The blow-up profile for a fast diffusion equation with a nonlinear boundary condition

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Abstract

We study positive solutions of a fast diffusion equation in the half-line with a nonlinear boundary condition,

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in {}_+ \times (0,T), \\ -(u^m)_x(0,t) = u^p(0,t) & t \in (0,T), \\ u(x,0) = u_0(x) & x \in {}_+, \end{cases}$$

where $0 < m < 1$ and $p > 0$ are parameters. We describe in terms of p and m when all solutions exist globally in time, when all solutions blow up in a finite time, and when there are both blowing up and global solutions. For blowing up solutions we find the blow-up rate and the blow-up set and we describe the asymptotic behaviour close to the blow-up time T in terms of a self-similar profile.

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1 Introduction and main results

We deal with the problem

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in \ {}_+ \times (0,T), \\ -(u^m)_x(0,t) = u^p(0,t) & t \in (0,T), \\ u(x,0) = u_0(x) & x \in \ {}_+, \end{cases} \tag{1.1}$$

where $0 < m < 1$ and $p > 0$ are parameters. We assume that u_0 is bounded, continuous and positive in ${}_+ = (0,\infty)$. Hence the solution that we are considering is positive for all its time interval of existence, see [Fi].

For every $m > 0$ problem (1.1) can be thought of as a model for nonlinear heat propagation. In this case u stands for the temperature and $-(u^m)_x$ represents the heat flux. Hence the boundary condition represents a nonlinear radiation law at the boundary. This kind of boundary conditions appear also in combustion problems when the reaction happens only at the boundary of the container, for example because of the presence of a solid catalyzer, see [MV] for a justification.

Local in time existence of positive classical solutions of this problem and comparison arguments can be easily established, see for instance [Fi]. The time T is the maximal existence time for the solution, which may be finite or infinite. If $T < \infty$, then u becomes unbounded in finite time and we say that it *blows up*. If $T = \infty$ we say that the solution is *global*.

In this article we are interested in the blow-up phenomenon, a subject that has deserved a great deal of attention in recent years, see for example the book [SGKM] and the surveys [L] and [DL]. For specific references about blow-up in problems with nonlinear boundary conditions see the surveys [CF2, FF].

As a precedent we have the work of Galaktionov and Levine [GL], where they study the same problem for the range of parameters $m \geq 1$. The authors show that if $0 < p \leq p_0 = (m+1)/2$, then for arbitrary initial data the solution is global in time, while for $p > (m+1)/2$ there are solutions with finite time blow-up. Thus, p_0 is the *critical global existence exponent*. Moreover, they prove that $p_c = m + 1$ is a *critical exponent of Fujita type*. By definition, this means that p_c has the following properties:

(i) if $p_0 < p \leq p_c$, then u blows up for all nontrivial u_0;

(ii) if $p > p_c$, then u is global in time for "small" u_0.

Our first theorem proves that the same result holds valid for $0 < m < 1$.

Theorem 1.1 *The critical exponents for problem* (1.1) *are given by* $p_0 = (m+1)/2$ *and* $p_c = m + 1$. *More precisely:*
(i) If $0 < p \leq (m+1)/2$ *every positive solution is global in time.*
(ii) If $(m+1)/2 < p \leq m + 1$ *every positive solution blows up in finite time.*
(iii) If $p > m + 1$ *there exist solutions that blow up in finite time and there exist also global solutions.*

The main difference between the cases $m < 1$ (the so-called fast diffusion equation) and $m > 1$ (the well-known porous medium equation) is the finite speed of propagation property. Solutions with compactly supported initial data u_0 stay compactly supported for $0 < t < T$ if $m > 1$. However, if $m < 1$, solutions become instantaneously positive everywhere. That is the reason why we are restricting ourselves to positive solutions. This introduces a technical difficulty. The results in [GL] for $m > 1$ are restricted to compactly supported initial data. This makes comparison with global supersolutions easier. In our case $m < 1$ and we have to take care of the decay of solutions at infinity. This is done thanks to the decay results for general solutions of the fast diffusion equation of Herrero and Pierre, [HP]. The linear case $m = 1$, also treated in [GL], is similar to the fast diffusion case, as solutions become instantaneously positive. However, the linearity of the equation provides a representation formula. In the nonlinear case we do not have such a tool, and the proofs have to be necessarily different.

We also remark that the critical exponents are not the same in the cases $m > 1$ or $m < 1$ if we consider problem (1.1) defined in a bounded interval $[0, L]$ with the boundary conditions $-(u^m)_x(0,t) = u^p(0,t)$, $-(u^m)_x(L,t) = 0$, see [Fi]. In fact, in this situation $p_0 = 1$ if $m > 1$ while $p_0 = (m+1)/2$ if $0 < m < 1$; the exponent p_c does not exist, since every nontrivial solution blows up for $p > p_0$. Therefore Theorem 1.1 is not so evident.

Once we have characterized for which exponents the solution to problem (1.1) can or cannot blow up, we want to study the way the blowing up solutions behave as approaching the blow-up time. This means that we must investigate where the solutions blow up (the *blow-up set*), the speed at which they blow up (the *blow-up rate*) and the shape of the solutions close to the blow-up time (the *blow-up profile*). To this purpose we consider from now on exponents $p > (m+1)/2$ and u a blowing up solution with blow-up time T.

We begin with the blow-up set, which is defined in the following way,

$$B(u) = \{x \in [0,\infty);\ \exists\, x_n \to x,\ t_n \nearrow T,\ \text{with } u(x_n,t_n) \to +\infty\}.$$

We have single-point blow-up for every $p > (m+1)/2$.

Theorem 1.2 *Let u be a blowing up solution of* (1.1). *Then* $B(u) = \{0\}$.

The argument that we use is valid for a bounded interval, improving the result in [Fi] as we do not require conditions on the initial data.

Our theorem is in contrast with the case $m > 1$, where the blow-up set is a single point if $p > m$, but is the whole half-line if $p < m$ and a bounded interval if $p = m$. Notice that in our case in order to have blowing up solutions we need $p > (m+1)/2$, and hence $p > m$.

To get the blow-up rate of blowing up solutions, we need an extra monotonicity assumption,

(H) $u_t > 0$ for t near T.

This hypothesis holds for example for solutions with smooth compatible initial data such that $(u_0^m)'' \geq 0$.

Theorem 1.3 *Let u be a solution of* (1.1) *with finite blow-up time T satisfying* (H). *As t approaches T we have*

$$\|u(\cdot,t)\|_\infty \sim (T-t)^{-1/(2p-m-1)}. \tag{1.2}$$

where $f \sim g$ means that there exist finite positive constants c_1, c_2 such that $c_1 g \leq f \leq c_2 g$.

The argument in the proof is local and hence applies to the case of a bounded interval. Therefore (1.2) holds for solutions of the latter problem, improving slightly the result of [Fi]. This contrasts with the case $m > p > 1$, where the rate is not the same in the half-line as in a bounded interval, see [Fi, QR].

Remark. Following [H] it is possible to obtain the blow-up rate without the monotonicity assumption (H). However, a restriction on the exponents arises and the rates are only valid for $(m+1)/2 < p \leq m+1$. A different approach to eliminate the monotonicity assumption, based on the study of the energy functional, has been recently proposed in [CF] for the case $m = 1$. However, this technique does not work in the nonlinear case $m < 1$. Indeed, the boundedness of the energy functional is required in the proof, and in order to get it we already use the rates, see Section 5.

Next we study the asymptotic behaviour of blowing up solutions, which is the main subject of this work. As is often the case in nonlinear problems of parabolic type, the characteristic properties of an equation, in this case the blow-up behaviour, are displayed by means of appropriate self-similar solutions, [Ba2].

The linear case $m = 1$ is considered in [FQ] and [HY]. The case $m > 1$ is studied in [PQR] by means of a self-similar solution constructed using results from [GP]. When $m < 1$ no such self-similar solution is available (the results in [GP] are restricted to $m > 1$) and we must establish its existence in the present paper.

In the case of problems with finite-time blow-up the expected self-similar solutions take the form

$$U(x,t) = (T-t)^{-\alpha} F(\xi), \qquad \xi = x(T-t)^{-\beta}. \tag{1.3}$$

The values of the similarity parameters α and β are automatically determined from the fact that $U(x,t)$ is a solution of (1.1); we easily get the values

$$\alpha = \frac{1}{2p-m-1}, \qquad \beta = \frac{p-m}{2p-m-1}\,. \tag{1.4}$$

Observe that in the blow-up range we have $\alpha, \beta > 0$. The profile $F = F(\xi)$ must satisfy the problem

$$\left\{ \begin{array}{ll} (F^m)'' - \beta\xi F' - \alpha F = 0 & \quad \xi \in (0,\infty), \\ -(F^m)'(0) = F^p(0). & \end{array} \right. \tag{1.5}$$

Theorem 1.4 *There exists a unique solution of problem* (1.5). *This solution is positive, strictly decreasing and satisfies*

$$F(\xi) \sim \xi^{-\alpha/\beta} \qquad as\ \xi \to \infty. \tag{1.6}$$

The precise decay (1.6) is crucial. Other decays would lead either to a self-similar solution with global blow-up (something which is impossible) or to a trivial asymptotic profile.

Next we show that the asymptotic behaviour of the solution $u(x,t)$ of problem (1.1) as t approaches the blow-up time T is described by the self-similar solution (1.3). Following the standard technique, we introduce the new rescaled function

$$g(\xi,\tau) = (T-t)^{\alpha} u(x,t), \qquad \xi = x(T-t)^{-\beta},$$

where α and β are given by (1.4) and

$$\tau = -\log(T-t)$$

is the new time. Then $g(\xi,\tau)$ satisfies the following parabolic problem

$$\left\{ \begin{array}{ll} g_\tau = (g^m)_{\xi\xi} - \beta\xi g_\xi - \alpha g & (\xi,\tau) \in \ {}_+ \times (-\log T, \infty), \\ -(g^m)_\xi(0,\tau) = g^p(0,\tau) & \tau \in (-\log T, \infty), \\ g(\xi, -\log T) = T^\alpha u_0(\xi T^{-\beta}) & \xi \in \ {}_+. \end{array} \right. \tag{1.7}$$

Therefore, the problem of the asymptotic behaviour of $u(x,t)$ near a finite blow-up time $T > 0$ is reduced to the problem of the stabilization of $g(\xi,\tau)$ as $\tau \to \infty$ to a stationary solution of (1.7), i.e., a solution of (1.5). We prove the following general result.

Theorem 1.5 *Let u a solution to problem* (1.1) *satisfying the rate* (1.2). *Then the rescaled orbits $g(\xi,\tau)$ tend to the stationary self-similar profile constructed in Theorem 1.4. Therefore, as t approaches T, we have single point blow-up in the precise form*

$$\lim_{t \nearrow T} (T-t)^{\alpha} |u(x,t) - U(x,t)| = 0, \tag{1.8}$$

uniformly in sets of the form $|x| \le c(T-t)^{\beta}$.

The self-similar solution constructed in Theorem 1.4 also gives the behaviour close to the blow-up time for solutions of problem (1.1) in a bounded interval $(0,L)$ with a Neumann boundary condition $(u^m)_x(L,t) = 0$ at the right boundary. This completes the analysis given in [Fi].

Organization of the paper. In Section 2 we prove Theorem 1.1; Theorems 1.2 and 1.3 are proven in Section 3; Theorem 1.4 is proven in Section 4; finally we prove Theorem 1.5 in Section 5 and give the appropriate modifications needed to adapt the proof to the case of a bounded interval.

2 Global existence and Fujita exponents

In this section we prove Theorem 1.1. The basic idea is to compare from below with blowing up subsolutions or from above with global in time supersolutions.

Proof of Theorem 1.1. (i) We look for a supersolution in self-similar form

$$\overline{u}(x,t) = e^{Lt}\varphi(\xi), \qquad \xi = xe^{Jt},$$

see [PQR]. We choose $\varphi(\xi) = (K + e^{-M\xi})^{1/m}$, with

$$J = \frac{1}{2}(1-m)L, \quad L = (K+1)^{(p-1)/(2m)}, \quad M = (K+1)^{p/m}, \quad K > 0.$$

We have $\overline{u}(x,0) \geq u_0(x)$ if we choose $K \geq \|u_0\|_\infty^m$. Hence a comparison argument gives $\overline{u}(x,t) \geq u(x,t)$ and we conclude that u is global.

(ii) We now construct blowing up subsolutions for $p > (m+1)/2$. Consider the function

$$\underline{u}(x,t) = (T-t)^{-\alpha}\varphi(\xi), \qquad \xi = x(T-t)^{-\beta},$$

with α and β as in (1.4), and with profile $\varphi(\xi) = (A + B\xi)^{-2/(1-m)}$. It is a subsolution if

$$B^2 \geq \frac{(1-m)^2}{2m(m+1)(2p-m-1)}, \qquad A^{\frac{2p-m-1}{1-m}} \leq \frac{1-m}{2mB},$$

i.e., B large and A small. The condition $p > (m+1)/2$ comes from the second inequality.

We have thus proven that $p_0 = (m+1)/2$ is the global existence exponent. To prove that every solution blows up in the range $(m+1)/2 < p < m+1$, we use this subsolution $\underline{u}$ and show that it can be put from below any solution u. We follow the same argument used in [GL], comparing our solution with the explicit Barenblatt solution (see below) and then comparing this solution with the above blowing up subsolution.

We first assume, without loss of generality, that u is nonincreasing in x, for if not we consider the (nonincreasing) solution w corresponding to the initial value the nonincreasing minorant of u_0, $w_0(x) = \inf\{u_0(y),\ 0 \leq y \leq x\}$. If w blows up in finite time the same does u. On the other hand, u satisfies, for every $\varepsilon > 0$ and $t_0 > 0$ fixed,

$$u(x,t_0) \geq \left(\frac{(C_m+\varepsilon)x^2}{t_0}\right)^{-1/(1-m)} \qquad \text{for } x \geq M,$$

with

$$C_m = (1-m)/(2m(m+1)) \tag{2.1}$$

and some $M > 0$ large, see [HP]. Also,

$$u(x,t_0) \geq u(M,t_0) \quad \text{for } 0 \leq x \leq M.$$

Now consider the Barenblatt solution

$$E(x,t) = t^{-1/(m+1)} G(x/t^{1/(m+1)}), \qquad G(\xi) = (a^2 + C_m \xi^2)^{-1/(1-m)}, \tag{2.2}$$

where C_m is given in (2.1) and $a > 0$ is a free constant. It is easy to choose $a > 0$ large and $0 < \tau < t_0$ such that $u(x,t_0) \geq E(x,\tau)$. As $\partial E/\partial x(0,\tau) = 0$, comparison implies that $u(x,t+t_0) \geq E(x,t+\tau)$ for every $t > 0$. The final step is to select $t_* > 0$ such that $E(x,t_* + \tau) \geq \underline{u}(x,0)$. This means

$$a^{-2/(1-m)}(t_* + \tau)^{-1/(m+1)} \geq T^{-\alpha} A^{-2/(1-m)},$$

$$((t_* + \tau)/C_m)^{1/(1-m)} \geq (T/B^2)^{1/(1-m)}.$$

This is possible if

$$T^{\alpha} \gg T^{1/(m+1)}$$

for arbitrarily large T, i.e. if and only if $p < m+1$.

It remains to deal with the case $p = m+1$. We follow the stationary state technique from [GL]. Assume by contradiction that there exists a global nontrivial solution. Without loss of generality, we can assume that $u_0(0) > 0$. Hence, using the spatial decay given in [HP], we can choose a and b such that

$$u_0(x) \geq G(x+b),$$

where C_m is given (2.1) and G is the Barenblatt profile given in (2.2). Now we make the following change of variables,

$$\theta(\xi,\tau) = (1+t)^{1/(m+1)} u(\xi(1+t)^{1/(m+1)}, t),$$

where $\tau = \log(1+t)$ denotes the new time. This function θ is a solution of

$$\begin{cases} \theta_\tau = (\theta^m)_{\xi\xi} + \dfrac{1}{m+1}(\xi\theta)_\xi & (\xi,\tau) \in {}_+ \times {}_+, \\ -(\theta^m)_\xi(0,\tau) = \theta^{m+1}(0,\tau) & \tau \in {}_+, \\ \theta(\xi,0) = u_0(\xi) & \xi \in {}_+. \end{cases}$$

Let us call $\underline{\theta}(\xi,\tau)$ the corresponding solution with initial data $G(\xi+b)$. It follows that $\theta \geq \underline{\theta}$ and therefore $\underline{\theta}$ is also global. It can be easily checked that $\underline{\theta}_\tau \geq 0$ (see [GL]). We will prove that for any $\xi > 0$,

$$+\infty > \lim_{\tau\to\infty} \underline{\theta}(\xi,\tau) = G(\xi) \not\equiv 0.$$

To see this we just observe that the limit of $\underline{\theta}(\xi,\tau)$ exists (finite or not) for every $\xi > 0$. If there exists a point $\xi_0 > 0$ with $\lim\limits_{\tau\to\infty} \underline{\theta}(\xi_0,\tau) = +\infty$ then we have that $\underline{\theta}(\xi,\tau) \to +\infty$

uniformly for $0<\xi<\xi_0$. Hence we can put a blowing up solution below $\underline{\theta}$, a contradiction with the fact that $\underline{\theta}$ is global. Arguing as in [GL] we can prove that G is a solution of

$$(G^m)_{\xi\xi}+\frac{1}{m+1}(\xi G)_\xi=0. \tag{2.3}$$

satisfying the boundary condition $-(G^m)'(0)=G^{m+1}(0)$. We end the argument by recalling that every solution of (2.3) is a Barenblatt profile, and this profile does not satisfy the boundary condition, leading to a contradiction.

(iii) Now we consider $p>m+1$ and show that besides the above blowing up solutions there are also nontrivial global in time solutions. We look for a global supersolution of the form

$$\overline{u}(x,t)=(t+\tau)^{-\alpha}f(\xi), \qquad \xi=\frac{x}{(t+\tau)^{\beta}},$$

where $\tau>0$ and α and β as before. As we need f to satisfy

$$\begin{cases} (f^m)''(\xi)+\beta\xi f'(\xi)+\alpha f(\xi)\le 0 & \xi\in \;_+, \\ -(f^m)'(0)\ge f^p(0), \end{cases} \tag{2.4}$$

we try $f(\xi)=LG(\xi+1)$, with G the profile of the Barenblatt solution (2.2). Since $p>m+1$, it is posible to choose L such that

$$\frac{m+1}{2p-m-1}<L^{m-1}<1.$$

Finally, with this choice of L one can check that it is posible to choose a large enough in order to verify (2.4). $\square$

3 The blow-up set and the blow-up rate

In this section, we prove Theorems 1.2 and 1.3. We are assuming $p>(m+1)/2$ and we consider a solution $u(x,t)$ of (1.1) that blows up at a finite time T.

Proof of Theorem 1.2. We perform comparison with the explicit supersolution

$$U_1(x,t)=\left(\frac{t+\tau}{C_m x^2}\right)^{1/(1-m)}$$

for $\tau>0$ large enough and C_m as before, if u_0 has the appropriate decay at infinity. If not we use the supersolution

$$U_2(x,t)=(t+\tau)^{1/(1-m)}w(x),$$

where w is a solution of the elliptic problem

$$\begin{cases} (w^m)''-\frac{1}{1-m}w=0 & 0<x<R, \\ w(0)=w(R)=\infty, \end{cases}$$

and τ and R are large enough, see [CV]. $\square$

For the rest of the section we assume that u satisfies hypothesis (H). As a consequence we have that $(u^m)_{xx} \geq 0$ and $u_x \leq 0$. Therefore, the following properties hold for t close to T,

$$u(0,t) = \max_{x\in[0,\infty)} u(x,t), \quad (u^m)_x(0,t) = \min_{x\in[0,\infty]} (u^m)_x(x,t).$$

Proof of Theorem 1.3. In this proof we follow the techniques used in [HY, QR] for the case $m \geq 1$.

We define $M(t) = u(0,t)$ and the function

$$\phi_M(r,s) = \frac{1}{M(t)} u(ar, bs+t),$$

where $a = M^{m-p}$, $b = M^{m+1-2p}$. Since $p > (m+1)/2 > m$, we have that a and b go to zero as $t \to T$.

The function ϕ_M is a solution of the following problem,

$$\begin{cases} (\phi_M)_s = (\phi_M^m)_{rr} & (r,s) \in \quad_+ \times (-t/b, 0), \\ -(\phi_M^m)_r(0,s) = \phi_M^p(0,s) & s \in (-t/b, 0). \end{cases}$$

Moreover, using that $u_t \geq 0$, we get that $0 \leq \phi_M \leq 1$ and $\phi_M(0,0) = 1$.

We claim that there exist two positive constants c and C such that

$$c \leq (\phi_M)_s(0,0) \leq C.$$

If we rewrite these inequalities in terms of $M(t)$, we obtain

$$c \leq M^{m-2p}(t) M'(t) \leq C.$$

Integrating and taking into account that $M(t) = u(0,t)$, we get

$$C_1 (T-t)^{-1/(2p-m-1)} \leq u(0,t) \leq C_2 (T-t)^{-1/(2p-m-1)},$$

and the proof is complete.

Now we prove the claim. First of all we show that $\phi_M(r,s)$ is bounded below away from zero in sets of the form $[0,r_0] \times [s_0, 0]$, with $s_0 < 0$ small and any $r_0 > 0$. In fact $\phi_M(0,s_0) \geq 1/2$ if $s_0 < 0$, $|s_0| < \epsilon$, for if not ϕ_M cannot reach the value 1 at $(0,0)$. On the other hand, the fact that $(\phi_M)_s \geq 0$ implies ϕ_M^m convex, and therefore, using the boundary condition, $\phi_M^m(r,s) \geq (1/2)^m - r$, so that, choosing r_1 small, we get $\phi_M \geq c > 0$ for $0 \leq r \leq r_1$, $s_0 \leq s \leq 0$. Now, using the decay rate of solutions of the fast diffusion equation given above, [HP], we obtain $\phi_M(r,s) \geq c r^{-2/(1-m)}$, $r > 0$, $s_0 \leq s \leq 0$.

The uniform bounds for ϕ_M in compact sets allow us to use standard parabolic estimates to conclude the upper bound for $(\phi_M)_s(0,0)$, see [Li].

To obtain the lower estimate assume by contradiction that there exists a sequence $M_j \to \infty$ such that

$$(\phi_{M_j})_s(0,0) \to 0.$$

Again the uniform bounds for ϕ_M give that every sequence $\phi_{M_j}(r,s)$ is equicontinuous in compact sets. Then, for some subsequence, which we write again as $M_j \to \infty$, we get

$$\phi_{M_j}(r,s) \to \Phi(r,s)$$

uniformly on compact sets. Our assumption implies for the limit

$$\Phi_s(0,0) = 0.$$

On the other hand, since $u_t \geq 0$, the function $w = \Phi_s$ is nonnegative and satisfies

$$w_s = (m\Phi^{m-1}w)_{rr}.$$

Moreover, at $r = 0$,

$$-(m\Phi^{m-1}w)_r(0,s) = p\Phi^{p-1}w(0,s).$$

Observe that $\Phi(0,0) = 1 > 0$ and that w has a minimum at $(0,0)$. So by Hopf's Lemma we obtain that $w \equiv 0$. Then Φ is a stationary solution of the fast diffusion equation, i.e.,

$$\Phi^m = c_1 r + c_2.$$

The boundary conditions imply $\Phi^m(r) = 1 - r$, which is a contradiction with the fact that Φ is a nonnegative function in all $\ _+$. The claim is proved and the theorem follows. $\square$

4 The self-similar profile

In this section we construct the self-similar profile giving the asymptotic behaviour. The construction is based in the following lemma.

Lemma 4.1 *Let $0 < m < 1$, $\alpha, \beta > 0$ with $\alpha/\beta < 2/(1-m)$, $V \in$, and consider the problem*

$$\begin{cases} (F^m)'' - \beta\xi F' - \alpha F = 0 & \xi \in \ _+, \\ F(0) = 1, \\ -(F^m)'(0) = V. \end{cases} \tag{4.1}$$

There exists a unique value $V = V_* > 0$ such that this problem has a classical bounded solution. The solution is unique, strictly decreasing and satisfies the decay rate (1.6).

The existence of a unique self-similar profile for our problem with the proper decay is now immediate.

Proof of Theorem 1.4. Let F_1 be the solution to problem (4.1) with $V = V_*$ for α and β as in (1.4). For this choice of α and β the condition $\alpha/\beta < 2/(1-m)$ reads $p > (m+1)/2$. Then the scaled function

$$F_\lambda(\xi) = \lambda F_1(\lambda^{(1-m)/2}\xi)$$

satisfies the equation for the profile plus the boundary conditions

$$F_\lambda(0) = \lambda, \qquad -(F_\lambda^m)'(0) = \lambda^{(m+1)/2}V_*.$$

Therefore it suffices to choose $\lambda = V_*^{2/(2p-m-1)}$. □

Proof of Lemma 4.1. We introduce the variables

$$X = \frac{\xi F'}{F}, \qquad Y = \frac{1}{m}\xi^2 F^{1-m}, \qquad \eta = \log \xi.$$

This kind of transformation goes back to [Ba1] and [J], and is used for instance in [Hu, QV]. The resulting system is

$$\begin{cases} \dfrac{dX}{d\eta} = X(1-mX) + Y\left(\alpha + \beta X\right), \\[2ex] \dfrac{dY}{d\eta} = Y(2 + (1-m)X). \end{cases} \tag{4.2}$$

We look for nonnegative profiles F, so we consider only the upper plane $\{Y > 0\}$. As we are interested in solutions with $F(0) = 1$ and $(F^m)'(0)$ finite, the orbits we are looking for start at the critical point $A = (0,0)$. The local analysis of this point is straightforward.

Proposition 4.1 *The linearization of* (4.2) *around* $A = (0,0)$ *has matrix*

$$\begin{pmatrix} 1 & \alpha \\ 0 & 2 \end{pmatrix}$$

with eigenvalues $\lambda_1 = 1$ *and* $\lambda_2 = 2$ *and corresponding eigenvectors* $e_1 = (1,0)$ *and* $e_2 = (\alpha, 1)$. *Thus* A *is a repeler.*

To study the point at infinity in this phase-plane, we perform the inversion change of variables

$$H = \frac{1}{Y}.$$

We arrive at

$$\begin{cases} \dfrac{dX}{d\eta} = X(1-mX) + \dfrac{\alpha + \beta X}{H}, \\[2ex] \dfrac{dH}{d\eta} = -H(2 + (1-m)X). \end{cases}$$

In order to eliminate the singularity we perform the nonlinear change of variable given implicitly by

$$\frac{d\eta}{d\tau} = H(\eta).$$

Observe that this change preserves the direction of the flow on the upper half-plane $\{H > 0\}$, which is the same, $\{Y > 0\}$. Then X, H satisfy

$$\begin{cases} \dfrac{dX}{d\tau} = HX(1 - mX) + (\alpha + \beta X), \\ \\ \dfrac{dH}{d\tau} = -H^2(2 + (1 - m)X). \end{cases} \tag{4.3}$$

The proper behaviour in these variables corresponds to the critical point $B = (-\alpha/\beta, 0)$. The local analysis around this point is again straightforward.

Proposition 4.2 *The critical point $B = (-\alpha/\beta, 0)$ is a saddle-node of system* (4.3). *The linearization of* (4.3) *around $B = (-\alpha/\beta, 0)$ has matrix*

$$\begin{pmatrix} \beta & -\frac{\alpha}{\beta^2}(\beta + m\alpha) \\ 0 & 0 \end{pmatrix}$$

with eigenvalues $\lambda_1 = \beta$ and $\lambda_2 = 0$ and corresponding eigenvectors $e_1 = (1, 0)$ and $e_2 = (1, \beta^3/(\alpha(\beta + m\alpha)))$. The point B is a repeler on the half-plane $\{H < 0\}$ and a saddle on the half-plane $\{H > 0\}$.

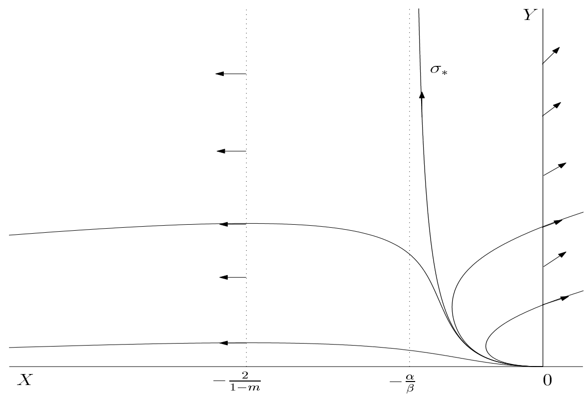

Figure 1: The trajectories in the XY–plane.

Existence of the connection. We are looking for an orbit connecting the critical points A and B. As there is a unique orbit σ_* arriving at B, we just have to trace back where it comes from. In the XY-plane the critical point B corresponds to $(-\alpha/\beta, +\infty)$. We observe that $dY/d\eta > 0$ for $X > -2/(1-m)$, $Y > 0$ and that $dX/d\eta > 0$ for $X = 0$,

$Y > 0$. Since $2/(1-m) > \alpha/\beta$ (this is equivalent to the condition $p > (m+1)/2$), then the orbit σ_* necessarily comes from A.

We now have to look more carefully at the behaviour of this trajectory near the point A. From Proposition 4.1, we have that in the second quadrant, $\{X < 0,\, Y > 0\}$, all the trajectories exit the origin tangent to the horizontal axis. Moreover, it is easy to check that they do this quadratically. In particular our trajectory exits A like $Y \sim \Lambda X^2$. This gives the value $V_* = 1/\sqrt{m\Lambda}$. Even more, $X < 0$ implies that the profile F_* corresponding to the orbit σ_* is strictly decreasing.

In order to give a complete understanding of the picture in the upper plane $\{Y > 0\}$, and prove that σ_* gives the unique profile, we consider all the trajectories starting at the origin $(X, Y) = (0, 0)$ and check that other choices of V give profiles that are either unbounded or defined only in a bounded interval.

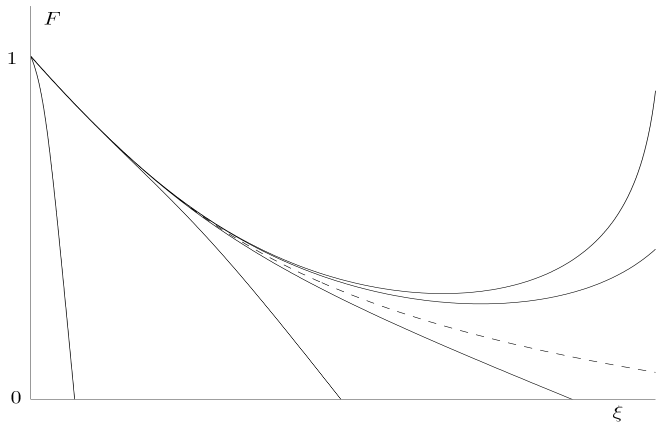

Figure 2: Solutions of (4.1) for different values of V.

First of all, from the equation of the profile (4.1), it is easy to see that once the function F satisfies $F'(\xi_0) \geq 0$ for some $\xi_0 \geq 0$, then $F'(\xi) > 0$ for every $\xi > \xi_0$. Therefore, integrating (4.1) we get the inequality

$$(F^m)' \geq c_1 F,$$

which implies that F becomes unbounded for a finite value of ξ. This corresponds to the trajectories in the XY-plane that exit the origin in the first quadrant ($V \leq 0$) and also those in the second quadrant that cross the vertical axis ($0 < V < V_*$).

We also observe that the trajectory σ_* is a separatrix in the second quadrant between those trajectories that cross the vertical axis and those trajectories that cross the vertical line $X = -2/(1-m)$. We now prove that these last trajectories, which correspond to taking $V > V_*$, give profiles that vanish at a finite value of ξ, thus not being defined in the whole $_+$. To see this let $Y = Y(X)$ be a trajectory passing through a point $(-2/(1-m), D)$, $D > 0$, at a time $\eta_0 \in$. The first equation in (4.2) gives, for $\eta > \eta_0$,

$$\frac{dX}{d\eta} \leq -mX^2,$$

since $\alpha + \beta X < 0$, $Y > 0$. This implies that there exists a finite η_∞ such that

$$\lim_{\eta \to \eta_\infty} X(\eta) = -\infty. \tag{4.4}$$

Also, since $dY/d\eta < 0$ and $dX/d\eta < 0$ for $\eta > \eta_0$, we have that there exists the limit

$$\lim_{\eta \to \eta_\infty} Y(\eta) = Y_\infty \geq 0.$$

If $Y_\infty > 0$ and (4.4), we get $F(\eta_\infty) > 0$ and $F'(\eta_\infty) = -\infty$, a contradiction with the regularity of positive solutions of (4.1). Therefore, from the relation between η and ξ, we deduce that there exists a finite ξ_∞ such that,

$$\lim_{\xi \to \xi_\infty} F(\xi) = 0.$$

This completes the proof. $\square$

5 Asymptotic behaviour

In this section we prove the stabilization result for the rescaled problem (1.7).

Proof of Theorem 1.5. Thanks to the blow-up rate (1.2) we know that g is bounded. The behaviour of u near $t = T$ is translated into the behaviour of g as $\tau \to \infty$. As expected, stationary solutions will play an outstanding role.

Using arguments similar to those in the proof of Theorem 1.3, we have that there exists a sequence $\tau_j \to \infty$ such that

$$\lim_{j \to \infty} g(\xi, \tau + \tau_j) = g_*(\xi, \tau) \tag{5.1}$$

uniformly in compact sets of $_+$. We want to prove that the function g_* does not depend on τ and therefore it coincides with the unique stationary solution F constructed in the previous section.

We now construct a Lyapunov function for g following the ideas of [Z] and [Ga], taking note of the boundary condition. Putting $h = g^m$, we get the problem

$$\begin{cases} h_\tau = a(h)(h_{\xi\xi} + b(\xi, h, h_\xi)), & (\xi, \tau) \in \ _+ \times \ _+, \\ -h_\xi(0, \tau) = h^{p/m}(0, \tau), & \tau \in \ _+, \end{cases}$$

where

$$a(h) = mh^{(m-1)/m}, \qquad b(\xi, h, z) = -\frac{\beta}{m}\xi h^{(1-m)/m} z - \alpha h^{1/m}.$$

We remark that the boundedness of h, together with $0 < m < 1$, imply $a(h) \geq a_0 > 0$.

Consider the function

$$L_h(\tau) = \int_0^\infty \Phi(\xi, h(\xi,\tau), h_\xi(\xi,\tau))\, d\xi - \frac{m}{m+p} h^{(m+p)/m}(0,\tau).$$

Differentiating and integrating by parts, we get

$$\begin{aligned}\frac{d}{d\tau}L_h(\tau) \; &= -(h^{p/m}(0,\tau) + \Phi_z(0, h, h_\xi))h_\tau(0,\tau) \\ &- \int_0^\infty \frac{1}{a}\Phi_{zz}(h_\tau)^2\, d\xi + \int_0^\infty (\Phi_h - \Phi_{\xi z} - \Phi_{hz}h_\xi + b\Phi_{zz})h_\tau\, d\xi.\end{aligned}$$

We can eliminate the last integral by choosing appropriately the function Φ using the method of characteristics. It is given by

$$\Phi(\xi, h, z) = \int_0^z (z-s)\rho(\xi, h, s)\, ds - \int_0^h b(\xi, s, 0)\rho(\xi, s, 0)\, ds,$$

where

$$\rho(\xi, h, z) = \exp\left(\int_0^\xi b_z(\eta)\, d\eta\right),$$

and b_z is evaluated along the characteristic $\phi(\eta, \xi, h, z)$, solution to

$$\begin{cases} \phi'' + b(\eta, \phi, \phi') = 0 & 0 < \eta < \xi, \\ \phi(\xi) = h, \; \phi'(\xi) = z, \end{cases} \tag{5.2}$$

(see [Ga]). Thus

$$\rho(\xi, h, z) = \exp\left(-\frac{\beta}{m}\int_0^\xi \eta\phi^{(1-m)/m}(\eta, \xi, h, z)\, d\eta\right).$$

Therefore, ρ is defined only on the domain spanned by characteristic lines. The calculation above allows to generate a global function ρ if equation $\phi'' = -b(\eta, \phi, \phi')$ allows to pass in a homeomorphic way from data at $\eta = 0$ to data at $\eta = \xi$. It is then clear that whenever $\beta \geq 0$, as in our case, we have $\rho \leq 1$. A lower estimate for ρ is crucial in our Lyapunov argument. If we could assert that all solutions ϕ involved in b_z are bounded then the last formula would imply a lower estimate for ρ of the form $\rho \leq \exp(-C\xi^2)$. This is in principle not the case for problem (5.2) and we meet problems if there are blow up solutions. This difficulty will be dealt with below, so we assume that $\rho \leq \exp(-C\xi^2)$.

We now calculate

$$\Phi_z(0, h, h_\xi) = \int_0^{h_\xi} \rho(0, h, s)\, ds = h_\xi(0,\tau) = -h^{p/m}(0,\tau).$$

Putting all together we get

$$\frac{d}{d\tau}L_h(\tau) = -\int_0^\infty \frac{1}{a(h)}\rho(\xi,h,h_\xi)(h_\tau)^2\,d\xi.$$

Since $b(\xi,h,z)\le 0$, we have that

$$L_h(\tau)\ge -C.$$

On the other hand, if we restrict ourselves to $0<\xi<L$ for any given L, we get $\rho\ge C(L)>0$. These estimates imply

$$\begin{aligned}\frac{4m}{(m+1)^2}\int_{\tau_1}^{\tau_2}\int_0^L((h^{(m+1)/(2m)})_\tau)^2\,d\xi\,d\tau \le{}& \frac{1}{C(L)}\int_{\tau_1}^{\tau_2}\int_0^\infty\frac{1}{a(h)}\rho(\xi,h,h_\xi)(h_\tau)^2\,d\xi\,d\tau\\ ={}& \frac{1}{C(L)}(L_h(\tau_1)-L_h(\tau_2))\le C.\end{aligned}$$

We conclude in a rather standard way the convergence of the orbits to a stationary solution to the limit problem (1.7), see for instance [ACP, Ga]. Indeed,

$$\begin{aligned}&\left\|h^{(m+1)/(2m)}(\cdot,\tau_j+\tau)-\ h^{(m+1)/(2m)}(\cdot,\tau_j)\right\|^2_{L^2([0,L])}=\\ &\qquad=\int_0^L\left|h^{(m+1)/(2m)}(\xi,\tau_j+\tau)-h^{(m+1)/(2m)}(\xi,\tau_j)\right|^2\,d\xi\\ &\qquad\le\tau\int_0^L\int_{\tau_j}^{\tau_j+\tau}\left|\frac{d}{d\tau}h^{(m+1)/(2m)}(\xi,s)\right|^2\,ds\,d\xi\to 0\end{aligned}$$

as $j\to\infty$, uniformly for bounded τ. Therefore, the sequence $h^{(m+1)/2m}(\xi,\tau_j+\tau)$ converges in the space $L^\infty([0,\tau]:L^2([0,L]))$ for every $\tau>0$. The limit does not depend on τ and is a stationary solution of (1.7). The uniqueness of the stationary solution to that problem, cf. Theorem 1.4, implies that the limit function g_* in (5.1) does not depend on τ. Hence (1.8).

Justification of the computation. If there exist solutions of equation (5.2) which blow up, the justification of the above process implies the study of the approximate problem

$$\begin{cases} h_\tau = a(h)(h_{\xi\xi}+\tilde b(\xi,h,h_\xi)) \quad \text{in} \quad {}_+\times\ {}_+,\\ (h^m)_\xi(0,\tau)=h^{p/m}(0,\tau),\end{cases}\tag{5.3}$$

where as before $a(h)=mh^{(m-1)/m}$, and $\tilde b$ is b corrected only for large values of h, $h\ge M$, so that the stationary solutions $\phi(\eta;\xi,h,z)$ are uniformly bounded for bounded data $\xi>0$, $h>0$, $z<0$ and depend continuously on the data ξ,h,z. Now, if we take a particular solution of the original evolution problem, from Theorem 1.3 we have that the function h is globally bounded, $h(\xi,\tau)\le C$ for all $\xi\in{}_+$ and $\tau>0$. Taking a constant $M\ge C$ in the correction performed above, we see that the solution under consideration is also a solution of the corrected problem (5.3). Therefore, the use of the corrected problem is justified and the calculations given above hold. $\square$

THE PROBLEM IN A BOUNDED INTERVAL. We define the Lyapunov functional as before, but now integrating in the interval $(0, R(\tau))$, where $R(\tau) = L(T-t)^{-\beta}$, (recall that $\tau = -\log(T-t)$). Then we have

$$L_h(\tau) = \int_0^{R(\tau)} \Phi(\xi, h(\xi,\tau), h_\xi(\xi,\tau))\, d\xi - \frac{m}{m+p} h^{(m+p)/m}(0,\tau).$$

We must show that the extra term that appears in the expression for the derivative of $L_h(\tau)$, namely

$$\Phi(R(\tau), h(R(\tau),\tau), h_\xi(R(\tau),\tau))R'(\tau),$$

is integrable. To this purpose we observe that $\phi(\eta, R(\tau), s, 0)$ is decreasing for $\eta \in (0, R(\tau))$ and that $v(\xi,\tau) \le C$. Hence

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} \Phi(R(\tau), h(R(\tau),\tau), h_\xi(R(\tau),\tau))\, R'(\tau)\, d\tau \\
&\quad = \int_{\tau_1}^{\tau_2} \int_0^{h(R(\tau),\tau)} R'(\tau) s^{1/m} \exp\left(-\frac{\beta}{m} \int_0^{R(\tau)} \eta\, (\phi(\eta, R(t), s, 0))^{(1-m)/m}\, d\eta \right) ds\, dt \\
&\quad \le \int_0^C \int_{\tau_1}^{\tau_2} R'(\tau) s^{1/m} \exp\left(-c\, s^{(1-m)/m} R(\tau)^2 \right) d\tau\, ds \le C,
\end{aligned}$$

where the last constant is independent of τ_1, τ_2. Therefore, following the same argument given above we conclude that (1.8) holds also in the case of a bounded interval.

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