STEKLOV EIGENVALUES FOR THE ∞-LAPLACIAN

J. GARCIA-AZORERO, J. J. MANFREDI, I. PERAL AND J. D. ROSSI

Abstract. We study the Steklov eigenvalue problem for the ∞-laplacian. To this end we consider the limit as $p \to \infty$ of solutions of $-\Delta_p u_p = 0$ in a domain Ω with $|\nabla u_p|^{p-2}\partial u_p/\partial\nu = \lambda |u|^{p-2}u$ on $\partial\Omega$. We obtain a limit problem that is satisfied in the viscosity sense and a geometric characterization of the second eigenvalue.

1. Introduction.

Let $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ be the $p-$laplacian. The *limit operator* $\lim_{p\to\infty}\Delta_p = \Delta_\infty$ is the ∞-Laplacian given by

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial x_i}\frac{\partial u}{\partial x_i}$$

in the viscosity sense. This operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function f, see [A], [ACJ], and [J].

Our concern in this paper is the study of the Steklov eigenvalue problem for the $\infty-$Laplacian. To this end we consider the $\infty-$Laplacian in a bounded smooth domain as limit of the $p-$laplacian as $p \to \infty$. Therefore our aim is to analyze the limit as $p \to \infty$ for the Steklov eigenvalue problem

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \lambda |u|^{p-2}u & \text{on } \partial\Omega,. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\frac{\partial}{\partial\nu}$ is the outer normal derivative. Steklov eigenvalues have been introduced in [S] for $p = 2$. For the existence of a sequence of variational eigenvalues

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see [S] for $p = 2$ and [FBR1] for general p. As happens for the eigenvalues for the Dirichlet problem for the $p-$Laplacian it is not known, in general, if this sequence constitutes the whole spectrum. Note that the first eigenvalue of (1.1) is $\lambda_{1,p} = 0$ with eigenfunction $u_{1,p} \equiv 1$. Hence we can trivially pass to the limit and obtain $\lambda_{1,\infty} = 0$ with eigenfunction $u_{1,\infty} \equiv 1$. Our main result in this paper shows that we can pass to the limit in the variational eigenvalues defined in [FBR1]. Since the first eigenvalue is isolated, [MR], there exists a second eigenvalue that has a variational characterization, [FBR2]. We can pass to the limit in this second eigenvalue and obtain a geometric characterization of the second Steklov eigenvalue for the $\infty-$Laplacian. Moreover we obtain a uniform limit of the sequence of eigenfunctions (along subsequences) and we find a limit eigenvalue problem that is satisfied in a viscosity sense which involves the $\infty-$Laplacian together with a boundary condition with the normal derivative $\frac{\partial u}{\partial \nu}$.

Theorem 1.1. *For the first eigenvalue of* (1.1) *we have,*

$$\lim_{p\to\infty} \lambda_{1,p}^{1/p} = \lambda_{1,\infty} = 0,$$

with eigenfunction given by $u_{1,\infty} = 1$.
For the second eigenvalue, it holds

$$\lim_{p\to\infty} \lambda_{2,p}^{1/p} = \lambda_{2,\infty} = \frac{2}{\operatorname{diam}(\Omega)}.$$

Moreover, given $u_{2,p}$ *eigenfunctions of* (1.1) *of eigenvalues* $\lambda_{2,p}$ *normalized by* $\|u_{2,p}\|_{L^\infty(\partial\Omega)} = 1$, *there exits a sequence* $p_i \to \infty$ *such that* $u_{2,p_i} \to u_{2,\infty}$, *in* $C^\alpha(\overline{\Omega})$. *The limit* $u_{2,\infty}$ *is a solution of*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ \Lambda(x, u, \nabla u) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

in the viscosity sense, where

$$\Lambda(x, u, \nabla u) \equiv \begin{cases} \min\left\{|\nabla u| - \lambda_{2,\infty}|u| \ , \ \frac{\partial u}{\partial \nu}\right\} & \text{if } u > 0, \\ \max\{\lambda_{2,\infty}|u| - |\nabla u| \ , \ \frac{\partial u}{\partial \nu}\} & \text{if } u < 0, \\ \frac{\partial u}{\partial \nu} & \text{if } u = 0. \end{cases}$$

For the k-th eigenvalue we have that if $\lambda_{k,p}$ is the k-th variational eigenvalue of (1.1) *with eigenfunction* $u_{k,p}$ *normalized by* $\|u_{k,p}\|_{L^\infty(\partial\Omega)} = 1$, *then for every sequence* $p_i \to \infty$ *there exists a subsequence such that*

$$\lim_{p_i\to\infty} \lambda_{k,p}^{1/p} = \lambda_{*,\infty}$$

and $u_{k,p_i} \to u_{*,\infty}$ *in* $C^\alpha(\overline{\Omega})$, *where* $u_{*,\infty}$ *and* $\lambda_{*,\infty}$ *is a solution of* (1.2).

We have a simple geometrical characterization of $\lambda_{2,\infty}$ as $2/\text{diam}(\Omega)$. From this characterization and the convergence of the eigenfunctions we conclude that the second Steklov eigenfunction in an annulus or a ball is not radial. Also we have that the domain that maximizes $\lambda_{2,\infty}$ among domains with fixed volume is a ball.

We end the introduction with a brief comment on the Dirichlet case. Eigenvalues of the $p-$Laplacian, $-\Delta_p u = \lambda |u|^{p-2}u$, with Dirichlet boundary conditions, $u = 0$ on $\partial\Omega$, have been extensively studied since [GAP]. The limit as $p \to \infty$ was studied in [JL], [JLM]. In these papers the authors prove results similar to ours. however our proofs are necessarily different due to the presence of the Neumann boundary condition. An anisotropic version of the Dirichlet problem was studied in [BK].

2. The Steklov eigenvalue problem

First, let us recall some well known results concerning the Steklov eigenvalue problem for the $p-$laplacian. To this end, we introduce a topological tool, the *genus*, see [K].

Definition 2.1. *Given a Banach Space X, we consider the class $\Sigma = \{A \subset X : A \text{ is closed}, A = -A\}$. Over this class we define the genus, $\gamma : \Sigma \to \mathbb{N} \cup \{\infty\}$, as*

$$\gamma(A) = \min\{k \in \mathbb{N} : \text{there exists } \varphi \in C(A, \mathbb{R}^k - \{0\}), \varphi(x) = -\varphi(-x)\}.$$

We have the following result whose proof can be obtained following [FBR1], we do not provide the details.

Theorem 2.1. *There exists a sequence of eigenvalues λ_n of* (1.1) *such that $\lambda_n \to +\infty$ as $n \to +\infty$. The so-called variational eigenvalues λ_k can be characterized by*

$$\frac{1}{\lambda_k} = \sup_{C \in C_k} \min_{u \in C} \frac{\|u\|^p_{L^p(\partial\Omega)}}{\|u\|^p_{W^{1,p}(\Omega)}}, \tag{2.1}$$

where $C_k = \{C \subset W^{1,p}(\Omega);\ C \text{ is compact, symmetric and } \gamma(C) \geq k\}$ and γ is the genus.

There exists a second eigenvalue for (1.1) and it coincides with the second variational eigenvalue $\lambda_{2,p}$, see [FBR2]. Moreover, the following characterization of the second eigenvalue $\lambda_{2,p}$ holds

$$\lambda_{2,p} = \inf_{u \in A} \left\{ \frac{\displaystyle\int_\Omega |\nabla u|^p \, dx}{\displaystyle\int_\Omega |u|^p \, d\sigma} \right\},$$

where $A = \{C \subset W^{1,p}(\Omega); \quad C \text{ is compact, symmetric and } \gamma(C) \geq 2\}$. Observe that every eigenfunction associated with λ_2 changes sign on $\partial\Omega$, see [MR].

Following [B] let us recall the definition of viscosity solution taking into account general boundary conditions.

Definition 2.2. *Consider the boundary value problem*

$$\begin{cases} F(x, \nabla u, D^2 u) = 0 & \text{in } \Omega, \\ B(x, u, \nabla u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\max\{B(x_0, \phi(x_0), \nabla\phi(x_0)), \; F(x_0, \nabla\phi(x_0), D^2\phi(x_0))\} \geq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

(2) *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\min\{B(x_0, \phi(x_0), \nabla\phi(x_0)), \; F(x_0, \nabla\phi(x_0), D^2\phi(x_0))\} \leq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \leq 0.$$

(3) *Finally, u is a viscosity solution if it is a super and a subsolution.*

In our case for the Steklov problem for the $p-$Laplacian we have

$$F_p(\eta, X) \equiv -Trace(A_p(\eta)X),$$

where

$$A_p(\eta) = Id + (p-2)\frac{\eta \otimes \eta}{|\eta|^2}, \quad \text{if } \eta \neq 0, \qquad A_p(0) = I_N,$$

and

$$B_p(x,u,\eta) \equiv |\eta|^{p-2} < \eta, \nu(x) > -\lambda |u|^{p-2}u. \tag{2.3}$$

With this notation we have,

$$\Delta_p u = F_p(\nabla u, D^2 u) \equiv - \left\{ \frac{|\nabla \phi(x_0)|^2 \Delta \phi(x_0)}{p-2} + \Delta_\infty \phi(x_0) \right\}.$$

Remark 2.1. *If B_p is monotone in the variable $\frac{\partial u}{\partial \nu}$ Definition 2.2 takes a simpler form, see* [B]. *This is indeed the case for* (2.3). *More concretely, if u is a supersolution of* (1.1) *and $\phi \in C^2(\overline{\Omega})$ is such that $u - \phi$ has a strict minimum at x_0 with $u(x_0) = \phi(x_0)$, then*

(1) *if $x_0 \in \Omega$, then*

$$- \left\{ \frac{|\nabla \phi(x_0)|^2 \Delta \phi(x_0)}{p-2} + \Delta_\infty \phi(x_0) \right\} \geq 0,$$

and

(2) *if $x_0 \in \partial\Omega$, then*

$$|\nabla \phi(x_0)|^{p-2} \langle \nabla \phi(x_0), \nu(x_0) \rangle \geq \lambda |\phi(x_0)|^{p-2} \phi(x_0).$$

Let us state a lemma that says that weak solutions of (1.1) are viscosity solutions.

Lemma 2.1. *A continuous weak solution of* (1.1) *is a viscosity solution.*

Proof. Let $x_0 \in \Omega$ and a test function ϕ such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0. We want to show that

$$-(p-2)|\nabla \phi|^{p-4} \Delta_\infty \phi(x_0) - |\nabla \phi|^{p-2} \Delta \phi(x_0) \geq 0.$$

Assume that this is not the case, then there exists a radius $r > 0$ such that

$$-(p-2)|\nabla \phi|^{p-4} \Delta_\infty \phi(x) - |\nabla \phi|^{p-2} \Delta \phi(x) < 0,$$

for every $x \in B(x_0, r)$. Set $m = \inf_{|x-x_0|=r} (u - \phi)(x)$ and let $\psi(x) = \phi(x) + m/2$. This function ψ verifies $\psi(x_0) > u(x_0)$ and

$$-\mathrm{div}(|\nabla \psi|^{p-2} \nabla \psi) < 0.$$

Multiplying by $(\psi - u)^+$ extended by zero outside $B(x_0, r)$ we get

$$\int_{\{\psi > u\}} |\nabla \psi|^{p-2} \nabla \psi \nabla (\psi - u) < 0.$$

Taking $(\psi - u)^+$ as test function in the weak form we get

$$\int_{\{\psi > u\}} |\nabla u|^{p-2} \nabla u \nabla (\psi - u) = 0.$$

Hence,

$$\begin{aligned} C(N,p)&\int_{\{\psi>u\}}|\nabla\psi-\nabla u|^p \\ &\le\int_{\{\psi>u\}}\langle|\nabla\psi|^{p-2}\nabla\psi-|\nabla u|^{p-2}\nabla u,\nabla(\psi-u)\rangle<0,\end{aligned}$$

a contradiction.

If $x_0\in\partial\Omega$ we want to prove

$$\begin{aligned}\max\{\, &|\nabla\phi(x_0)|^{p-2}<\nabla\phi(x_0),\nu(x_0)>-\lambda|\phi(x_0)|^{p-2}\phi(x_0),\\ &-(p-2)|\nabla\phi|^{p-4}\Delta_\infty\phi(x_0)-|\nabla\phi|^{p-2}\Delta\phi(x_0)\}\ge 0.\end{aligned}$$

Assume that this is not the case. We proceed as before and we obtain

$$\int_{\{\psi>u\}}|\nabla\psi|^{p-2}\nabla\psi\nabla(\psi-u)<\int_{\partial\Omega\cap\{\psi>u\}}\lambda|u|^{p-2}u(\psi-u),$$

and

$$\int_{\{\psi>u\}}|\nabla u|^{p-2}\nabla u\nabla(\psi-u)\ge\int_{\partial\Omega\cap\{\psi>u\}}\lambda|u|^{p-2}u(\psi-u).$$

Therefore,

$$\begin{aligned} C(N,p)&\int_{\{\psi>u\}}|\nabla\psi-\nabla u|^p \\ &\le\int_{\{\psi>u\}}\langle|\nabla\psi|^{p-2}\nabla\psi-|\nabla u|^{p-2}\nabla u,\nabla(\psi-u)\rangle<0,\end{aligned}$$

again a contradiction. This proves that u is a viscosity supersolution. The proof of the fact that u is a viscosity subsolution runs as above, we omit the details. □

With all these preliminaries we are ready to pass to the limit as $p\to\infty$ in the eigenvalue problem.

Since $u_{1,p}\equiv 1$ is the first eigenfunction of (1.1) associated to $\lambda_{1,p}=0$ we can trivially pass to the limit and obtain

$$\lim_{p\to\infty}\lambda_{1,p}^{1/p}=0=\lambda_{1,\infty}$$

and

$$\lim_{p\to\infty}u_{1,p}=1=u_{1,\infty}.$$

Now let us prove a geometrical characterization of the second Steklov eigenvalue for the $\infty-$laplacian, defined by,

$$\lambda_{2,\infty}=\inf\left\{\frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}}\ :\ u\in C\subset W^{1,\infty}(\Omega)\text{ with }\gamma(C)\ge 2\right\}. \tag{2.4}$$

We have

Lemma 2.2. *$\lambda_{2,\infty}$ has the following geometrical characterization*

$$\lambda_{2,\infty} = \frac{2}{\mathrm{diam}(\Omega)}.$$

Proof. Let

$$R = \sup\left\{ r \ : \ \exists x_0, x_1 \in \overline{\Omega} \text{ with } B(x_0, r) \cap B(x_1, r) = \emptyset \right\} = \frac{\mathrm{diam}(\Omega)}{2}.$$

We can take as a test function in (2.4) the combination of two cones centered at x_0 and x_1 with radius R, that is, if

$$C_0(x) = \left(1 - \frac{|x - x_0|}{R}\right)_+, \qquad C_1(x) = \left(1 - \frac{|x - x_1|}{R}\right)_+,$$

we consider

$$\phi(x) = C_0(x) - C_1(x).$$

We obtain

$$\lambda_{2,\infty} \le \frac{1}{R} = \frac{2}{\mathrm{diam}(\Omega)}.$$

To prove the reverse inequality, let us take a function u in $W^{1,\infty}(\Omega)$ that changes sign and such that

$$\lambda_{2,\infty} \ge \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\partial\Omega)}} - \varepsilon.$$

Now u^+ and u^- have disjoint supports and we may normalize with $\|u^+\|_{L^\infty(\partial\Omega)} = \|u^-\|_{L^\infty(\partial\Omega)} = 1$ then

$$\|\nabla u^+\|_{L^\infty(\Omega)} \ge 1/R, \qquad \text{or} \qquad \|\nabla u^-\|_{L^\infty(\Omega)} \ge 1/R.$$

Therefore,

$$\lambda_{2,\infty} \ge \frac{1}{R} - \varepsilon = \frac{2}{\mathrm{diam}(\Omega)} - \varepsilon.$$

Since this holds for every ε, the proof is complete. □

Lemma 2.3.

$$\limsup_{p\to\infty} \lambda_{2,p}^{1/p} \le \lambda_{2,\infty}.$$

Proof. As above, let $C_0(x)$ and $C_1(x)$ two cones centered at x_0 and x_1 of radius R as above, that is

$$C_0(x) = \left(1 - \frac{|x - x_0|}{R}\right)_+, \qquad C_1(x) = -\left(1 - \frac{|x - x_1|}{R}\right)_+.$$

Let us normalize a function $v = aC_0 - bC_1$ $(a, b > 0)$ by $\|v\|_{L^\infty(\partial\Omega)} = 1$, then

$$\lambda_{2,p}^{1/p} \leq \frac{\left(\int_\Omega |\nabla v|^p\right)^{1/p}}{\left(\int_{\partial\Omega} |v|^p\right)^{1/p}}.$$

Hence

$$\limsup \lambda_{2,p}^{1/p} \leq \frac{1}{R} = \lambda_{2,\infty},$$

as we wanted to prove. □

Lemma 2.4. *Given $u_{2,p}$ eigenfunctions of* (1.1) *of eigenvalues $\lambda_{2,p}$ normalized by $\|u_{2,p}\|_{L^\infty(\partial\Omega)} = 1$, there exits a sequence $p_i \to \infty$ such that*

$$u_{2,p_i} \to u_{2,\infty}, \qquad \text{in } C^\alpha(\overline{\Omega}).$$

The limit $u_{2,\infty}$ verifies $\|u_{2,\infty}\|_{L^\infty(\partial\Omega)} = 1$ and it changes sign in $\partial\Omega$. Moreover it is a minimizer of (2.4) *and*

$$\lim_{p\to\infty} \lambda_{2,p}^{1/p} = \lambda_{2,\infty}.$$

Proof. If $q < p$,

$$\begin{aligned}\left(\int_\Omega |\nabla u_{2,p}|^q\right)^{1/q} &\leq |\Omega|^{(1/q)-(1/p)} \left(\int_\Omega |\nabla u_{2,p}|^p\right)^{1/p} \\ &= \lambda_{2,p}^{1/p} |\Omega|^{(1/q)-(1/p)} \left(\int_{\partial\Omega} |u_{2,p}|^p\right)^{1/p} \leq \lambda_{2,p}^{1/p} |\Omega|^{(1/q)-(1/p)} |\partial\Omega|^{1/p}.\end{aligned} \tag{2.5}$$

Therefore, by Lemma 2.3, we get that there exists a constant C independent of p such that

$$\left(\int_\Omega |\nabla u_{2,p}|^q\right)^{1/q} \leq C. \tag{2.6}$$

Hence, as $u_{2,p}$ are uniformly bounded in $W^{1,q}(\Omega)$ we can take a subsequence such that it converges weakly in $W^{1,q}(\Omega)$ (and hence in $C^\alpha(\overline{\Omega})$ if $q > N$) to a limit $u_{2,\infty}$. Since this can be done for any q we obtain that $u_{2,\infty} \in W^{1,\infty}(\Omega)$. Indeed, from (2.5), we get

$$\left(\int_\Omega |\nabla u_{2,\infty}|^q\right)^{1/q} \leq \limsup_{p_i\to\infty} \left(\int_\Omega |\nabla u_{2,p_i}|^q\right)^{1/q} \leq \lambda_{2,\infty} |\Omega|^{1/q}. \tag{2.7}$$

Hence, taking limit as $q \to \infty$ in (2.7) we get

$$\|\nabla u_{2,\infty}\|_{L^\infty(\Omega)} \leq \liminf_{p\to\infty} \lambda_{2,p}^{1/p} \leq \lambda_{2,\infty}. \tag{2.8}$$

From the convergence in $C^\alpha(\overline{\Omega})$ of the sequence u_{2,p_i} and the normalization

$$\|u_{2,p_i}\|_{L^\infty(\partial\Omega)} = 1$$

we obtain that

$$\|u_{2,\infty}\|_{L^\infty(\partial\Omega)} = 1. \tag{2.9}$$

To end the proof we need to check that $u_{2,\infty}$ changes sign. Assume that $u_{2,\infty} \geq 0$. Hence u_{2,p_i}^- converges uniformly to zero in $\overline{\Omega}$. From (2.9) there exists a point $x_0 \in \partial\Omega$ such that $u_{2,\infty}(x_0) = 1$. At level p we have,

$$\int_{\partial\Omega} |u|^{p-2}u = 0,$$

then

$$\int_{\partial\Omega} (u^+)^{p-1} = \int_{\partial\Omega} (u^-)^{p-1}.$$

Therefore,

$$\begin{aligned} &|\partial\Omega|^{(1/r)-(1/(p_i-1))} \left(\int_{\partial\Omega} |u_{2,p_i}^+|^r \right)^{1/r} \leq \left(\int_{\partial\Omega} |u_{2,p_i}^+|^{p_i-1} \right)^{1/(p_i-1)} \\ &= \left(\int_{\partial\Omega} |u_{2,p_i}^-|^{p_i-1} \right)^{1/(p_i-1)} \leq |\partial\Omega|^{1/(p_i-1)} \|u_{2,p_i}^-\|_{L^\infty(\partial\Omega)}. \end{aligned} \tag{2.10}$$

From the uniform convergence of u_{2,p_i} and (2.8) taking limit as $p_i \to \infty$ we get that

$$|\partial\Omega|^{1/r} \left(\int_{\partial\Omega} |u_{2,\infty}^+|^r \right)^{1/r} \leq 0. \tag{2.11}$$

A contradiction. This proves that $u_{2,\infty}$ changes sign and verifies (2.8) and (2.9), hence, from the definition of $\lambda_{2,\infty}$ we obtain

$$\lambda_{2,\infty} \leq \liminf_{p\to\infty} \lambda_{2,p}^{1/p}.$$

This fact and Lemma 2.3 end the proof. $\square$

Now let us analyze the equation satisfied by $u_{2,\infty}$. Let

$$\Lambda(x,u,\eta) \equiv \begin{cases} \min\{|\eta| - \lambda_{2,\infty}|u| \ , \ <\eta,\nu(x)>\} & \text{if } u > 0, \\ \max\{\lambda_{2,\infty}|u| - |\eta| \ , \ <\eta,\nu(x)>\} & \text{if } u > 0, \\ <\eta,\nu(x)> & \text{if } u = 0, \end{cases}$$

Lemma 2.5. *The limit $u_{2,\infty}$ is a viscosity solution of*

$$\begin{cases} \Delta_\infty u_{2,\infty} = 0 & \text{in } \Omega, \\ \Lambda(x,u,\nabla u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{2.12}$$

Proof. First, let us check that $-\Delta_\infty u_{2,\infty} = 0$ in the viscosity sense in Ω. Let us recall the standard proof. Let ϕ be a smooth test function such that $v_\infty - \phi$ has a strict maximum at $x_0 \in \Omega$. Since u_{p_i} converges uniformly to v_∞ we get that $u_{p_i} - \phi$ has a maximum at some point $x_i \in \Omega$ with $x_i \to x_0$. Now we use the fact that u_{p_i} is a viscosity solution of

$$-\Delta_p u_p = 0$$

and we obtain

$$-(p_i - 2)|\nabla\phi|^{p_i-4}\Delta_\infty\phi(x_i) - |\nabla\phi|^{p_i-2}\Delta\phi(x_i) \le 0. \tag{2.13}$$

If $\nabla\phi(x_0) = 0$ we get $-\Delta_\infty\phi(x_0) \le 0$. If this is not the case, we have that $\nabla\phi(x_i) \neq 0$ for large i and then

$$-\Delta_\infty\phi(x_i) \le \frac{1}{p_i - 2}|\nabla\phi|^2\Delta\phi(x_i) \to 0, \quad \text{as } i \to \infty.$$

We conclude that

$$-\Delta_\infty\phi(x_0) \le 0.$$

That is v_∞ is a viscosity subsolution of $-\Delta_\infty u_\infty = 0$.

Now we check the boundary condition.

Assume that $u_{2,\infty} - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ such that $u_{2,\infty}(x_0) = \phi(x_0) > 0$. Using the uniform convergence of u_{2,p_i} to $u_{2,\infty}$ we obtain that $u_{2,p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty\phi(x_0) \ge 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i) \ge \lambda_{2,p_i}|\phi|^{p_i-2}(x_i)\phi(x_i).$$

If $\nabla\phi(x_0) = 0$, then

$$\frac{\partial\phi}{\partial\nu}(x_0) = 0.$$

If $\nabla\phi(x_0) \neq 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i) \ge \lambda_{2,p_i}^{1/(p_i-1)}\left(\frac{\lambda_{2,p_i}^{1/(p_i-1)}|\phi|}{|\nabla\phi|}(x_i)\right)^{p_i-2}\phi(x_i).$$

Using that $\lambda_{2,p}^{1/(p-1)} \to \lambda_{2,\infty}$ as $p \to \infty$ we conclude that

$$\frac{\lambda_{2,\infty}|\phi|}{|\nabla\phi|}(x_0) \le 1.$$

Moreover,

$$\frac{\partial \phi}{\partial \nu}(x_0) \geq 0.$$

Hence, if $u_{2,\infty} - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $\phi(x_0) = u_{2,\infty}(x_0) > 0$, we have

$$\max\left\{ \min\{-\lambda_{2,\infty}|\phi| + |\nabla\phi|(x_0), \frac{\partial \phi}{\partial \nu}(x_0)\} \ , -\Delta_\infty \phi(x_0) \right\} \geq 0. \tag{2.14}$$

Now assume that $u_{2,\infty} - \phi$ has a strict maximum at $x_0 \in \partial\Omega$ with $u_{2,\infty}(x_0) = \phi(x_0) > 0$. Using the uniform convergence of u_{2,p_i} to $u_{2,\infty}$ we obtain that $u_{2,p_i} - \phi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty u_{2,\infty}(x_0) \leq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \leq \lambda_{2,p_i}|\phi|^{p_i-2}(x_i)\phi(x_i).$$

If $\nabla\phi(x_0) = 0$, then

$$\frac{\partial \phi}{\partial \nu}(x_0) = 0.$$

If $\nabla\phi(x_0) \neq 0$ we obtain

$$\frac{\partial \phi}{\partial \nu}(x_i) \leq \lambda_{2,p_i}^{1/(p_i-1)} \left(\frac{\lambda_{2,p_i}^{1/(p_i-1)}|\phi|}{|\nabla\phi|}(x_i) \right)^{p_i-2} \phi(x_i).$$

If $\lambda_{2,\infty}|\phi|(x_0) < |\nabla\phi|(x_0)$, then

$$\frac{\partial \phi}{\partial \nu}(x_0) \leq 0.$$

Hence,

$$\min\left\{ \min\{-\lambda_{2,\infty}|\phi| + |\nabla\phi|(x_0), \frac{\partial \phi}{\partial \nu}(x_0)\} \ , -\Delta_\infty \phi(x_0) \right\} \leq 0. \tag{2.15}$$

Now assume that $u_{2,\infty} - \phi$ has a strict maximum at x_0 such that $u_{2,\infty}(x_0) = \phi(x_0) < 0$ Using the uniform convergence of u_{2,p_i} to $u_{2,\infty}$ we obtain that $u_{2,p_i} - \phi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty u_{2,\infty}(x_0) \leq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \leq \lambda_{2,p_i}|\phi|^{p_i-2}(x_i)\phi(x_i).$$

If $\nabla\phi(x_0)=0$, then

$$\frac{\partial\phi}{\partial\nu}(x_0)=0.$$

If $\nabla\phi(x_0)\neq 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i)\le\lambda_{2,p_i}^{1/(p_i-1)}\left(\frac{\lambda_{2,p_i}^{1/(p_i-1)}|\phi|}{|\nabla\phi|}(x_i)\right)^{p_i-2}\phi(x_i).$$

Using that $\lambda_{2,p}^{1/(p-1)}\to\lambda_{2,\infty}$ as $p\to\infty$ we conclude that

$$\frac{\lambda_{2,\infty}|\phi|}{|\nabla\phi|}(x_0)\le 1.$$

Moreover,

$$\frac{\partial\phi}{\partial\nu}(x_0)\le 0.$$

Hence,

$$\min\left\{\max\{\lambda_{2,\infty}|\phi|-|\nabla\phi|(x_0),\frac{\partial\phi}{\partial\nu}(x_0)\}\ ,-\Delta_\infty\phi(x_0)\right\}\le 0. \tag{2.16}$$

Now assume that $u_{2,\infty}-\phi$ has a strict minimum at $x_0\in\partial\Omega$ with $u_{2,\infty}(x_0)=\phi(x_0)<0$. Using the uniform convergence of u_{2,p_i} to $u_{2,\infty}$ we obtain that $u_{2,p_i}-\phi$ has a minimum at some point $x_i\in\overline{\Omega}$ with $x_i\to x_0$. If $x_i\in\Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty\phi(x_0)\ge 0.$$

On the other hand if $x_i\in\partial\Omega$ we have

$$|\nabla\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i)\ge\lambda_{2,p_i}|\phi|^{p_i-2}(x_i)\phi(x_i).$$

If $\nabla\phi(x_0)=0$, then

$$\frac{\partial\phi}{\partial\nu}(x_0)=0.$$

If $\nabla\phi(x_0)\neq 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i)\ge\lambda_{2,p_i}^{1/(p_i-1)}\left(\frac{\lambda_{2,p_i}^{1/(p_i-1)}|\phi|}{|\nabla\phi|}(x_i)\right)^{p_i-2}\phi(x_i).$$

If $\lambda_{2,\infty}|\phi|(x_0)<|\nabla\phi|(x_0)$, then

$$\frac{\partial\phi}{\partial\nu}(x_0)\ge 0.$$

Hence,

$$\max\left\{\max\{\lambda_{2,\infty}|\phi|-|\nabla\phi|(x_0),\frac{\partial\phi}{\partial\nu}(x_0)\}\ ,-\Delta_\infty\phi(x_0)\right\}\ge 0. \tag{2.17}$$

Assume that $u_{2,\infty} - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ such that $u_{2,\infty}(x_0) = \phi(x_0) = 0$. Using the uniform convergence of u_{2,p_i} to $u_{2,\infty}$ we obtain that $u_{2,p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \geq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i) \geq \lambda_{2,p_i}|\phi|^{p_i-2}(x_i)\phi(x_i).$$

If $\nabla\phi(x_0) = 0$, then

$$\frac{\partial\phi}{\partial\nu}(x_0) = 0.$$

If $\nabla\phi(x_0) \neq 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i) \geq \lambda_{2,p_i}^{1/(p_i-1)}\left(\frac{\lambda_{2,p_i}^{1/(p_i-1)}|\phi|}{|\nabla\phi|}(x_i)\right)^{p_i-2}\phi(x_i).$$

As $0 = \lambda_{2,\infty}|\phi|(x_0) < |\nabla\phi|(x_0)$, then

$$\frac{\partial\phi}{\partial\nu}(x_0) \geq 0.$$

Hence,

$$\max\left\{\frac{\partial\phi}{\partial\nu}(x_0)\ ,-\Delta_\infty\phi(x_0)\right\} \geq 0. \tag{2.18}$$

Finally, assume that $u_{2,\infty} - \phi$ has a strict maximum at x_0 with $u_{2,\infty}(x_0) = \phi(x_0) = 0$ Using the uniform convergence of u_{2,p_i} to $u_{2,\infty}$ we obtain that $u_{2,p_i} - \phi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty u_{2,\infty}(x_0) \leq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|\nabla\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i) \leq \lambda_{2,p_i}|\phi|^{p_i-2}(x_i)\phi(x_i).$$

If $\nabla\phi(x_0) = 0$, then

$$\frac{\partial\phi}{\partial\nu}(x_0) = 0.$$

If $\nabla\phi(x_0) \neq 0$ we obtain

$$\frac{\partial\phi}{\partial\nu}(x_i) \leq \lambda_{2,p_i}^{1/(p_i-1)}\left(\frac{\lambda_{2,p_i}^{1/(p_i-1)}|\phi|}{|\nabla\phi|}(x_i)\right)^{p_i-2}\phi(x_i).$$

As $0 = \lambda_{2,\infty}|\phi|(x_0) < |\nabla\phi|(x_0)$, then

$$\frac{\partial\phi}{\partial\nu}(x_0) \leq 0.$$

Hence,

$$\min\left\{\frac{\partial\phi}{\partial\nu}(x_0)\ ,-\Delta_\infty\phi(x_0)\right\} \leq 0. \tag{2.19}$$

Inequalities (2.14)-(2.19) prove the result. □

With the same ideas used to deal with the second eigenvalue we can prove the following lema.

Lemma 2.6. *Let $\lambda_{k,p}$ be the k-th variational eigenvalue of* (1.1) *with eigenfunction $u_{k,p}$ normalized by $\|u_{k,p}\|_{L^\infty(\partial\Omega)} = 1$. Then for every sequence $p_i \to \infty$ there exists a subsequence such that*

$$\lim_{p_i\to\infty} \lambda_{k,p}^{1/p} = \lambda_{*,\infty},$$

$$u_{k,p_i} \to u_{*,\infty}, \qquad in\ C^\alpha(\overline{\Omega}),$$

where $(u_{*,\infty}, \lambda_{*,\infty})$ is a solution of (1.2).

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Departamento de Matemáticas, U. Autonoma de Madrid,
28049 Madrid, Spain.
E-mail address: jesus.azorero@uam.es, ireneo.peral@uam.es

Department of Mathematics,
University of Pittsburgh. Pittsburgh, Pennsylvania 15260.
E-mail address: manfredi@math.pitt.edu

Consejo Superior de Investigaciones Científicas (CSIC),
Serrano 123, Madrid, Spain,
on leave from Departamento de Matemática, FCEyN UBA (1428)
Buenos Aires, Argentina.
E-mail address: jrossi@dm.uba.ar