THE NEUMANN PROBLEM FOR THE ∞-LAPLACIAN AND THE MONGE-KANTOROVICH MASS TRANSFER PROBLEM

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Abstract. We consider the natural Neumann boundary condition for the ∞-Laplacian. We study the limit as $p \to \infty$ of solutions of $-\Delta_p u_p = 0$ in a domain Ω with $|Du_p|^{p-2}\partial u_p/\partial\nu = g$ on $\partial\Omega$. We obtain a natural minimization problem that is verified by a limit point of $\{u_p\}$ and a limit problem that is satisfied in the viscosity sense. It turns out that the limit variational problem is related to the Monge-Kantorovich mass transfer problems when the measures are supported on $\partial\Omega$.

1. Introduction.

In this paper we study the natural Neumann boundary conditions that appear when one considers the ∞-Laplacian in a smooth bounded domain as limit of the Neumann problem for the p-Laplacian as $p \to \infty$. This problem is related to the Monge-Kantorovich mass tranfer problem when the involved measures are supported on the boundary of the domain.

Let $\Delta_p u = \operatorname{div}\left(|Du|^{p-2}Du\right)$ be the p-Laplacian. The ∞-Laplacian is the limit operator $\Delta_\infty = \lim_{p\to\infty}\Delta_p$ given by

$$\Delta_\infty u = \sum_{i,j=1}^{N} \frac{\partial u}{\partial x_j}\frac{\partial^2 u}{\partial x_j \partial x_i}\frac{\partial u}{\partial x_i}$$

in the viscosity sense. This operator appears naturally when one considers absolutely minimizing Lipschitz extensions of a boundary function f; see [A], [ACJ], and [J]. A fundamental result of Jensen [J] establishes that the Dirichlet problem for Δ_∞ is well posed in the viscosity sense.

When considering the Neumann problem, boundary conditions that involve the outer normal derivative, $\partial u/\partial\nu$ have been addressed from the point of view of viscosity solutions for fully nonlinear equations in [B] and [ILi]. In these references it is proved that there exist viscosity solutions and comparison principles between them when appropriate hypothesis are satisfied. In particular strict monotonicity with respect to u is needed, a property that does not hold in our case of interest.

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We study the Neumann problem for the ∞-Laplacian obtained as the limit as $p \to \infty$ of the problems

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ |Du|^{p-2}\frac{\partial u}{\partial \nu} = g & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

Here Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and $\frac{\partial}{\partial \nu}$ is the outer normal derivative. The boundary data g is a continuous function that necessarily verifies the compatibility condition

$$\int_{\partial\Omega} g = 0,$$

otherwise there is no solution to (1.1). Imposing the normalization

$$\int_{\Omega} u = 0 \tag{1.2}$$

there exists a unique solution to problem (1.1) that we denote by u_p. This solution can also be obtained by a variational principle. In fact, we can write

$$\int_{\partial\Omega} u_p\, g = \max\left\{ \int_{\partial\Omega} wg \colon w \in W^{1,p}(\Omega),\ \int_{\Omega} w = 0\,,\ \int_{\Omega} |Dw|^p \leq 1 \right\}.$$

Our first result states that there exist limit points of u_p as $p \to \infty$ and that they are maximizers of a variational problem that is a natural limit of these variational problems.

Observe that for $q > N$ the set $\{u_p\}_{p>q}$ is bounded in $C^{1-p/q}(\overline{\Omega})$. Let v_∞ be a uniform limit of a subsequence $\{u_{p_i}\}$, $p_i \to \infty$.

Theorem 1.1. *A limit function v_∞ is a solution to the maximization problem*

$$\int_{\partial\Omega} v_\infty g = \max\left\{ \int_{\partial\Omega} wg \colon w \in W^{1,\infty}(\Omega),\ \int_{\Omega} w = 0,\ \|Dw\|_\infty \leq 1 \right\}. \tag{1.3}$$

An equivalent dual statement is the minimization problem

$$\|Dv_\infty\|_\infty = \min\left\{ \|Dw\|_\infty \colon w \in W^{1,\infty}(\Omega),\ \int_{\Omega} w = 0,\ \int_{\partial\Omega} wg \geq 1 \right\}. \tag{1.4}$$

The maximization problem (1.3) is also obtained by applying the Kantorovich optimality principle to a mass transfer problem for the measures $\mu^+ = g^+ \mathcal{H}^{N-1} \llcorner \partial\Omega$ and $\mu^- = g^- \mathcal{H}^{N-1} \llcorner \partial\Omega$ that are concentrated on $\partial\Omega$. The mass transfer compatibility condition $\mu^+(\partial\Omega) = \mu^-(\partial\Omega)$ holds since g has zero average on $\partial\Omega$. The maximizers of (1.3) are called maximal Kantorovich potentials [Am].

Evans and Gangbo [EG] have considered mass transfer optimization problems between absolutely continuous measures that appear as limits of p-Laplacian problems. A very general approach is discussed in [BBP], where a problem related to but different from ours is discussed (see Remark 4.3 in [BBP].)

Our next results discusses the equation that v_∞ satisfies in the viscosity sense.

Theorem 1.2. *A limit v_∞ is a solution of*

$$\begin{cases} \Delta_\infty u = 0 & \text{in } \Omega, \\ B(x, u, Du) = 0, & \text{on } \partial\Omega, \end{cases} \tag{1.5}$$

in the viscosity sense. Here

$$B(x,u,Du) \equiv \begin{cases} \min\left\{|Du|-1\ ,\ \frac{\partial u}{\partial \nu}\right\} & \text{if } g>0, \\ \max\{1-|Du|\ ,\ \frac{\partial u}{\partial \nu}\} & \text{if } g<0, \\ H(|Du|)\frac{\partial u}{\partial \nu} & \text{if } g=0, \end{cases}$$

and $H(a)$ is given by

$$H(a) = \begin{cases} 1 & \text{if } a \geq 1, \\ 0 & \text{if } 0 \leq a < 1. \end{cases}$$

Observe that the normalization (1.2) is not necessary to obtain existence of a solution to (1.1) nor to (1.5) since both problems are invariant by adding a constant to the solution u.

The next question to consider is whether we have uniqueness of viscosity solutions of (1.5). Examples discussed in Section §3 show that this is not the case. Nevertheless we can say something about uniqueness under some favorable geometric assumptions on g and Ω. We will need that a limit is infinite harmonic in Ω by Theorem 1.2, a maximizer of (1.3), and some geometric assumptions on g and Ω. The proof is based on some tools from [EG].

To state our uniqueness result let us describe the required geometrical hypothesis on the boundary data. Let $\partial\Omega_+ = \operatorname{supp} g^+$ and $\partial\Omega_- = \operatorname{supp} g^-$. For a given v_∞ a maximizer in (1.3) following [EG] we define the transport set as

$$T(v_\infty) = \left\{ \begin{array}{r} z \in \overline{\Omega} : \exists x \in \partial\Omega_+, y \in \partial\Omega_-, \quad v_\infty(z) = v_\infty(x) - |x-z| \\ \text{and} \quad v_\infty(z) = v_\infty(y) + |y-z| \end{array} \right\}. \tag{1.6}$$

Observe that this set T is closed. We have the following property

Proposition 1.1. *Suppose that Ω is a convex domain. Let v_∞ be a maximizer of (1.3) with $\Delta_\infty v_\infty = 0$, then*

$$|Dv_\infty(x)| = 1, \qquad \text{for a.e. } x \in T(v_\infty).$$

Define a transport ray by $R_x = \{z \mid |v_\infty(x) - v_\infty(z)| = |x-z|\}$. Notice that two transport rays cannot intersect in Ω unless they are identical. Indeed, assume $z \in T$ then there exist $x, y \in \overline{\Omega}$ such that $v_\infty(x) - v_\infty(z) = |x-z|$ and $v_\infty(z) - v_\infty(y) = |z-y|$, then $|x-y| \leq |x-z| + |z-y| = v_\infty(x) - v_\infty(y)$. If x, y and z are not colinear we contradict the Lipschitz condition verified by v_∞.

Our geometric hypothesis for uniqueness is then

$$T(v_\infty) = \overline{\Omega}.$$

We have:

Theorem 1.3. *Assume that we have a convex domain Ω and a boundary datum g on $\partial\Omega$ such that every maximizer v_∞ with $\Delta_\infty v_\infty = 0$ verifies $T(v_\infty) = \overline{\Omega}$, then there exists a unique infinite harmonic solution, u_∞ to (1.3). Hence, the limit*

$$\lim_{p\to\infty} u_p = u_\infty, \qquad \text{uniformly in } \Omega \tag{1.7}$$

exists.

To illustrate our results we present some examples. In an interval $\Omega = (-L, L)$ with $g(L) = -g(-L) > 0$ the limit of the solutions of (1.1), u_p, turns out to be $u_\infty(x) = x$. It is easy to check that this function is indeed the unique solution of the maximization problem (1.3) and of the problem (1.5).

This example can be easily generalized to the case where Ω is an annulus,

$$\Omega = \{r_1 < |x| < r_2\},$$

and the function g is a positive constant g_1 on $|x| = r_1$ and a negative constant g_2 on $|x| = r_2$ with the constraint

$$\int_{\partial\Omega} g = \int_{|x|=r_1} g + \int_{|x|=r_2} g = 0.$$

The solutions u_p of (1.1) in the annulus converge uniformly as $p \to \infty$ to a cone

$$u_\infty(x) = C - |x|.$$

However one can modify the function g on $|x| = r_2$ in such a way it does not change its sign and that the cone does not maximize (1.3), see Section §3. Hence, there is no uniqueness for (1.5) even for non-vanishing boundary data.

An example of a domain and boundary data such that uniqueness of the limit holds is a disk in $\mathbb{R}^2$, $D = \{|(x,y)| < 1\}$ with $g(x,y) > 0$ for $x > 0$ and $g(x,y) < 0$ for $x < 0$ with $\int_{\partial D} g = 0$. The details are in Section §3.

2. The Neumann problem

In this section we prove that there exists a limit, v_∞, of the solutions at level p, u_p. It satisfies a variational principle (1.3) and it is a solution to (1.5).

Recall from the introduction that we call u_p the solution of (1.1) with the normalization (1.2). As we have mentioned, this solution can be obtained by a variational principle. Indeed, consider the minimum in S of the following functional

$$J_p(u) = \int_\Omega |Du|^p - \int_{\partial\Omega} ug$$

where S is given by

$$S = \left\{ u \in W^{1,p}(\Omega) \ : \ \int_\Omega u = 0 \right\}.$$

Lemma 2.1. *The functional J_p attains a minimun in S. Moreover there is a unique minimizer.*

Proof. It is standard. For the sake of completeness we provide the proof when $p \geq 2$. The functional attains a minimum in S since for every r, $1 \leq r < p(N-1)/(N-p)$, the embedding $S \hookrightarrow L^r(\partial\Omega)$ is compact.

Next, let us show that if u and v are weak solutions of (1.1) then they agree up to an additive constant. The proof of this fact is just to multiply the equation by $u - v$ and integrate. We get

$$\begin{aligned} 0 = & \int_\Omega < |Du|^{p-2}Du - |Dv|^{p-2}Dv, Du - Dv > \\ & - \int_{\partial\Omega} (u-v)(|Du|^{p-2}\partial_\nu u - |Dv|^{p-2}\partial_\nu v) \\ = & \int_\Omega < |Du|^{p-2}Du - |Dv|^{p-2}Dv, Du - Dv > \\ \geq & \ C(N,p) \int_\Omega |Du - Dv|^p. \end{aligned}$$

Hence $Du = Dv$. □

An alternative variational formulation that is equivalent to the previous one is to consider the maximization problem

$$M_p = \max \left\{ \int_{\partial\Omega} wg \, : \, w \in W^{1,p}(\Omega) \, : \, \int_\Omega w = 0 \, , \, \int_\Omega |Dw|^p \le 1 \right\}.$$

Denoting a maximizer by $\tilde{u}_p$ we have

$$\Delta_p \tilde{u}_p = 0$$

with the boundary condition

$$|D\tilde{u}_p|^{p-2} \frac{\partial \tilde{u}_p}{\partial \nu} = \frac{g}{M_p}.$$

Hence, it holds

$$u_p \equiv M_p^{1/(p-1)} \tilde{u}_p.$$

The quantity M_p is uniformly bounded in $p \in [2, \infty)$. To see this fact we use the trace inequality to obtain

$$M_p = \int_{\partial\Omega} \tilde{u}_p g \le \|g\|_\infty \int_{\partial\Omega} |\tilde{u}_p| \le C_1 \|g\|_\infty \int_\Omega |D\tilde{u}_p| \le C_1 \|g\|_\infty.$$

Suppose that we have a sequence $\{u_p\}$ of solutions to (1.1). We derive some estimates on the family u_p. Since we are interested in large values of p we may assume that $p > N$ and hence $u_p \in C^\alpha(\overline{\Omega})$. Multiplying the equation by u_p and integrating we obtain,

$$\int_\Omega |Du_p|^p = \int_{\partial\Omega} u_p \, g \le \left(\int_{\partial\Omega} |u_p|^p \right)^{1/p} \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} \tag{2.1}$$

where p' is the exponent conjugate to p, that is $1/p' + 1/p = 1$. Recall the following trace inequality, see for example [E],

$$\int_{\partial\Omega} |\phi|^p d\sigma \le Cp \left(\int_\Omega |\phi|^p + |D\phi|^p dx \right),$$

where C is a constant that does not depend on p. Going back to (2.1), we get,

$$\int_\Omega |Du_p|^p \le \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} C^{1/p} p^{1/p} \left(\int_\Omega |u_p|^p + |Du_p|^p dx \right)^{1/p}.$$

On the other hand, for large p we have

$$|u_p(x) - u_p(y)| \le C_p |x - y|^{1 - \frac{N}{p}} \left(\int_\Omega |Du_p|^p dx \right)^{1/p}.$$

Since we are assuming that $\int_\Omega u_p = 0$, we may choose a point y such that $u_p(y) = 0$, and hence

$$|u_p(x)| \le C(p, \Omega) \left(\int_\Omega |Du_p|^p dx \right)^{1/p}.$$

The arguments in [E], pag. 266-267, show that the constant $C(p, \Omega)$ can be chosen uniformly in p. Hence, we obtain

$$\int_\Omega |Du_p|^p \le \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p'} C^{1/p} p^{1/p} (C_2^p + 1)^{1/p} \left(\int_\Omega |Du_p|^p dx \right)^{1/p}.$$

Taking into account that $p' = p/(p-1)$, for large values of p we get

$$\left(\int_\Omega |Du_p|^p \right)^{1/p} \leq \alpha_p \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p}$$

where $\alpha_p \to 1$ as $p \to \infty$. Next, fix m, and take $p > m$. We have,

$$\left(\int_\Omega |Du_p|^m \right)^{1/m} \leq |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_\Omega |Du_p|^p \right)^{1/p} \leq |\Omega|^{\frac{1}{m}-\frac{1}{p}} \left(\int_{\partial\Omega} |g|^{p'} \right)^{1/p},$$

where $|\Omega|^{\frac{1}{m}-\frac{1}{p}} \to |\Omega|^{\frac{1}{m}}$ as $p \to \infty$. Hence, there exists a weak limit in $W^{1,m}(\Omega)$ that we will denote by v_∞. This weak limit has to verify

$$\left(\int_\Omega |Dv_\infty|^m \right)^{1/m} \leq |\Omega|^{\frac{1}{m}}.$$

As the above inequality holds for every m, we get that $v_\infty \in W^{1,\infty}(\Omega)$ and moreover, taking the limit $m \to \infty$,

$$|Dv_\infty| \leq 1, \qquad \text{a.e. } x \in \Omega.$$

Lemma 2.2. *The subsequence u_{p_i} converge to v_∞ uniformly in $\overline{\Omega}$.*

Proof. From our previous estimates we know that

$$\left(\int_\Omega |Du_p|^p dx \right)^{1/p} \leq C,$$

uniformly in p. Therefore we conclude that u_p is bounded (independently of p) and has a uniform modulus of continuity. Hence u_p converges uniformly to v_∞. ☐

Proof of Theorem 1.1. Multiplying by u_p, passing to the limit, and using Lemma 2.2, we obtain,

$$\lim_{p\to\infty} \int_\Omega |Du_p|^p = \lim_{p\to\infty} \int_{\partial\Omega} u_p g = \int_{\partial\Omega} v_\infty g.$$

If we multiply (1.1) by a test function w, we have, for large enough p,

$$\begin{aligned} \int_{\partial\Omega} wg &\leq \left(\int_\Omega |Du_p|^p \right)^{(p-1)/p} \left(\int_\Omega |Dw|^p \right)^{1/p} \\ &\leq \left(\int_{\partial\Omega} v_\infty g d\sigma + \delta \right)^{(p-1)/p} \left(\int_\Omega |Dw|^p \right)^{1/p}. \end{aligned}$$

As the previous inequality holds for every $\delta > 0$, passing to the limit as $p \to \infty$ we conclude,

$$\int_{\partial\Omega} wg \leq \left(\int_{\partial\Omega} v_\infty g \right) \|Dw\|_\infty.$$

Hence, the function v_∞ verifies,

$$\int_{\partial\Omega} v_\infty g = \max \left\{ \int_{\partial\Omega} wg \,:\, w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \|Dw\|_\infty \leq 1 \right\},$$

or equivalently,

$$\|Dv_\infty\|_\infty = \min \left\{ \|Dw\|_\infty \,:\, w \in W^{1,\infty}(\Omega), \int_\Omega w = 0, \int_{\partial\Omega} wg \leq 1 \right\}.$$

This ends the proof. ☐

On the other hand, taking as a test function in the maximization problem v_∞ itself we obtain the following corollary.

Corollary 2.1. *If $g \not\equiv 0$, then $\|Dv_\infty\|_{L^\infty(\Omega)} = 1$.*

Following [B] let us recall the definition of viscosity solution taking into account general boundary conditions.

Definition 2.1. *Consider the boundary value problem*

$$\left\{ \begin{array}{ll} F(x, Du, D^2u) = 0 & \text{in } \Omega, \\ B(x, u, Du) = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.2}$$

(1) *A lower semi-continuous function u is a viscosity supersolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\max\{B(x_0, \phi(x_0), D\phi(x_0)),\ F(x_0, D\phi(x_0), D^2\phi(x_0))\} \geq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \geq 0.$$

(2) *An upper semi-continuous function u is a subsolution if for every $\phi \in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality*

$$\min\{B(x_0, \phi(x_0), D\phi(x_0)),\ F(x_0, D\phi(x_0), D^2\phi(x_0))\} \leq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F(x_0, D\phi(x_0), D^2\phi(x_0)) \leq 0.$$

(3) *Finally, u is a viscosity solution if it is a super and a subsolution.*

We will use the following notation

$$F_p(\eta, X) \equiv -Trace(A_p(\eta)X),$$

where

$$A_p(\eta) = Id + (p-2)\frac{\eta \otimes \eta}{|\eta|^2}, \quad \text{if } \eta \neq 0, \qquad A_p(0) = I_N,$$

and the notation

$$B_p(x, u, \eta) \equiv |\eta|^{p-2} < \eta, \nu(x) > -g(x). \tag{2.3}$$

If we have a weak solution of (1.1) that is continuous in $\overline{\Omega}$ then it is a viscosity solution. This is the content of our next result.

Lemma 2.3. *Let u be a continuous weak solution of* (1.1) *for $p > 2$. Then u is a viscosity solution of*

$$\left\{ \begin{array}{ll} F_p(Du, D^2u) = 0 & \text{in } \Omega, \\ B_p(x, u, Du) = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{2.4}$$

Proof. Let $x_0 \in \Omega$ and a test function ϕ such that $u(x_0) = \phi(x_0)$ and $u - \phi$ has a strict minimum at x_0. We want to show that

$$-(p-2)|D\phi|^{p-4}\Delta_\infty\phi(x_0) - |D\phi|^{p-2}\Delta\phi(x_0) \geq 0.$$

Assume that this is not the case, then there exists a radius $r > 0$ such that

$$-(p-2)|D\phi|^{p-4}\Delta_\infty\phi(x) - |D\phi|^{p-2}\Delta\phi(x) < 0,$$

for every $x \in B(x_0, r)$. Set $m = \inf_{|x-x_0|=r}(u-\phi)(x)$ and let $\psi(x) = \phi(x) + m/2$. This function ψ verifies $\psi(x_0) > u(x_0)$ and

$$-\mathrm{div}(|D\psi|^{p-2}D\psi) < 0.$$

Multiplying by $(\psi - u)^+$ extended by zero outside $B(x_0, r)$ we get

$$\int_{\{\psi>u\}} |D\psi|^{p-2} D\psi D(\psi - u) < 0.$$

Taking $(\psi - u)^+$ as test function in the weak form of (1.1) we get

$$\int_{\{\psi>u\}} |Du|^{p-2} Du D(\psi - u) = 0.$$

Hence,

$$\begin{aligned} C(N,p) &\int_{\{\psi>u\}} |D\psi - Du|^p \\ &\leq \int_{\{\psi>u\}} \langle |D\psi|^{p-2}D\psi - |Du|^{p-2}Du, D(\psi-u)\rangle < 0, \end{aligned}$$

a contradiction.

If $x_0 \in \partial\Omega$ we want to prove

$$\begin{aligned} \max\{\, &|D\phi(x_0)|^{p-2} < D\phi(x_0), \nu(x_0) > -g(x_0), \\ &-(p-2)|D\phi|^{p-4}\Delta_\infty\phi(x_0) - |D\phi|^{p-2}\Delta\phi(x_0)\} \geq 0. \end{aligned}$$

Assume that this is not the case. We proceed as before and we obtain

$$\int_{\{\psi>u\}} |D\psi|^{p-2} D\psi D(\psi - u) < \int_{\partial\Omega\cap\{\psi>u\}} g(\psi - u),$$

and

$$\int_{\{\psi>u\}} |Du|^{p-2} Du D(\psi - u) \geq \int_{\partial\Omega\cap\{\psi>u\}} g(\psi - u).$$

Therefore,

$$\begin{aligned} C(N,p) &\int_{\{\psi>u\}} |D\psi - Du|^p \\ &\leq \int_{\{\psi>u\}} \langle |D\psi|^{p-2}D\psi - |Du|^{p-2}Du, D(\psi-u)\rangle < 0, \end{aligned}$$

again a contradiction. This proves that u is a viscosity supersolution. The proof of the fact that u is a viscosity subsolution runs as above, we omit the details. $\square$

Remark 2.1. *If B_p is monotone in the variable $\frac{\partial u}{\partial \nu}$ Definition 2.1 takes a simpler form, see* [B]. *This is indeed the case for* (2.3). *More concretely, if u is a supersolution and $\phi \in C^2(\overline{\Omega})$ is such that $u - \phi$ has a strict minimum at x_0 with $u(x_0) = \phi(x_0)$, then*

(1) *if $x_0 \in \Omega$, then*

$$-\left\{ \frac{|D\phi(x_0)|^2 \Delta\phi(x_0)}{p-2} + \Delta_\infty\phi(x_0) \right\} \geq 0,$$

and if

(2) *If $x_0 \in \partial\Omega$, then*

$$|D\phi(x_0)|^{p-2}\langle D\phi(x_0), \nu(x_0)\rangle \geq g(x_0).$$

Lemma 2.4. *The limit* $\lim_{p_i\to\infty} u_{p_i} = v_\infty$ *verifies*

$$|Dv_\infty| \le 1, \qquad \text{in } \Omega \text{ in the viscosity sense.} \tag{2.5}$$

Proof. See [BBM], Proposition 5.1. ☐

We are now ready to prove our result concerning the equation satisfied by $\lim_{p_i\to\infty} u_{p_i} = v_\infty$.

Proof of Theorem 1.2. First, let us check that $-\Delta_\infty u_\infty = 0$ in the viscosity sense in Ω. Let us recall the standard proof. Let ϕ be a smooth test function such that $v_\infty - \phi$ has a strict maximum at $x_0 \in \Omega$. Since u_{p_i} converges uniformly to v_∞ we get that $u_{p_i} - \phi$ has a maximum at some point $x_i \in \Omega$ with $x_i \to x_0$. Next we use the fact that u_{p_i} is a viscosity solution of

$$-\Delta_p u_p = 0$$

and we obtain

$$-(p_i - 2)|D\phi|^{p_i-4}\Delta_\infty\phi(x_i) - |D\phi|^{p_i-2}\Delta\phi(x_i) \le 0. \tag{2.6}$$

If $D\phi(x_0) = 0$ we get $-\Delta_\infty\phi(x_0) \le 0$. If this is not the case, we have that $D\phi(x_i) \ne 0$ for large i and then

$$-\Delta_\infty\phi(x_i) \le \frac{1}{p_i - 2}|D\phi|^2\Delta\phi(x_i) \to 0, \quad \text{as } i \to \infty.$$

We conclude that

$$-\Delta_\infty\phi(x_0) \le 0.$$

That is v_∞ is a viscosity subsolution of $-\Delta_\infty u_\infty = 0$.

A similar argument shows that v_∞ is also a supersolution and therefore a solution of $-\Delta_\infty v_\infty = 0$ in Ω.

Let us check the boundary condition. There are six cases to be considered. Assume that $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty\phi(x_0) \ge 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial\phi}{\partial\nu}(x_i) \ge g(x_i).$$

Since $g(x_0) > 0$, we have $D\phi(x_0) \ne 0$, and we obtain

$$|D\phi|(x_0) \ge 1.$$

Moreover, we also have

$$\frac{\partial\phi}{\partial\nu}(x_0) \ge 0.$$

Hence, if $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$, we have

$$\max\left\{\min\{-1 + |D\phi|(x_0), \frac{\partial\phi}{\partial\nu}(x_0)\}\ , -\Delta_\infty\phi(x_0)\right\} \ge 0. \tag{2.7}$$

Next assume that $v_\infty - \phi$ has a strict maximum at $x_0 \in \partial\Omega$ with $g(x_0) > 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \le 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \le g(x_i).$$

If $1 < |D\phi|(x_0)|$ we have

$$\frac{\partial \phi}{\partial \nu}(x_0) \le 0.$$

Hence, the following inequality holds

$$\min\left\{\min\{-1 + |D\phi|(x_0), \frac{\partial \phi}{\partial \nu}(x_0)\} \ , -\Delta_\infty \phi(x_0)\right\} \le 0. \tag{2.8}$$

For the following case assume that $v_\infty - \phi$ has a strict maximum at x_0 with $g(x_0) < 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \le 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \le g(x_i).$$

As $g(x_0) < 0$, $D\phi(x_0) \neq 0$ and we obtain

$$|D\phi|(x_0) \ge 1,$$

and

$$\frac{\partial \phi}{\partial \nu}(x_0) \le 0.$$

Hence, the following inequality holds

$$\min\left\{\max\{1 - |D\phi|(x_0), \frac{\partial \phi}{\partial \nu}(x_0)\} \ , -\Delta_\infty \phi(x_0)\right\} \le 0. \tag{2.9}$$

Now assume that $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) < 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \ge 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \ge g(x_i).$$

If $1 < |D\phi|(x_0)$ we have

$$\frac{\partial \phi}{\partial \nu}(x_0) \ge 0.$$

Hence, the following inequality holds.

$$\max\left\{\max\{1 - |D\phi|(x_0), \frac{\partial \phi}{\partial \nu}(x_0)\} \ , -\Delta_\infty \phi(x_0)\right\} \ge 0. \tag{2.10}$$

For the next case assume that $v_\infty - \phi$ has a strict minimum at $x_0 \in \partial\Omega$ with $g(x_0) = 0$. Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a minimum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \geq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \geq g(x_i).$$

If $D\phi(x_0) = 0$, then we have

$$\frac{\partial \phi}{\partial \nu}(x_0) = 0.$$

If $D\phi(x_0) \neq 0$ we obtain

$$\frac{\partial \phi}{\partial \nu}(x_i) \geq \left(\frac{1}{|D\phi|}(x_i)\right)^{p_i-2} g(x_i).$$

If $|D\phi(x_0)| \geq 1$ then we have

$$\frac{\partial \phi}{\partial \nu}(x_0) \geq 0.$$

Therefore, the following inequality holds

$$\max\left\{ H(|D\phi|(x_0))\frac{\partial \phi}{\partial \nu}(x_0)\ , -\Delta_\infty \phi(x_0)\right\} \geq 0. \tag{2.11}$$

Finally, assume that $v_\infty - \phi$ has a strict maximum at x_0 with $g(x_0) = 0$ Using the uniform convergence of u_{p_i} to v_∞ we obtain that $u_{p_i} - \phi$ has a maximum at some point $x_i \in \overline{\Omega}$ with $x_i \to x_0$. If $x_i \in \Omega$ for infinitely many i, we can argue as before and obtain

$$-\Delta_\infty \phi(x_0) \leq 0.$$

On the other hand if $x_i \in \partial\Omega$ we have

$$|D\phi|^{p_i-2}(x_i)\frac{\partial \phi}{\partial \nu}(x_i) \leq g(x_i).$$

If $D\phi(x_0) = 0$, then we have

$$\frac{\partial \phi}{\partial \nu}(x_0) = 0.$$

If $|D\phi(x_0)| \geq 1$ we obtain

$$\frac{\partial \phi}{\partial \nu}(x_0) \leq 0.$$

Hence, the following inequality holds

$$\min\left\{ H(|D\phi|(x_0))\frac{\partial \phi}{\partial \nu}(x_0)\ , -\Delta_\infty \phi(x_0)\right\} \leq 0. \tag{2.12}$$

□

Remark 2.2. *The function v_∞ is a viscosity solution of $\Delta_\infty v_\infty = 0$ in Ω and therefore it is an absolutely minimizing function,* [ACJ]. *It is a minimizer of the Lipschitz constant of u among functions that coincides with v_∞ on $\partial\Omega'$ in every subdomain Ω' of Ω. Therefore we can rewrite the maximization problem* (1.3) *as a maximization problem on $\partial\Omega$: $v_\infty|_{\partial\Omega}$ is a function that has Lipschitz constant less or equal than one on $\partial\Omega$ and maximizes $\int_{\partial\Omega} ug$.*

Remark 2.3. *If* u_p *is the solution of* (1.1) *with boundary data* g *and* $\hat{u}_p$ *is the solution with boundary data* $\hat{g} = \lambda g$, $\lambda > 0$, *then*

$$u(x) = \lambda^{-1/(p-1)} \hat{u}(x).$$

Therefore the limit v_∞ *is the same if we consider any positive multiple of* g *as boundary data and the same subsequence.*

As a consequence the limit problem must be invariant by scalar multiplication of the data g*. One could naively conjecture that the limits depends only on the sign of* g*, however this conjecture is not true as we will see in Example 2 below.*

3. Proof of the Uniqueness Theorem

We proceed now with the proof of Theorem 1.3. For this purpose, we first prove a crucial Lemma based upon ideas from [EG]. Let us call $d_\Omega(x,y)$ the usual distance function for points in Ω, that is,

$$d_\Omega(x,y) = \inf\{\text{length}(\gamma), \gamma\colon [0,1] \mapsto \Omega,\ \gamma(0) = x,\ \gamma(1) = y\}.$$

Observe that $d_\Omega(x,y) = |x-y|$ if the segment from x to y lies in Ω.

Lemma 3.1. *Let* v_∞ *be a maximizer of* (1.3)*. Set* $\partial\Omega_+ = \operatorname{supp} g^+$ *and* $\partial\Omega_- = \operatorname{supp} g^-$*. Define*

$$v^*(x) = \inf_{y \in \partial\Omega_-} \{v_\infty(y) + d_\Omega(x,y)\} \tag{3.13}$$

and

$$v_*(x) = \sup_{y \in \partial\Omega_+} \{v_\infty(y) - d_\Omega(x,y)\}. \tag{3.14}$$

Then, we have

$$v^*(x) \ge v_\infty(x) \ge v_*(x), \qquad \text{for all } x \in \overline{\Omega}, \tag{3.15}$$

$$v_\infty(x) = \inf_{y \in \partial\Omega_-} \{v_\infty(y) + d_\Omega(x,y)\}, \qquad \text{for all } x \in \partial\Omega_+, \tag{3.16}$$

and

$$v_\infty(x) = \sup_{y \in \partial\Omega_+} \{v_\infty(y) - d_\Omega(x,y)\}, \qquad \text{for all } x \in \partial\Omega_-. \tag{3.17}$$

Proof. First, observe that (3.15) can be easily deduced from the fact that v_∞ verifies

$$|v_\infty(x) - v_\infty(y)| \le d_\Omega(x,y), \qquad \text{for all } x, y \in \overline{\Omega}. \tag{3.18}$$

We prove (3.16), the proof of (3.17) being analogous. As we have that v_∞ verifies (3.18) we obtain

$$v_\infty(x) \le \inf_{y \in \partial\Omega_-} \{v_\infty(y) + d_\Omega(x,y)\}, \qquad \text{for all } x \in \overline{\Omega}. \tag{3.19}$$

The function $v^*(x)$ verifies

$$|v^*(x) - v^*(y)| \le d_\Omega(x,y), \qquad \text{for all } x, y \in \overline{\Omega},$$

$$v_\infty(x) \le v^*(x) \qquad \text{for all } x \in \overline{\Omega}$$

and

$$v_\infty(x) = v^*(x) \qquad \text{for all } x \in \partial\Omega_-.$$

Using that v_∞ is a maximizer, we get

$$\int_{\partial\Omega} g v_\infty \geq \int_{\partial\Omega} g v^*.$$

Hence

$$\int_{\partial\Omega_+} g(v_\infty - v^*) \geq 0,$$

and, since $g > 0$ on $\partial\Omega_+$ we conclude that

$$v_\infty(x) = v^*(x), \qquad \text{for all } x \in \partial\Omega_+,$$

as we wanted to prove. □

Remark 3.1. *When Ω is convex, the proof of the lemma, the definition of v^*, v_* and the definition of the transport set T given in* (1.6) *imply*

$$\partial\Omega_+ \cup \partial\Omega_- \subset T = \{v^*(x) = v_\infty(x) = v_*(x)\}.$$

See Lemma 3.1 in [EG] *for a detailed proof.*

Lemma 3.2. *We have that*

$$\begin{aligned} &|Dv^*| = 1, && \text{in } \overline{\Omega} \setminus \partial\Omega_-, \\ &|Dv_*| = 1, && \text{in } \overline{\Omega} \setminus \partial\Omega_+. \end{aligned} \tag{3.20}$$

Proof. Let us check the first property (the second being similar). By definition $|Dv^*(x)| \leq 1$ *a.e*; moreover there exists $y_x \in \partial\Omega_-$ such that

$$v^*(x) = v_\infty(y_x) + |x - y_x|.$$

Take z a point in the segment defined by x, y_x. We claim that

$$v^*(z) = v_\infty(y_x) + |z - y_x|.$$

We argue by contradiction, if

$$v^*(z) < v_\infty(y_x) + |z - y_x|$$

we obtain

$$v^*(x) = v_\infty(y_x) + |x - y_x| > v_\infty(y_z) + |z - y_z| + |x - z| \geq v_\infty(y_z) + |x - y_z|,$$

because x, z, y_x are co-linear. This is a contradiction which proves the claim. Property (3.20) follows immediately from the claim. □

Proof of Proposition 1.1. If follows from Remark 3.1 and Lemma 3.2 that

$$|Dv_\infty| = 1, \qquad \text{a.e. in } T. \tag{3.21}$$

□

Proof of Theorem 1.3. Assume that we have a datum g on $\partial\Omega$ such that every maximizer v_∞ verifies $T(v_\infty) = \overline{\Omega}$. If we have two maximizers of (1.3), v_∞ and w_∞, then

$$z_\infty = \frac{v_\infty + w_\infty}{2}$$

is also a maximizer, and hence, by (3.21),

$$|Dv_\infty| = |Dw_\infty| = |Dz_\infty| = 1, \qquad \text{a.e. } \Omega.$$

This implies that

$$Dv_\infty = Dw_\infty, \qquad \text{a.e. } \Omega,$$

and hence, using our normalization constraint,

$$\int_\Omega v_\infty = \int_\Omega w_\infty = 0,$$

we obtain

$$v_\infty = w_\infty.$$

□

Remark 3.2. *Observe that, in general, the variational problem* (1.3) *does not have a unique solution. To see this fact, observe that given v_∞ a maximizer it does not have to coincide with v^* or with v_* in Ω, see* [ACJ].

4. Examples

Example: The Annulus. Let Ω be the annulus

$$\Omega = \{r_1 < |x| < r_2\}.$$

Let us begin with a function g_0 that is a positive constant g_1 on $|x| = r_1$ and a negative constant g_2 on $|x| = r_2$ satisfyting the constraint

$$\int_{\partial\Omega} g_0 = \int_{|x|=r_1} g_1 + \int_{|x|=r_2} g_2 = 0.$$

As we stated in the introduction, the limit v_∞ is the cone,

$$v_\infty(x) = C(x) = \left(\frac{1}{|\Omega|}\int_\Omega |y|\right) - |x|. \tag{4.22}$$

To check this fact we observe that, by uniqueness, the solutions u_p of (1.1) are radial hence the limit v_∞ must be a radial function. Direct integration shows that it must be a cone with gradient one.

Note however that the cone (4.22) may not be a maximizer of (1.3) for another nonradial boundary datum g with $\mathrm{sign}(g) = \mathrm{sign}(g_0)$. In fact, consider a cone with the vertex slightly displaced,

$$C_{x_0}(x) = C - |x - x_0|. \tag{4.23}$$

One may concentrate g on $|x| = r_2$ near a point $\bar{x}$ and on $|x| = r_1$ near a point $\hat{x}$ preserving the total integral and the sign. It is easy to show that in this case the centered cone given by (4.22) does not maximizes (1.3) since for a suitable g we obtain

$$\int_{\partial\Omega} g(x)C(x)\,dx < \int_{\partial\Omega} g(x)C_{x_0}(x)\,dx.$$

Since this can de done without altering the sign of g we have that there is no uniqueness for the limit problem (1.5). Moreover, the limit v_∞ depends on the shape of g not only on its sign (see Remark 2.3.)

Observe that we have not used Theorem 1.3 to conclude uniqueness, but rather we have used an ad-hoc argument based on the simple nature of the boundary function. When boundary function is constant, subject to the condition of zero average, the transport rays are radial so that the hypothesis of Theorem 1.3 are trivially satisfied.

Example: The Disk. Now let us present a more interesting and non-trivial example of a domain and boundary data such that uniqueness holds. Let Ω be a disk in $\mathbb{R}^2$, $D = \{|(x, y)| < 1\}$ with boundary datum $g(x, y) > 0$ for $x > 0$ and

$g(x,y) < 0$ for $x < 0$ with $\int_{\partial D} g = 0$. Let v_∞ a maximizer of (1.3) and define for $x \in T$ the transport ray as

$$R_x = \{z\,;\, |v_\infty(x) - v_\infty(z)| = |x - z|\}.$$

Recall that wo transport rays cannot intersect in Ω unless they are identical. Moreover the endpoints of every transport ray $R_x = [a\ b]$ satisfy $a \in \partial\Omega_+$ and $b \in \partial\Omega_-$. The union of the transport rays is the transport set T (see Lemma 3.2 in [EG].) From these properties we conclude that there exists a monotone continuous function from $\partial\Omega_+$ to $\partial\Omega_-$ that sends a point $a \in \partial\Omega_+$ to the endpoint of the ray, $b \in \partial\Omega_-$. A monotonicity argument shows that $T(v_\infty) = \overline{\Omega}$. Hence we may apply Theorem 1.3 to obtain the existence of the limit $\lim_{p\to\infty} u_p$.

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