) \in \Omega\}$. Our main result says that there is a limit as $\epsilon \to 0$ of the solutions to our problem and that this limit, when we take its mean value in the x_2-direction, is a solution to a limit nonlocal problem in the projected set $\Omega_1 \subset \mathbb{R}^{N_1}$.

1. Introduction

Our main goal in this paper is to study parabolic nonlocal problems with non-singular kernels and Neumann conditions in thin domains.

We consider the following nonlocal diffusion equation

$$\left\{ \begin{aligned} u_t(t,x) &= \int_\Omega J_\epsilon(x-y)(u^\epsilon(t,y) - u^\epsilon(t,x))dy + f(t,x), \\ u(0,x) &= u_0(x), \end{aligned} \right. \qquad x \in \Omega,\ t \in \mathbb{R}, \tag{1.1}$$

where

$$J_\epsilon(x) = J(x_1, \epsilon x_2),$$

and $\epsilon > 0$ is a parameter. The domain $\Omega \subset \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$ is a bounded domain. We take the initial condition u_0 in $L^2(\Omega)$, and the local forcing term f in $C(\mathbb{R}; L^2(\Omega))$. Also, we denote by $x = (x_1, x_2)$ a point in $\mathbb{R}^{N_1} \times \mathbb{R}^{N_2}$. Along the whole paper, the function J satisfies the hypotheses

$$\textbf{(H)} \qquad \begin{gathered} J \in \mathcal{C}(\mathbb{R}^N, \mathbb{R}) \text{ is non-negative with } J(0) > 0,\ J(-x) = J(x) \text{ for every } x \in \mathbb{R}^N, \text{ and} \\ \int_{\mathbb{R}^N} J(x)\, dx = 1. \end{gathered}$$

Existence and uniqueness of solutions $u^\epsilon : \mathbb{R} \times \Omega \mapsto \mathbb{R}$ with $u^\epsilon \in C^1([a,b], L^2(\Omega))$ for every bounded interval $[a,b] \subset \mathbb{R}$, can be easily obtained (we will include some comments on this in Section 2).

Notice that, since the kernel J is smooth, there is no regularizing effect in (1.1) and therefore the problem is well posed for $t \in \mathbb{R}$ and not only for $t \in \mathbb{R}_+$ as happens for local equations like the heat equation.

It is worth noting that we are calling (1.1) as a nonlocal thin domain problem due to its equivalence with the nonlocal problem

$$\left\{ \begin{aligned} v_t(t,z) &= \frac{1}{\epsilon^{N_2}} \int_{\Omega_\epsilon} J(z-w)(v(w) - v(z))\, dw + f^\epsilon(t,z), \\ v(0,z) &= v_0^\epsilon(z), \end{aligned} \right. \qquad z \in \Omega_\epsilon,\ t \in \mathbb{R}. \tag{1.2}$$

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Here the domain $\Omega_\epsilon \subset \mathbb{R}^N$ is assumed to be a general thin domain defined as

$$\Omega_\epsilon = \{(x_1, \epsilon x_2) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \,:\, (x_1, x_2) \in \Omega\}.$$

Also, we are taking here the initial condition

$$v_0^\epsilon(z_1, z_2) = u_0(x_1, \epsilon^{-1} x_2) \quad \text{ in } L^2(\Omega_\epsilon),$$

and the family of forcing terms

$$f^\epsilon(t, z_1, z_2) = f(t, z_1, \epsilon^{-1} z_2) \quad \text{ in } C(\mathbb{R}; L^2(\Omega_\epsilon)).$$

Notice that the equivalence between problems (1.1) and (1.2) is a direct consequence of the simple change of variable

$$(x_1, x_2) \in \Omega \mapsto (x_1, \epsilon x_2) \in \Omega_\epsilon.$$

Furthermore, it is not difficult to see that (1.1), and then (1.2), are nonlocal singular problems since the bounded domain Ω_ϵ degenerates to

$$\Omega_1 := \pi_1(\Omega) \subset \mathbb{R}^{N_2}$$

when the positive parameter ϵ goes to zero. Here, the open set $\pi_1(\Omega)$ is given by the projection map onto the N_1 first coordinates

$$\pi_1 : \mathbb{R}^N = \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \mapsto \mathbb{R}^{N_1}, \qquad \pi_1(z_1, z_2) = z_1.$$

We also note that our model is in agreement with pioneering and recent references on thin domain problems, see [10, 22, 24, 26], which use the factor $1/\epsilon^{N_2}$ as in (1.2) to preserve the relative size of the open set Ω_ϵ. The convenience of using this approach is clear since the solutions of (1.1) are defined in the fixed domain Ω which allows us to analyze its asymptotic behavior as $\epsilon \to 0$ in a fixed space of functions.

Now let us state our main result. It says that there is a limit as $\epsilon \to 0$ of the solutions to our problem and that this limit, when we take its mean value in the x_2-direction, is a solution to a limit nonlocal problem in the projected set $\Omega_1 = \pi_1(\Omega) \subset \mathbb{R}^{N_1}$.

Theorem 1.1. *Let $\{u^\epsilon\}_{\epsilon>0}$ be the family of solutions given by problem (1.1). Then, there exists $u^* : \mathbb{R} \times \Omega \mapsto \mathbb{R}$, $u^* \in C^1([a,b], L^2(\Omega))$ for any closed interval $[a,b] \subset \mathbb{R}$, such that we have*

$$\sup_{t \in [a,b]} \|u^\epsilon(t,\cdot) - u^*(t,\cdot)\|_{L^2(\Omega)} \to 0,$$

and

$$\sup_{t \in [a,b]} \|\left(U^\epsilon(t,\cdot) - U^*(t,\cdot)\right)|\Gamma(\cdot)|\|_{L^2(\Omega_1)} \to 0$$

as $\epsilon \to 0$. Here the functions U^ϵ and U^* are given by the mean value of u^ϵ and u^* in the x_2-direction, that is,

$$U^\epsilon(t, x_1) = \frac{1}{|\Gamma(x_1)|} \int_{\Gamma(x_1)} u^\epsilon(t, x_1, x_2)\, dx_2, \quad \text{and}$$

$$U^*(t, x_1) = \frac{1}{|\Gamma(x_1)|} \int_{\Gamma(x_1)} u^*(t, x_1, x_2)\, dx_2, \qquad \text{a.e. in } \Omega_1,$$

where $\Gamma(x_1)$ denotes the transversal section of Ω for each $x_1 \in \Omega_1$, that is,

$$\Gamma(x_1) = \{(x_1, x_2) \in \Omega \,:\, x_1 \in \Omega_1\}.$$

Furthermore, we have that U^* satisfies the following nonlocal equation in Ω_1

$$U_t^*(t, x_1) = \frac{1}{|\Gamma(x_1)|} \int_{\Omega_1} J_*(x_1, y_1)(U^*(t, y_1) - U^*(t, x_1))\, dy_1 + \frac{1}{|\Gamma(x_1)|} \int_{\Gamma(x_1)} f(t, x)\, dx_2 \tag{1.3}$$

with initial condition

$$U^*(0, x_1) = \frac{1}{|\Gamma(x_1)|} \int_{\Gamma(x_1)} u_0(x)\, dx_2$$

where the limit kernel J_* *is given by*

$$J_*(x_1, y_1) = J(x_1 - y_1, 0)\,|\Gamma(x_1)|\,|\Gamma(y_1)|.$$

Remark 1.1. As we have already mentioned, since the kernel J is smooth, there is no regularizing effect for this problem and therefore to obtain strong convergence in L^2-norm is not straightforward. Notice that we have strong convergence in L^2 even when we consider the averages of solutions on the transversal sections of Ω. These convergences are uniform in bounded intervals of time. In addition we remark that, with the same arguments used to prove Theorem 1.1, one can obtain weak convergence in $L^\infty-$norm (or in C^k-norm) when we take initial conditions in L^∞ (C^k) and forcing terms $f \in C(\mathbb{R}; L^\infty(\Omega))$ ($f \in C(\mathbb{R}; C^k(\Omega))$).

Also, we get at the limit a kernel J_* which is symmetric and non-negative, but the limit nonlocal problem is not of convolution type. We call equation (1.3) as the *limit equation* of problem (1.1). Let us point its dependence with respect on the geometry of open set Ω given by the term $|\Gamma(\cdot)|$, the Lebesgue measure of the transversal section of Ω. In some sense, we see here how the geometry of the thin domain Ω_ϵ affects the original problem (1.2) as it becomes thinner and thinner in the vertical direction x_2.

The study of equations in thin domains occurs in applications as they can be found in mathematical models for ocean dynamics (where one is dealing with fluid regions which are thin compared to the horizontal length scales), lubrication, nanotechnology, blood circulation, material engineering, meteorology, etc. Many techniques and methods have been developed in order to understand the effect of the geometry and thickness of the domain on the solutions of such singular problems. From pioneering works to recent ones we can mention [27, 19, 14, 5, 3] concerned with elliptic and parabolic equations, as well as [2, 12, 4, 15, 18] where the authors considered Stokes and Navier-Stokes equations from fluid mechanics.

Concerning references for nonlocal evolution problems with smooth kernels we refer to [6, 7, 8, 13, 25], the book [1] and references therein. This kind of equations have been considered recently in connection with real applications (for example to peridynamics, a recent model for elasticity, biology, etc.), we quote for instance [9, 16, 17, 20, 28, 29]. Let us point out that since we are integrating in Ω the nonlocal problem considered here is a nonlocal analogous to the classical elliptic problem for the Laplacian with homogeneous Neumann boundary conditions, that is,

$$\begin{cases} u_t = \Delta u + f, \\ \dfrac{\partial u}{\partial n} = 0. \end{cases}$$

In fact, in [7] it is proved that solutions to the nonlocal problem (1.1) converge, as a rescaling parameter that controls the size of the support of J goes to zero, to the solution to the local problem.

This paper can be viewed as a natural continuation of [21] where the authors deal with the elliptic nonlocal problem in a product domain, that is, when the thin domain Ω_ϵ is of the form $\Omega_\epsilon = \Omega_1 \times \epsilon\Omega_2$. Notice that in the case of a product domain we have that the measure of the transversal section is constant $|\Gamma(\cdot)| = |\Omega_2|$ and hence the effect in the operator that appears in the limit equation is just the multiplication by a constant. Our results here apply also to the elliptic case, extending the previous results to general domains.

The paper is organized as follows: in Section 2 we include some preliminary results (we discuss existence and uniqueness of the solutions to (1.1) and obtain a crucial estimate that is uniform in $\epsilon > 0$); while in Section 3 we deal with the proof of our main result concerning the limit as $\epsilon \to 0$ for the Neumann problem, Theorem 1.1.

2. Preliminary results

In this section we discuss existence and uniqueness of the solutions of the nonlocal Neumann problem (1.1). We also get a uniform in $\epsilon > 0$ estimate.

Proposition 2.1. *Let us assume hypotheses* (**H**) *and take $f \in C(\mathbb{R}; L^2(\Omega))$. Then, for each $u_0 \in L^2(\Omega)$, the nonlocal problem (1.1) possesses a unique global solution $u^\epsilon : \mathbb{R} \times \Omega \mapsto \mathbb{R}$ such that, for all bounded interval $[a,b] \subset \mathbb{R}$, we have*

$$u^\epsilon \in C^1([a,b], L^2(\Omega))$$

and satisfies the equation in an integral sense, that is,

$$\begin{aligned} u^\epsilon(t,x) = e^{-A_\epsilon(x)t} u_0(x) + \int_0^t e^{-A_\epsilon(x)(t-s)} \int_\Omega J_\epsilon(x-y)\, u^\epsilon(s,y)\, dyds \\ + \int_0^t e^{-A_\epsilon(x)(t-s)} f(s,x)\, ds \end{aligned} \tag{2.4}$$

for $(t,x) \in \mathbb{R} \times \Omega$, where $A_\epsilon \in L^\infty(\Omega)$ is the positive function

$$A_\epsilon(x) = \int_\Omega J_\epsilon(x-y)\, dy, \quad x \in \Omega.$$

Moreover, there exist positive constants α and C, independent of ϵ, and ϵ_0 such that

$$\begin{aligned} \left\| u^\epsilon(t,\cdot) - \frac{1}{|\Omega|}\int_\Omega u_0(x)\, dx \right\|_{L^2(\Omega)} \le e^{-\alpha t} \Bigg[\left\| u_0 - \frac{1}{|\Omega|}\int_\Omega u_0(x)\, dx \right\|_{L^2(\Omega)} \\ + C \int_0^t \|f(s,\cdot)\|^2_{L^2(\Omega)}\, ds \Bigg] \end{aligned} \tag{2.5}$$

for any $\epsilon \in (0, \epsilon_0]$.

If we also assume that $u_0 \in L^\infty(\Omega)$ and

$$\int_0^\infty \|e^{\gamma s} f(s,\cdot)\|_{L^\infty(\Omega)} ds < \infty \tag{2.6}$$

for $\gamma = \|J\|_{L^\infty(\mathbb{R}^N)}\, |\Omega|$, then

$$\left\| u^\epsilon(t,\cdot) - \frac{1}{|\Omega|}\int_\Omega u_0(x)\, dx \right\|_{L^\infty(\Omega)} \le C\, e^{-mt} \tag{2.7}$$

for some constant $m > 0$ and any $t \ge 0$.

Proof. First we note that the solutions of (1.1) can be written in the integral form as in (2.4). In fact, for each $\epsilon > 0$, the solution can be obtained as the fixed point of the map

$$\begin{aligned} F(u)(t,x) = e^{-A_\epsilon(x)t} u_0(x) + \int_0^t e^{-A_\epsilon(x)(t-s)} \int_\Omega J_\epsilon(x-y)\, u^\epsilon(s,y)\, dyds \\ + \int_0^t e^{-A_\epsilon(x)(t-s)} f(s,x)\, ds. \end{aligned}$$

The proof that F possesses a fixed point globally defined with $t \in \mathbb{R}$ is analogous to that one given, for instance, in [25, Section 4]. See also [1, Section 3.2.1]. We omit the details here.

Thus, let us prove estimate (2.5). Consider

$$H(t) = \frac{1}{2}\int_\Omega \left(u^\epsilon(t,\cdot) - \frac{1}{|\Omega|}\int_\Omega u_0(x)\, dx \right)^2 dx.$$

Then, due to Young's inequality, for any $\delta > 0$ we have

$$
\begin{aligned}
H'(t) &= \int_\Omega \left(u(t,x) - \frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx \right) \int_\Omega J_\epsilon(x-y)(u(t,y)-u(t,x))^2 dy dx \\
&\quad + \int_\Omega \left(u(t,x) - \frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx \right) f(t,x)\, dx \\
&= -\frac{1}{2} \int_\Omega \int_\Omega J_\epsilon(x-y)(u(t,y)-u(t,x))^2 dy dx \\
&\quad + \int_\Omega \left(u(t,x) - \frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx \right) f(t,x)\, dx \\
&\leq 2(\delta^2 - \lambda_1^\epsilon) H(t) + \delta^{-2} \|f(t,\cdot)\|^2_{L^2(\Omega)}
\end{aligned} \tag{2.8}
$$

where λ_1^ϵ is the first nontrivial eigenvalue of the nonlocal Neumann problem given by

$$
\lambda_1^\epsilon = \inf_{u \in W} \frac{\displaystyle \frac{1}{2} \int_\Omega \int_\Omega J_\epsilon(x-y)(u(y)-u(x))^2 dy\, dx}{\displaystyle \int_\Omega u^2(x)\, dx}
$$

in the space

$$
W = \left\{ u \in L^2(\Omega) \, : \, \int_\Omega u(x)\, dx = 0 \right\}.
$$

For the positivity of λ_1^ϵ we refer to [1].

Note that here we are using the identity

$$
\int_\Omega \varphi(x) \int_\Omega J_\epsilon(x-y)(\phi(y)-\phi(x))\, dy dx = -\frac{1}{2} \int_\Omega \int_\Omega J_\epsilon(x-y)(\varphi(y)-\varphi(x))(\phi(y)-\phi(x))\, dy dx
$$

that holds since $J_\epsilon(-x) = J_\epsilon(x)$.

Hence, we can integrate (2.8) obtaining

$$
H(t) \leq e^{2(\delta^2 - \lambda_1^\epsilon) t} \left[H(0) + \delta^{-2} \int_0^t \|f(s,\cdot)\|^2_{L^2(\Omega)} ds \right].
$$

Also, we know from [21, Lemma 2.1] that the family of eigenvalues λ_1^ϵ satisfies $\lambda_1^\epsilon \to \lambda_1$ as $\epsilon \to 0$ with λ_1 the first non trial eigenvalue of the Neumann Laplacian that is strictly positive. Thus, we conclude the proof of (2.5) taking some $\epsilon_0 > 0$, and δ small enough, in order to guarantee that $\delta^2 - \lambda_1^\epsilon < 0$ for all $\epsilon \in (0, \epsilon_0]$.

Now, let us show (2.7). If we let

$$
w^\epsilon(t,x) = u^\epsilon(t,x) - \frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx,
$$

it is not difficult to see that w^ϵ satisfies (1.1) with initial datum $w_0(x) = u_0(x) - \frac{1}{|\Omega|}\int_\Omega u_0(x)\, dx$, and then, it is given by (2.4)

$$
\begin{aligned}
w^\epsilon(t,x) &= e^{-A_\epsilon(x)t} w_0(x) + \int_0^t e^{-A_\epsilon(x)(t-s)}\, f(s,x)\, ds \\
&\quad + \int_0^t e^{-A_\epsilon(x)(t-s)} \int_\Omega J_\epsilon(x-y) w(s,x)\, dy ds.
\end{aligned}
$$

Next, since J is continuous with $J(0) > 0$, and Ω is a bounded domain, there exists a positive constant m such that

$$
0 < m \leq A_\epsilon(x) = \int_\Omega J_\epsilon(x-y)\, dy \leq \|J\|_{L^\infty} |\Omega| = \gamma,
$$

whenever $\epsilon \in (0, \epsilon_1]$ for any fixed $\epsilon_1 > 0$. Hence, we obtain

$$\|e^{A_\epsilon(\cdot)t} w^\epsilon(t,\cdot)\|_{L^\infty(\Omega)} \leq \|w_0\|_{L^\infty(\Omega)} + \int_0^\infty \|e^{\gamma s} f(s,\cdot)\|_{L^\infty(\Omega)}\, ds$$
$$+\gamma \int_0^t \|e^{A_\epsilon(\cdot)s} w(s,\cdot)\|_{L^\infty(\Omega)}\, ds.$$

Therefore, from [11, Gronwall's inequality] and (2.6), there exists a positive constant C, independent of ϵ and t, such that

$$\|e^{A_\epsilon(\cdot)t} w^\epsilon(t,\cdot)\|_{L^\infty(\Omega)} \leq C, \quad \forall t > 0 \text{ and } \epsilon \in (0, \epsilon_1].$$

From this, we conclude the result. □

3. Proof of Theorem 1.1

In this section we prove our main result, Theorem 1.1.

Proof of Theorem 1.1. First we note that the existence of the family of solutions u^ϵ of (1.1) is guaranteed by Proposition 2.1. Also, for any bounded interval $[a,b] \subset \mathbb{R}$, we get that there exists a positive constant K, independent of ϵ, such that

$$\sup_{t\in[a,b]} \|u^\epsilon(t,\cdot)\|_{L^2(\Omega)} \leq K. \tag{3.9}$$

That is, the family u^ϵ is uniformly bounded in $L^\infty\left([a,b]; L^2(\Omega)\right)$. Thus, since $L^1\left([a,b]; L^2(\Omega)\right)$ is a separable Banach space, we can extract a subsequence, still denoted by u^ϵ, such that

$$u^\epsilon \rightharpoonup u^* \text{ weakly}^* \text{ in } L^\infty([a,b]; L^2(\Omega)), \tag{3.10}$$

for some $u^* \in L^\infty\left([a,b]; L^2(\Omega)\right)$. In order to simplify the notation, we assume, from now on that $[a,b] = [0,T]$ with $T > 0$. Now we pass to the limit in the variational formulation of the expression (2.4). For any $\varphi \in L^2(\Omega)$, we have to pass to the limit in

$$\begin{aligned} \int_\Omega \varphi(x)\, u^\epsilon(t,x)\, dx = & \int_\Omega \varphi(x)\, e^{-A_\epsilon(x)t}\, u_0(x)\, dx \\ & + \int_\Omega \varphi(x) \int_0^t e^{-A_\epsilon(x)(t-s)}\, f(s,x)\, dsdx \\ & + \int_\Omega \varphi(x) \int_0^t e^{-A_\epsilon(x)(t-s)} \int_\Omega J_\epsilon(x-y)\, u^\epsilon(s,y)\, dydsdx \\ & = I_1^\epsilon + I_2^\epsilon + I_3^\epsilon. \end{aligned} \tag{3.11}$$

First, let us see that for any compact set $\mathcal{K} \subset \mathbb{R}^N$, we have

$$J_\epsilon(x_1,x_2) = J(x_1, \epsilon x_2) \to J(x_1, 0) \quad \text{uniformly in } \mathcal{K} \tag{3.12}$$

as $\epsilon \to 0$. In fact, since J is a continuous function in $\mathbb{R}^N$, given any $\delta > 0$, we have that there exists $\epsilon_0 > 0$ such that

$$|J(x_1, \epsilon x_2) - J(x_1, 0)|\, dz \leq \delta$$

whenever $\epsilon |x_2| \leq \epsilon_0$ and $(x_1, x_2) \in \mathcal{K}$.

Let

$$A_\epsilon(x) = \int_\Omega J_\epsilon(x-y)\, dy, \quad x \in \Omega.$$

It follows from (3.12) that

$$A_\epsilon \to A_0 \quad \text{strongly in } L^\infty(\Omega) \tag{3.13}$$

with A_0 strictly positive in Ω. Indeed, since we assume $J(0) > 0$, there exit $\epsilon_1 > 0$ and $m > 0$ such that

$$|\Omega| \|J\|_\infty \geq \int_\Omega J_\epsilon(x-y)\, dy \geq m > 0, \text{ whenever } \epsilon \in [0, \epsilon_1]. \tag{3.14}$$

Hence, we can argue as in (3.12) to obtain

$$|A_\epsilon(x) - A_0(x)| \leq \int_\Omega |J(x_1 - y_1, \epsilon(x_2 - y_2)) - J(x_1 - y_1, 0)|\, dz \leq \delta\, |\Omega|$$

whenever $\epsilon |x_2 - y_2| \leq \epsilon_0$ and $(x - y) \in \mathcal{K}$ where $\mathcal{K}$ is any comapact in $\mathbb{R}^N$ fixed. Thus, we get (3.13) with

$$|\Omega| \|J\|_\infty \geq A_0(x) \geq m > 0 \qquad \forall x \in \Omega.$$

Now, since the exponential function is uniformly continuous in any bounded closed interval in $\mathbb{R}$, it follows from (3.13) that

$$e^{-A_\epsilon(x)t} \to e^{-A_0(x)t} \quad \text{ uniformly in } (t, x) \in [0, T] \times \Omega \tag{3.15}$$

as $\epsilon \to 0$.

Thus, for all $t \in [0, T]$, we obtain from (3.15) that

$$I_1^\epsilon = \int_\Omega \varphi(x)\, e^{-A_\epsilon(x)t}\, u_0(x)\, dx \qquad \text{and} \qquad I_2^\epsilon = \int_\Omega \varphi(x) \int_0^t e^{-A_\epsilon(x)(t-s)}\, f(s, x)\, ds dx$$

verify

$$I_1^\epsilon \to \int_\Omega \varphi(x)\, e^{-A_0(x)t}\, u_0(x)\, dx \qquad \text{and} \qquad I_2^\epsilon \to \int_\Omega \varphi(x) \int_0^t e^{-A_0(x)(t-s)}\, f(s, x)\, ds dx \tag{3.16}$$

for any test function $\varphi \in L^2(\Omega)$.

Next, let us consider the following sequence

$$\mathcal{U}_\epsilon(t, x) = \int_0^t e^{-A_\epsilon(x)(t-s)} \int_\Omega J_\epsilon(x - y) u^\epsilon(s, y) dy ds$$

defined for any $(t, x) \in \mathbb{R} \times \Omega$. Thus, since the sequences u^ϵ and J_ϵ satisfy (3.10) and (3.12) respectively, we obtain from (3.15)

$$\mathcal{U}_\epsilon(t, x) \to \mathcal{U}_0(t, x) = \int_0^t e^{-A_0(x)(t-s)} \int_\Omega J(x_1 - y_1, 0)\, u^*(s, y) dy ds$$

for any $(t, x) \in \mathbb{R} \times \Omega$. Furthermore, for all $t \in [0, T]$, we have from (3.9) and (3.14) that

$$|\mathcal{U}_\epsilon(t, x)| \leq \int_0^t \|J_\epsilon\|_{L^2(\Omega)} \|u^\epsilon(s, \cdot)\|_{L^2(\Omega)}\, ds \leq T\, K\, |\Omega|^{1/2}.$$

Therefore, due to Convergence Dominated Theorem, it follows from (3.9) and (3.12) that

$$\mathcal{U}_\epsilon(t, \cdot) \rightharpoonup \mathcal{U}_0(t, \cdot) \text{ weakly in } L^2(\Omega) \tag{3.17}$$

for each $t \in [0, T]$. Indeed, we have

$$\mathcal{U}_\epsilon(t, \cdot) \to \mathcal{U}_0(t, \cdot) \quad \text{ strongly in } L^2(\Omega) \tag{3.18}$$

since

$$|\mathcal{U}_\epsilon(t, x)|^2 \leq T^2 K^2 |\Omega|,$$

and hence, we can take limit again, due to Dominated Convergence Theorem, to obtain

$$\|\mathcal{U}_\epsilon(t, \cdot)\|_{L^2(\Omega)} \to \|\mathcal{U}_0(t, \cdot)\|_{L^2(\Omega)} \tag{3.19}$$

for all $t \in [0, T]$. Therefore, the strong convergence (3.18) follows from (3.17) and (3.19) since we are working here in a Hilbert space.

Now, we are ready to analyze the term

$$I_3^\epsilon = \int_\Omega \varphi(x) \int_0^t e^{-A_\epsilon(x)(t-s)} \int_\Omega J_\epsilon(x-y)\, u^\epsilon(s,y)\, dyds dx.$$

From (3.18) we have

$$\begin{aligned} I_3^\epsilon &= \int_\Omega \varphi(x) \mathcal{U}_\epsilon(t,x)\, dx \\ &\to \int_\Omega \varphi(x) \mathcal{U}_0(t,x)\, dx \\ &= \int_\Omega \varphi(x) \int_0^t e^{-A_0(x)(t-s)} \int_\Omega J(x_1-y_1,0)\, u^*(s,y)\, dydxds \end{aligned}$$

for any $\varphi \in L^2(\Omega)$. Thus, we can pass to the limit in (3.11). Using (3.16) and (3.20), we obtain

$$\begin{aligned} \int_\Omega \varphi(x)\, u^*(t,x)\, dx = \int_\Omega \varphi(x) \left[e^{-A_0(x)t}\, u_0(x) + \int_0^t e^{-A_0(x)(t-s)}\, f(s,x)\, ds \right] dx \\ + \int_\Omega \varphi(x) \int_0^t e^{-A_0(x)(t-s)} \int_\Omega J(x_1-y_1,0)\, u^*(s,y)\, dyds dx, \end{aligned}$$

which implies

$$\begin{aligned} u^*(t,x) = e^{-A_0(x)t}\, u_0(x) + \int_0^t e^{-A_0(x)(t-s)}\, f(s,x)\, ds \\ + \int_0^t e^{-A_0(x)(t-s)} \int_\Omega J(x_1-y_1,0)\, u^*(s,y)\, dyds \end{aligned} \tag{3.21}$$

for all $t \in [0,T]$ and a.e. x in Ω.

Hence, it follows from Theorem 2.1 that $u^* \in C^1([0,T]; L^2(\Omega))$ is unique and satisfies

$$\begin{aligned} u_t^*(t,x) = \int_\Omega J(x_1-y_1,0)(u^*(t,y) - u^*(t,x)) dy + f(t,x) \\ u^*(0,x) = u_0(x). \end{aligned} \tag{3.22}$$

That is, the sequence u^ϵ is weak convergent, and converges to a function u^* solution of the nonlocal equation (3.22). Notice that, in despite of $J(\cdot,0)$ does not satisfy the condition $\int_\Omega J(x_1,0)\, dx = 1$, we have that Theorem 2.1 can still be applied here.

We can also obtain the limit equation (1.3). Taking test functions $\varphi(x_1,x_2) = \varphi(x_1)$ in (3.22), (that is, test functions only depending on the first variable $x_1 \in \Omega_1$), we get

$$\begin{aligned} &\int_{\Omega_1} \varphi(x_1) \int_{\Gamma(x_1)} u_t^*(t,x)\, dx_2 dx_1 = \int_{\Omega_1} \varphi(x_1) \int_{\Gamma(x_1)} f(t,x)\, dx_2 dx_1 \\ &+ \int_{\Omega_1} \varphi(x_1) \int_{\Omega_1} J(x_1-y_1,0) \left[|\Gamma(x_1)| \int_{\Gamma(y_1)} u^*(t,y)\, dy_2 - |\Gamma(y_1)| \int_{\Gamma(x_1)} u^*(t,x)\, dx_2 \right] dy_1 dx_1. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \int_{\Gamma(x_1)} u_t^*(t,x)\, dx_2 = \int_{\Gamma(x_1)} f(t,x)\, dx_2 \\ + \int_{\Omega_1} J(x_1-y_1,0) \left[|\Gamma(x_1)| \int_{\Gamma(y_1)} u^*(t,y)\, dy_2 - |\Gamma(y_1)| \int_{\Gamma(x_1)} u^*(t,x)\, dx_2 \right] dy_1 \end{aligned}$$

for all $t \in \mathbb{R}$ and a.e. $x_1 \in \Omega_1$. Hence, we obtain the limit problem (1.3) observing that

$$|\Gamma(x_1)| \neq 0 \quad \text{a.e. } x_1 \in \Omega_1.$$

Next, let us prove that we have strong convergence, that is,

$$\sup_{t\in[0,T]} \|u^\epsilon(t,\cdot) - u^*(t,\cdot)\|_{L^2(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0. \tag{3.23}$$

We first observe that, due to (2.4) and (3.21), we have

$$\begin{aligned}
|u^\epsilon(t,x) - u^*(t,x)|^2 \;=\; & \Big| \left(e^{-A_\epsilon(x)t} - e^{-A_0(x)t}\right) u_0(x) \\
& + \int_0^t \left(e^{-A_\epsilon(x)(t-s)} - e^{-A_0(x)(t-s)}\right) f(s,x)\, ds \\
& + \int_0^t \left(e^{-A_\epsilon(x)(t-s)} - e^{-A_0(x)(t-s)}\right) \int_\Omega J_\epsilon(x-y)\, u^\epsilon(s,y)\, dyds \\
& + \int_0^t e^{-A_0(x)(t-s)} \int_\Omega \left(J_\epsilon(x-y) - J(x_1-y_1,0)\right) u^\epsilon(s,y)\, dyds \\
& + \int_0^t e^{-A_0(x)(t-s)} \int_\Omega J(x_1-y_1,0) \left(u^\epsilon(s,y) - u^*(s,y)\right)\, dyds \Big|^2 .
\end{aligned}$$

Thus,

$$\begin{aligned}
|u^\epsilon(t,x) - u^*(t,x)|^2 \le 16 \Bigg[& C^2_{T,\Omega}(\epsilon) \left(|u_0(x)|^2 + t^2 \int_0^t (f(s,x))^2\, ds \right) \\
& + C^2_{T,\Omega}(\epsilon)\, \|J_\epsilon(x-\cdot)\|^2_{L^2(\Omega)} \int_0^t \int_\Omega (u^\epsilon(s,y))^2\, dyds + M^2_{\mathcal{B}}(\epsilon) \int_0^t \int_\Omega (u^\epsilon(s,y))^2\, dyds \\
& + \|J_\epsilon(x-\cdot)\|^2_{L^2(\Omega)} \int_0^t \int_\Omega \left(u^\epsilon(s,y) - u^*(s,y)\right)^2\, dyds \Bigg]
\end{aligned}$$

where

$$C_{T,\Omega}(\epsilon) = \sup_{(t,x)\in[0,T]\times\Omega} \left| e^{-A_\epsilon(x)t} - e^{-A_0(x)t} \right|$$

and

$$M_{\mathcal{B}}(\epsilon) = \sup_{x\in\mathcal{B}} |J_\epsilon(x) - J(x_1,0)|.$$

Here, the set $\mathcal{B}$ denotes a ball in $\mathbb{R}^N$ with radius bigger than twice the diameter of the bounded set Ω. We also have from (3.12) and (3.15) that

$$C_{T,\Omega}(\epsilon) \to 0 \qquad \text{and} \qquad M_{\mathcal{B}}(\epsilon) \to 0 \tag{3.27}$$

as $\epsilon \to 0$.

Hence, we can integrate in Ω the previous inequality obtaining

$$\begin{aligned}
\|u^\epsilon(t,\cdot) - u^*(t,\cdot)\|^2_{L^2(\Omega)} \le 16 \Bigg[& C^2_{T,\Omega}(\epsilon) \left(\|u_0(x)\|^2_{L^2(\Omega)} + t^2 \int_0^t \|f(s,\cdot)\|^2_{L^2(\Omega)}\, ds \right) \\
& + C^2_{T,\Omega}(\epsilon) \left(\int_\Omega \int_\Omega J_\epsilon(x-y)^2\, dydx \right) \int_0^t \int_\Omega (u^\epsilon(s,y))^2\, dyds \\
& + M^2_{\mathcal{B}}(\epsilon)\, |\Omega| \int_0^t \int_\Omega (u^\epsilon(s,y))^2\, dyds \\
& + \left(\int_\Omega \int_\Omega J(x_1-y_1,0)^2\, dydx \right) \int_0^t \|u^\epsilon(s,\cdot) - u^*(s,\cdot)\|^2_{L^2(\Omega)}\, ds \Bigg].
\end{aligned}$$

Consequently, we get (3.23) from (3.9), (3.27) and [11, Gronwall's inequality].

Finally, we conclude the proof obtaining

$$\sup_{t\in[0,T]} \| \left(U^\epsilon(t,\cdot) - U^*(t,\cdot)\right) |\Gamma(\cdot)| \|_{L^2(\Omega)} \to 0, \qquad \text{as } \epsilon \to 0.$$

In fact, this limit is due to (3.23) and the following estimate

$$\begin{aligned}\int_{\Omega_1} |\Gamma(x_1)|^2 \, (U^\epsilon(t,x_1) - U^*(t,x_1))^2 \, dx_1 &= \int_{\Omega_1} \left(\int_{\Gamma(x_1)} (u^\epsilon(t,x_1,x_2) - u^*(t,x_1,x_2)) \, dx_2 \right)^2 dx_1 \\ &\le \int_{\Omega_1} |\Gamma(x_1)| \int_{\Gamma(x_1)} (u^\epsilon(t,x_1,x_2) - u^*(t,x_1,x_2))^2 \, dx_2 dx_1 \\ &\le \sup_{x_1 \in \Omega_1} |\Gamma(x_1)| \int_\Omega (u^\epsilon(t,x) - u^*(t,x))^2 \, dx.\end{aligned}$$

The proof is now complete. □

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