Numerical blow-up for a nonlinear problem with a nonlinear boundary condition

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Abstract

In this paper we study numerical approximations for positive solutions of a nonlinear diffusion equation with a nonlinear boundary condition,

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in (0,L) \times [0,T), \\ (u^m)_x(0,t) = 0 & t \in [0,T), \\ (u^m)_x(L,t) = u^p(L,t) & t \in [0,T), \\ u(x,0) = u_0(x) & x \in (0,L), \end{cases}$$

where m and p are positive parameters.

We describe in terms of p and m when solutions of a semidiscretization in space exist globally in time and when they blow up in finite time. We also find the blow-up rates and the blow-up sets. In particular we prove that regional blow-up is not reproduced by the numerical scheme, however in appropriate variables we can get the correct behaviour when the mesh parameter goes to zero.

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1 Introduction.

In this paper we deal with a numerical approximation for the following problem,

$$\begin{cases} u_t = (u^m)_{xx} & (x,t) \in (0,L) \times [0,T), \\ (u^m)_x(0,t) = 0 & t \in [0,T), \\ (u^m)_x(L,t) = u^p(L,t) & t \in [0,T), \\ u(x,0) = u_0(x) & x \in (0,L), \end{cases} \tag{1.1}$$

where m and p are positive parameters. We assume that u_0 is positive.

In many problems, like (1.1), solutions exist only for a finite period of time, $T < \infty$, in this case u becomes unbounded in finite time and we say that it has blow-up, or it is defined for all positive t, $T = \infty$, in that case we call it a global solution. See [FF], [L], [P], [SGKM] for references on blow-up problems.

We are interested in numerical approximations of (1.1). Since the solution u may develop a singularity in finite time, it is an interesting question what can be said about numerical approximations of this kind of problems. For previous work on numerical approximations of blowing up solutions we refer to [ALM1], [ALM2], [BB2], [BK], [BHR], [C], [FBR], [LR], [N], [NU] the survey [BB] and references therein. For (1.1) in the case $m = 1$, that is for linear diffusion, we refer to [DER]. Up to our knowledge this is the first numerical study of (1.1).

In our problem one has a nonlinear source term at the boundary $x = L$ and a nonlinear diffusion in the equation. The behaviour of the solutions is determined by the different influence of both terms. This problem was analyzed in [Fi] and the behaviour varies if m is greater or less than one. Let us summarize the known results for solutions of (1.1):

If $m > 1$ the equation is known as *porous medium equation*, it is proved in [Fi] that for every positive initial data the solution blows up if and only if $p > 1$. Moreover the blow-up rate for increasing initial data is given by,

- $\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{p-1}}$ if $1 < p \le m$,
- $\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{2p-m-1}}$ if $p > m$.

For this problem the blow-up set, $B(u)$, i.e., the set of points $x \in [0,L]$ where $u(x,t)$ becomes unbounded, is given by,

- $B(u) = [0,L]$ if $1 < p < m$ (global blow-up),
- $B(u) = [0,L]$ if $p = m$ and $L \le \frac{2m}{m-1}$ (global blow-up),
- $B(u) = [L - \frac{2m}{m-1}, L]$ if $p = m$ and $L > \frac{2m}{m-1}$ (regional blow-up),
- $B(u) = \{L\}$ if $p > m$ (single point blow-up).

This is proved in [Fi] for the cases $1 < p < m$ and $p > m$. For the case $p = m$ we perform the analysis in the Appendix and give a more precise description of blow-up showing the existence and uniqueness of a profile that gives the asymptotic behaviour of blowing up solutions when we consider self-similar variables.

In the case $0 < m \le 1$ (known as *fast diffusion equation* if $0 < m < 1$ or *heat equation* for $m = 1$) the existence of blowing up solutions depends on m and p. In fact every positive solution blows up if $p > \frac{m+1}{2}$, and if $p \le \frac{m+1}{2}$ every solution is global (see [Fi], [WD]). In this case the blow-up rate is given by

- $\|u(\cdot,t)\|_{L^\infty} \sim (T-t)^{-\frac{1}{2p-m-1}}$ if $p > \frac{m+1}{2}$,

and the blow-up set is

- $B(u) = \{L\}$ if $p > \frac{m+1}{2}$ (single point blow-up).

Now we introduce the numerical scheme. We discretize using piecewise linear finite elements with mass lumping in a uniform mesh for the space variable, it is well known that this discretization in space coincides with the classic central finite difference second order scheme.

We denote with $U(t) = (u_1(t), ..., u_{N+1}(t))$ the values of the numerical approximation at the nodes $x_k = (k-1)h$, $h = L/N$, at time t. Then $U(t)$ is a solution of the following problem (see [Ci]):

$$\begin{cases} MU'(t) = -AU^m(t) + BU^p(t), \\ U(0) = u_0^I, \end{cases} \tag{1.2}$$

where M is the mass matrix obtained with lumping, A is the stiffness matrix and u_0^I is the Lagrange interpolation of the initial data, u_0. Writing this equation explicitly we obtain the following ODE system,

$$\begin{cases} u_1'(t) = \frac{2}{h^2}(u_2^m(t) - u_1^m(t)), \\ \\ u_k'(t) = \frac{1}{h^2}(u_{k+1}^m(t) - 2u_k^m(t) + u_{k-1}^m(t)), \qquad\qquad 2 \le k \le N, \\ \\ u_{N+1}'(t) = \frac{2}{h^2}(u_N^m(t) - u_{N+1}^m(t)) + \frac{2}{h}u_{N+1}^p(t), \\ \\ u_k(0) = u_0(x_k) > 0, \qquad 1 \le k \le N+1. \end{cases} \tag{1.3}$$

To begin our analysis we prove that numerical approximations given by (1.3) converge uniformly if we consider a regular bounded solution of the continuous problem. Hence our scheme is uniformly convergent in sets of the form $[0,L] \times [0, T-\tau]$.

Our main results concern the behaviour of the numerical approximations given by (1.3). Significant differences appear between the continuous and the discrete problem.

First we prove that positive solutions of the numerical problem blow up if and only if $p > 1$. Hence, in the case $0 < m < 1$ with $\frac{m+1}{2} < p \le 1$ the continuos solutions blow up in finite time while the numerical approximations are globally defined.

Next, we turn our attention to the blow-up rate. For increasing initial data we find that the blow-up rate for the numerical scheme is given by

- $\|U(t)\|_\infty \sim (T_h - t)^{-\frac{1}{p-1}}$ if $p > 1$.

Therefore the blow-up rate does not coincide in the range of parameters given by $p > m > 1$ or $p > 1$ with $0 < m \le 1$, and coincides in the range $m > 1$ with $1 < p \le m$.

Concerning the blow-up set for the numerical approximations we prove that

- $B(U) = [0, L]$ if $1 < p \le m$ (global blow-up).

- $B(U) = [L - Kh, L]$ if $p > \max\{m, 1\}$. Where the constant K depends only on m and p. K is the integer that verifies

$$\frac{\sum_{i=0}^{K+1} m^i}{\sum_{i=0}^{K} m^i} < p \le \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i}.$$

In the range $1 < p < m$ the numerical and the continuous blow-up sets coincide.

In the range $p > \max\{m, 1\}$, the blow-up set can be larger than a single point, $x = L$, but our results show that

$$B(U) = [L - Kh, L] = B(u) + [-Kh, 0],$$

and hence

$$B(U) \to B(u), \qquad \text{as } h \to 0.$$

In the case $p = m > 1$ with L large a mayor difference appears concerning the blow-up sets. The continuous problem blows up in the subinterval $[L - \frac{2m}{m-1}, L]$, strictly contained in $[0, L]$, but the numerical solution blows up in the whole interval $[0, L]$. There is no regional blow-up for the numerical scheme.

However, in this case we can recover regional blow-up for the numerical scheme taking into account self-similar variables and letting h go to zero. We prove that the numerical method blows up at nodes that lies in $[0, L - \frac{2m}{m-1})$ at a blow-up rate that goes to zero as h goes to zero while for nodes that belongs to $[L - \frac{2m}{m-1}, L]$ the blow-up rate does not vanish as h goes to zero. Hence we have recovered the regional blow-up by looking at the different blow-up rates of the solution at different nodes when the mesh parameter goes to zero.

To summarize our results; we observe that when computing numerical approximations of a blow-up problem significant differences appear. The continuous problem can blow up while the numerical scheme has global solutions. Moreover, in case that both problems blow up the blow-up rate and the blow-up set can be different. We remark that regional blow-up is impossible for a numerical scheme with a fixed mesh, however if we look at the right variables we can recover regional blow-up when the mesh parameter goes to zero.

Organization of the paper. In Section 2 we describe some properties of the numerical scheme and prove the convergence of the method. In Section 3 we prove the numerical blow-up results and find the numerical blow-up rates. Section 4 is devoted to the numerical blow-up sets. In Section 5 we present some numerical experiments. Finally we include an Appendix with some results on stability and uniqueness of the asymptotic profile for solutions of the continuous problem (1.1) in self-similar variables.

2 Properties of the numerical scheme

In this section we collect some preliminary results on our numerical method. In particular we prove convergence for regular solutions.

First, we want to prove a comparison Lemma. To do this we need the following definition,

Definition: We will call $\overline{U}$ a supersolution if it satisfies

$$\begin{cases} \overline{u}_1'(t) \geq \frac{2}{h^2}(\overline{u}_2^m(t) - \overline{u}_1^m(t)), \\ \overline{u}_k'(t) \geq \frac{1}{h^2}(\overline{u}_{k+1}^m(t) - 2\overline{u}_k^m(t) + \overline{u}_{k-1}^m(t)), \qquad 2 \leq k \leq N, \\ \overline{u}_{N+1}'(t) \geq \frac{2}{h^2}(\overline{u}_N^m(t) - \overline{u}_{N+1}^m(t)) + \frac{2}{h}\overline{u}_{N+1}^p(t), \\ \overline{u}_k(0) \geq u_0(x_k), \qquad 1 \leq k \leq N+1. \end{cases} \tag{2.1}$$

Analogously, we say that $\underline{U}$ is a subsolution if it satisfies (2.1) with the reverse inequalities.

Lemma 2.1 *Let $\overline{U}$ and $\underline{U}$ be a superolution and a subsolution respectively, then*

$$\overline{U}(t) \geq U(t) \geq \underline{U}(t).$$

Proof. By an approximation procedure we can assume that we have strict inequalities in (2.1). Let us prove that $\overline{U}(t) > U(t)$. We argue by contradiction. Let us assume that there exists a first time t_0 and a node j such that $\overline{u}_j(t_0) = u_j(t_0)$ then if $1 < j < N+1$,

$$0 \geq \overline{u}_j'(t_0) - u_j'(t_0) > \frac{1}{h^2}(\overline{u}_{j+1}^m(t_0) - u_{j+1}^m(t_0) + \overline{u}_{j-1}^m(t_0) - u_{j-1}^m(t_0)) \geq 0,$$

a contradiction. In case $j = 1$ we get

$$0 \geq \overline{u}_1'(t) - u_1'(t_0) > \frac{2}{h^2}(\overline{u}_2^m(t_0) - u_2^m(t_0)) \geq 0,$$

again a contradiction. Finally, if $j = N+1$ we have

$$0 \geq \overline{u}_{N+1}'(t_0) - u_{N+1}'(t_0) > \frac{2}{h^2}(\overline{u}_N^m(t_0) - u_N^m(t_0)) \geq 0.$$

This contradiction proves that $\overline{U}(t) > U(t)$.

The inequality $U(t) \geq \underline{U}(t)$ can be handled in a similar way. $\square$

Now we prove our convergence result for regular solutions up to $t = T - \tau$, for any $\tau > 0$.

Theorem 2.1 *Let $u(x,t) \in C^{4,1}([0,L] \times [0,T-\tau])$ be a positive solution of (1.1) and $U(t)$ the numerical approximation given by (1.3). Then, there exists a constant C, that depends on the $C^{4,1}([0,L] \times [0,T-\tau])$ norm of u, such that for every h small enough it holds*

$$\max_{t \in [0,T-\tau]} \max_k |u(x_k,t) - u_k(t)| \leq Ch^2.$$

Proof. In the course of this proof we will denote by C_i a constant independent on h which can be different in different occurrences.

If we rewrite the system (1.3) in terms of $Z = U^m$ we obtain

$$\begin{cases} (z_1^{1/m})'(t) = \frac{2}{h^2}(z_2(t) - z_1(t)), \\ (z_k^{1/m})'(t) = \frac{1}{h^2}(z_{k+1}(t) - 2z_k(t) + z_{k-1}(t)), \qquad 2 \le k \le N, \\ (z_{N+1}^{1/m})'(t) = \frac{2}{h^2}(z_N(t) - z_{N+1}(t)) + \frac{2}{h} z_{N+1}^{p/m}(t), \\ z_k(0) = u_0^{1/m}(x_k) > 0, \qquad 1 \le k \le N+1. \end{cases}$$

Let $v_k(t) = u^m(x_k, t)$ where u is the solution of the continuous problem (1.1). We define the error function as

$$e_k = z_k - v_k.$$

Let

$$t_0 = \max\{t \in [0, T-\tau] : |e_k|(t) \le c/2\}$$

where $c = \min\{v_k(t) : 1 \le k \le N+1 \text{ and } 0 \le t \le T-\tau\}$. We perform the following calculations with $t \in [0, t_0]$ and we will prove at the end that $t_0 = T - \tau$ for every h small enough.

The error function satisfies that, for $2 \le k \le N$,

$$\begin{aligned} \frac{1}{m} z_k^{(1-m)/m} e_k' = & \frac{1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) - \frac{1}{m}(z_k^{(1-m)/m} - v_k^{(1-m)/m})\, v_k' + C_1 h^2 \\ \le & \frac{1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) + C_1 h^2 + C_2 \xi_k^{(1-2m)/m)} |e_k|, \end{aligned}$$

where ξ_k is an intermediate value between z_k and v_k. Taking into account that there exist constants, c and C, such that $c \le z_k(t) \le C$ for every $t \in [0, t_0]$ we have

$$e_k' \le \frac{C_1}{h^2}(e_{k+1} - 2e_k + e_{k-1}) + C_2|e_k| + C_3 h^2. \tag{2.2}$$

Making analogous calculations for the first and the last nodes, we get

$$e_1' \le \frac{C_1}{h}(e_2 - e_1) + C_2|e_1| + C_3 h^2, \tag{2.3}$$

$$e_{N+1}' \le \frac{C_1}{h^2}(e_N - e_{N+1}) + C_2 h^2 - C_3 \xi^{(1-2m)/m} e_{N+1} + \frac{C_4}{h} \eta^{(p-m)/m} e_{N+1}, \tag{2.4}$$

where η is an intermediate value between z_{N+1} and v_{N+1}. Now we use a comparison argument. We remark that the system (2.2), (2.3), (2.4), has a comparison principle, that can be proved as in Lemma 2.1. Let us look for a supersolution of the form

$$w_k(t) = \varphi(t) \left(e^{C x_k} - C\, x_k \right),$$

where $\varphi(t)$ is a solution of

$$\begin{cases} \varphi'(t) = C_1\varphi(t) + C_2h^2, \\ \varphi(0) = C_3h^2. \end{cases}$$

That is

$$\varphi(t) = h^2\left(C_1e^{C_2t} + C_3\right).$$

A direct calculation shows that we can choose the constant C large but independent of h such that, $W(t)$ is a supersolution. Hence,

$$e_k(t) \le w_k(t) \le C_1h^2e^{C_2T}.$$

Arguing in the same way with $-e_k$ we obtain

$$|e_k(t)| \le w_k(t) \le C_1h^2e^{C_2T}.$$

From this inequality it is easy to see that $t_0 = T - \tau$ for every h small enough and the Theorem is proved. $\square$

To finish this Section, we state a Lemma that says that an increasing in space initial data gives an increasing solution. This will be used in the following Sections to ensure that the maximum of $U(t)$ is attained at the last node, x_{N+1}.

Lemma 2.2 *Let U be a solution of* (1.3) *with $u_k(0) \le u_{k+1}(0)$, $k = 1, ..., N$ then*

$$u_k(t) < u_{k+1}(t), \qquad 1 \le k \le N.$$

Proof. We argue by contradiction, let us assume there exists a first time t_0 and two consecutive nodes where the Lemma fails, let us call them $j, j+1$. We can assume that $u_{j-1}(t_0) < u_j(t_0)$. Indeed, if all the nodes have the same value then all of them have null derivative except the last one, which verifies $u'_{N+1}(t_0) > 0$, but this is not possible. Then we can assume that

$$u_{j-1}(t_0) < u_j(t_0) = u_{j+1}(t_0) \le u_{j+2}(t_0),$$

hence we obtain

$$u'_j(t_0) < 0, \qquad u'_{j+1}(t_0) \ge 0,$$

which is a contradiction. We observe that if at time $t = 0$ we have this situation then instantaneously we have $u_j(t) < u_{j+1}(t)$. $\square$

3 Numerical blow-up.

In this Section we prove that solutions of the numerical problem blow up if and only if $p > 1$ and we find the blow-up rate in this case.

Theorem 3.1 *If $p > 1$ then every positive solution of* (1.3) *blows up. Moreover, if $U(0)$ is increasing, then there exist two constants, $C_1 = C_1(h)$ and $C_2 = C_2(h)$, such that*

$$\frac{C_1}{(T_h - t)^{1/(p-1)}} \le \|U(t)\|_\infty \le \frac{C_2}{(T_h - t)^{1/(p-1)}}.$$

where T_h is the blow-up time of U.

Proof. Let us begin with an increasing initial data $U(0)$. By Lemma 2.2 $U(t)$ is increasing and then its maximum will be $u_{N+1}(t)$.

We define the function

$$w(t) := \frac{1}{2}u_1(t) + \sum_{j=2}^{N} u_j(t) + \frac{1}{2}u_{N+1}(t),$$

which satisfies

$$w'(t) = \frac{1}{h}u_{N+1}^p(t).$$

Hence there exists two positive constants, $c = c(h)$ and $C = C(h)$, such that

$$cw(t) \le u_{N+1}(t) \le Cw(t).$$

Therefore, the function $w(t)$ verifies

$$w'(t) \le Cw^p(t), \qquad w'(t) \ge cw^p(t). \tag{3.1}$$

From the second inequality we get that w (and hence U) blows up if $p > 1$ at a time T_h, and from the first one, if $p \le 1$, w (and therefore U) must be global. This blow-up result is valid for every initial data as we can use a comparison argument with an increasing super or subsolution.

Now we find the blow-up rate. By integration of (3.1) we obtain that

$$\frac{C_1}{(T_h - t)^{1/(p-1)}} \le w(t) \le \frac{C_2}{(T_h - t)^{1/(p-1)}}.$$

In terms of u_{N+1} we get

$$\frac{C_1}{(T_h - t)^{1/(p-1)}} \le u_{N+1}(t) = \|U(t)\|_\infty \le \frac{C_2}{(T_h - t)^{1/(p-1)}}.$$

Note that the blow-up phenomena occurs if and only if $p > 1$ and the blow-up rate is independent of the parameter m. □

We remark that this rate coincides with the blow-up rate of the continuous problem (1.1) if $1 < p \le m$, but it is different in the case $p > m > 1$ or $0 < m \le 1$ and $p > 1$.

4 Numerical blow-up set.

Now we turn our interest to the blow-up set of the numerical solution. We consider an increasing initial data, and for a fixed h we want to look at the set of nodes, x_k, such that $u_k(t) \to +\infty$ as $t \nearrow T_h$. To do this we introduce the self-similar variables given by

$$\begin{cases} y_k(s) = (T_h - t)^{\frac{1}{p-1}} u_k(t), \\ \\ (T_h - t) = e^{-s}. \end{cases} \tag{4.1}$$

In this new variables $Y = (y_k(s))$, problem (1.3) reads,

$$\begin{cases} y_1'(s) = \frac{2}{h^2} e^{\frac{m-p}{p-1}s}(y_2^m(s) - y_1^m(s)) - \frac{1}{p-1} y_1(s), \\ \\ y_k'(s) = \frac{2}{h^2} e^{\frac{m-p}{p-1}s}(y_{k+1}^m(s) - 2y_k^m(s) + y_{k-1}^m(s)) - \frac{1}{p-1} y_k(s), & 2 \le k \le N, \\ \\ y_{N+1}'(s) = \frac{2}{h^2} e^{\frac{m-p}{p-1}s}(y_N^m(s) - y_{N+1}^m(s)) + \frac{2}{h} y_{N+1}^p(s) - \frac{1}{p-1} y_{N+1}(s), \\ \\ y_k(-\ln(T_h)) = (T_h)^{\frac{1}{p-1}} u_0(x_k), \qquad 1 \le k \le N+1. \end{cases} \tag{4.2}$$

We observe that, from the blow-up rates proved in Theorem 3.1, the vector Y is bounded and there exists a subsequence such that

$$\lim_{s_j \to \infty} y_{N+1}(s_j) = a_{N+1} \neq 0.$$

Theorem 4.1 *If $1 < p < m$ then we have uniform global blow-up, that is there exists a constant C independent of k such that*

$$u_k(t) \sim \frac{C}{(T_h - t)^{\frac{1}{p-1}}} \qquad 1 \le k \le N,$$

therefore

$$B(U) = [0, L].$$

Proof. Since $y_{N+1}(s)$ is bounded we have that $y_{N+1}'(s)$ can not go to infinity as $s \to \infty$. From (4.2) we have

$$y_{N+1}'(s) = \frac{2}{h^2} e^{\frac{m-p}{p-1}s}(y_N^m(s) - y_{N+1}^m(s)) + \frac{2}{h} y_{N+1}^p(s) - \frac{1}{p-1} y_{N+1}(s).$$

As $p < m$ there exists a subsequence $\{s_j\}$, such that

$$y_N^m(s_j) - y_{N+1}^m(s_j) \to 0.$$

Now, we can take again a subsequence such that $y_N(s_j) \to a_{N+1}$. Applying the same argument with y_k we obtain that $y_{k-1}(s_j) \to a_{N+1}$ for all $2 \le k < N$.

Therefore, in the context of the variable $U(t)$ we have that all nodes blow up in finite time T_h with the same rate. □

Theorem 4.2 *If $p > \max\{m, 1\}$, then U blows up at exactly K nodes near L, i.e.*

$$B(U) = [L - Kh, L],$$

where K is determined only by p and m in the following way: K is the unique integer such that

$$\frac{\sum_{i=0}^{K+1} m^i}{\sum_{i=0}^{K} m^i} < p \le \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i}.$$

Moreover, the asymptotic behaviour of the blowing up nodes is given by

$$u_{N+1-j}(t) \sim (T_h - t)^{\gamma_j}, \qquad \gamma_j = -\frac{m^j}{p-1} + \sum_{l=0}^{j-1} m^l, \qquad \text{if} \quad p \neq \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i} \quad \text{or} \quad j \neq K$$

and by

$$u_K(t) \sim -\ln(T_h - t), \qquad \text{if} \quad p = \frac{\sum_{i=0}^{K} m^i}{\sum_{i=0}^{K-1} m^i}.$$

Proof. As Y is bounded and $p > m$, we have that for s large enough

$$y'_{N+1}(s) \sim \frac{2}{h} y^p_{N+1}(s) - \frac{1}{p-1} y_{N+1}(s).$$

So, for s large the function $y_{N+1}(s)$ is monotone and it has a positive limit as $s \to \infty$.

On the other hand, if we return to the variable $U(t)$, we have that

$$\begin{aligned} u'_N(t) = {} & \frac{1}{h^2}(u^m_{N+1}(t) - 2u^m_N(t) + u^m_{N-1}(t)) \leq \frac{1}{h^2}(u^m_{N+1}(t) + u^m_{N-1}(t)) \\ \leq {} & C(T_h - t)^{-\frac{m}{p-1}} \end{aligned}$$

and by integration

$$u_N(t) \leq \begin{cases} C(T_h - t)^{\frac{p-1-m}{p-1}} & p < m+1, \\ -C \ln(T_h - t) & p = m+1, \\ C & p > m+1. \end{cases}$$

If we translate this inequality in terms of Y, we get

$$0 \leq y_N(s) \leq \begin{cases} C\, e^{-\frac{p-m}{p-1}s} & p < m+1, \\ C\, s\, e^{-\frac{1}{p-1}s} & p = m+1, \\ C\, e^{-\frac{1}{p-1}s} & p > m+1. \end{cases}$$

Notice that in all cases $y_N(s) \to 0$. Repeating the argument we obtain that

$$y_k(s) \to 0, \qquad \text{for all} \quad k \leq N.$$

Moreover, for s large

$$\begin{aligned} y'_N(s) = {} & \frac{2}{h^2} e^{\frac{m-p}{p-1}s}(y^m_{N+1}(s) - 2y^m_N(s) + y^m_{N-1}(s)) - \frac{1}{p-1} y_N(s) \\ \sim {} & C e^{\frac{m-p}{p-1}s} - \frac{1}{p-1} y_N(s). \end{aligned}$$

Integrating we obtain the asymptotic behaviour of y_N,

$$y_N(s) \sim \begin{cases} C_1 e^{-\frac{1}{p-1}s} + C_2 e^{\frac{m-p}{p-1}s} \\ C_1 e^{-\frac{1}{p-1}s} + C_2 s e^{-\frac{1}{p-1}s} \end{cases} \sim \begin{cases} C_2 e^{\frac{m-p}{p-1}s} & p < m+1, \\ C_2 s e^{-\frac{1}{p-1}s} & p = m+1. \end{cases}$$

Translating this behaviour to the context of $U(t)$,

$$u_N(t) \sim \begin{cases} C(T_h - t)^{\frac{p-m-1}{p-1}} & p < m+1, \\ -C \ln(T_h - t) & p = m+1. \end{cases}$$

Therefore, if $p \leq 1 + m$ the node $u_N(t)$ blows up with a different rate than u_{N+1}, and for $p > m + 1$ it is bounded.

Repeating this argument with the nodes $N-1, N-2, \dots$ we obtain the asymptotic behaviour of each node and the result follows. □

Theorem 4.3 *If $p = m$ every node is a blow-up point. Moreover in the self-similar variables,*

$$Y(s) \to W, \qquad \text{as} \quad s \to \infty,$$

where $W = (w_1, \dots, w_{N+1})$ is the unique positive stationary solution of (4.2). Hence the asymptotic behaviour of $u_k(t)$ is given by

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}} w_k.$$

Proof. In this case, if we write the numerical problem (4.2) in matrix form (1.2), we have a Lyapunov functional. In fact,

$$\Phi(Y)(s) = -\frac{1}{2}\langle A^{1/2}Y^m, A^{1/2}Y^m\rangle - \frac{1}{2m}BY^{2m} + \frac{1}{(p-1)(m+1)}MY^{m+1}$$

satisfies

$$\Phi'(Y)(s) = \langle MY', Y^{m-1}Y'\rangle \geq 0.$$

Hence, the orbit $Y(t)$ goes to a stationary state, see [H]. Therefore we turn our attention to the study of stationary solutions of (4.2), which satisfies

$$\begin{cases} 0 = \frac{2}{h^2}(w_2^m - w_1^m) - \frac{1}{m-1}w_1, & \\ 0 = \frac{1}{h^2}(w_{k+1}^m - 2w_k^m + w_{k-1}^m) - \frac{1}{m-1}w_k, & 2 \leq k \leq N, \\ 0 = \frac{2}{h^2}(w_N^m - w_{N+1}^m) + \frac{2}{h}w_{N+1}^m - \frac{1}{m-1}w_{N+1}. & \end{cases} \tag{4.3}$$

Moreover, as we begin with an increasing data $U(0)$ then for a fixed s the vector $Y(s)$ is positive and increasing, hence we have to look to nonnegative and nondecreasing stationary solutions.

On the other hand, from the blow-up rate we know that $y_{N+1}(s) \geq C(h) > 0$, then W is not the trivial solution. We claim that w_k must be positive for all k. To prove this claim just suppose that there exists some j with $w_j = 0$. Since W is non-negative and non-decreasing we have that $w_1 = 0$, but in this case $w_2 = 0$ and therefore $w_k = 0$ for all k. This contradiction proves the claim.

In order to prove the uniqueness of W, we observe that the functions w_k are increasing with respect to w_1. Indeed, from (4.3) we have that w_2 is increasing. Now, let us suppose that w_j is increasing for $j = 2, \dots, J$

$$\begin{aligned} w_{J+1}^m - w_J^m &= w_J^m - w_{J-1}^m + \frac{h^2}{m-1} w_J \\ &= w_2^m - w_1^m + \frac{h^2}{m-1} \sum_{l=2}^{l=J} w_l. \end{aligned}$$

Therefore,

$$w_{J+1}^m = w_J^m + \frac{h^2}{2(m-1)} w_1 + \frac{h^2}{m-1} \sum_{l=2}^{l=J} w_l.$$

So the function w_{J+1} is an increasing function of w_1.

Therefore, if there exists two stationary solutions, $\widetilde{W}$ and W, then we can assume that $\widetilde{W} > W$. But this is imposible since in the original variable t the corresponding solutions $\widetilde{U}(t)$ and $U(t)$ given by

$$\widetilde{U}(t) = (T_h - t)^{-\frac{1}{p-1}} \widetilde{W}, \qquad U(t) = (T_h - t)^{-\frac{1}{p-1}} W$$

verify $\widetilde{U}(t) > U(t)$ and have the same blow-up time. $\square$

Now our goal is to recover regional blow-up by looking carefully at the behaviour of the stationary solution as the parameter h goes to zero.

When h goes to zero we expect that $Z = W^m$ converges to a solution of the following problem

$$\begin{cases} z_{xx} - \frac{1}{m-1} z^{\frac{1}{m}} = 0, & x \in (0, L), \\ z_x(0) = 0, \\ z_x(L) = z(L). \end{cases} \tag{4.4}$$

This is the content of our next Lemma. We remark that since $m > 1$ a non Lipschitz function appears in (4.4).

Lemma 4.1 *Let W be the solution of* (4.3) *and let $z(x)$ be the unique stationary solution of* (4.4). *Then*

$$Z = W^m \to z(x), \qquad \text{as } h \to 0.$$

Proof. Multiplying the equation satisfied by Z by

$$\frac{(z_{k+1} - z_k) + (z_k - z_{k-1})}{2}$$

and summing we get

$$0 = \frac{(z_{N+1} - z_N)^2}{h^2} - \frac{(z_2 - z_1)^2}{h^2} - \frac{1}{m-1} \sum_{l=2}^{N} z_l^{\frac{1}{m}} \frac{(z_{l+1} - z_{l-1})}{2}.$$

Hence

$$0 = \frac{(z_{N+1} - z_N)^2}{h^2} - \frac{(z_2 - z_1)^2}{h^2} - \frac{m}{m^2-1}(z_{N+1}^{\frac{m+1}{m}} - z_1^{\frac{m+1}{m}}) + O(h).$$

Using the first and the last equations of (4.3) we get that z_{N+1} and z_1 must verify the following polynomial,

$$0 = \frac{1}{2} z_{N+1}^2 - \frac{m}{m^2-1} z_{N+1}^{\frac{1+m}{m}} + \frac{m}{m^2-1} z_1^{\frac{1+m}{m}} + O(h). \tag{4.5}$$

Therefore, z_1 must be bounded. Then we can take a subsequence of h such that $z_1 \to \Gamma \geq 0$.

We consider two different cases,

$$L < \frac{2m}{m-1} \qquad \text{and} \qquad L \geq \frac{2m}{m-1}.$$

In the first case, if $\Gamma > 0$ we consider the auxiliary initial value problem

$$\begin{cases} z'' = \frac{1}{m-1} z^{1/m}, & z \in [0, L], \\ z(0) = \Gamma, \\ z'(0) = 0. \end{cases}$$

This problem has a unique solution. Moreover, for $x \in [0, L]$ the function $z(x)$ is increasing. Then, by general theory (see [JR]) we get that in this case

$$Z = W^m \to z(x), \qquad \text{as } h \to 0.$$

Moreover by (4.5) we have that

$$0 = \frac{1}{2} z(L)^2 - \frac{m}{m^2-1} z(L)^{\frac{1+m}{m}} + \frac{m}{m^2-1} \Gamma^{\frac{1+m}{m}}.$$

So, the constant Γ must be the only constant such that z is a solution of

$$\begin{cases} z'' = \frac{1}{m-1} z^{1/m}, & x \in [0, L], \\ z'(0) = 0, \\ z'(L) = z^{p/m}(L). \end{cases}$$

For the existence and uniqueness results for this problem we refer to the Appendix.

On the other hand, if we assume that $\Gamma = 0$ we arrive to a contradiction. By (4.5) we obtain that $z_{N+1} \to A$, where A is the only positive root of

$$0 = \frac{1}{2} z(L)^2 - \frac{m}{m^2-1} z(L)^{\frac{1+m}{m}}.$$

That is

$$A = \left(\frac{2m}{m^2-1} \right)^{\frac{m}{m-1}}. \tag{4.6}$$

Hence, we consider the problem

$$\left\{\begin{array}{ll} z'' = \frac{1}{m-1} z1/m & x \in [0, L], \\ zL) = A, & \\ z'(L) = -A^{p/m}. & \end{array}\right. \tag{4.7}$$

By classical theory we have that in the interval where w is positive, $(L - \frac{2m}{m-1}, L]$,

$$Z = W^m \to z(x), \qquad \text{as } h \to 0.$$

Since $L < \frac{2m}{m-1}$ we have that $z(x)$ is positive in $[0, L]$ (see the Appendix). Hence, $z_1 \to z(0) > 0$, which is a contradiction. This proves that Γ must be positive.

In the second case, $L \geq \frac{2m}{m-1}$. If $\Gamma > 0$ we arrive to a contradiction, as by the previous argument $z_{N+1} \to z(L)$ with $z(L)$ is a root of

$$0 = \frac{1}{2} z(L)^2 - \frac{m}{m^2 - 1} z(L)^{\frac{1+m}{m}} + \frac{m}{m^2 - 1} \Gamma^{\frac{1+m}{m}}.$$

But this contradicts the uniqueness result given in the Appendix. Hence $\Gamma = 0$ and therefore

$$z_{N+1} \to z(L) = A,$$

where A is given by (4.6). Now we consider the problem (4.7). When the solution z is positive we can again apply the standard theory to find that

$$Z = W^m \to z(x), \qquad \text{as } h \to 0.$$

On the other hand, since Z is increasing, we have that for h small enough,

$$z_k \leq w\left(L - \frac{2m}{m-1} + \delta\right) \leq \varepsilon, \qquad \forall\, x_k < L - \frac{2m}{m-1} + \delta.$$

Then, $Z = W^m \to z(x)$ in all the interval $[0, L]$. $\square$

Remark. As a consequence of Lemma 4.1 if $L > \frac{2m}{m-1}$ we have that

$$W \to 0, \qquad \text{in } \left[0, L - \frac{2m}{m-1}\right],$$

and we recover the regional blow-up in the sense that the constants that appear in the blow-up rate for the nodes that lie in $[0, L - \frac{2m}{m-1}]$ go to zero as h goes to zero, i.e.,

$$u_k(t) \sim (T_h - t)^{-\frac{1}{p-1}} w_k$$

with $w_k \to 0$ as $h \to 0$ for every k such that $x_k \in [0, L - \frac{2m}{m-1}]$.

5 Numerical experiments.

In this Section we present some numerical experiments. Our goal is to show that the results presented in the previous sections can be observed when one perform numerical computations. For the numerical experiments we use an adaptive ODE solver to integrate (1.3).

We start with the case $p < m$, in **Figure 1** global blow-up can be appreciated.

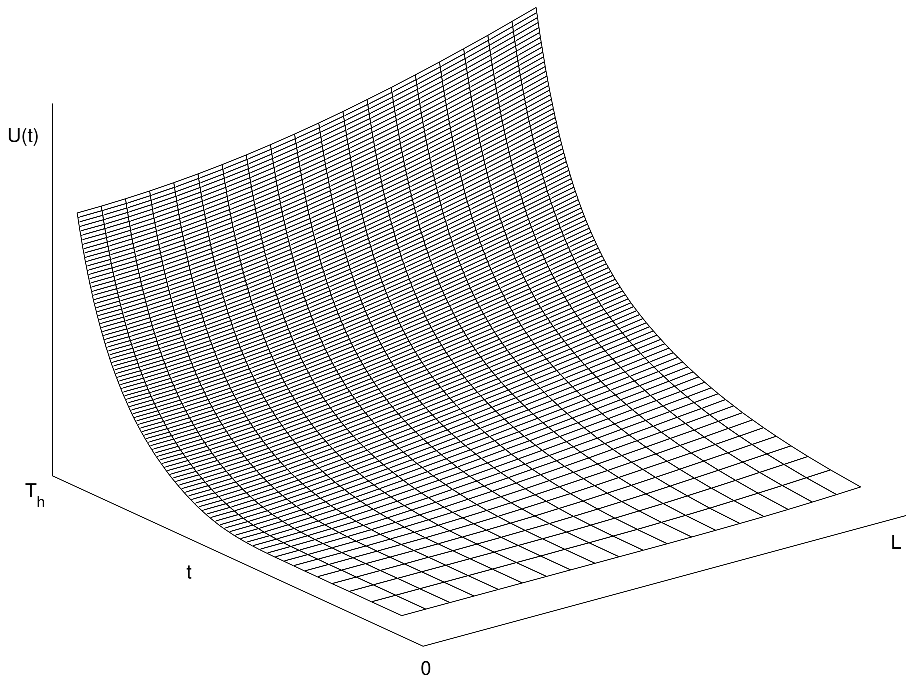

Figure 1.

Next, in the case $p = m$ we look for regional blow-up. In **Figure 2** the numerical solution blows up in every node but the behavior of the stationary solution imposes for $x_i < L_c = L - \frac{2m}{m-1}$ a minor rate.

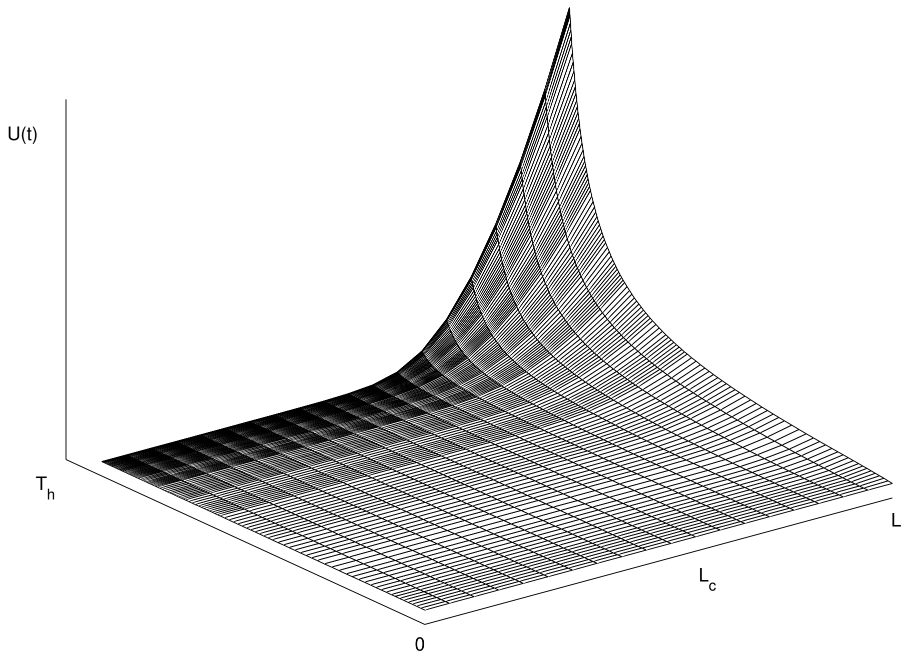

Figure 2.

The next picture (**Figure 3**) shows the profile of the numerical solution near the numerical blow-up time, T_h, in self similar variables. As it was proved this profile is close to the continuous stationary profile, which is drawn in the same picture.

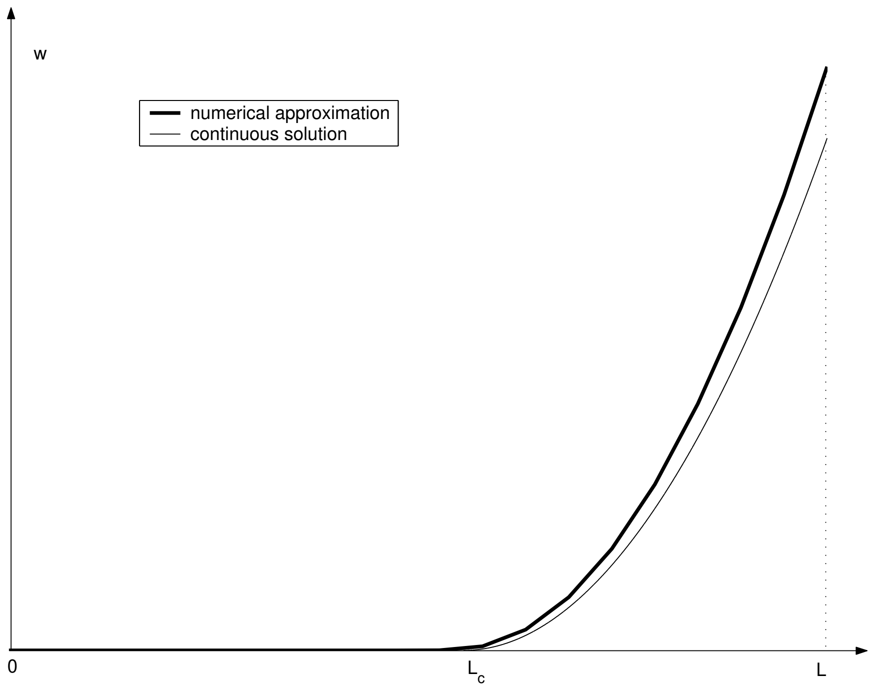

Figure 3.

Figure 4 shows the evolution of the first node (the minimum), which is blowing up.

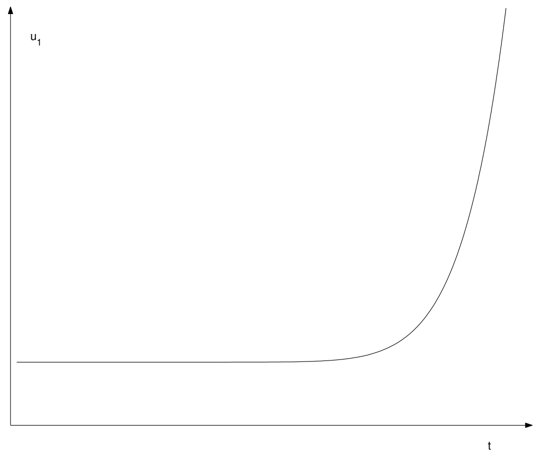

Figure 4.

Finally, we consider the case $p > m$, **Figures 5, 6** and **7**. When $p = 2$ and $m = 3$, Theorem 4.2 says that just two nodes blow up. The first picture shows the numerical solution, $U(t)$. The second one, the profile at time t_0 close to T_h. In the last one, we draw the evolution of the nodes $N-1$ and N, $u_{N-1}(t)$ is bounded while $u_N(t)$ is blowing up.

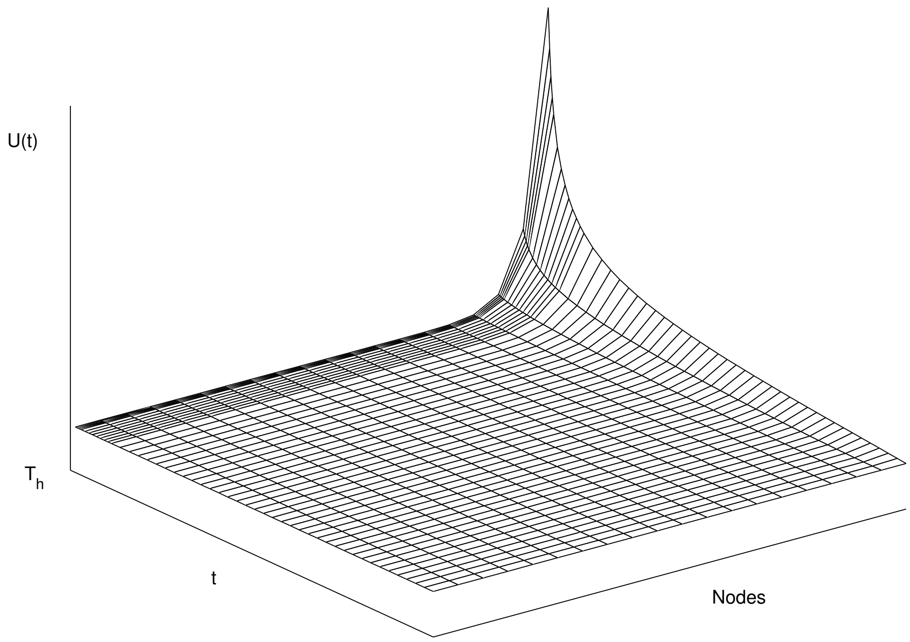

Figure 5.

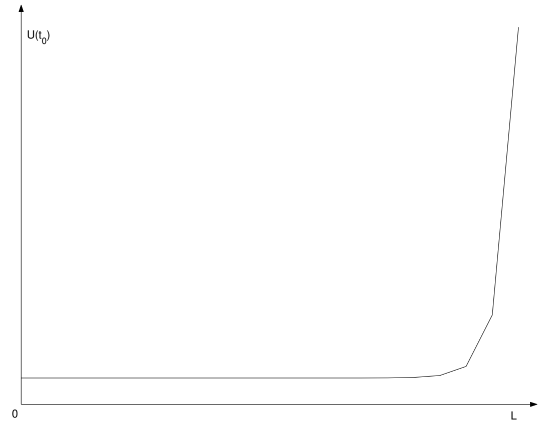

Figure 6.

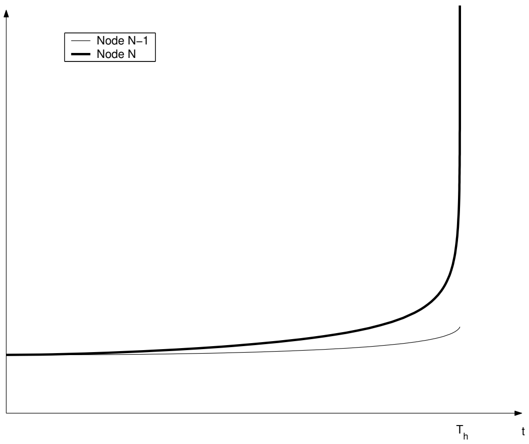

Figure 7.

6 Appendix.

In this Appendix we study the continuous problem (1.1) in the range of parameters $p = m$. In this case the blow-up rate is $(T-t)^{-\frac{1}{p-1}}$, [Fi]. We concentrate in the blow-up set and in addition we find the blow-up profile in self-similar variables.

Lemma 6.1 *In the region $p = m > 1$ the blow-up set is given by*

$$B(u) = \begin{cases} [0, L] & \text{if } L < \frac{2m}{m-1}, \\ \\ \left[L - \frac{2m}{m-1}, L\right] & \text{if } L \geq \frac{2m}{m-1}. \end{cases}$$

Proof. The proof of this result is given in several steps.
First, we introduce the self-similar variables

$$v(x,\tau) = (T-t)^{\frac{1}{m-1}} u(x,t), \qquad \tau = -\ln(T-t). \tag{6.1}$$

This function $v(x,\tau)$ satisfies the following problem,

$$\begin{cases} v_\tau = (v^m)_{xx} - \frac{1}{m-1} v & (x,\tau) \in (0,L) \times (0,+\infty), \\ (v^m)_x(0,\tau) = 0 & \tau \in (0,+\infty), \\ (v^m)_x(L,\tau) = v^m(L,\tau) & \tau \in (0,+\infty), \\ v(x,0) = T^{\frac{1}{m-1}} u_0(x) & x \in (0,L). \end{cases} \tag{6.2}$$

The second step consists in proving that $v(x,\tau)$ converges (in terms of ω-limits) to a stationary state. For this we multiply the equation by $(v^m)_\tau$ and integrate with respect to x to obtain

$$\int_0^L (v^m)_t v_t = -\frac{d}{dt} F(v),$$

where

$$F(v) = \int_0^L \frac{(v^m)_x^2}{2}(s,\tau)\, ds + \frac{m}{m^2-1} \int_0^L v^{m-1}(s,\tau)\, ds - \frac{1}{2} v^{2m}(L,\tau).$$

Hence $F(v)$ is a Lyapunov functional for problem (6.2).

Therefore, we have that the $\omega-$limit set of $v(x,\tau)$ consists of nontrivial stationary solutions of (6.2). Now we claim that for each L there exists only one nonnegative and nontrivial stationary solution.
Accepting this claim, we have that the $\omega-$limit set has only one element, $w(x)$, hence

$$v(x,\tau) \to w(x), \qquad \tau \to \infty.$$

The third step is to prove the claim. We consider the stationary problem

$$\begin{cases} (w^m)_{xx} - \frac{1}{m-1} w = 0, & x \in (0,L), \\ (w^m)_x(0) = 0, \\ (w^m)_x(L) = w^m(L). \end{cases} \tag{6.3}$$

If $L \geq \frac{2m}{m-1}$ this problem has a unique compactly supported solution (see [V]),

$$w(x) = \left(\frac{m-1}{2m(m+1)}\right)^{\frac{1}{m-1}} \left(\frac{2m}{m-1} + (x-L)\right)_+^{\frac{2}{m-1}}, \tag{6.4}$$

and if $L < \frac{2m}{m-1}$ there is no compactly supported solution.

To study the existence of other solutions of (6.3) we use ideas from [ChFQ]. We set $\widetilde{w} = w^m$ and look for a shooting argument beginning with $\widetilde{w}(0) = \rho$, $\widetilde{w}_x(0) = 0$. Integrating the equation we get

$$\frac{\widetilde{w}_x^2}{2} - \frac{m}{m^2-1} \widetilde{w}^{\frac{1+m}{m}} = -\frac{m}{m^2-1} \rho^{\frac{1+m}{m}}.$$

Evaluating at $x = L$ and using the boundary condition, we observe that $\widetilde{w}(L)$ must be a root of the following polinomial

$$P(z) = \frac{z^2}{2} - \frac{m}{m^2-1} z^{\frac{1+m}{m}} + \frac{m}{m^2-1} \rho^{\frac{1+m}{m}}.$$

As

$$P'(z) = z - \frac{1}{m-1} z^{\frac{1}{m}},$$

P attains a unique minimum at the point

$$z_0 = \left(\frac{1}{m-1}\right)^{\frac{m}{m-1}}.$$

In order to guarantee that $P(z)$ has a root we need that $P(z_0) \leq 0$, this imposes the following restriction on ρ,

$$\rho \leq \rho_c = \left(\frac{1}{m-1}\right)^{\frac{m}{m-1}} \left(1 - \frac{m+1}{2m}\right)^{\frac{m}{m+1}}.$$

For those values of ρ we have two roots $R_1(\rho)$, $R_2(\rho)$. As $P(\rho)\geq 0$ we conclude that

$$\rho\leq R_1(\rho)\leq R_2(\rho).$$

On the other hand, if we integrate again the equation we have, using that $\widetilde{w}_x\geq 0$,

$$\int_\rho^{w(x)}\frac{ds}{\sqrt{s^{\frac{1+m}{m}}-\rho^{\frac{1+m}{m}}}}=\sqrt{\frac{2m}{m^2-1}}x.$$

At $x=L$ this reads

$$\int_\rho^{w(L)}\frac{ds}{\sqrt{s^{\frac{1+m}{m}}-\rho^{\frac{1+m}{m}}}}=\sqrt{\frac{2m}{m^2-1}}L.$$

Hence for each root $R_i(\rho)$ we have a solution of (6.3) in an interval $[0,L_i(\rho)]$ where $L_i(\rho)$ is given by

$$L_i(\rho)=\sqrt{\frac{m^2-1}{2m}}\,\rho^{\frac{m-1}{2m}}\int_1^{\frac{R_i(\rho)}{\rho}}\frac{ds}{\sqrt{s^{\frac{1+m}{m}}-1}}.$$

Now we observe that, as $R_1(\rho)\leq R_2(\rho)$ we have that $L_1(\rho)\leq L_2(\rho)$.

In order to prove the uniqueness of the solution of (6.3) in the interval $[0,L]$ we study the monotonicity of the functions $L_i(\rho)$.

First we observe that $L_1(\rho)$ is increasing with ρ because $R_1(\rho)/\rho$ increases with ρ, in fact a direct calculation shows that

$$\frac{d}{d\rho}\left(\frac{R_i(\rho)}{\rho}\right)=-\left(\frac{m-1}{2m}\right)\frac{R_i^2(\rho)}{\rho^2((m-1)R_i(\rho)-R_i^{\frac{1}{m}}(\rho))}.$$

Notice that the signs of $(m-1)R_i(\rho)-R_i^{\frac{1}{m}}(\rho)$ and $P'(R_i(\rho))$ are the same. To see that $L_2(\rho)$ is decreasing we just observe, after differentiating and simplifying the resulting expression that this is equivalent to

$$\psi(\rho)=\int_1^a\frac{dv}{\sqrt{v^{\frac{1+m}{m}}-1}}-\frac{a}{\sqrt{a^{\frac{1+m}{m}}-1}((m-1)-R_2^{\frac{1-m}{m}})}\leq 0$$

where we have set $a=R_2(\rho)/\rho$. In fact one can check that

$$\frac{d}{d\rho}L_2(\rho)=\sqrt{\frac{m-1}{2m(m+1)}}\,\rho^{-\frac{m+1}{2m}}\psi(\rho).$$

We note that $R_2(\rho)\geq(m-1)^{-\frac{m}{m-1}}$ and also $a(\rho)$ is a decreasing function of ρ, so $a(\rho)\geq a(\rho_c)=(\frac{2m}{m-1})^{\frac{m}{m+1}}$. In that range of values it is easy to check that $\psi'(\rho)<0$ for $\rho>0$ and also that $\psi(0)=0$, therefore $\psi(\rho)<0$ and we conclude that $L_2(\rho)$ is decreasing.

We want to get a bound for $L_2(\rho)$, $0\leq\rho\leq\rho_c$. As

$$R_2(\rho)\to\left(\frac{2m}{m^2-1}\right)^{\frac{m}{m-1}},\qquad\rho\to 0,$$

a direct computation shows that

$$\lim_{\rho\to 0} L_2(\rho) = \frac{2m}{m-1}.$$

Therefore L_2 attains a maximum at $\rho = 0$ that is $L_2(0) = \frac{2m}{m-1}$, which is exactly the support of the explicit solution (6.4).

Since $R_1(\rho_c) = R_2(\rho_c)$ we have $L_1(\rho_c) = L_2(\rho_c)$. Also, as $R_1(\rho) \to 0$ when $\rho \to 0$, one can easily check that $L_1(0) = 0$.

We conclude that for $L \geq \frac{2m}{m-1}$ we have a unique solution of (6.3) that is given by (6.4) and has compact support, and for $L < \frac{2m}{m-1}$, problem (6.3) has only one positive solution. This proves the claim and finishes our study of the stationary problem.

In the fourth step we show that $v(x,\tau)$ goes to zero exponentially in every compact set included in $[0, \frac{2m}{m-1})$. More precisely:

Let v a solution of (6.2) that goes to zero in an interval of the form $[a-\delta, b+\delta]$, then there exists a constant C such that

$$v(x,\tau) \leq Ce^{-\frac{1}{m-1}\tau} \qquad x \in [a,b].$$

In order to see this we use ideas from [CDE], [V]. We will only sketch the arguments and refer to those works for details.

Since $v(x,\tau)$ goes to zero in $[a-\delta, b+\delta]$, we can use a solution of

$$\begin{cases} p_\tau = (p^m)_{xx} - \frac{1}{m-1}p & (x,\tau) \in [a-\delta, b+\delta] \times (0,+\infty), \\ p(a-\delta,\tau) = \varepsilon & \tau \in (0,+\infty), \\ p(b+\delta,\tau) = \varepsilon & \tau \in (0,+\infty), \\ v(x,0) = \varepsilon & x \in (a-\delta, b+\delta). \end{cases} \tag{6.5}$$

as a supersolution of our problem.

To finish the proof we only have to observe that solutions p of (6.5) goes exponentially to a stationary solution that has the form

$$h(x) = \begin{cases} C(m)(a-x)^{\frac{2}{m-1}} & x < a, \\ 0 & a \leq x \leq b, \\ C(m)(x-b)^{\frac{2}{m-1}} & x > b. \end{cases}$$

If ε is small, $h(x)$ has a dead core that includes the interval $[a,b]$ and the result follows.

Finally, if we translate the behaviour of $v(x,\tau)$ in terms of the function $u(x,t)$, we obtain the desired result. $\square$

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