FRACTIONAL p–LAPLACIAN EVOLUTION EQUATIONS

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ABSTRACT. In this paper we study the fractional $p-$Laplacian evolution equation given by

$$u_t(t,x) = \int_A \frac{1}{|x-y|^{N+sp}} |u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))dy \qquad \text{for } x\in\Omega,\ t>0,$$

$0<s<1$, $p\geq 1$. In a bounded domain Ω we deal with the Dirichlet problem by taking $A=\mathbb{R}^N$ and $u=0$ in $\mathbb{R}^N\setminus\Omega$, and the Neumann problem by taking $A=\Omega$. We include here the limit case $p=1$ that has the extra difficulty of giving a meaning to $\frac{u(y)-u(x)}{|u(y)-u(x)|}$ when $u(y)=u(x)$. We also consider the Cauchy problem in the whole $\mathbb{R}^N$ by taking $A=\Omega=\mathbb{R}^N$. We find existence and uniqueness of strong solutions for each of the above mentioned problems. We also study the asymptotic behaviour of these solutions as $t\to\infty$. Finally, we recover the local $p-$Laplacian evolution equation with Dirichlet or Neumann boundary conditions, and for the Cauchy problem, by taking the limit as $s\to 1$ in the nonlocal problems multiplied by a suitable scaling constant.

1. Introduction

The interest on the fractional Laplacian operators and nonlocal operators has constantly increased over the last few years. These operators arise in a number of applications such as: continuum mechanics, phase transition phenomena, population dynamics, image process, game theory and Lévy processes, see [8], [15], [20], [21], [25] and the references therein. Recently, motivated by some situations arising in game theory, nonlinear generalizations of the fractional Laplacian have been introduced, see [9], [15]. Our aim here is to study some evolution equations associated to a nonlinear version of the fractional Laplacian, the fractional p–Laplacian, for $1\leq p<+\infty$.

Let Ω be an open set in $\mathbb{R}^N$. For any $p\in[1,\infty)$ and any $0<s<1$, let us denote by

$$[u]_{W^{s,p}(\Omega)} = \left(\int_\Omega\int_\Omega \frac{|u(y)-u(x)|^p}{|x-y|^{N+sp}}\,dxdy\right)^{\frac{1}{p}},$$

the (s,p)–Gagliardo seminorm of a measurable function u in Ω. We consider the fractional Sobolev space

$$W^{s,p}(\Omega) = \{u\in L^p(\Omega) \ : \ [u]_{W^{s,p}(\Omega)}<\infty\},$$

which is a Banach space respect to the norm

$$\|u\|_{W^{s,p}(\Omega)} := [u]_{W^{s,p}(\Omega)} + \|u\|_{L^p(\Omega)}.$$

We denote by $W_0^{s,p}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in the norm $\|\cdot\|_{W^{s,p}(\Omega)}$. Functions in the space $W_0^{s,p}(\Omega)$ can be defined in the whole space $W_0^{s,p}(\mathbb{R}^N)$ by extending then by zero outside Ω, we will consider such extensions. We refer to [19] where one can find a description of most of the useful properties of the fractional Sobolev spaces (see also [12]).

We will write, as usual, $p_s^\star = \frac{Np}{N-sp}$, to denote the fractional critical exponent for $1\leq p<\frac{N}{s}$. For Ω bounded and smooth and $1\leq r\leq p_s^\star$ we have the continuous immersion $W^{s,p}(\Omega)\hookrightarrow L^r(\Omega)$, that is compact for $1\leq r<p_s^\star$.

Recall that a function $u\in L^1(\Omega)$ whose gradient Du in the sense of distributions is a vector valued Radon measure with finite total variation in Ω is called a *function of bounded variation.* The class of such functions will

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be denoted by $BV(\Omega)$. For a function $u \in BV(\Omega)$, we will denote by $|Du|$, the total variation of the measure Du. If $\Omega \subset \mathbb{R}^N$ is an open set with Lipchitz bounday and

$$\overline{u}(x) := \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \mathbb{R}^N \setminus \overline{\Omega}, \end{cases}$$

by [1, Corollary 3.89], we have $\overline{u} \in BV(\mathbb{R}^N)$ and

$$|D\overline{u}|(\mathbb{R}^N) = |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1}. \tag{1.1}$$

In [12] it is proved that for every $u \in BV(\mathbb{R}^N)$,

$$[u]_{W^{s,1}(\mathbb{R}^N)} \leq \frac{2N\omega_N}{(1-s)s} \left[|Du|(\mathbb{R}^N)\right]^s \|u\|^{1-s}_{L^1(\mathbb{R}^N)}, \tag{1.2}$$

where ω_N is the volume of the unit ball of $\mathbb{R}^N$. Consequently, $BV(\mathbb{R}^N)$ is contained in $W^{s,1}(\mathbb{R}^N)$. For further information concerning functions of bounded variation we refer to [1].

Through Calculus of Variations one arrives to the local p–Laplacian operator, $\mathrm{div}(|\nabla u|^{p-2}\nabla u)$, for $1 < p < \infty$, as the Euler-Lagrange equation associated with the L^p–norm of the gradient of a function. Using an equivalent framework one may define the fractional p–Laplacian (or p–s–Laplacian), $\Delta^s_p u$, by means of the Euler-Lagrange equation of the L^p–norm of the s–derivative of a function, concretely of the energy functional

$$\frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^p}{|x-y|^{N+sp}} \, dxdy,$$

that is well defined for $u \in W^{s,p}(\mathbb{R}^N)$. In this way, $\Delta^s_p u$ is given by

$$\begin{aligned} \Delta^s_p u(x) &:= \text{P.V.} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \, |u(y)-u(x)|^{p-2}(u(y)-u(x)) \, dy \\ &= \lim_{\epsilon \downarrow 0} \int_{\mathbb{R}^N \setminus B_\epsilon(x)} \frac{1}{|x-y|^{N+sp}} \, |u(y)-u(x)|^{p-2}(u(y)-u(x)) \, dy, \qquad x \in \mathbb{R}^N. \end{aligned} \tag{1.3}$$

This fractional version of the p–Laplacian is studied through energy and test function methods by A. Chambolle, E. Lindgren and R. Monneau in [16]. The viscosity version of this non local operator was given by H. Ishii and G. Nakamura in [23] and C. Bjorland, L. Caffarelli and A. Figalli in [9]. In [28] on can find results for the evolution problem with Neumann conditions for the case $p \geq 2$ using Galerkin's method. Here we deal with the Dirichlet, Neumann and Cauchy problems associated with the fractional p-Laplacian using semigroup theory. Note that this theory is well suited for the problem under consideration since it gives existence and uniqueness of strong solutions under very weak conditions.

If we assume that the integral in the definition of $\Delta^s_p u$ exists, then for $\varphi \in W^{s,p}(\mathbb{R}^N)$, due to the symmetry of the kernel, we have the following integration by parts formula

$$\begin{aligned} \int_{\mathbb{R}^N} \Delta^s_p u(x)\varphi(x) \, dx &= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \, |u(y)-u(x)|^{p-2}(u(y)-u(x)) \, dy \, \varphi(x) \, dx \\ &= -\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \, |u(y)-u(x)|^{p-2}(u(y)-u(x))(\varphi(y)-\varphi(x)) \, dydx, \end{aligned}$$

which leads to the following definition: Let $f \in L^1(\mathbb{R}^N)$, we say that $u \in W^{s,p}(\mathbb{R}^N)$ is a *weak solution* to the problem

$$-\Delta^s_p u = f,$$

if

$$\frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \, |u(y)-u(x)|^{p-2}(u(y)-u(x))(\varphi(y)-\varphi(x)) \, dydx = \int_{\mathbb{R}^N} f(x)\varphi(x)dx, \tag{1.4}$$

for all $\varphi \in W^{s,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Now let us turn our attention to the case $p = 1$. Formally, the *fractional* 1*–Laplacian operator of order* s of a function $u \in W^{s,1}(\mathbb{R}^N)$ is defined as

$$\Delta_1^s u(x) := \text{P.V.} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \frac{u(y)-u(x)}{|u(y)-u(x)|}\, dy, \qquad x \in \mathbb{R}^N.$$

Note that in this formula one has to give a meaning to $\frac{u(y)-u(x)}{|u(y)-u(x)|}$ when $u(y) = u(x)$. Here, to overcome this difficulty, we follow the same idea that we used in [3] and [4] (see also [5]) to study a similar problem but with a non-singular kernel, that is, we replace $\frac{u(y)-u(x)}{|u(y)-u(x)|}$ by an antisymmetric L^∞–function $\eta(x,y)$ such that $\|\eta(\cdot,\cdot)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \le 1$ and

$$\eta(x,y) \in \text{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

where $\text{sign}(r)$ is the multivalued sign of r. We give the following definition of a weak solution: we say that $u \in W^{s,1}(\mathbb{R}^N)$ is a *weak solution* to the problem

$$-\Delta_1^s u = f,$$

if there exists η as above such that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,\eta(x,y)(\varphi(y)-\varphi(x))\,dydx = \int_{\mathbb{R}^N} f(x)\varphi(x)dx \qquad \text{for all } \varphi \in W^{s,1}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N).$$

In this paper, we focus our attention on the evolution problems associated with these fractional operators. We will consider Dirichlet or Neumann boundary conditions for problems posed in Ω, being Ω a bounded Lipschitz domain in $\mathbb{R}^N$, and we will also consider the Cauchy problem in the whole space $\mathbb{R}^N$. Among other results we prove:

Theorem 1.1. *Assume that* $1 \le p < \infty$. *For every* $u_0 \in L^2(\Omega)$ *there exists a unique strong solution of the Dirichlet problem*

$$\begin{cases} u_t(t,x) = \Delta_p^s u(t,x) & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,T)\times(\mathbb{R}^N\setminus\Omega), \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.5}$$

for any $T > 0$. *Moreover, a contraction principle holds: if* $u_{i,0} \in L^2(\Omega)$ *and* u_i *are solutions of the Dirichlet problem* (1.5) *in* $(0,T)$ *with initial data* $u_{i,0}$, $i = 1,2$, *respectively, then*

$$\int_\Omega (u_1(t)-u_2(t))^+ \le \int_\Omega (u_{1,0}-u_{2,0})^+ \quad \text{for every } t \in (0,T).$$

In addition, for $q \ge p \ge 1$ *and for* $u_0 \in L^\infty(\Omega)$ *if* $q > p$ *and* $u_0 \in L^2(\Omega)$ *if* $q = p$, *we have the decay bound*

$$\|u(t)\|_{L^q(\Omega)}^q \le C\frac{\|u_0\|_{L^\infty(\Omega)}^{q-p}\|u_0\|_{L^2(\Omega)}^2}{t} \qquad \forall\, t > 0,$$

where $C = C(\Omega, N, s, p)$.

Similar existence and uniqueness results (we refer to Section 5 for the precise statements) are also obtained for Neumann boundary conditions, that is, when we consider

$$\begin{cases} u_t(t,x) = \Delta_{\Omega,p}^s u(t,x) & \text{in } (0,T)\times\Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases}$$

where $\Delta_{\Omega,p}^s u$ stands for the fractional p–Laplacian in Ω (this operator is defined as in (1.3) but integrating in Ω). In this case the asymptotic behaviour is given by the convergence to the mean value of the initial condition, $(u_0)_\Omega$. For instance, we show that for $u_0 \in L^2(\Omega)$ and $p = 1$ it holds that

$$\|u(t)-(u_0)_\Omega\|_{L^1(\Omega)} \le C\frac{\|u_0\|_{L^2(\Omega)}}{t} \qquad \forall\, t > 0.$$

With respect to the Cauchy problem for the fractional 1–Laplacian we prove:

Theorem 1.2. *For every $u_0 \in L^2(\mathbb{R}^N)$ there exists a unique strong solution of the Cauchy problem*

$$\begin{cases} u_t(t,x) = \Delta_1^s u(t,x) & \text{in } (0,T)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1.6}$$

for any $T > 0$. Moreover, if $u_{i,0} \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and u_i are solutions of Cauchy problem (1.6) *in* $(0,T)$ *with initial data $u_{i,0}$, $i = 1,2$, respectively, then*

$$\int_{\mathbb{R}^N} (u_1(t) - u_2(t))^+ \le \int_{\mathbb{R}^N} (u_{1,0} - u_{2,0})^+ \qquad \text{for every } t \in (0,T).$$

Here we also consider the limit as $s \to 1$ in these nonlocal fractional $p-$Laplacian evolution problems. We show that, after multiplying by an adequate scale factor $L_{p,s} \sim (1-s)$, the solutions to our fractional p-Laplacian evolution problem (for the Cauchy problem and for Dirichlet or Neumann conditions) converge as $s \nearrow 1$ to the solutions of the corresponding evolution problems for the classical p-Laplacian, $u_t = \Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ (for the Cauchy problem and for classical Dirichlet boundary conditions, $u|_{\partial\Omega} = 0$, or Neumann boundary conditions, $|\nabla u|^{p-2}\nabla u \cdot \nu|_{\partial\Omega} = 0$). First results in this direction are obtained in [23] for a similar problem for $p > 1$ in the stationary case (see also [9] and [12]) and [3], [4] and [5] for nonlocal evolution problems with non degenerate kernels.

Let us finish this introduction with some notations and results from the theory of completely accretive operators (see [7]) that will be used in what follows. We denote by J_0 and P_0 the following sets of functions:

$$J_0 := \{ j : \mathbb{R} \to [0,+\infty], \text{ convex and lower semi-continuos with } j(0) = 0 \},$$

$$P_0 := \{ q \in C^\infty(\mathbb{R}) : 0 \le q' \le 1, \operatorname{supp}(q') \text{ is compact, and } 0 \notin \operatorname{supp}(q) \}.$$

In [7] the following relation for $u, v \in L^1(\Omega)$ is defined,

$$u \ll v \ \text{ if and only if } \int_\Omega j(u)\,dx \le \int_\Omega j(v)\,dx \quad \text{ for all } j \in J_0,$$

and the following facts are proved.

Proposition 1.3. Let Ω be a bounded domain in $\mathbb{R}^N$.
(i) For any $u, v \in L^1(\Omega)$, if $\int_\Omega uq(u) \le \int_\Omega vq(u)$ for all $q \in P_0$, then $u \ll v$.
(ii) If $u, v \in L^1(\Omega)$ and $u \ll v$, then $\|u\|_{L^r(\Omega)} \le \|v\|_{L^r(\Omega)}$ for any $r \in [1,+\infty]$.
(iii) If $v \in L^1(\Omega)$, then $\{u \in L^1(\Omega) : u \ll v\}$ is a weakly compact subset of $L^1(\Omega)$.

An operator $A \subset L^1(\Omega) \times L^1(\Omega)$ is *completely accretive* if given $(u_i, v_i) \in A$, $i = 1,2$, then

$$\int_\Omega (v_1 - v_2) q(u_1 - u_2) \ge 0,$$

for every $q \in P_0$.

The paper is organized as follows: in Section 2 we consider the Dirichlet problem for the fractional $p-$Laplacian for $p > 1$; in Section 3 we deal with the fractional $1-$Laplacian with Dirichlet boundary conditions; in Section 4 we consider the Cauchy problem for the fractional 1–Laplacian; while in Section 5 and Section 6 we deal with the Neumann problems for the fractional $p-$Laplacian for $p > 1$ and for $p = 1$, respectively. Finally, in Section 7, we study the convergence of these nonlocal evolution problems, with a rescaling factor of order $1 - s$ in front of the fractional p–s–Laplacian, to classical local evolution problems for the p–Laplacian, as the parameter s goes to 1.

2. The Dirichlet problem for the fractional p–Laplacian

As mentioned in the Introduction, Ω will be a bounded Lipschitz domain in $\mathbb{R}^N$. We will study in this section the Dirichlet problem

$$\begin{cases} u_t(t,x) = \Delta_p^s u(t,x) & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,T)\times(\mathbb{R}^N \setminus \Omega), \\ u(0,x) = u_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

in the case $1 < p < \infty$. Since our approach to this problem is the Nonlinear Semigroup Theory (see [17]) we will first deal with the study of the following Dirichlet problem:

$$\begin{cases} u(x) - \Delta_p^s u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{2.2}$$

Therefore, we start our analysis introducing which is the concept of weak solution to

$$\begin{cases} -\Delta_p^s u(x) = f(x) & \text{in } \Omega, \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.3}$$

for a given datum f. Notice that the Dirichlet condition $u = 0$ is posed in the whole complement of Ω, as usual when dealing with nonlocal operators. The integration by part formula (1.4) leads to the following definition:

Definition 2.1. Let $f \in L^2(\Omega)$. We say that $u \in W_0^{s,p}(\Omega)$ is a *weak solution* of the Dirichlet problem (2.3) for the datum f if

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} |u(y)-u(x)|^{p-2}(u(y)-u(x))(\varphi(y)-\varphi(x))\, dydx = \int_\Omega f(x)\varphi(x)\, dx, \tag{2.4}$$

for all $\varphi \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$.

To study the Dirichlet problem (2.2) (and hence problem (2.1)) we consider the energy functional $\mathcal{D}_p^s : L^2(\Omega) \to [0,\infty[$ given by

$$\mathcal{D}_p^s(u) := \begin{cases} \dfrac{1}{2p}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} |u(y)-u(x)|^p\, dxdy & \text{if } u \in W_0^{s,p}(\Omega) \cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W_0^{s,p}(\Omega). \end{cases}$$

By Fatou's Lemma we have that $\mathcal{D}_p^s$ is lower semicontinuous in $L^2(\Omega)$. Then, since $\mathcal{D}_p^s$ is convex, we have that the subdifferential $\partial\mathcal{D}_p^s$ is a maximal monotone operator in $L^2(\Omega)$. To characterize the subdifferential $\partial\mathcal{D}_p^s$ we introduce the following operator:

Definition 2.2. We define in $L^2(\Omega) \times L^2(\Omega)$ the operator $D_{p,s}$ as:

$$(u,v) \in D_{p,s} \iff u, v \in L^2(\Omega) \text{ and } u \text{ is a weak solution of the Dirichlet problem (2.3) for the datum } v.$$

In the following result we prove that operator $D_{p,s}$ satisfies adequate conditions to apply the Nonlinear Semigroup Theory to solve problem (2.1), briefly this theory says that problem (2.2) has an unique solution for any $f \in L^2(\Omega)$ and that there is an L^q–contraction principle for any $q \geq 1$. See [17] and [7] for definitions and results from such theory (or the Appendix in [5] for a detailed overview).

Theorem 2.3. *The operator $D_{p,s}$ is m–completely accretive in $L^2(\Omega)$ with dense domain. Moreover,*

$$D_{p,s} = \partial\mathcal{D}_p^s. \tag{2.5}$$

Proof. Given $(u_i, v_i) \in \mathrm{Dom}(D_{p,s})$, $i = 1, 2$, and $q \in P_0$, since $q \in P_0$ and $u_1, u_2 \in W_0^{s,p}(\Omega)$, we have $q(u_1 - u_2) \in W_0^{s,p}(\Omega) \cap L^\infty(\Omega)$. Then we can take $q(u_1 - u_2)$ as test function in (2.4) and we get

$$\begin{aligned}
&\int_\Omega (v_1(x) - v_2(x))q(u_1(x) - u_2(x))dx \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} |u_1(y)-u_1(x)|^{p-2}(u_1(y)-u_1(x))(q(u_1(y)-u_2(y)) - q(u_1(x)-u_2(x)))\, dydx \\
&\quad\quad - \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} |u_2(y)-u_2(x)|^{p-2}(u_2(y)-u_2(x))(q(u_1(y)-u_2(y)) - q(u_1(x)-u_2(x)))\, dydx \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}} \left[q(u_1(y)-u_2(y)) - q(u_1(x)-u_2(x))\right] \times \\
&\quad\quad\quad \left[|u_1(y)-u_1(x)|^{p-2}(u_1(y)-u_1(x)) - |u_2(y)-u_2(x)|^{p-2}(u_2(y)-u_2(x))\right]\, dxdy \geq 0.
\end{aligned}$$

Therefore, the operator $D_{p,s}$ is completely accretive.

To see that $D_{p,s}$ is m–completely accretive in $L^2(\Omega)$, we need to show that it satisfies the range condition

$$L^2(\Omega) \subset R(I + D_{p,s}). \tag{2.6}$$

Given $f \in L^2(\Omega)$, we consider the variational problem

$$\min_{u\in L^2(\Omega)} \mathcal{D}_p^s(u) + \frac{1}{2}\int_\Omega u^2 - \int_\Omega fu \tag{2.7}$$

The existence of a unique minimizer u of the variational problem (2.7) is proved via a standard application of the direct method in the Calculus of Variations. Indeed, take a miminizing sequence $u_n \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$. We can assume that

$$\mathcal{D}_p^s(u_n) + \frac{1}{2}\int_\Omega u_n^2 - \int_\Omega fu_n \le M, \qquad \forall n \in \mathbb{N}.$$

Then, by Young's inequality, we have

$$\mathcal{D}_p^s(u_n) + \frac{1}{4}\int_\Omega u_n^2 \le M + 4\int_\Omega f^2, \qquad \forall n \in \mathbb{N}. \tag{2.8}$$

Therefore,

$$\|u_n\|_{W_0^{s,p}(\Omega)} \le C, \qquad \forall n \in \mathbb{N}.$$

Hence, by the compact embedding theorem [19, Theorem 7.1], we can assume, taking a subsequence if necessary, that $u_n \to u$ in $L^p(\Omega)$, and by the reflexivity of $W_0^{s,p}(\Omega)$, we get that $u \in W_0^{s,p}(\Omega)$. Moreover, by (2.8), we have $\{u_n\}$ is bounded in $L^2(\Omega)$, and consequently $u \in L^2(\Omega)$. By Fatou's lemma we deduce that u is actually a minimizer of the variational problem (2.7). The uniqueness follows by the strictly convexity of the functional. Now, to derive the Euler-Lagrange equation satisfied by u. Fix a function $v \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$, then the function

$$\varphi(t) := \mathcal{D}_p^s(u+tv) + \frac{1}{2}\int_\Omega (u+tv)^2 - \int_\Omega f(u+tv)$$

has a minimum at $t = 0$, and consequently

$$0 = \varphi'(0) = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\,|u(y)-u(x)|^{p-2}(u(y)-u(x))(v(y)-v(x))\,dydx - \int_\Omega (u(x)-f(x))v(x)dx.$$

Then, we have $(u, f-u) \in D_{p,s}$ and the range condition (2.6) is satisfied.

Let us now see that $\mathrm{Dom}(D_{p,s})$ is dense in $L^2(\Omega)$. To this end, is enough to show that

$$W_0^{s,1}(\Omega) \cap L^2(\Omega) \subset \overline{\mathrm{Dom}(D_{p,s})}^{L^2(\Omega)}.$$

So, let us take $v \in W_0^{s,1}(\Omega) \cap L^2(\Omega)$. By (2.6) and having in mind that $D_{p,s}$ is accretive, there exists $u_n \in \mathrm{Dom}(D_{p,s})$ such that $(u_n, n(v-u_n)) \in D_{p,s}$. Hence

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\,|u_n(y)-u_n(x)|^{p-2}(u_n(y)-u_n(x))(\varphi(y)-\varphi(x))\,dydx = n\int_\Omega (v(x)-u_n(x))\varphi(x)\,dx,$$

for all $\varphi \in W_0^{s,p}(\Omega) \cap L^2(\Omega)$. Then, taking $\varphi = v - u_n$, and applying Young's inequality, we obtain that

$$\begin{aligned}
&\int_\Omega (v(x)-u_n(x))^2dx \\
&\quad= \frac{1}{2n}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\,|u_n(y)-u_n(x)|^{p-2}(u_n(y)-u_n(x))(v(y)-v(x))\,dydx \\
&\qquad-\frac{1}{2n}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\,|u_n(y)-u_n(x)|^{p}\,dydx \\
&\quad\le \frac{1}{2n}\frac{1}{p'}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\,|u_n(y)-u_n(x)|^{p}\,dydx + \frac{1}{2n}\frac{1}{p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+ps}}|v(y)-v(x)|^pdydx \\
&\qquad-\frac{1}{2n}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\,|u_n(y)-u_n(x)|^{p}\,dydx \\
&\quad\le \frac{1}{2n}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+ps}}|v(y)-v(x)|^pdydx = \frac{1}{n}[v]^p_{W^{s,p}(\mathbb{R}^N)},
\end{aligned}$$

from where it follows that $u_n \to v$ in $L^2(\Omega)$.

Finally, let us see that (2.5) holds. Given $(u,v) \in D_{p,s}$, we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\, |u(y)-u(x)|^{p-2}(u(y)-u(x))(\varphi(y)-\varphi(x))\, dydx = \int_\Omega v(x)\varphi(x)\, dx \tag{2.9}$$

for all $\varphi \in W_0^{s,p}(\Omega)\cap L^2(\Omega)$. Then, given $w \in W_0^{s,p}(\Omega)\cap L^2(\Omega)$, taking $\varphi = w-u$ in (2.9) and having in mind the numerical inequality

$$p|r|^{p-2}r(s-r) \le |s|^p - |r|^p \qquad \forall\, s,r\in\mathbb{R},$$

we obtain

$$\begin{aligned}
&\int_\Omega v(x)(w(x)-u(x))\, dx \\
&\quad = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\, |u(y)-u(x)|^{p-2}(u(y)-u(x))\left((w(y)-w(x))-(u(y)-u(x)\right)\, dydx \\
&\quad \le \mathcal{D}_p^s(w) - \mathcal{D}_p^s(u).
\end{aligned}$$

Therefore, $(u,v)\in\partial\mathcal{D}_p^s$, and consequently $D_{p,s}\subset\partial\mathcal{D}_p^s$. Then, since $D_{p,s}$ is m–completely accretive in $L^2(\Omega)$, we get (2.5). □

Now, let us introduce our definition of solution to the evolution problem (2.1).

Definition 2.4. Given $u_0 \in L^2(\Omega)$, we say that u is a *solution* of the Dirichlet problem (2.1) in $[0,T]$, if $u \in W^{1,1}(0,T;L^2(\Omega))$, $u(0,\cdot)=u_0(\cdot)$, and for almost all $t\in(0,T)$

$$\begin{cases} u_t(t,\cdot) = \Delta_p^s u(t,\cdot) & \text{in } \Omega, \\ u(t,\cdot)=0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

in the sense of Definition 2.1. In other words, $u(t,\cdot)\in W_0^{s.p}(\Omega)$ and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+sp}}\, |u(t,y)-u(t,x)|^{p-2}(u(t,y)-u(t,x))(\varphi(y)-\varphi(x))\, dydx = -\int_\Omega u_t(t,x)\varphi(x)dx, \tag{2.10}$$

for all $\varphi \in W_0^{s,p}(\Omega)\cap L^2(\Omega)$.

We have the following result about existence and uniqueness of solution of the Dirichlet problem (2.1).

Theorem 2.5. *For every $u_0\in L^2(\Omega)$ there exists a unique solution of the Dirichlet problem* (2.1) *in* $(0,T)$ *for any $T>0$. Moreover, if $u_{i,0}\in L^2(\Omega)$ and u_i are solutions of the Dirichlet problem* (2.1) *in* $(0,T)$ *with initial data $u_{i,0}$, $i=1,2$, respectively, then*

$$\int_\Omega (u_1(t)-u_2(t))^+ \le \int_\Omega (u_{1,0}-u_{2,0})^+ \qquad \textit{for every } t\in(0,T). \tag{2.11}$$

Proof. By the theory of maximal monotone operators (see [13]), and having in mind the characterization of the subdifferential of $\mathcal{D}_p^s$ obtained in Theorem 2.3, for every $u_0\in L^2(\Omega)$ there exists a unique strong solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + D_{p,s}(u(t)) \ni 0, & t\in(0,T), \\ u(0)=u_0. \end{cases} \tag{2.12}$$

Now, the concept of solution of the Dirichlet problem (2.1) coincides with the concept of strong solution of (2.12), and the proof of the existence and uniqueness concludes. The contraction principle (2.11) holds since the operator $D_{1,s}$ is completely accretive. □

With respect to the asymptotic behaviour of the solutions of the Dirichlet problem (2.1) we have the following result.

Theorem 2.6. *Let $q\ge p$. Let $u(t)$ be the solution of the Dirichlet problem* (2.1) *for the initial datum $u_0 \in L^\infty(\Omega)$, if $q>p$ and $u_0\in L^2(\Omega)$ if $q=p$. Then the L^q-norm of the solution goes to zero as $t\to\infty$ since we have the following estimate:*

$$\|u(t)\|^q_{L^q(\Omega)} \le C\frac{\|u_0\|^{q-p}_{L^\infty(\Omega)}\|u_0\|^2_{L^2(\Omega)}}{t} \qquad \forall\, t>0,$$

where $C = C(\Omega, N, s, p)$.

Proof. By the complete accretiveness of the operator $D_{p,s}$, and since $0 \in D_{p,s_p}(0)$, the $L^q(\Omega)$–norm of $u(\cdot,t)$ is decreasing with t.

Now, we use the following Sobolev-Poincaré inequality:

$$\int_\Omega |u(t,x)|^p\,dx \le C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(t,y)-u(t,x)|^p}{|x-y|^{N+sp}}\,dy\,dx,$$

that holds for any $1 \le p < \infty$, that is valid since the first eigenvalue of this operator is positive, see [24].

Using this Sobolev-Poincaré inequality, we get

$$\int_\Omega |u(t,x)|^q\,dx \le C\|u_0\|_{L^\infty(\Omega)}^{q-p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(t,y)-u(t,x)|^p}{|x-y|^{N+sp}}\,dy\,dx,$$

where $C = C(\Omega, N, s, p)$. Consequently,

$$\begin{aligned} t\int_\Omega |u(t,x)|^q\,dx &\le \int_0^t\int_\Omega |u(s,x)|^q\,dx\,ds \\ &\le C\|u_0\|_{L^\infty(\Omega)}^{q-p}\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(s,y)-u(s,x)|^p}{|x-y|^{N+sp}}\,dy\,dx\,ds. \end{aligned} \tag{2.13}$$

On the other hand, taking $u(t,x)$ as test function in (2.10), and integrating in space and time, we get

$$\int_0^t\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(s,y)-u(s,x)|^p}{|x-y|^{N+sp}}\,dy\,dx\,ds = \int_\Omega |u_0(x)|^2\,dx - \int_\Omega |u(t,x)|^2 \le \|u_0\|^2_{L^2(\Omega)}. \tag{2.14}$$

Therefore, putting together (2.13) and (2.14), we get

$$\int_\Omega |u(t,x)|^q\,dx \le C\frac{\|u_0\|_{L^\infty(\Omega)}^{q-p}\|u_0\|^2_{L^2(\Omega)}}{t},$$

as we wanted to prove. □

3. The Dirichlet problem for the fractional 1–Laplacian

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. Formally, as mentioned in the Introduction, the *fractional 1–Laplacian operator of order* s of a function $u \in W^{s,1}(\Omega)$ is defined as

$$\Delta_1^s u(x) := \text{P.V.}\int_\Omega \frac{1}{|x-y|^{N+s}}\,\frac{u(y)-u(x)}{|u(y)-u(x)|}\,dy, \quad x \in \Omega.$$

Solutions to the homogeneous Dirichlet problem associated with this operator Δ_1^s will be in a larger space than $W_0^{s,1}(\Omega)$, they live in the space

$$\mathcal{W}_0^{s,1}(\Omega) := \{u \in L^1(\Omega) : [u]_{W^{s,1}(\mathbb{R}^N)} < \infty \text{ and } u = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\}.$$

Definition 3.1. Given $v \in L^2(\Omega)$, we say that $u \in \mathcal{W}_0^{s,1}(\Omega)$ is a *weak solution* to the Dirichlet problem

$$\begin{cases} -\Delta_1^s u(x) = v(x) & \text{in } \Omega \\ u(x) = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.1}$$

if there exists $\eta \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $\eta(x,y) = -\eta(y,x)$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $\|\eta\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \le 1$, such that

$$\eta(x,y) \in \operatorname{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N,$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x))\,dydx = \int_\Omega v(x)\varphi(x)\,dx \quad \text{for all } \varphi \in \mathcal{W}_0^{s,1}(\Omega) \cap L^2(\Omega),$$

Consider now the Dirichlet problem for the fractional 1–Laplacian

$$\begin{cases} u_t(t,x) = \Delta_1^s u(t,x) & \text{in } (0,T)\times\Omega, \\ u(t,x) = 0 & \text{in } (0,T)\times(\mathbb{R}^N\setminus\Omega), \\ u(0,x) = u_0(x) & \text{in } \Omega. \end{cases} \tag{3.2}$$

Our concept of solution for this problem is the following:

Definition 3.2. Given $u_0 \in L^2(\Omega)$, we say that u is a *solution* of problem Dirichlet (3.2) in $[0,T]$, if $u \in W^{1,1}(0,T;L^2(\Omega))$, $u(0,\cdot) = u_0$, and for almost all $t \in (0,T)$

$$\begin{cases} u_t(t,\cdot) = \Delta_1^s u(t,\cdot) & \text{in } \Omega, \\ u(t,\cdot) = 0 & \text{in } \mathbb{R}^N\setminus\Omega, \end{cases}$$

in the sense of Definition 3.1. In other words, if there exists $\eta(t,\cdot,\cdot) \in L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, $\eta(t,x,y) = -\eta(t,y,x)$ for almost all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, $\|\eta(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \le 1$, such that

$$\eta(t,x,y) \in \operatorname{sign}(u(t,y)-u(t,x)) \quad \text{a.e. } (t,x,y)\in\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^+$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\eta(t,x,y)(\varphi(y)-\varphi(x))\,dydx = -\int_\Omega u_t(t,x)\varphi(x)\,dx \qquad \forall\varphi\in\mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega),$$

To study the Dirichlet problem (3.2) we consider the energy functional $\mathcal{D}_1^s : L^2(\Omega)\to[0,\infty[$ given by

$$\mathcal{D}_1^s(u) := \begin{cases} \displaystyle\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}|u(y)-u(x)|\,dxdy & \text{if } u\in\mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega), \\ +\infty & \text{if } u\in L^2(\Omega)\setminus\mathcal{W}_0^{s,1}(\Omega). \end{cases}$$

By Fatou's Lemma we have that $\mathcal{D}_1^s$ is lower semi-continuous in $L^2(\Omega)$. Then, since $\mathcal{D}_1^s$ is convex, we have that the subdifferential $\partial\mathcal{D}_1^s$ is a maximal monotone operator in $L^2(\Omega)$. To characterize the subdifferential $\partial\mathcal{D}_1^s$ we introduce the following operator.

Definition 3.3. We define in $L^2(\Omega)\times L^2(\Omega)$ the operator $D_{1,s}$ as:

$$(u,v)\in D_{1,s} \iff u,v\in L^2(\Omega) \text{ and } u \text{ is a weak solution to problem (3.1).}$$

Theorem 3.4. *The operator $D_{1,s}$ is m–completely accretive in $L^2(\Omega)$ with dense domain. Moreover,*

$$D_{1,s} = \partial\mathcal{D}_1^s. \tag{3.3}$$

Proof. Given $(u_i,v_i)\in\operatorname{Dom}(D_{1,s})$, $i=1,2$, there exists $\eta_i\in L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, $\eta_i(x,y) = -\eta_i(y,x)$ for almost all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, $\|\eta_i\|_{L^\infty(\Omega\times\Omega)}\le 1$, such that

$$\eta_i(x,y)\in\operatorname{sign}(u_i(y)-u_i(x)) \quad \text{for a.e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N,$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{1}{|x-y|^{N+s}}\eta_i(x,y)(\varphi(y)-\varphi(x))\,dydx = \int_\Omega v_i(x)\varphi(x)\,dx \qquad \text{for all } \varphi\in\mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega).$$

Given $q \in P_0$, taking $\varphi(x) := q(u_1(x) - u_2(x))$ as test function, we have

$$\begin{aligned}
&\int_\Omega (v_1(x) - v_2(x))q(u_1(x) - u_2(x))\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}(\eta_1(x,y) - \eta_2(x,y))\left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))\right)\,dxdy \\
&= \frac{1}{2}\int\int_{\{(x,y): u_1(y)\neq u_1(x), u_2(y)=u_2(x)\}} \frac{1}{|x-y|^{N+s}}(\eta_1(x,y) - \eta_2(x,y)) \\
&\qquad\qquad \times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))\right)\,dx\,dy \\
&= \frac{1}{2}\int\int_{\{(x,y): u_1(y)= u_1(x), u_2(y)\neq u_2(x)\}} \frac{1}{|x-y|^{N+s}}(\eta_1(x,y) - \eta_2(x,y)) \\
&\qquad\qquad \times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))\right)\,dx\,dy \\
&= \frac{1}{2}\int\int_{\{(x,y): u_1(y)\neq u_1(x), u_2(y)\neq u_2(x)\}} \frac{1}{|x-y|^{N+s}}(\eta_1(x,y) - \eta_2(x,y)) \\
&\qquad\qquad \times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x))\right)\,dx\,dy.
\end{aligned}$$

Note that the last three integrals are nonnegative. Hence

$$\int_\Omega (v_1(x) - v_2(x))q(u_1(x) - u_2(x))\,dx \geq 0,$$

from where it follows that $D_{1,s}$ is a completely accretive operator.

Let us see that the operator $D_{1,s}$ satisfies the range condition

$$L^2(\Omega) \subset R(I + D_{1,s}). \tag{3.4}$$

From now on C denotes a constant independent of p that may change from one line to another. For $1 < p < \frac{N}{s}$, take $s_p := \frac{N}{(p_s^\star)'}$. We have $0 < s_p < 1$ for all $1 < p < (N^\star)' = \frac{N}{N+s-1} \leq \frac{N}{s}$. Then, given $f \in L^2(\Omega)$, for $1 < p < (N^\star)'$, applying Theorem 2.3, there exists $u_p \in W_0^{s_p,p}(\Omega)$ such that $(u_p, f - u_p) \in D_{p,s_p}$. Now, since $N + s_p p = (N+s)p$, we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p}}\,|u_p(y) - u_p(x)|^{p-2}(u_p(y) - u_p(x))(\varphi(y) - \varphi(x))\,dydx = \int_\Omega (f(x) - u_p(x))\varphi(x)dx, \tag{3.5}$$

for all $\varphi \in W_0^{s_p,p}(\Omega) \cap L^2(\Omega)$. Moreover, since D_{p,s_p} is completely accretive and $0 \in D_{p,s_p}(0)$, $u_p \ll f$ and

$$\|u_p\|_{L^q(\Omega)} \leq \|f\|_{L^q(\Omega)} \qquad \forall 1 < p < (N^\star)', \quad \text{for any } 1 \leq q \leq 2. \tag{3.6}$$

By (3.6), there exists a sequence $p_n \downarrow 1$, such that

$$u_{p_n} \rightharpoonup u \quad \text{weakly in } L^2(\Omega), \text{ and } \|u\|_{L^2(\Omega)} \leq \|f\|_{L^2(\Omega)}.$$

On the other hand, taking $\varphi = u_p$ in (3.5) we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p}}\,|u_p(y) - u_p(x)|^p\,dydx = \int_\Omega (f(x) - u_p(x))u_p(x)dx \leq C \qquad \forall 1 < p < (N^\star)'. \tag{3.7}$$

Now, since

$$\int_\Omega\int_\Omega \frac{1}{|x-y|^{N+s}}\,|u_p(y) - u_p(x)|\,dydx \leq \left(\int_\Omega\int_\Omega \frac{1}{|x-y|^{(N+s)p}}\,|u_p(y) - u_p(x)|^p\,dydx\right)^{1/p} |\Omega \times \Omega|^{1/p'},$$

from (3.7) we get

$$\|u_p\|_{W^{s,1}(\Omega)} \leq C \qquad \forall 1 < p < (N^\star)'.$$

Hence, by the compact embedding Theorem [19, Theorem 7.1] and [12, Theorem 2.7], we have that for a subsequence of $\{p_n\}$, denoted equal,

$$u_{p_n} \to u \quad \text{strongly in } L^1(\Omega) \quad \text{and} \quad u \in \mathcal{W}_0^{s,1}(\Omega).$$

For $k > 0$ we set

$$C_{p,k} := \left\{ (x,y) \in \mathbb{R}^N \times \mathbb{R}^N : \left| \frac{u_p(y) - u_p(x)}{|x-y|^{N+s}} \right| > k \right\}.$$

Then, by (3.7),

$$|C_{p,k}| \leq \frac{C}{k^p}. \tag{3.8}$$

On the other hand,

$$\left| \left| \frac{u_p(y) - u_p(x)}{|x-y|^{N+s}} \right|^{p-2} \frac{u_p(y) - u_p(x)}{|x-y|^{N+s}} \chi_{\mathbb{R}^N \times \mathbb{R}^N \setminus C_{p,k}}(x,y) \right| \leq k^{p-1} \qquad \forall (x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Therefore, for any $k \in \mathbb{N}$ there exists a subsequence of $\{p_n\}_n$, denoted by $\{p_{n_j^k}\}_j$, such that

$$\left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}-2} \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \chi_{\mathbb{R}^N \times \mathbb{R}^N \setminus C_{p_{n_j^k},k}}(x,y) \stackrel{j\to\infty}{\rightharpoonup} \eta_k(x,y),$$

weakly* in $L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, with η_k antisymmetric such that $\|\eta_k\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \leq 1$. Now there exist a subsequence of $\{\eta_k\}_k$, $\{\eta_{k_j}\}_j$ such that,

$$\eta_{k_j} \stackrel{j\to\infty}{\rightharpoonup} \eta \quad \text{weakly}^* \text{ in } L^\infty(\mathbb{R}^N \times \mathbb{R}^N),$$

with η antisymmetric and

$$\|\eta\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \leq 1.$$

Now let us see how to pass to the limit in (3.5): Let us first take $\varphi \in \mathcal{D}(\Omega)$. Then, for a fixed $1 < q_0 < \frac{N}{N+s-1}$, for the extended φ as 0 outside Ω, $\varphi \in W^{r_0,q_0}(\mathbb{R}^N)$ with $r_0 = \frac{(N+s)q_0 - N}{q_0} < 1$.

Let us fix $k \in \mathbb{N}$. From (3.5) we have

$$\begin{aligned} \int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}-2} \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \chi_{\mathbb{R}^N \times \mathbb{R}^N \setminus C_{p_{n_j^k},k}}(x,y) \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+s}}\, dydx \\ - \int_\Omega (f - u_{p_{n_j^k}})\varphi \end{aligned} \tag{3.9}$$

$$= -\frac{1}{2} \int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}-2} \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \chi_{C_{p_{n_j^k},k}}(x,y) \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+s}}\, dydx,$$

Now, for $p_{n_j^k} < q_0$, using Hölder's inequality, (3.7) and (3.8),

$$\begin{aligned}
&\left| \int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}-2} \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \chi_{C_{p_{n_j^k},k}}(x,y) \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+s}}\, dydx \right| \\
&\quad \le \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}} dydx \right)^{(p_{n_j^k}-1)/p_{n_j^k}} \left(\int_{C_{p_{n_j^k},k}} \left| \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}} dydx \right)^{1/p_{n_j^k}} \\
&\quad \le \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}} dydx \right)^{(p_{n_j^k}-1)/p_{n_j^k}} \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{\varphi(y)-\varphi(x)}{|x-y|^{N+s}} \right|^{q_0} dydx \right)^{1/q_0} |C_{p_{n_j^k},k}|^{\frac{q_0-p_{n_j^k}}{p_{n_j^k}q_0}} \\
&\quad = \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \left| \frac{u_{p_{n_j^k}}(y) - u_{p_{n_j^k}}(x)}{|x-y|^{N+s}} \right|^{p_{n_j^k}} dydx \right)^{(p_{n_j^k}-1)/p_{n_j^k}} \left(\int_{\mathbb{R}^N\times\mathbb{R}^N} \frac{|\varphi(y)-\varphi(x)|^{q_0}}{|x-y|^{N+r_0q_0}} dydx \right)^{1/q_0} |C_{p_{n_j^k},k}|^{\frac{q_0-p_{n_j^k}}{p_{n_j^k}q_0}} \\
&\quad \le \frac{C_\varphi}{k^{\frac{q_0-p_{n_j^k}}{q_0}}}.
\end{aligned}$$

Therefore, taking limits as $j\to\infty$ in (3.9), we get

$$\left| \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,\eta_k(x,y)(\varphi(y)-\varphi(x))\,dydx - \int_\Omega (f(x)-u(x))\varphi(x)dx \right| \le \frac{C_\varphi}{k}.$$

In particular,

$$\left| \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,\eta_{k_j}(x,y)(\varphi(y)-\varphi(x))\,dydx - \int_\Omega (f(x)-u(x))\varphi(x)dx \right| \le \frac{C_\varphi}{k_j}.$$

Therefore, taking now the limit as $j\to\infty$, we obtain that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,\eta(x,y)(\varphi(y)-\varphi(x))\,dydx - \int_\Omega (f(x)-u(x))\varphi(x)dx = 0. \tag{3.10}$$

Suppose now that $\varphi\in\mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega)$. As in [12, Lemma 2.3], there exists $\varphi_n\in\mathcal{D}(\mathbb{R}^N)$ such that

$$\varphi_n\to\varphi \quad \text{in } L^2(\Omega) \quad \text{as } n\to+\infty,$$

and

$$[\varphi_n]_{W^{s,1}(\mathbb{R}^N)} \to [\varphi]_{W^{s,1}(\mathbb{R}^N)} \quad \text{as } n\to+\infty.$$

By Fatou's Lemma and (3.10), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{|x-y|^{N+s}}\,|\varphi(y)-\varphi(x)| - \frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x)) \right) dydx \\
&\qquad \le \liminf_{n\to\infty} \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\left(\frac{1}{|x-y|^{N+s}}\,|\varphi_n(y)-\varphi_n(x)| - \frac{1}{|x-y|^{N+s}}\,\eta(x,y)(\varphi_n(y)-\varphi_n(x)) \right) dydx \\
&\qquad = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}|\varphi(y)-\varphi(x)|dydx - \int_\Omega v\varphi,
\end{aligned}$$

which implies

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x))\,dydx \ge \int_\Omega v(x)\varphi(x)\,dx \qquad \text{for all } \varphi\in\mathcal{W}_0^{s,1}(\Omega)\cap L^2(\Omega),$$

and hence we obtain an equality, since the above inequality is also true for $-\varphi$.

To finish the proof of (3.4), we only need to show that

$$\eta(x,y)\in\operatorname{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N. \tag{3.11}$$

By (3.7) for p_n, and taking $\varphi = u$ in (3.10), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p_n}}\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n}\,dydx = \int_\Omega (f(x)-u_{p_n}(x))u_{p_n}(x)dx\\
&\quad= \int_\Omega (f(x)-u(x))u(x)dx - \int_\Omega f(x)(u(x)-u_{p_n}(x))dx\\
&\qquad +2\int_\Omega u(x)((u(x)-u_{p_n}(x))dx - \int_\Omega (u(x)-u_{p_n}(x))^2dx\\
&\quad\le \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,\eta(x,y)(u(y)-u(x))\,dydx - \int_\Omega f(x)(u(x)-u_{p_n}(x))dx\\
&\qquad +2\int_\Omega u(x)((u(x)-u_{p_n}(x))dx.
\end{aligned}$$

Then, taking limit as $n\to\infty$, we get

$$\begin{aligned}
&\limsup_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p_n}}\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n}\,dydx\\
&\qquad\le \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,\eta(x,y)(u(y)-u(x))\,dydx.
\end{aligned}$$

On the other hand, given $\epsilon>0$ we can find $A\supset\Omega$ with $|A|<+\infty$ such that

$$\int_{\mathbb{R}^N\setminus A} \frac{1}{|x-y|^{N+s}}dy \le \frac{\epsilon}{\|f\|_{L^1(\Omega)}} \qquad \forall\, x\in\Omega.$$

Then,

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx\\
&\quad= \int_\Omega\int_{\mathbb{R}^N\setminus A} \frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx + \frac{1}{2}\int_A\int_A \frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx\\
&\quad= \int_\Omega |u_p(x)|\left(\int_{\mathbb{R}^N\setminus A}\frac{1}{|x-y|^{N+s}}dy\right)dx + \frac{1}{2}\int_A\int_A \frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx\\
&\quad\le \epsilon + \frac{1}{2}\int_A\int_A \frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx.
\end{aligned}$$

By the lower semi-continuity in $L^1(\mathbb{R}^N)$ of $[\,\cdot\,]_{W^{s,1}(\Omega)}$, we have

$$\begin{aligned}
&\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,|u(y)-u(x)|\,dydx \le \liminf_{n\to\infty}\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx\\
&\quad\le \epsilon + \liminf_{n\to\infty}\frac{1}{2}\int_A\int_A \frac{1}{|x-y|^{N+s}}\,|u_{p_n}(y)-u_{p_n}(x)|\,dydx\\
&\quad\le \epsilon + \liminf_{n\to\infty}\frac{1}{2}\left(\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{(N+s)p_n}}\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n}\,dydx\right)^{1/p_n}|A\times A|^{1/{p_n}'}\\
&\quad\le \epsilon + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,\eta(x,y)(u(y)-u(x))\,dydx.
\end{aligned}$$

Therefore,

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,|u(y)-u(x)|\,dydx \le \epsilon + \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,\eta(x,y)(u(y)-u(x))\,dydx,$$

from where it follows (3.11), since ϵ is arbitrary.

Let us see that $\mathrm{Dom}(D_{1,s})$ is dense in $L^2(\Omega)$. To see this fact it is enough to show that

$$\mathcal{D}(\Omega)\subset \overline{\mathrm{Dom}(D_{1,s})}^{L^2(\Omega)}.$$

In fact, given $v \in \mathcal{D}(\Omega) \cap L^\infty(\Omega)$, by (3.4) and having in mind that $D_{1,s}$ is accretive, there exists $u_n \in \mathrm{Dom}(B_{1,s})$ such that $(u_n, n(v-u_n)) \in D_{1,s}$. Hence, there exists $\eta_n \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $\eta_n(x,y) = -\eta_n(y,x)$ for almost all $(x,y) \in \Omega \times \Omega$, $\|\eta_n\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} \leq 1$, such that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta_n(x,y)(\varphi(y)-\varphi(x))\,dydx = n\int_\Omega (v(x)-u_n(x))\varphi(x)\,dx \quad \text{for all } \varphi \in \mathcal{W}_0^{s,1}(\Omega) \cap L^2(\Omega),$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}|u_n(y)-u_n(x)|\,dydx = n\int_\Omega (v(x)-u_n(x))u_n(x)\,dx.$$

Then,

$$\begin{aligned}\int_\Omega (v(x)-u_n(x))^2 dx &= \frac{1}{2n}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta_n(x,y)(v(y)-u_n(y)-(v(x)-u_n(x))\,dydx \\ &\leq \frac{1}{2n}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}|v(y)-v(x)|dydx = \frac{1}{n}[v]_{W^{s,1}(\Omega)},\end{aligned}$$

from where it follows that $u_n \to v$ in $L^2(\Omega)$.

Finally, let us see that (3.3) holds. Given $(u,v) \in D_{1,s}$, there exists $\eta \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $\eta(x,y) = -\eta(y,x)$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $\|\eta\|_{L^\infty(\mathbb{R}^N \times \mathbb{R}^N)} \leq 1$, such that

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x))\,dydx = \int_\Omega v(x)\varphi(x)\,dx \quad \text{for all } \varphi \in \mathcal{W}_0^{s,1}(\Omega) \cap L^2(\Omega),$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}|u(y)-u(x)|\,dydx = \int_\Omega v(x)u(x)\,dx.$$

Then, given $w \in \mathcal{W}_0^{s,1}(\Omega) \cap L^2(\Omega)$, we have

$$\int_\Omega v(x)(w(x)-u(x))\,dx = \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(x,y)\,(w(y)-w(x))\;dydx - \mathcal{D}_1^s(u) \leq \mathcal{D}_1^s(w) - \mathcal{D}_1^s(u).$$

Therefore, $(u,v) \in \partial\mathcal{D}_1^s$, and consequently $D_{1,s} \subset \partial\mathcal{D}_1^s$. Then, since $D_{1,s}$ is m–accretive in $L^2(\Omega)$, we have

$$\partial\mathcal{D}_1^s = D_{1,s}.$$

□

Working as in the proof of Theorem 2.5 we get the following result about existence and uniqueness of solutions to the Dirichlet problem (3.2).

Theorem 3.5. *For every $u_0 \in L^2(\Omega)$ there exists a unique solution of the Dirichlet problem* (3.2) *in* $(0,T)$ *for any $T > 0$. Moreover, if $u_{i,0} \in L^2(\Omega)$ and u_i are solutions of the Dirichlet problem* (3.2) *in* $(0,T)$ *with initial data $u_{i,0}$, $i = 1,2$, respectively, then*

$$\int_\Omega (u_1(t)-u_2(t))^+ \leq \int_\Omega (u_{1,0}-u_{2,0})^+ \quad \textit{for every } t \in (0,T).$$

Respect to the asymptotic behaviour of the solutions of the Dirichlet problem for the fractional 1−Laplacian (3.2) we have the following result, whose proof is similar to the one that we made for the fractional p−Laplacian.

Theorem 3.6. *Let $q \geq 1$ and $u(t)$ be the solution of the Dirichlet problem* (3.2) *for the initial datum $u_0 \in L^\infty(\Omega)$ if $q > 1$, and $u_0 \in L^2(\Omega)$ if $q = 1$. Then the L^q-norm of the solution goes to zero as $t \to \infty$ and the following estimate holds:*

$$\|u(t)\|_{L^q(\Omega)}^q \leq C\frac{\|u_0\|_{L^\infty(\Omega)}^{q-1}\|u_0\|_{L^2(\Omega)}^2}{t} \qquad \forall\, t > 0,$$

where $C = C(\Omega, N, s)$.

4. The Cauchy problem for the fractional 1–Laplacian

In this section we consider the Cauchy problem for the fractional p–Laplacian

$$\begin{cases} u_t(t,x) = \Delta_p^s u(t,x) & \text{in } (0,T)\times\mathbb{R}^N, \\ u(0,x) = u_0(x) & \text{in } \mathbb{R}^N. \end{cases} \tag{4.1}$$

We will write down the proofs for the more singular case $p=1$. The case $p>1$ can be studied in a similar way, we leave to the reader the details of the definition of solutions and the proof of the existence and uniqueness result for this simpler case.

Definition 4.1. Given $v \in L^2(\mathbb{R}^N)$, we say that $u \in W^{s,1}(\mathbb{R}^N)$ is a *weak solution* to the problem

$$-\Delta_1^s u(x) = v(x) \quad \text{in } \mathbb{R}^N$$

if there exists $\eta \in L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, $\eta(x,y) = -\eta(y,x)$ for almost all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, $\|\eta\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \le 1$, satisfying

$$\eta(x,y) \in \operatorname{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N,$$

and

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x))\,dydx = \int_{\mathbb{R}^N} v(x)\varphi(x)\,dx \quad \text{for all } \varphi \in W^{s,1}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N).$$

Our concept of solution for the Cauchy problem (4.1) is the following.

Definition 4.2. Given $u_0 \in L^2(\mathbb{R}^N)$, we say that u is a *solution* of problem (4.1) in the interval $[0,T]$, if $u \in W^{1,1}(0,T;L^2(\mathbb{R}^N))$, $u(0,\cdot) = u_0$ and satisfies

$$u_t(t,\cdot) = \Delta_1^s u(t,\cdot) \quad \text{in } \mathbb{R}^N, \quad \text{for almost all } t\in(0,T),$$

in the sense of Definition 4.1. In other words, if there exists $\eta(t,\cdot,\cdot) \in L^\infty(\mathbb{R}^N\times\mathbb{R}^N)$, $\eta(t,x,y) = -\eta(t,y,x)$ for almost all $(x,y)\in\mathbb{R}^N\times\mathbb{R}^N$, $\|\eta(t,\cdot,\cdot)\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \le 1$, satisfying

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(t,x,y)(\varphi(y)-\varphi(x))\,dy = -\int_{\mathbb{R}^N} u_t(t,x)\varphi(x)\,dx \quad \text{for all } \varphi \in W^{s,1}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N),$$

and

$$\eta(t,x,y) \in \operatorname{sign}(u(t,y)-u(t,x)) \quad \text{a.e. } (x,y)\in\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^+.$$

To study the Cauchy problem (4.1) we consider the energy functional $\mathcal{C}_1^s : L^2(\mathbb{R}^N) \to [0,\infty]$ given by

$$\mathcal{C}_1^s(u) := \begin{cases} \displaystyle\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\,|u(y)-u(x)|\,dxdy & \text{if } u \in W^{s,1}(\mathbb{R}^N)\cap L^2(\mathbb{R}^N), \\ +\infty & \text{if } u\in L^2(\mathbb{R}^N)\setminus W^{s,1}(\mathbb{R}^N). \end{cases}$$

By Fatou's Lemma we have that $\mathcal{C}_1^s$ is lower semi-continuous in $L^2(\mathbb{R}^N)$. Then, since $\mathcal{C}_1^s$ is convex, we have that the subdifferential $\partial\mathcal{C}_1^s$ is a maximal monotone operator in $L^2(\mathbb{R}^N)$. To characterize the subdifferential $\partial\mathcal{C}_1^s$ we introduce the following operator in $L^2(\mathbb{R}^N)$:

Definition 4.3. We define in $L^2(\mathbb{R}^N)\times L^2(\mathbb{R}^N)$ the operator $C_{1,s}$ as:

$$(u,v)\in C_{1,s} \iff u,v\in L^2(\mathbb{R}^N) \text{ and } u\in W^{s,1}(\mathbb{R}^N) \text{ is a weak solution of the problem } -\Delta_1^s u = v \text{ in } \mathbb{R}^N.$$

Theorem 4.4. *The operator $C_{1,s}$ is m-completely accretive in $L^2(\mathbb{R}^N)$ with dense domain and moreover*

$$\partial\mathcal{C}_1^s = C_{1,s}\,.$$

Proof. The proof of the completely accretivity of the operator $C_{1,s}$ and the density of the domain is the same than the one given in Theorem 3.4 for the operator $D_{1,s}$. Also with the same proof of Theorem 3.4, we can show that $C_{1,s}\subset\partial\mathcal{C}_1^s$. Then, to finish the proof we only need to show that $C_{1,s}$ satisfies the range condition

$$L^2(\mathbb{R}^N)\subset R(I+C_{1,s}). \tag{4.2}$$

We take $f \in L^2(\mathbb{R}^N)$, and for every $n \in \mathbb{N}$, we set $f_n := f|_{B_n(0)}$. Then, as consequence of Theorem 3.4, we have there exists $u_n \in W_0^{s,1}(B_n(0))$ and $\eta_n \in L^\infty(\mathbb{R}^N \times \mathbb{R}^N)$, $\eta_n(x,y) = -\eta_n(y,x)$ for almost all $(x,y) \in \mathbb{R}^N \times \mathbb{R}^N$, $\|\eta_n\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \leq 1$, satisfying

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta_n(x,y)(\varphi(y)-\varphi(x))\,dydx = \int_{B_n(0)} (f_n(x)-u_n(x))\varphi(x)\,dx \tag{4.3}$$

for all $\varphi \in W_0^{s,1}(B_n(0)) \cap L^2(B_n(0))$, and

$$\eta_n(x,y) \in \operatorname{sign}(u_n(y)-u_n(x)) \quad \text{a.e. } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{4.4}$$

Taking a subsequence, if necessary, we can assume that

$$\eta_n \stackrel{*}{\rightharpoonup} \eta \quad \text{in } L^\infty(\mathbb{R}^N), \qquad \text{with } \|\eta\|_{L^\infty(\mathbb{R}^N\times\mathbb{R}^N)} \leq 1. \tag{4.5}$$

Moreover, η is antisymmetric.

We also have $u_n \ll f_n$, which implies that

$$\|u_n\|_{L^2(\mathbb{R}^N)} \leq \|f\|_{L^2(\mathbb{R}^N)} \qquad \forall\, n \in \mathbb{N},$$

and consequently

$$u_n \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^N).$$

Taking u_n as test function in (4.3), we obtain that

$$\|u_n\|_{W^{s,1}(\mathbb{R}^N)} \leq C \qquad \forall\, n \in \mathbb{N}. \tag{4.6}$$

Then, by (4.6) and the compact embedding theorem [19, Theorem 7.1], using a diagonal procedure, we can assume that

$$u_n \to u \quad \text{a.e. in } \mathbb{R}^N. \tag{4.7}$$

By (4.7), (4.6), applying Fatou's Lemma, we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}|u(y)-u(x)|\,dydx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}|\overline{u}_n(y)-\overline{u}_n(x)|\,dydx \leq C,$$

from where it follows that $u \in W^{s,1}(\mathbb{R}^N)$.

Given $\varphi \in C_c^\infty(\mathbb{R}^N)$, let $n_0 \in \mathbb{N}$ be such that $\operatorname{supp}(\varphi) \subset B_{n_0}(0)$. Then, by (4.3), for any $n \geq n_0$, we have

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta_n(x,y)(\varphi(y)-\varphi(x))\,dydx = \int_{B_n(0)} (f_n(x)-u_n(x))\varphi(x)\,dx. \tag{4.8}$$

Then, taking limit in (4.8) as $n \to \infty$, we get

$$\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}}\eta(x,y)(\varphi(y)-\varphi(x))\,dydx = \int_{\mathbb{R}^N} (f(x)-u(x))\varphi(x)\,dx.$$

Finally, by (4.4), (4.5) and (4.7), we have

$$\eta(x,y) \in \operatorname{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Therefore, $(u, f-u) \in \mathcal{C}_{1,s}$, and we have proved the range condition (4.2). □

Working as in the proof of Theorem 2.5, we get the following result about existence and uniqueness of solution of the Cauchy problem (4.1).

Theorem 4.5. *For every $u_0 \in L^2(\mathbb{R}^N)$ there exists a unique solution of the Cauchy problem (4.1) in $(0,T)$ for any $T > 0$. Moreover, if $u_{i,0} \in L^2(\mathbb{R}^N) \cap L^1(\mathbb{R}^N)$ and u_i are solutions of Cauchy problem (4.1) in $(0,T)$ with initial data $u_{i,0}$, $i = 1,2$, respectively, then*

$$\int_{\mathbb{R}^N} (u_1(t)-u_2(t))^+ \leq \int_{\mathbb{R}^N} (u_{1,0}-u_{2,0})^+ \qquad \textit{for every } t \in (0,T).$$

5. The Neumann problem for the fractional p–Laplacian

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^N$. For $1 < p < \infty$, the *fractional p–Laplacian operator of order s with Neumann boundary condition* applied on a function $u \in W^{s,p}(\Omega)$ is given by

$$\Delta^s_{\Omega,p} u(x) := \text{P.V.} \int_\Omega \frac{1}{|x-y|^{N+sp}} \, |u(y) - u(x)|^{p-2}(u(y) - u(x)) \, dy, \quad x \in \Omega.$$

Integrating by parts, as in the previous cases, we arrive to the following definition.

Definition 5.1. Let $f \in L^2(\Omega)$. We say that $u \in W^{s,p}(\Omega)$ is a *weak solution* of the Neumann problem

$$-\Delta^s_{\Omega,p} u = f \quad \text{in} \ \ \Omega$$

if

$$\frac{1}{2} \int_\Omega \int_\Omega \frac{1}{|x-y|^{N+sp}} \, |u(y) - u(x)|^{p-2}(u(y) - u(x))(\varphi(y) - \varphi(x)) \, dydx = \int_\Omega f(x)\varphi(x) dx,$$

for all $\varphi \in W^{s,p}(\Omega) \cap L^2(\Omega)$.

Consider now the Neumann evolution problem for the fractional p–Laplacian

$$\begin{cases} u_t(t,x) = \Delta^s_{\Omega,p} u(t,x), & \text{in} \ \ (0,T) \times \Omega \\ u(0,x) = u_0(x), & \text{in} \ \ \Omega. \end{cases} \tag{5.1}$$

Definition 5.2. Given $u_0 \in L^2(\Omega)$, we say that u is a *solution* of problem (5.1) in $[0,T]$, if $u \in W^{1,1}(0,T; L^2(\Omega))$, $u(0,\cdot) = u_0$, and

$$u_t(t,\cdot) = \Delta^s_{\Omega,p} u(t,\cdot) \quad \text{in } \Omega, \quad \text{for almost all } t \in (0,T),$$

in the sense of Definition 5.1. In other words, $u \in C([0,T]; L^2(\Omega)) \cap W^{1,1}(0,T; L^2(\Omega))$, $u(0,\cdot) = u_0$ and

$$\begin{aligned} \frac{1}{2} \int_\Omega \int_\Omega \frac{1}{|x-y|^{N+sp}} \, |u(t,y) - u(t,x)|^{p-2}(u(t,y) - u(t,x))(\varphi(y) - \varphi(x)) \, dy \\ = -\int_\Omega u_t(t,x)\varphi(x) \, dx \qquad \forall \varphi \in W^{s,p}(\Omega) \cap L^2(\Omega). \end{aligned} \tag{5.2}$$

To study the Neumann problem (5.1) we consider the energy functional $\mathcal{N}^s_p : L^2(\Omega) \to [0, \infty[$ defined as

$$\mathcal{N}^s_p(u) := \begin{cases} \dfrac{1}{2p} \displaystyle\int_\Omega \int_\Omega \frac{1}{|x-y|^{N+sp}} \, |u(y) - u(x)|^p \, dxdy & \text{if } u \in W^{s,p}(\Omega) \cap L^2(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W^{s,p}(\Omega). \end{cases}$$

We also consider the following operator.

Definition 5.3. *We define in $L^2(\Omega) \times L^2(\Omega)$ the operator $N_{p,s}$ as:*

$$(u,v) \in N_{p,s} \iff u, v \in L^2(\Omega), \ u \in W^{s,p}(\Omega), \text{ and } u \text{ is a weak solution of the Neumann problem } -\Delta^s_{\Omega,p} u = v \text{ in } \Omega.$$

Working as in the proof of Theorem 2.3, we can establish the following result.

Theorem 5.4. *The operator $N_{p,s}$ is m–completely accretive in $L^2(\Omega)$ with dense domain. Moreover,*

$$N_{p,s} = \partial \mathcal{N}^s_p.$$

Observe that the concept of solution of the Neumann problem (5.1) coincides with the concept of strong solution of the abstract Cauchy problem associated with the operator $N_{p,s}$. Then, by Theorem 5.4, working as in the proof of Theorem 2.5, we can establish the following existence and uniqueness result.

Theorem 5.5. *For every $u_0 \in L^2(\Omega)$ there exists a unique solution of the Neumann problem* (5.1)*in $(0,T)$ for any $T > 0$. Moreover, if $u_{i,0} \in L^2(\Omega)$ and u_i are solutions of the Neumann problem* (5.1) *in $(0,T)$ with initial data $u_{i,0}$, $i = 1,2$, respectively, then*

$$\int_\Omega (u_1(t) - u_2(t))^+ \leq \int_\Omega (u_{1,0} - u_{2,0})^+ \quad \textit{for every } t \in (0,T).$$

To study the asymptotic behaviour of the solutions of the Neumann problem (5.1) we use the following fractional Poincaré inequality (see [11], [22]): given $1 \le p < \infty$, there exists a constant C such that,

$$\int_\Omega |v(x) - v_\Omega|^p \le C[v]^p_{W^{s,p}(\Omega)} \qquad \forall\, v \in W^{s,1}(\Omega), \tag{5.3}$$

where v_Ω is the mean value of v in Ω, that is,

$$v_\Omega = \frac{1}{|\Omega|}\int_\Omega v(x)dx.$$

Theorem 5.6. *Let $u_0 \in L^2(\Omega)$. Let $u(t)$ the solution of the Neumann problem* (5.1)*. Then,*

$$\|u(t) - (u_0)_\Omega\|_{L^p(\Omega)} \le \left(2C\frac{\|u_0\|_{L^2(\Omega}}{t}\right)^{1/p} \qquad \forall\, t > 0,$$

where C is the constant in the fractional Poincaré inequality (5.3)*.*

Proof. Taking $\varphi = 1$ as test function in (5.2), we obtain that

$$\int_\Omega u_t(t,x)dx = 0 \quad \text{for evey } t \ge 0,$$

from where it follows that the function

$$t \mapsto \int_\Omega u(t,x)dx \quad \text{is constant,}$$

and consequently we have conservation of mass, i.e,

$$\int_\Omega u(t)\,dx = \int_\Omega u_0\,dx \quad \text{for evey } t \ge 0.$$

On the other hand, taking $u(t)$ as test function in (5.2), we get

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega |u(t,x)|^2\,dx = -\int_\Omega u_t(t,x)u(t,x)\,dx = \frac{1}{2}\int_\Omega\int_\Omega \frac{1}{|x-y|^{N+s}}|u(t,y) - u(t,x)|^p\,dydx,$$

which implies

$$\int_0^t [u(\tau)]^p_{W^{s,p}(\Omega)}d\tau \le 2\|u_0\|^2_{L^2(\Omega)} \quad \text{for evey } t \ge 0.$$

We set

$$w(t,x) := u(t,x) - (u_0)_\Omega.$$

Then, as the solution preserve the total mass, using the fractional Poincaré inequality (5.3), we have

$$\int_\Omega |w(t,x)|^p dx \le C[u(t,\cdot)]_{W^{s,p}(\Omega)}.$$

Since $N_{p,s}$ is a completely accretive operator and $0 \in N_{p,s}(0)$, we have

$$\int_\Omega |w(t,x)|^p dx \le \int_\Omega |w(\tau,x)|^p dx \qquad \text{if } t \ge \tau.$$

Then,

$$t\int_\Omega |w(t,x)|^p dx \le \int_0^t\int_\Omega |w(\tau,x)|^p dx d\tau \le C\int_0^t [u(\tau)]^p_{W^{s,1}(\Omega)}d\tau \le 2C\|u_0\|^2_{L^2(\Omega)},$$

which concludes the proof. □

6. The Neumann problem for the fractional 1–Laplacian

Let us now consider the case $p = 1$. Formally, the *fractional* 1*–Laplacian operator of order s with Neumann boundary condition* applied on a function $u \in W^{s,1}(\Omega)$ is given by

$$\Delta^s_{\Omega,1} u(x) := \text{P.V.} \int_\Omega \frac{1}{|x-y|^{N+s}} \frac{u(y)-u(x)}{|u(y)-u(x)|}\, dy, \quad x \in \Omega.$$

Definition 6.1. Given $f \in L^2(\Omega)$, we say that $u \in W^{s,1}(\Omega)$ is a *weak solution* of the Neumann problem

$$-\Delta^s_1 u = f \quad \text{in } \Omega,$$

if there exists $\eta \in L^\infty(\Omega\times\Omega)$, $\eta(x,y) = -\eta(y,x)$ for almost all $(x,y) \in \Omega\times\Omega$, $\|\eta\|_{L^\infty(\Omega\times\Omega)} \le 1$, satisfying

$$\eta(x,y) \in \text{sign}(u(y)-u(x)) \quad \text{a.e. } (x,y) \in \Omega\times\Omega,$$

and

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{1}{|x-y|^{N+s}} \eta(x,y)(\varphi(y)-\varphi(x))\, dydx = \int_\Omega f(x)\varphi(x)\, dx \quad \text{for all } \varphi \in W^{s,1}(\Omega)\cap L^2(\Omega).$$

Consider now the Neumann problem for the fractional 1–Laplacian

$$\begin{cases} u_t(t,x) = \Delta^s_{\Omega,1} u(t,x), & \text{in } (0,T)\times\Omega \\ u(0,x) = u_0(x), & \text{in } \Omega. \end{cases} \tag{6.1}$$

Definition 6.2. Given $u_0 \in L^2(\Omega)$, we say that u is a *solution* of problem (6.1) in $[0,T]$, if $u \in C([0,T];L^2(\Omega)) \cap W^{1,1}(0,T;L^2(\Omega))$, $u(0,\cdot) = u_0$, and

$$u_t(t,\cdot) = \Delta^s_{\Omega,1} u(t,\cdot) \quad \text{in } \Omega, \quad \text{for almost all } t \in (0,T),$$

in the sense of Definition 6.1. In other words, if there exists $\eta(t,\cdot,\cdot) \in L^\infty(\Omega\times\Omega)$, $\eta(t,x,y) = -\eta(t,y,x)$ for almost all $(x,y)\in\Omega\times\Omega$, $\|\eta(t,\cdot,\cdot)\|_{L^\infty(\Omega\times\Omega)} \le 1$, satisfying

$$\eta(t,x,y) \in \text{sign}(u(t,y)-u(t,x)) \quad \text{a.e. } (x,y) \in \Omega\times\Omega,$$

and

$$\frac{1}{2}\int_\Omega\int_\Omega \frac{1}{|x-y|^{N+s}} \eta(t,x,y)(\varphi(y)-\varphi(x))\, dy = -\int_\Omega u_t(t,x)\varphi(x)\, dx \qquad \forall \varphi \in W^{s,1}(\Omega)\cap L^2(\Omega).$$

To study the Neumann problem (6.1) we consider the energy functional $\mathcal{N}^s_1 : L^2(\Omega) \to [0,\infty[$ defined as

$$\mathcal{N}^s_1(u) := \begin{cases} \displaystyle\frac{1}{2}\int_\Omega\int_\Omega \frac{1}{|x-y|^{N+s}}\, |u(y)-u(x)|\, dxdy & \text{if } u \in W^{s,1}(\Omega)\cap L^2(\Omega) \\ +\infty & \text{if } u \in L^2(\Omega)\setminus W^{s,1}(\Omega). \end{cases}$$

By Fatou's Lemma we have that $\mathcal{N}^s_1$ is lower semi-continuous in $L^2(\Omega)$. Then, since $\mathcal{N}^s_1$ is convex, we have that the subdifferential $\partial\mathcal{N}^s_1$ is a maximal monotone operator in $L^2(\Omega)$. To characterize the subdifferential of the operator $\mathcal{N}^s_1$ we introduce the following operator.

Definition 6.3. *We define in $L^2(\Omega)\times L^2(\Omega)$ the operator $N_{1,s}$ as:*

$$(u,v) \in N_{1,s} \iff u,v \in L^2(\Omega),\ u \in W^{s,1}(\Omega),\ \text{and } u \text{ is a weak solution of the Neumann problem } -\Delta^s_{\Omega,1} u = v \text{ in } \Omega.$$

Working as in the proof of Theorem 3.4, we can establish the following result.

Theorem 6.4. *The operator $N_{1,s}$ is m–completely accretive in $L^2(\Omega)$ with dense domain. Moreover,*

$$N_{1,s} = \partial\mathcal{N}^s_1.$$

Observe that the concept of solution of the Neumann problem (6.1) coincides with the concept of strong solution of the abstract Cauchy problem associated with the operator $N_{1,s}$. Then, by Theorem 6.4, working as in the proof of Theorem 2.5, we can establish the following existence and uniqueness result.

Theorem 6.5. *For every $u_0 \in L^2(\Omega)$ there exists a unique solution of the Neumann problem* (6.1) *in* $(0,T)$ *for any* $T > 0$.

Finally, with a similar proof of the one of Theorem 5.6, we can obtain the following result concerning the asymptotic behaviour of the solutions of the the Neumann problem (6.1).

Theorem 6.6. *Let $u_0 \in L^2(\Omega)$ and $u(t)$ the solution of the Neumann problem* (6.1)*. Then,*

$$\|u(t) - (u_0)_\Omega\|_{L^1(\Omega)} \leq 2C \frac{\|u_0\|_2}{t} \quad \forall\, t > 0,$$

where C is the constant in the fractional Poincaré inequality (5.3)*.*

7. The limit as $s \to 1$.

In this section we show that, with an adequate rescale factor, $L_{p,s}$, the solutions to fractional p-Laplacian evolution problem converge as $s \nearrow 1$ to the solutions of the corresponding evolution problems for the classical p-Laplacian.

7.1. The Neumann problem.

First we consider the Neumann problem,

$$\begin{cases} u_t(t,x) = L_{p,s} \Delta^s_{\Omega,p} u(t,x), & (t,x) \in (0,T) \times \Omega, \\ u(0,x) = u_0(x), & x \in \Omega, \end{cases} \tag{7.1}$$

where the scaling factor is given by:

$$L_{p,s} = \frac{2}{K_{p,N}}(1-s), \qquad K_{p,N} = \frac{1}{|S^{N-1}|} \int_{S^{N-1}} |e_1 \cdot \sigma|^p d\mathcal{H}^{N-1}(\sigma). \tag{7.2}$$

We denote by ν the unitary exterior normal vector to $\partial\Omega$. We have the following result.

Theorem 7.1. *For $p \geq 1$. Given $s_n \to 1^-$, let u_n be the solution of* (7.1) *for $s = s_n$. Then, if u is the solution of the Neumann p–Laplacian problem*

$$\begin{cases} u_t(t,x) = \Delta_p u(t,x), & \text{in } (0,T) \times \Omega, \\ |\nabla u(t,x)|^{p-2} \nabla u(t,x) \cdot \nu(x) = 0, & \text{on } (0,T) \times \partial\Omega, \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases}$$

we have

$$\lim_{n \to \infty} \sup_{t \in [0,T]} \|u_n(t) - u(t)\|_{L^2(\Omega)} = 0.$$

Note that we have full convergence as $s \to 1$ (without the need of considering subsequences) since the solution to the limit problem is unique.

Proof. Consider the energy functionals $\Psi^{\Omega,p}_{s_n}, \Psi^p : L^2(\Omega) \to [0,\infty]$ defined as

$$\Psi^{\Omega,p}_{s_n}(u) := L_p(s_n) \mathcal{N}^{s_n}_p(u) = \begin{cases} \dfrac{1 - s_n}{pK_{p,N}} \displaystyle\int_\Omega \int_\Omega \frac{|u(y) - u(x)|^p}{|x-y|^{N+s_n p}} \, dxdy & \text{if } u \in W^{s_n,p}(\Omega) \cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W^{s_n,p}(\Omega), \end{cases}$$

for $p \geq 1$,

$$\Psi^{\Omega,p}(u) := \begin{cases} \dfrac{1}{p} \displaystyle\int_\Omega |\nabla u|^p & \text{if } u \in W^{1,p}(\Omega) \cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus W^{1,p}(\Omega), \end{cases}$$

for $p > 1$, and

$$\Psi^{\Omega,1}(u) := \begin{cases} |Du|(\Omega) & \text{if } u \in BV(\Omega) \cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega) \setminus BV(\Omega), \end{cases}$$

for $p = 1$, where $|Du|$ is the total variation of the measure Du. Then, for $u_0 \in L^2(\Omega)$, by Theorems 5.5 and 6.5, we have that u_n is the strong solution of the abstract Cauchy problem

$$\begin{cases} u_n'(t) + \partial\Psi^{\Omega,p}_{s_n}(u_n(t)) \ni 0, & \text{a.e. } t \in (0,T), \\ u_n(0) = u_0, \end{cases}$$

and also u is the strong solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Psi^{\Omega,p}(u(t)) \ni 0, & \text{a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases}$$

(see [2] for $p = 1$). Consequently, by classical convergence results of the nonlinear semigroup theory due to Brezis–Pazy ([14]) and Attouch ([6]), to prove the theorem it is enough to show the Mosco convergence (see [26]) of the functionals $\Psi^{\Omega,p}_{s_n}$ to $\Psi^{\Omega,p}$. That is, we need to check that

$$\forall\, u \in \mathrm{Dom}(\Psi^{\Omega,p}) \ \ \exists u_n \in \mathrm{Dom}(\Psi^{\Omega,p}_{s_n}) : u_n \to u \ \text{ and } \ \Psi^{\Omega,p}(u) \geq \limsup_{n\to\infty} \Psi^{\Omega,p}_{s_n}(u_n); \tag{7.3}$$

and

$$\text{if } u_n \rightharpoonup u \quad \text{then} \ \ \Psi^{\Omega,p}(u) \leq \liminf_{n\to\infty} \Psi^{\Omega,p}_{s_n}(u_n). \tag{7.4}$$

In [10] and [18] it is proved that, for $u \in W^{1,p}(\Omega)$ if $p > 1$, and for $u \in BV(\Omega)$ if $p = 1$,

$$\lim_{n\to\infty} \Psi^{\Omega,p}_{s_n}(u) = \Psi^{\Omega,p}(u), \tag{7.5}$$

from where (7.3) follows. To prove (7.4) we can suppose that $\{\Psi^{\Omega,p}_{s_n}(u_n) \ : \ n \in \mathbb{N}\}$ is bounded. Then, by [10, Theorem 4], we may assume that

$$u_n \text{ converges strongly in } L^p \text{ to a function } u \in W^{1,p}(\Omega) \text{ if } p > 1,\ u \in BV(\Omega) \text{ if } p = 1.$$

Therefore (7.4) follows by the Γ-convergence in L^1 of these functionals, that was proved in [27, Theorem 8]. □

7.2. **The Dirichlet problem.**

Consider now the Diriclet problem,

$$\begin{cases} u_t(t,x) = L_{p,s}\Delta_p^s u(t,x), & \text{in } \ (0,T)\times\Omega, \\ u(t,x) = 0, & \text{in } \ (0,T)\times(\mathbb{R}^N\setminus\Omega), \\ u(0,x) = u_0(x), & \text{in } \ \Omega, \end{cases} \tag{7.6}$$

with $L_{p,s}$ as in (7.2). We have the following result.

Theorem 7.2. *Let $p \geq 1$. Given $s_n \to 1^-$, let u_n be the solution of (7.6) for $s = s_n$. Then, if u is the solution of the Dirichlet p–Laplacian problem*

$$\begin{cases} u_t(t,x) = \Delta_p u(t,x), & \text{in } \ (0,T)\times\Omega, \\ u(t,x) = 0, & \text{on } \ (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x), & \text{in } \ \Omega, \end{cases}$$

we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t) - u(t)\|_{L^2(\Omega)} = 0.$$

Note again that here we have full convergence as $s \to 1$ (without the need of considering subsequences) since the solution to the limit problem is unique.

Proof. Following the same idea as above, for $p > 1$ consider now the energy functionals $\Phi^{\Omega,p}_{s_n}, \Phi^{\Omega,p} : L^2(\Omega) \to [0,\infty[$ defined as

$$\Phi^{\Omega,p}_{s_n}(u) := L_{p,s_n}\mathcal{D}^{s_n}_1(u) = \begin{cases} \dfrac{1-s_n}{pK_{p,N}} \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^p}{|x-y|^{N+s_np}}\,dxdy & \text{if } u \in W_0^{s_n,p}(\Omega)\cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus W_0^{s_n,p}(\Omega), \end{cases}$$

and

$$\Phi^{\Omega,p}(u) := \begin{cases} \dfrac{1}{p}\displaystyle\int_\Omega |\nabla u|^p & \text{if } u \in W_0^{1,p}(\Omega)\cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus W_0^{1,p}(\Omega), \end{cases}$$

and for $p=1$ consider

$$\Phi_{s_n}^{\Omega,1}(u) := L_{1,s_n}\mathcal{D}_1^{s_n}(u) = \begin{cases} \dfrac{1-s_n}{K_{1,N}}\displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{|u(y)-u(x)|}{|x-y|^{N+s_n}}\,dxdy & \text{if } u\in \mathcal{W}_0^{s_n,1}(\Omega)\cap L^2(\Omega), \\ +\infty & \text{if } u \in L^2(\Omega)\setminus \mathcal{W}_0^{s_n,1}(\Omega), \end{cases}$$

and

$$\Phi^{\Omega,1}(u) := \begin{cases} |Du|(\Omega) + \displaystyle\int_{\partial\Omega}|u| & \text{if } u\in BV(\Omega)\cap L^2(\Omega), \\ +\infty & \text{if } u\in L^2(\Omega)\setminus BV(\Omega). \end{cases}$$

Then, for $u_0\in L^2(\Omega)$, by Theorems 2.5 and 3.5, we have that u_n is the strong solution of the abstract Cauchy problem

$$\begin{cases} u_n'(t) + \partial\Phi_{s_n}^{\Omega,p}(u_n(t)) \ni 0, & \text{a.e. } t\in(0,T), \\ u_n(0)=u_0, \end{cases}$$

and also u is the strong solution of the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Phi^{\Omega,p}(u(t)) \ni 0, & \text{a.e. } t\in(0,T), \\ u(0)=u_0. \end{cases}$$

Let us check the Mosco convergence of the functionals $\Phi_{s_n}^{\Omega,p}$ to $\Phi^{\Omega,p}$, that is, we have to show that

$$\forall\, u\in \mathrm{Dom}(\Phi^{\Omega,p}) \;\; \exists u_n \in \mathrm{Dom}(\Phi_{s_n}^{\Omega,p}) : u_n\to u \;\text{ and }\; \Phi^{\Omega,p}(u) \geq \limsup_{n\to\infty}\Phi_{s_n}^{\Omega,p}(u_n); \tag{7.7}$$

and

$$\text{if } u_n \rightharpoonup u \quad \text{then} \;\; \Phi^{\Omega,p}(u) \leq \liminf_{n\to\infty}\Phi_{s_n}^{\Omega,p}(u_n). \tag{7.8}$$

Set $\tilde\Omega := \Omega + B(0,1)$. Observe that, using the notation of the Neumann case,

$$\Phi_{s_n}^{\Omega,p}(u) = \Psi_{s_n}^{\tilde\Omega,p}(u) + \frac{2(1-s_n)}{pK_{p,N}}\int_\Omega\left(\int_{\mathbb{R}^N\setminus\tilde\Omega}\frac{1}{|x-y|^{N+s_np}}dy\right)|u(x)|^pdx, \tag{7.9}$$

and that

$$\lim_{n\to\infty}\frac{2(1-s_n)}{pK_{p,N}}\int_\Omega\left(\int_{\mathbb{R}^N\setminus\tilde\Omega}\frac{1}{|x-y|^{N+s_np}}dy\right)|u(x)|^pdx = 0. \tag{7.10}$$

Let us first assume that $p>1$. Given $u\in\mathrm{Dom}(\Phi^{\Omega,p}) = W_0^{1,p}(\Omega)\cap L^2(\Omega)$, we consider $u_n = u\chi_\Omega$, then $u_n\in W^{1,p}(\tilde\Omega)\cap L^2(\tilde\Omega)$. Hence, working as in the proof of (7.5), we have

$$\lim_{n\to\infty}\Psi_{s_n}^{\tilde\Omega,p}(u) = \frac{1}{p}\int_{\tilde\Omega}|\nabla u|^p = \frac{1}{p}\int_\Omega|\nabla u|^p = \Phi^{\Omega,p}(u).$$

Therefore, by (7.9) and (7.10), we get (7.7).

To prove (7.8) we can suppose that $\{\Phi_{s_n}^{\Omega,p}(u_n) \;:\; n\in\mathbb{N}\}$ is bounded. Therefore, $\{\Psi_{s_n}^{\tilde\Omega,p}(u_n) \;:\; n\in\mathbb{N}\}$ is also bounded, and consequently, as above,

$$u_n\to u \quad \text{strongly in } L^p(\tilde\Omega).$$

Hence, by the results of [27] on the Γ–convergence of $\Psi_{s_n}^{\tilde\Omega,p}$, and having in mind (7.10), we get

$$\frac{1}{p}\int_\Omega|\nabla u|^p \leq \frac{1}{p}\int_{\tilde\Omega}|\nabla u|^p \leq \liminf_n \Phi_{s_n}^{\Omega,p}(u_n),$$

as we wanted to show and we concludes the proof for the case $p>1$.

Let us now do the proof for the singular case $p = 1$ which is a little different from the previous case since the boundary values are taken now in a weaker sense, see the expression of $\Phi^{\Omega,1}$ above (see [2] for more details on how the Dirichlet boundary conditions must be considered for the Dirichlet problem for the total variational flow).

Given $u \in \mathrm{Dom}(\Phi^{\Omega,1}) = BV(\Omega) \cap L^2(\Omega)$, we consider $u_n = u\chi_\Omega$, then by (1.1) and (1.2), $u_n \in \mathcal{W}_0^{s_n,1}(\Omega) \cap L^2(\Omega)$. Hence, working as in the proof of (7.5), we have

$$\lim_{n\to\infty} \Psi_{s_n}^{\tilde{\Omega},1}(u_n) = |Du|(\Omega) + \int_{\partial\Omega} |u| d\mathcal{H}^{N-1} = \Phi^{\Omega,1}(u).$$

Therefore, by (7.9) and (7.10), we get (7.7).

To prove (7.8) we can suppose that $\{\Phi_{s_n}^{\Omega,1}(u_n) \ : \ n \in \mathbb{N}\}$ is bounded. Therefore, $\{\Psi_{s_n}^{\tilde{\Omega},1}(u_n) \ : \ n \in \mathbb{N}\}$ is also bounded and consequently, as above,

$$u_n \to u \quad \text{strongly in } L^1(\tilde{\Omega}).$$

Hence, by the results of [27] on the Γ–convergence of $\Psi_{s_n}^{\tilde{\Omega},1}$, and having in mind (7.10), we get

$$|Du|(\tilde{\Omega}) \leq \liminf_n \Phi_{s_n}^{\Omega,1}(u_n).$$

Now, by (1.1),

$$|Du|(\tilde{\Omega}) = |Du|(\Omega) + \int_{\partial\Omega} |u|.$$

Then, from (7.9) and (7.10), we get (7.8). □

7.3. **The Cauchy problem.** Finally, let us deal with the Cauchy problem,

$$\begin{cases} u_t(t,x) = L_{p,s}\Delta_p^s u(t,x), & \text{in } \ (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & \text{in } \ \mathbb{R}^N, \end{cases} \tag{7.11}$$

with $L_{p,s}$ as in (7.2). We have the following result.

Theorem 7.3. *Let $p \geq 1$. Given $s_n \to 1^-$, let u_n be the solution of* (7.11) *for $s = s_n$. Then, if u is the solution of the Cauchy p–Laplacian problem*

$$\begin{cases} u_t(t,x) = \Delta_p u(t,x), & \textit{in } \ (0,T) \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & \textit{in } \ \mathbb{R}^N, \end{cases}$$

we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|u_n(t) - u(t)\|_{L^2(\mathbb{R}^N)} = 0.$$

Proof. As above, consider the energy functionals $\Phi_{s_n}^p, \Phi^p : L^2(\mathbb{R}^N) \to [0,\infty]$ defined as

$$\Phi_{s_n}^p(u) := \begin{cases} \dfrac{1-s_n}{pK_{p,N}} \displaystyle\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|u(y)-u(x)|^p}{|x-y|^{N+s_n p}}\, dxdy & \text{if } u \in W^{s_n,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus W^{s_n,p}(\mathbb{R}^N), \end{cases}$$

for $p \geq 1$, and the usual counterparts associated with the local problems,

$$\Phi^p(u) := \begin{cases} \dfrac{1}{p} \displaystyle\int_{\mathbb{R}^N} |\nabla u|^p & \text{if } u \in W^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus W^{1,p}(\mathbb{R}^N), \end{cases}$$

for $p > 1$, and

$$\Phi^1(u) := \begin{cases} |Du|(\mathbb{R}^N) & \text{if } u \in BV(\mathbb{R}^N) \cap L^2(\mathbb{R}^N), \\ +\infty & \text{if } u \in L^2(\mathbb{R}^N) \setminus BV(\mathbb{R}^N), \end{cases}$$

for $p=1$. By Theorem 4.5, for $u_0 \in L^2(\mathbb{R}^N)$, we have that u_n is the strong solution to the abstract Cauchy problem

$$\begin{cases} u_n'(t) + \partial \Phi^p_{s_n}(u_n(t)) \ni 0, & \text{a.e. } t \in (0,T), \\ u_n(0) = u_0, \end{cases}$$

and also u is the strong solution to the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial \Phi^p(u(t)) \ni 0, & \text{a.e. } t \in (0,T), \\ u(0) = u_0, \end{cases}$$

As in the proof of Theorem 7.1, we only need to check the Mosco convergence of the functionals $\Phi^p_{s_n}$ to Φ^p, that is, we have to show that

$$\forall\, u \in \mathrm{Dom}(\Phi^p) \ \ \exists u_n \in \mathrm{Dom}(\Phi^p_{s_n}) : u_n \to u \ \text{ and } \ \Phi^p(u) \geq \limsup_{n\to\infty} \Phi^p_{s_n}(u_n); \tag{7.12}$$

and

$$\text{if } u_n \rightharpoonup u \quad \text{then} \ \ \Phi^p(u) \leq \liminf_{n\to\infty} \Phi^p_{s_n}(u_n). \tag{7.13}$$

Let us begin with the proof of (7.13) for $p>1$. Again we can suppose that $\{\Phi^p_{s_n}(u_n) \ : \ n \in \mathbb{N}\}$ is bounded, therefore, for a fixed but arbitrary ball $B(0,R)$, it also bounded $\{\Phi^{B(0,R),p}_{s_n}(u_n) \ : \ n \in \mathbb{N}\}$, and consequently, as in the previous case, we get that

$$\frac{1}{p}\int_{B(0,R)} |\nabla u|^p \leq \liminf_{n\to\infty} \Phi^{B(0,R),p}_{s_n}(u_n),$$

which implies

$$\frac{1}{p}\int_{B(0,R)} |\nabla u|^p \leq \liminf_{n\to\infty} \Phi^{p}_{s_n}(u_n)$$

and we conclude using the monotone convergence Theorem.

Let us now prove (7.12). Take $u \in W^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N)$, and consider a sequence $v_m \in C^\infty_c(\mathbb{R}^N)$ such that

$$v_m \to v \quad \text{in } W^{1,p}(\mathbb{R}^N) \cap L^2(\mathbb{R}^N).$$

In particular, we have

$$\frac{1}{p}\int_{\mathbb{R}^N} |\nabla v_m|^p \to \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p.$$

For each $m \in \mathbb{N}$, we have $v_m \in W^{1,p}_0(B(0,R_m))$, for some ball $B(0,R_m)$, therefore, by the proof of Theorem 7.2, we have

$$\lim_{n\to\infty} \Phi^{B(0,R_m),p}_{s_n}(v_m) = \frac{1}{p}\int_{B(0,R_m)} |\nabla v_m|^p,$$

which implies

$$\lim_{n\to\infty} \Phi^{p}_{s_n}(v_m) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla v_m|^p.$$

Hence,

$$\lim_{m\to\infty}\lim_{n\to\infty} \Phi^{p}_{s_n}(v_m) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p.$$

Therefore, there exists a subsequence $\{u_n\} = \{v_{m_n}\}$ of $\{v_m\}$ such that

$$\lim_{n\to\infty} \Phi^{p}_{s_n}(u_n) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p,$$

and the proof of (7.12) concludes.

The proof for the case $p=1$ is similar, therefore we omit the details. □

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