NONLOCAL APPROXIMATIONS TO FOKKER-PLANCK EQUATIONS

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Abstract. We show that solutions to a classical Fokker-Plank equation can be approximated by solutions to nonlocal evolution problems when a rescaling parameter that controls the size of the nonlocality goes to zero.

1. Introduction

Nonlocal reaction-diffusion equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} K(x,y)u(y,t)dy - u(x,t), \tag{1}$$

where $K : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ is a nonnegative smooth kernel (usually assumed to be symmetric, but here this may not be the case) such that $\int K(x,y)dx = 1$, and variations of it, have been recently studied to model diffusion process. If $u(y,t)$ is thought of as a density of a population at location y at time t and $K(x,y)$ as the probability distribution of jumping from y to x, then the rate at which individuals are arriving to x is $\int K(x,y)u(y,t)dy$. On the other hand, the rate at which individuals are leaving location x to travel to other places is $-\int K(y,x)u(x,t)dy = -u(x,t)$. In the absence of external sources this implies that the density satisfies equation (1).

New in this work is to consider kernels of the form

$$K(x,y) = J\big(\mathcal{M}(y)(x-y)\big) \det \mathcal{M}(y). \tag{2}$$

Here $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative radial function such that

$$J \in \mathcal{C}_c(\mathbb{R}^n), \quad \int_{\mathbb{R}^N} J(z)dz = 1 \quad \text{and} \quad \int_{\mathbb{R}^N} J(z)z_N^2 dz = C(J) < \infty \tag{3}$$

and $\mathcal{M}(y)$ is a $N \times N$ real matrix with smooth and bounded coefficients such that $\det \mathcal{M}(y) \geq \gamma > 0$. Note that, for this kind of kernels, we have a mass preserving property, that is,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\big(\mathcal{M}(y)(x-y)\big) \det \mathcal{M}(y)\, u(y)dydx = \int_{\mathbb{R}^N} u(x)dx, \qquad \forall u \in \mathcal{C}(\mathbb{R}^N).$$

Our main goal in this work is to show that solutions to the nonlocal problem (1) with kernels of the form (2) adequately rescaled approximate solutions to the classical local Fokker-Plank equation.

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In more detail, consider the following local diffusion problem

$$(4) \qquad \begin{cases} v_t(x,t) = \displaystyle\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(a_{ij}(x) v(x,t) \right), & x \in \mathbb{R}^N, t \in [0,T], \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $A(x) = (a_{ij}(x))$ is a real positive-definite matrix.

Throughout the paper, we make the following assumptions on the matrix A: $A(x) = (a_{ij}(x))$ is a real $N \times N$ symmetric and positive-definite matrix with smooth coefficients such that

$$\delta \|\xi\|^2 \leq \sum_{i,j} a_{ij}(x) \xi_i \xi_j \leq \beta \|\xi\|^2, \qquad \forall x, \xi \in \mathbb{R}^N,$$

for some constants $0 < \delta < \beta$ and we will also assume that

$$(5) \qquad \max_x \left\{ \sum_{i,j} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j} \right\} < \infty.$$

Given $A(x)$, we let $B(x) = (b_{ij}(x))$ be a real $N \times N$ matrix with strictly positive determinant and smooth coefficients satisfying $B(x)B^t(x) = A(x)$, $x \in \mathbb{R}^N$. Note that such decomposition is possible since A is a positive-definite matrix (e.g. using Cholesky factorization).

Now, let us consider the following nonlocal equation

$$(6) \qquad \begin{cases} u_t^\varepsilon = \dfrac{C}{\varepsilon^2} \left\{ \displaystyle\int_{\mathbb{R}^N} K_\varepsilon(x,y) u(y,t) dy - u(x,t) \right\}, & x \in \mathbb{R}^N, t \in [0,T], \\ u^\varepsilon(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

where $C^{-1} = \frac{1}{2} \int J(z) z_N^2 dz$ is a constant that depends only on J and the kernel K_ε is given by

$$K_\varepsilon(x,y) = \frac{1}{\varepsilon^N} J\Big(B^{-1}(y) \frac{(x-y)}{\varepsilon} \Big) \det B^{-1}(y),$$

with B as above, that is, such that $BB^t = A$ and J satisfying (3).

As we have mentioned, our aim is to show that solutions of (6) converge uniformly to solutions of (4). Our main result reads as follows:

Theorem 1.1. *Let v be a classical solution of Fokker-Planck equation (4) with initial datum $v_0 \in \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For every $\varepsilon > 0$, consider u^ε the solution of the nonlocal equation (6). Then,*

$$\sup_{t \in [0,T]} \|u^\varepsilon(\cdot,t) - v(\cdot,t)\|_{L^\infty} \to 0,$$

as $\varepsilon \to 0$.

Now, let us comment briefly on previous results concerning approximations of local PDEs by nonlocal problems.

Kernels of the form (2) cover a wide variety of nonlocal diffusion problems treated in the past twenty years. For example, taking the simplest case $\mathcal{M}(y) = Id$, equation (1) reduces to the following convolution type diffusion problem

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y) u(y,t) dy - u(x,t).$$

This model has been treated by several authors in different contexts, see for example [1, 2, 6] and the references given therein. In addition, in [5] the authors prove that, under an appropriate rescaling of the kernel, that is, solutions to

$$u^{\varepsilon}_t(x,t)=\frac{C}{\varepsilon^2}\left\{\int_{\mathbb{R}^N}\frac{1}{\varepsilon^N}J\Big(\frac{x-y}{\varepsilon}\Big)u(y,t)dy-u(x,t)\right\},$$

converge, as $\varepsilon\to 0$, to solutions to the local heat equation, $v_t=\Delta v$.

Another example is the kernel (2) with $\mathcal{M}(y)=g^{-1}(y)Id$, being g a positive scalar function. In this case (1) takes the form

$$u_t(x,t)=\int_{\mathbb{R}^N}J\left(\frac{x-y}{g(y)}\right)\frac{u(y,t)}{g^N(y)}dy-u(x,t).$$

Note that in this evolution problem the step size, $g(y)$, depends on the position y. Such kind of diffusion kernel was introduced in [3] in order to model a non-homogeneous dispersal process. See also [4] and [7]. For this problem in [13] the authors prove that under appropriate rescaling of the kernel, i.e. when the problem takes the form

$$u^{\varepsilon}_t(x,t)=\frac{C}{\varepsilon^2}\left\{\int_{\mathbb{R}^N}\frac{1}{\varepsilon^N}J\Big(\frac{x-y}{\varepsilon g(y)}\Big)\frac{u(y,t)}{g^N(y)}dy-u(x,t)\right\},\tag{7}$$

solutions converge to solutions to the local equation

$$v_t(x,t)=\sum_i(g^2(x)v(x,t))_{x_ix_i}.$$

Closely related to this work is [11] where we find kernels for nonlocal evolution problems that, when appropriately rescaled as above, have solutions that approximate solutions to local problems with spatial dependence in divergence form,

$$v_t(x,t)=\sum_{i,j}(a_{ij}(x)v_{x_j})_{x_i}(x,t)$$

or in non-divergence form,

$$v_t(x,t)=\sum_{i,j}a_{ij}(x)v_{x_ix_j}(x,t).$$

Notations. Given $A(x)=(a_{ij}(x))$ we denote by $a^t_{ij}(x)$ and $a^{-1}_{ij}(x)$ the coefficients of the matrices $A^t(x)$, $A^{-1}(x)$ respectively. Also, for any given function $f:\mathbb{R}^N\to\mathbb{R}$ we denote by $f'_i(s)=\frac{\partial f(s)}{\partial s_i}$ and by $[f]_+(s)=\max\{0,f(s)\}$.

The paper is organized as follows: in Section 2 we show existence, uniqueness and a comparison principle for the nonlocal problem and in Section 3 we prove the convergence of the solutions as the scaling parameter ε goes to zero.

2. Existence, uniqueness and comparison principle

We start this section proving the comparison principle for our problem

$$(P)\quad\begin{cases} u_t(x,t)=\displaystyle\int_{\mathbb{R}^N}J\left(\mathcal{M}(y)(x-y)\right)\det\mathcal{M}(y)\,u(y,t)dy-u(x,t), & x\in\mathbb{R}^N, t>0,\\ u(x,0)=u_0(x), & x\in\mathbb{R}^N.\end{cases}$$

For this purpose, we first set the notion of sub and supersolution for (P).

Definition 2.1. *A function $u \in \mathcal{C}^1\left([0,\infty),\mathcal{C}(\mathbb{R}^N)\right)$ is a subsolution of problem (P) if it satisfies*

$$\begin{cases} u_t(x,t) \leq \displaystyle\int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right)\det\mathcal{M}(y)\,u(y,t)dy - u(x,t), & x\in\mathbb{R}^N, t>0,\\ u(x,0)\leq u_0(x), & x\in\mathbb{R}^N. \end{cases}$$

As usual, a supersolution is defined analogously by replacing " $\leq$ " by " $\geq$ ".

Theorem 2.2. *[Comparison Principle] Let u, v be a subsolution and supersolution respectively of problem (P). Then $u \leq v$.*

Proof. To prove this result we follow closely [13, Theorem 2.5]. Set $w = u - v$, then

$$\begin{cases} w_t(x,t) \leq \displaystyle\int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right)\det\mathcal{M}(y)w(y,t)dy - w(x,t) & x\in\mathbb{R}^N, t>0,\\ w(x,0)\leq 0, & x\in\mathbb{R}^N. \end{cases} \tag{8}$$

Let us consider the following function

$$s(x,t) = \begin{cases} 1, & \text{if } w(x,t)\geq 0,\\ 0, & \text{if } w(x,t)<0. \end{cases}$$

Multiplying (8) by $s(x,t)$ and taking into account that $w_t(x,t)s(x,t) = \left([w]_+\right)_t(x,t)$ and $w(y,t)\leq [w]_+(y,t)$, we obtain, dropping the positive term $w(x,t)s(x,t)$, that

$$\left([w]_+\right)_t(x,t) \leq \int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right)\det\mathcal{M}(y)\,[w]_+(y,t)dy,$$

integrating in $\mathbb{R}^N$ and by using the mass preserving property, we get

$$\int_{\mathbb{R}^N}\left([w]_+\right)_t(x,t) \leq \int_{\mathbb{R}^N}[w]_+(y,t)dy.$$

Finally, integrating in $(0,t)$ and since $[w]_+(x,0)=0$ we can assert, using Fubini's theorem, that

$$h(t) \leq \int_0^t h(s)ds, \tag{9}$$

where

$$h(t) = \int_{\mathbb{R}^N}[w]_+(x,t)dx.$$

Hence, applying Gronwall's Lemma in (9), we conclude that

$$h(t)\leq 0.$$

Now, since $[w]_+(x,t)\geq 0$ and by the continuity of $[w]_+$, we get that $[w]_+(x,t)=0$ and, consequently,

$$u(x,t)\leq v(x,t)$$

for all $x\in\mathbb{R}^N,\ t>0$. □

Note that the previous proof works locally in time, that is, a supersolution v and a subsolution u defined both for $t\in[0,T]$ verify $u(x,t)\leq v(x,t)$ for all $x\in\mathbb{R}^N,\ 0\leq t\leq T$.

Definition 2.3. *By a solution of the problem* (P)*, we mean a function* $u \in \mathcal{C}\left([0,\infty);\mathcal{C}(\mathbb{R}^N)\right)$ *that satisfies*

$$u(x,t) = \int_0^t \int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det \mathcal{M}(y)\, u(y,s) dy ds - \int_0^t u(x,s) ds + u_0(x),$$

for all $x \in \mathbb{R}^N$, $t \in [0,\infty)$*. Consequently, due to this integral expression, we can assert that* $u \in \mathcal{C}^1\left([0,\infty);\mathcal{C}(\mathbb{R}^N)\right)$.

Now, we prove existence and uniqueness of a solution which is bounded in $\mathbb{R}^N$.

Theorem 2.4. *[Existence] For every continuous and bounded initial data* u_0 *there exists a unique solution* $u \in \mathcal{C}\left([0,\infty);\mathcal{C}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\right)$ *of problem* (P).

Proof. For $T>0$ we consider the Banach space

$$X = \mathcal{C}\left([0,T];\mathcal{C}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)\right),$$

with the norm

$$\|w\| = \max_{0\le t\le T} e^{-k(M+1)t}\|w(\cdot,t)\|_{L^\infty}.$$

Here $M = \max_{x\in\mathbb{R}^N} \det \mathcal{M}(x) > 0$ and k is any value greater than one.

Now, let Y be the closed ball of X with radius $k\|u_0\|_\infty$ and centered at the origin. Note that Y is a complete metric space with the induced metric $d(w_1,w_2) = \|w_1 - w_2\|$.

In order to establish the existence and uniqueness of solutions of (P) via Banach contraction principle, we define the operator $\mathcal{T} : Y \longrightarrow Y$ by

$$\mathcal{T}(w)(x,t) = \int_0^t \int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det \mathcal{M}(y)\, w(y,s) dy ds - \int_0^t w(x,s) ds + u_0(x).$$

Let us first prove that this operator is well defined. Clearly $\mathcal{T}(w)$ is belongs to X and satisfies

$$\begin{aligned}\|\mathcal{T}(w)(\cdot,t)\|_{L^\infty} &\le \max_x \left| \int_0^t \int_{\mathbb{R}^N} J\left(\mathcal{M}(y)(x-y)\right) \det \mathcal{M}(y)\, w(y,s) dy ds \right| \\ &\quad + \int_0^t \|w(\cdot,s)\|_{L^\infty} ds + \|u_0\|_{L^\infty} \le (M+1)\int_0^t \|w(\cdot,s)\|_{L^\infty} ds + \|u_0\|_{L^\infty}.\end{aligned}$$

Since $\|w\| \le k\|u_0\|_{L^\infty}$, we obtain that, for $0\le t\le T$,

$$\|\mathcal{T}(w)(\cdot,t)\|_{L^\infty} \le e^{k(M+1)T}\|u_0\|_{L^\infty},$$

therefore, for T small, $\|\mathcal{T}(w)\| \le k\|u_0\|_{L^\infty}$ and $\mathcal{T}(w)$ belongs to Y.

Now, let us show that the operator $\mathcal{T}$ is a contraction. we have

$$d(\mathcal{T}(w_1),\mathcal{T}(w_2)) \le \max_{0\le t\le T} e^{-k(M+1)t}(M+1)\int_0^t \|w_1(\cdot,s) - w_2(\cdot,s)\|_{L^\infty} ds.$$

Arguing as above, we obtain

$$d(\mathcal{T}(w_1),\mathcal{T}(w_2)) \le \max_{0\le t\le T} \frac{1}{k}\|w_1 - w_2\|\left(1 - e^{-k(M+1)t}\right) \le \frac{1}{k} d(w_1,w_2).$$

Hence, using Banach's Fixed Point Theorem there exists u a fix point of $\mathcal{T}$, that is the unique solution of problem (P) for $t \in [0,T]$ and belongs to Y. Finally, since from the comparison principle we have that

$$-\|u_0\|_{L^\infty} e^{(\max_x \int K(x,y)dy-1)t} \le u(x,t) \le \|u_0\|_{L^\infty} e^{(\max_x \int K(x,y)dy-1)t},$$

we obtain a global solution, $u \in \mathcal{C}\left([0,\infty); \mathcal{C}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)\right)$. □

3. Approximations of the Fokker-Planck equation by nonlocal problems

In this section we prove our main result, that is, that solutions of the Fokker-Planck equation can be approximated by solutions of the nonlocal problem by rescaling the kernel.

Recall that the general Fokker-Planck equation is given by

$$(F\!-\!P) \qquad \begin{cases} v_t(x,t) = \displaystyle\sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(a_{ij}(x) v(x,t)\right), & x \in \mathbb{R}^N, t \in [0,T], \\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

We will call a solution to the Cauchy problem for the Fokker-Planck equation $(F\!-\!P)$ a *classical solution* if $v \in \mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N, [0,T]\right)$. Note that the regularity of v is related with smoothness of $a_{ij}(x)$ and the initial datum v_0; see [8, 10].

We first need to prove the following technical lemmas.

Lemma 3.1. *Let J be a function satisfying hypothesis (3). Then, the following properties are satisfied:*

(1)

$$\int_{\mathbb{R}^N} J'_j(w) w_p w_q w_r dw = \begin{cases} -3C(J), & \text{if } p = q = r = j, \\ -C(J), & \text{if } \begin{cases} p = j \text{ and } r = q \neq j, \text{ or} \\ q = j \text{ and } r = p \neq j, \text{ or} \\ r = j \text{ and } p = q \neq j, \end{cases} \\ 0, & \text{in other case.} \end{cases}$$

(2)

$$\int_{\mathbb{R}^N} J''_{jj}(w) w_l w_n w_s w_t dw = \begin{cases} 12C(J), & \text{if } l = n = s = t = j, \\ 2C(J), & \begin{cases} \text{if two indexes are equal to } j \\ \text{and the others two are equal} \\ \text{to each other and different to } j. \end{cases} \\ 0, & \text{in other case.} \end{cases}$$

(3) For $j \neq p$

$$\int_{\mathbb{R}^N} J''_{jp}(w) w_l w_n w_s w_t dw = \begin{cases} 3C(J), & \left\{ \begin{array}{l} \textit{if three indexes are equal to } j \\ \textit{and the other one are equal} \\ \textit{to } p, \textit{ or viceversa.} \end{array} \right. \\ \\ C(J), & \left\{ \begin{array}{l} \textit{if one index is equal to } j, \textit{ another} \\ \textit{index is equal to } p, \textit{ and the} \\ \textit{others two are equal to each} \\ \textit{other but different to } j \textit{ and } p. \end{array} \right. \\ \\ 0, & \textit{in other case.} \end{cases}$$

Proof. *(1)* If $p = q = r = j$ and since J has compact support, integrating by parts it follows that

$$\int_{\mathbb{R}^N} J'_j(w) w_j^3 dw = -3 \int_{\mathbb{R}^N} J(w) w_j^2 dw = -3C(J).$$

Similarly, if one of the indexes is equal to j and the others two are equal between them and different from j, integrating by parts respect to the variable j, we obtain

$$\int_{\mathbb{R}^N} J'_j(w) w_j w_s^2 dw = -\int_{\mathbb{R}^N} J(w) w_s^2 dw = -C(J),$$

for $s = p, q, r$. Finally, in the same way we show that is zero occurs in any different case.

(2) For the first case, integrating by parts twice, we get

$$\int_{\mathbb{R}^N} J''_{jj}(w) w_j^4 dw = -4 \int_{\mathbb{R}^N} J'_j(w) w_j^3 dw = 12 \int_{\mathbb{R}^N} J(w) w_j^2 dw = 12C(J).$$

We proceed likewise, if two indexes are equal to j and the other two are equal between them and different from j (there are 6 cases). For example, taking $l = n = j$ and $s = t \neq j$, we obtain integrating by parts twice

$$\int_{\mathbb{R}^N} J''_{jj}(w) w_j^2 w_s^2 dw = -2 \int_{\mathbb{R}^N} J'_j(w) w_j w_s^2 dw = 2 \int_{\mathbb{R}^N} J(w) w_j^2 dw = 2C(J).$$

Finally, the proof in any other case follows similarly and is left to the reader.

(3) We apply the same reasoning, integrating by parts twice, respect to the variable p and j. First, if three indexes are equal to j and the other one is equal to p (there are 8 cases) we get, taking for example $l = n = s = j$ and $t = p$, that

$$\int_{\mathbb{R}^N} J''_{jp}(w) w_j^3 w_p dw = -\int_{\mathbb{R}^N} J'_j(w) w_j^3 dw = 3 \int_{\mathbb{R}^N} J(w) w_j^2 dw = 3C(J).$$

Analogously, if one index is equal to j, another index is equal to p, and the other two are equalbetween them but different from j and p (there are 12 cases) we have, choosing for example $l = j$, $n = p$ and $s = t \neq j, p$, that

$$\int_{\mathbb{R}^N} J''_{jp}(w) w_j w_p w_s^2 dw = -\int_{\mathbb{R}^N} J'_j(w) w_j w_s^2 dw = \int_{\mathbb{R}^N} J(w) w_s^2 dw = C(J).$$

We leave it to the reader to verify that in any other case the integral expression is equal to zero. $\square$

Lemma 3.2. *Let $A(x) = (a_{ij}(x))$ be a $N\times N$ non-singular real matrix with smooth coefficients $a_{ij} : \mathbb{R}^N \to \mathbb{R}$, $i,j = 1\dots N$, then the following properties are satisfied:*

(1)
$$\sum_k \left(a_{ik}^{-1}\right)'_m (x) a_{kj}(x) = -\sum_k a_{ik}^{-1}(x) \left(a_{kj}\right)'_m (x),$$

(2)
$$\sum_k (a_{jk}^{-1})''_{mp}(x) a_{kq}(x) = -\sum_k \Big\{ (a_{jk}^{-1})'_m(x)(a_{kq})'_p(x) + (a_{jk}^{-1})'_p(x)(a_{kq})'_m(x) + a_{jk}^{-1}(x) \left(a_{kq}\right)''_{mp}(x) \Big\},$$

(3)
$$\sum_{j,k} a_{jk}^{-1}(x)(a_{kj})'_m(x) = \det A^{-1}(x) \left(\det A(x)\right)'_m .$$

Proof.

(1) It follows by computing the derivate of $\sum_k a_{ik}^{-1}(x) a_{kj}(x) = \delta_{ij}$.

(2) It is easy to prove when we compute the derivate of the expression in *(1)*.

(3) See [9] for a simple and original proof. □

Also the following propositions will be needed in the proof of our main theorem. To simplify the notation, in what follows we let
$$J_\varepsilon(s) = \frac{1}{\varepsilon^N} J\Big(\frac{s}{\varepsilon}\Big).$$

Proposition 3.3. *Let u be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N \times [0,T]\right)$ function and let $\mathcal{L}^1_\varepsilon$ and Λ be the operators given by*
$$\mathcal{L}^1_\varepsilon(u(x,t)) = \frac{C}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon\left(B^{-1}(y)(x-y)\right) \det B^{-1}(y)(u(y,t) - u(x,t))dy,$$
$$\Lambda(u(x,t)) = \sum_{i,j} \frac{\partial^2 u(x,t)}{\partial x_i \partial x_j} a_{ij}(x) + 2\sum_{i,j} \frac{\partial u(x,t)}{\partial x_i}\frac{\partial a_{ij}(x)}{\partial x_j}.$$
Then,
$$\sup_{t\in[0,T]} \| \left(\mathcal{L}^1_\varepsilon - \Lambda\right)(u(\cdot,t))\|_{L^\infty} \to 0 \ \ as \ \ \varepsilon \to 0.$$

Proof. Under the change variables $y = x - \varepsilon z$ and by a simple Taylor expansion we obtain
$$\mathcal{L}^1_\varepsilon(u(x,t)) = \sum_{i,j} \frac{\partial^2 u(x,t)}{\partial x_i\partial x_j} H^1_\varepsilon(x) + \sum_i \frac{\partial u(x,t)}{\partial x_i} H^2_\varepsilon(x) + O(\varepsilon^\alpha),$$
being
$$H^1_\varepsilon(x) = \frac{C}{2}\int_{\mathbb{R}^N} J\left(B^{-1}(x-\varepsilon z)z\right) \det B^{-1}(x-\varepsilon z) z_i z_j dz,$$
and
$$H^2_\varepsilon(x) = -\frac{C}{\varepsilon}\int_{\mathbb{R}^N} J\left(B^{-1}(x-\varepsilon z)z\right) \det B^{-1}(x-\varepsilon z) z_i dz.$$

First, we claim that
$$H^1_\varepsilon(x) \to a_{ij}(x)$$

as $\varepsilon \to 0$. Indeed, changing variables as $\omega = B^{-1}(x)z$ we get

$$\begin{aligned}\lim_{\varepsilon\to 0} H^1_\varepsilon(x) &= \frac{C}{2}\det B^{-1}(x)\int_{\mathbb{R}^N} J\left(B^{-1}(x)z\right) z_i z_j dz \\ &= \frac{C}{2}\sum_{k,m} b_{ik}(x)b_{jm}(x)\int_{\mathbb{R}^N} J(w)w_k w_m dw.\end{aligned}$$

Taking into account that

$$\int_{\mathbb{R}^N} J(w)w_k w_m dw = 0$$

if $k \neq m$ and the value of the constant C, we get that

$$\lim_{\varepsilon\to 0} H^1_\varepsilon(x) = \sum_k b_{ik}(x)b_{jk}(x) = \sum_k b_{ik}(x)b^t_{kj}(x) = a_{ij}(x).$$

Now, we claim that

$$H^2_\varepsilon(x) \to 2\sum_j \frac{\partial a_{ij}(x)}{\partial x_j}$$

as $\varepsilon \to 0$. Indeed, since J is a radial function, it follows that

$$\int_{\mathbb{R}^N} J(B^{-1}(x)z)z_i dz = 0.$$

Therefore, $\lim_{\varepsilon\to 0} H^2_\varepsilon(x) = \frac{0}{0}$ and we can use L'Hopital rule to obtain

$$\lim_{\varepsilon\to 0} H^2_\varepsilon(x) = \lim_{\varepsilon\to 0} -C\int_{\mathbb{R}^N} \left(F^1_\varepsilon(x,z) + F^2_\varepsilon(x,z)\right) dz,$$

where

$$F^1_\varepsilon(x,z) = \frac{\partial}{\partial\varepsilon}\left(J(B^{-1}(x-\varepsilon z)z)\right)\det B^{-1}(x-\varepsilon z)z_i,$$

and

$$F^2_\varepsilon(x,z) = J(B^{-1}(x-\varepsilon z)z)\frac{\partial}{\partial\varepsilon}\left(\det B^{-1}(x-\varepsilon z)\right) z_i.$$

To compute the first part, we note that

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\left(J(B^{-1}(x-\varepsilon z)z)\right) &= \sum_j \left\{ J'_j(B^{-1}(x-\varepsilon z)z)\frac{\partial}{\partial\varepsilon}\sum_k b^{-1}_{jk}(x-\varepsilon z)z_k \right\} \\ &= \sum_{j,k,m} J'_j(B^{-1}(x-\varepsilon z)z)\left(b^{-1}_{jk}\right)'_m (x-\varepsilon z)z_k(-z_m).\end{aligned} \tag{10}$$

In this way we obtain

$$\lim_{\varepsilon\to 0} -C\int_{\mathbb{R}^N} F^1_\varepsilon(x,z)dz = C\det B^{-1}(x)\sum_{j,k,m}\left(b^{-1}_{jk}\right)'_m(x)\int_{\mathbb{R}^N} J'_j(B^{-1}(x)z)z_k z_m z_i dz. \tag{11}$$

Now, we change variables as $w = B^{-1}(x)z$ to obtain

$$C\sum_{j,k,m,p,q,r}\left(b^{-1}_{jk}\right)'_m(x)b_{kp}(x)b_{mq}(x)b_{ir}(x)\int_{\mathbb{R}^N} J'_j(w)w_p w_q w_r dw.$$

Using property *(1)* from Lemma 3.1 we get

$$
\begin{aligned}
&= -6 \sum_{j,k,m} (b_{jk}^{-1})'_m(x) b_{kj}(x) b_{mj}(x) b_{ij}(x) \\
&\quad -2 \sum_{j,k,m,q\neq j} (b_{jk}^{-1})'_m(x) b_{kj}(x) b_{mq}(x) b_{iq}(x) \\
&\quad -2 \sum_{j,k,m,p\neq j} (b_{jk}^{-1})'_m(x) b_{kp}(x) b_{mj}(x) b_{ip}(x) \\
&\quad -2 \sum_{j,k,m,p\neq j} (b_{jk}^{-1})'_m(x) b_{kp}(x) b_{mp}(x) b_{ij}(x) \\
&= -2 \sum_{j,k,m,p} (b_{jk}^{-1})'_m(x) \left[b_{kj}(x) b_{mp}(x) b_{ip}(x) \right. \\
&\qquad\qquad + b_{kp}(x) b_{mj}(x) b_{ip}(x) \\
&\qquad\qquad \left. + b_{kp}(x) b_{mp}(x) b_{ij}(x) \right],
\end{aligned} \tag{12}
$$

which by property *(1)* from Lemma 3.2 turns out to be equal to

$$
\begin{aligned}
&2 \sum_{j,k,m,p} b_{jk}^{-1}(x) (b_{kj})'_m(x) b_{mp}(x) b_{ip}(x) \\
&\quad + 2 \sum_{k,p} (b_{kp})'_k(x) b_{ip}(x) + 2 \sum_{m,p} (b_{ip})'_m(x) b_{mp}(x) \\
&= 2 \sum_{j,k,m,p} b_{jk}^{-1}(x) (b_{kj})'_m(x) b_{mp}(x) b_{ip}(x) + 2 \sum_j \frac{\partial a_{ij}(x)}{\partial x_j},
\end{aligned} \tag{13}
$$

where in the last equality we have used that $a_{ij}(x) = \sum_p b_{ip}(x) b_{jp}(x)$ and we have replaced the indexes k and p by j.

To conclude the claim, we have to compute the second part and to verify that it is cancelled with the first term of the last part of (13). To be more specific, we need to show that

$$
\lim_{\varepsilon\to 0} C \int_{\mathbb{R}^N} F_\varepsilon^2(x,z)\,dz = 2 \sum_{j,k,m,p} b_{jk}^{-1}(x) (b_{kj})'_m(x) b_{mp}(x) b_{ip}(x).
$$

In fact, by virtue of

$$
\begin{aligned}
\frac{\partial}{\partial \varepsilon} \left(\det B^{-1}(x-\varepsilon z) \right) &= \sum_m \left(\det B^{-1}(x-\varepsilon z) \right)'_m (-z_m) \\
&= \sum_m \det B^{-2}(x-\varepsilon z) \left(\det B(x-\varepsilon z) \right)'_m z_m,
\end{aligned} \tag{14}
$$

we have that

$$
\lim_{\varepsilon\to 0} C \int_{\mathbb{R}^N} F_\varepsilon^2(x,z)\,dz = C \det B^{-2}(x) \sum_m \left(\det B(x) \right)'_m \int_{\mathbb{R}^N} J\left(B^{-1}(x) z \right) z_m z_i\,dz
$$

changing variables again $w = B^{-1}(x) z$

$$
= 2 \det B^{-1}(x) \sum_{m,p} \left(\det B(x) \right)'_m b_{mp}(x) b_{ip}(x)
$$

and finally, using property *(3)* from Lemma 3.2 we get

$$
= 2 \sum_{j,k,m,p} b_{jk}^{-1}(x) (b_{kj})'_m(x) b_{mp}(x) b_{ip}(x)
$$

and the proof is finished. □

Proposition 3.4. *Let u be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N\times[0,T]\right)$ function and let $\mathcal{L}^2_\varepsilon$, Γ be the operators defined as*

$$\mathcal{L}^2_\varepsilon(u(x,t))=\frac{C}{\varepsilon^2}\left[\int_{\mathbb{R}^N}J_\varepsilon\left(B^{-1}(y)(x-y)\right)\det B^{-1}(y)dy-1\right]u(x,t),$$

and

$$\Gamma(u(x,t))=\sum_{i,j}\frac{\partial^2 a_{ij}(x)}{\partial x_i\partial x_j}u(x,t),$$

Then,

$$\sup_{t\in[0,T]}\|\left(\mathcal{L}^2_\varepsilon-\Gamma\right)(u(x,t))\|_{L^\infty}\to 0\ \ as\ \ \varepsilon\to 0.$$

Proof. Under the change variables $y=x-\varepsilon z$ we obtain

$$\mathcal{L}^2_\varepsilon(u(x,t))=\frac{C}{\varepsilon^2}\left[\int_{\mathbb{R}^N}J\left(B^{-1}(x-\varepsilon z)z\right)\det B^{-1}(x-\varepsilon z)dz-1\right]u(x,t). \tag{15}$$

Note that

$$\lim_{\varepsilon\to0}\int_{\mathbb{R}^N}J\left(B^{-1}(x-\varepsilon z)z\right)\det B^{-1}(x-\varepsilon z)dz=\int_{\mathbb{R}^N}J\left(B^{-1}(x)z\right)\det B^{-1}(x)dz=1.$$

Therefore, using L'Hopital rule in (15) we get

$$\lim_{\varepsilon\to0}\mathcal{L}^2_\varepsilon(u(x,t))=\frac{C}{2\varepsilon}\int_{\mathbb{R}^N}\left(G^1_\varepsilon(x,z)+G^2_\varepsilon(x,z)\right)dz\,u(x,t), \tag{16}$$

where

$$G^1_\varepsilon(x,z)=\frac{\partial}{\partial\varepsilon}\left(J(B^{-1}(x-\varepsilon z)z)\right)\det B^{-1}(x-\varepsilon z),$$

and

$$G^2_\varepsilon(x,z)=J(B^{-1}(x-\varepsilon z)z)\frac{\partial}{\partial\varepsilon}\left(\det B^{-1}(x-\varepsilon z)\right).$$

Now, the proof splits naturally into two parts:

Part 1: To compute

$$\lim_{\varepsilon\to0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N}G^1_\varepsilon(x,z)dz.$$

Using equality (10), it is equivalent to compute

$$\lim_{\varepsilon\to0}\frac{C}{2\varepsilon}\sum_{j,k,m}\int_{\mathbb{R}^N}J'_j(B^{-1}(x-\varepsilon z)z)\,(b^{-1}_{jk})'_m(x-\varepsilon z)\,z_k(-z_m)\det B^{-1}(x-\varepsilon z)dz.$$

Taking into account that

$$\int_{\mathbb{R}^N}J'_j(w)w_qw_pdw=0,$$

a simple computation gives that the above expression is $\frac{0}{0}$ and we can use L'Hopital rule again, to obtain

$$\lim_{\varepsilon\to0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N}G^1_\varepsilon(x,z)dz=\lim_{\varepsilon\to0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}\left(A^1_\varepsilon(x,z)+A^2_\varepsilon(x,z)+A^3_\varepsilon(x,z)\right)dz, \tag{17}$$

where

$$A^1_\varepsilon(x,z)=\frac{\partial}{\partial\varepsilon}\left[J'_j(B^{-1}(x-\varepsilon z)z)\right](b^{-1}_{jk})'_m(x-\varepsilon z)\det B^{-1}(x-\varepsilon z)z_k(-z_m)dz,$$

$$A^2_\varepsilon(x,z) = J'_j(B^{-1}(x-\varepsilon z)z)\,\frac{\partial}{\partial\varepsilon}\left[(b^{-1}_{jk})'_m(x-\varepsilon z)\right]\det B^{-1}(x-\varepsilon z)z_k(-z_m)dz$$

and

$$A^3_\varepsilon(x,z) = J'_j(B^{-1}(x-\varepsilon z)z)\,(b^{-1}_{jk})'_m(x-\varepsilon z)\,\frac{\partial}{\partial\varepsilon}\left[\det B^{-1}(x-\varepsilon z)\right]z_k(-z_m)dz.$$

Therefore, the Part 1 will be splitter again into three steps:

Part 1.a: Compute

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}A^1_\varepsilon(x,z)dz.$$

By an argument similar to (10), we have

$$\frac{\partial}{\partial\varepsilon}\left[J'_j(B^{-1}(x-\varepsilon z)z)\right] = \sum_{p,q,r}J''_{jp}(B^{-1}(x-\varepsilon z)z)\,(b^{-1}_{pq})'_r(x-\varepsilon z)\,z_q(-z_r),$$

thus

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}A^1_\varepsilon(x,z)dz\\ &=\frac{C}{2}\sum_{j,k,m,p,q,r}(b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)\det B^{-1}(x)\int_{\mathbb{R}^N}J''_{jp}(B^{-1}(x)z)\,z_qz_rz_kz_mdz.\end{aligned}$$

Now we change variables as $w = B^{-1}(x)z$ to obtain

$$\begin{aligned}\frac{C}{2}\sum_{j,k,m,p,q,r,l,n,s,t}&(b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)b_{ql}(x)b_{rn}(x)b_{ks}(x)b_{mt}(x)\\ &\times\int_{\mathbb{R}^N}J''_{jp}(w)w_lw_nw_sw_tdw.\end{aligned}$$

Finally, by properties *(2)* and *(3)* from Lemma 3.1, proceeding with similar arguments applied in (12) with easy modifications, we obtain that

$$\begin{aligned}&\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}A^1_\varepsilon(x,z)dz =\\ &\sum_{j,k,m,p,q,r,s}(b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)\,[b_{qj}(x)b_{rp}(x)b_{ks}(x)b_{ms}(x)\\ &\qquad+b_{qj}(x)b_{rs}(x)b_{kp}(x)b_{ms}(x)+b_{qj}(x)b_{rs}(x)b_{ks}(x)b_{mp}(x)\\ &\qquad+b_{qp}(x)b_{rj}(x)b_{ks}(x)b_{ms}(x)+b_{qs}(x)b_{rj}(x)b_{kp}(x)b_{ms}(x)\\ &\qquad+b_{qs}(x)b_{rj}(x)b_{ks}(x)b_{mp}(x)+b_{qp}(x)b_{rs}(x)b_{kj}(x)b_{ms}(x)\\ &\qquad+b_{qs}(x)b_{rp}(x)b_{kj}(x)b_{ms}(x)+b_{qs}(x)b_{rs}(x)b_{kj}(x)b_{mp}(x)\\ &\qquad+b_{qp}(x)b_{rs}(x)b_{ks}(x)b_{mj}(x)+b_{qs}(x)b_{rp}(x)b_{ks}(x)b_{mj}(x)\\ &\qquad+b_{qs}(x)b_{rs}(x)b_{kp}(x)b_{mj}(x)]\,.\end{aligned}\tag{18}$$

Part 1.b: Compute

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}A^2_\varepsilon(x,z)dz.$$

Since

$$\frac{\partial}{\partial \varepsilon}\left[(b_{jk}^{-1})'_m(x-\varepsilon z)\right] = \sum_p (b_{jk}^{-1})''_{mp}(x-\varepsilon z)(-z_p),$$

it follows, letting $\varepsilon \to 0$ and changing variables $w = B^{-1}(x)z$, that

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^2(x,z)dz = \frac{C}{2}\sum_{j,k,m,p,q,r,s}(b_{jk}^{-1})''_{mp}(x)b_{pq}(x)b_{kr}(x)b_{ms}(x) \times \int_{\mathbb{R}^N} J'_j(w)w_q w_r w_s dw$$

which due to property *(1)* from Lemma 3.1 and arguing as in (12) is equal to

$$-\sum_{j,k,m,p,q}(b_{jk}^{-1})''_{mp}(x)\left[b_{pq}(x)b_{kq}(x)b_{mj}(x) + b_{pj}(x)b_{kq}(x)b_{mq}(x) + b_{pq}(x)b_{kj}(x)b_{mq}(x)\right].$$

Thus, using *(2)* from Lemma 3.2, we get

$$\begin{aligned}
\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^2(x,z)dz = \sum_{j,k,m,p,q} &\Big\{(b_{jk}^{-1})'_m(x)(b_{kq})'_p(x)b_{pq}(x)b_{mj}(x)\\
&+(b_{jk}^{-1})'_p(x)(b_{kq})'_m(x)b_{pq}(x)b_{mj}(x) + (b_{kq})''_{mp}(x)b_{mj}(x)b_{jk}^{-1}(x)b_{pq}(x)\\
&+(b_{jk}^{-1})'_m(x)(b_{kq})'_p(x)b_{pj}(x)b_{mq}(x) + (b_{jk}^{-1})'_p(x)(b_{kq})'_m(x)b_{pj}(x)b_{mq}(x)\\
&+(b_{kq})''_{mp}(x)b_{pj}(x)b_{jk}^{-1}(x)b_{mq}(x) + (b_{jk}^{-1})'_m(x)(b_{kj})'_p(x)b_{pq}(x)b_{mq}(x)\\
&+(b_{jk}^{-1})'_p(x)(b_{kj})'_m(x)b_{pq}(x)b_{mq}(x) + (b_{kj})''_{mp}(x)b_{pq}(x)b_{jk}^{-1}(x)b_{mq}(x).
\end{aligned} \tag{19}$$

Note that, thanks to *(1)* from Lemma 3.2, some terms from expressions (18) and (19) cancel. In fact, the 12^{th} term of (18) verifies

$$\begin{aligned}
&\sum_{j,k,m,p,q,r,s}(b_{pq}^{-1})'_r(x)(b_{jk}^{-1})'_m(x)b_{qs}(x)b_{rs}(x)b_{kp}(x)b_{mj}(x)\\
&\qquad = -\sum_{j,k,m,p,q,r,s} b_{pq}^{-1}(x)(b_{jk}^{-1})'_m(x)(b_{qs})'_r(x)b_{rs}(x)b_{kp}(x)b_{mj}(x)
\end{aligned}$$

and since

$$\sum_p b_{kp}(x)b_{pq}^{-1}(x) = 1$$

if $k = q$ and vanishes if $k \neq q$ we obtain

$$-\sum_{j,k,m,r,s}(b_{jk}^{-1})'_m(x)(b_{ks})'_r(x)b_{rs}(x)b_{mj}(x).$$

Replacing s by q and r by p, this last expression is cancelled by the 1^{st} term of (19). We leave it to the reader to verify that, in the same way, the 2^{nd}, 4^{th}, 5^{th} and 7^{th} terms of expression (19) are cancelled by the 5^{th}, 3^{rd}, 1^{st} and 2^{nd} terms of expression (18) respectively. Hence, from Part 1.b only the 3^{rd}, 6^{th}, 8^{th} and 9^{th} terms remain.

Part 1.c: Compute

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N} A_\varepsilon^3(x,z)dz.$$

By equality (14) we obtain

$$\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}A^3_\varepsilon(x,z)dz$$
$$= -\frac{C}{2}\det B^{-2}(x)\sum_{j,k,m,p}(b^{-1}_{jk})'_m(x)(\det B(x))'_p\int_{\mathbb{R}^N}J'_j(B^{-1}(x)z)z_kz_mz_pdz. \tag{20}$$

Furthermore, thanks to the result obtained from equality (11) in (13), inside the proof of Proposition 3.3 we get

$$\begin{aligned} &C\det B^{-1}(x)\textstyle\sum_{j,k,m}(b^{-1}_{jk})'_m(x)\displaystyle\int_{\mathbb{R}^N}J'_j(B^{-1}(x)z)z_kz_mz_pdz \\ &= 2\sum_{j,k,m,s}b^{-1}_{jk}(x)(b_{kj})'_m(x)b_{ms}(x)b_{ps}(x) \\ &\quad+2\sum_{k,j}(b_{jk})'_j(x)b_{pk}(x)+2\sum_{k,j}(b_{pk})'_j(x)b_{jk}(x). \end{aligned}\tag{21}$$

In addition, we have [9], that is,

$$\det B^{-1}(x)(\det B(x))'_p=\sum_{q,r}b^{-1}_{qr}(x)(b_{rq})'_p(x). \tag{22}$$

Replacing (21) and (22) in equality (20), we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}A^3_\varepsilon(x,z)dz \\ &= -\sum_{j,k,m,p,q,r,s}b^{-1}_{qr}(x)(b_{rq})'_p(x)b^{-1}_{jk}(x)(b_{kj})'_m(x)b_{ms}(x)b_{ps}(x) \\ &\quad-\sum_{j,k,m,p,q,r}b^{-1}_{qr}(x)(b_{rq})'_p(x)\left\{(b_{jk})'_j(x)b_{pk}(x)+(b_{pk})'_j(x)b_{jk}(x)\right\}. \end{aligned}\tag{23}$$

Note that above expression is cancelled with the 7^{th}, 10^{th} and 4^{th} terms of expression (18).

Summarizing, we conclude Part 1 of the proof as follows:

$$\begin{aligned} &\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N}G^1_\varepsilon(x,z)dz=\lim_{\varepsilon\to 0}\frac{C}{2}\sum_{j,k,m}\int_{\mathbb{R}^N}\big(A^1_\varepsilon(x,z)+A^2_\varepsilon(x,z)+A^3_\varepsilon(x,z)\big)\,dz \\ &=\sum_{j,k,m,p,q,r,s}(b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)\,[b_{qs}(x)b_{rj}(x)b_{ks}(x)b_{mp}(x) \\ &\quad+b_{qs}(x)b_{rp}(x)b_{kj}(x)b_{ms}(x)+b_{qs}(x)b_{rs}(x)b_{kj}(x)b_{mp}(x) \\ &\quad+b_{qs}(x)b_{rp}(x)b_{ks}(x)b_{mj}(x)] \\ &\quad+\sum_{j,k,m,p,q}\Big\{(b_{kq})''_{mp}(x)b_{mj}(x)b^{-1}_{jk}(x)b_{pq}(x) \\ &\quad+(b_{kq})''_{mp}(x)b_{pj}(x)b^{-1}_{jk}(x)b_{mq}(x)+(b^{-1}_{jk})'_p(x)(b_{kj})'_m(x)b_{pq}(x)b_{mq}(x) \\ &\quad+(b_{kj})''_{mp}(x)b_{pq}(x)b^{-1}_{jk}(x)b_{mq}(x)\Big\}. \end{aligned}\tag{24}$$

Part 2: We have to compute

$$\lim_{\varepsilon\to 0}\frac{C}{2\varepsilon}\int_{\mathbb{R}^N}G^2_\varepsilon(x,z)dz.$$

Which due to relation (14), it is equivalent to compute

$$\lim_{\varepsilon\to 0} \frac{C}{2\varepsilon} \sum_p \int_{\mathbb{R}^N} J(B^{-1}(x-\varepsilon z)z) \, \frac{(\det B(x-\varepsilon z))'_p}{\det B^2(x-\varepsilon z)} \, z_p \, dz.$$

Note that since

$$\int_{\mathbb{R}^N} J(B^{-1}(x)z) \, z_p \, dz = 0,$$

letting $\varepsilon \to 0$, we have that the above expression is $\frac{0}{0}$. Consequently, by L'Hopital rule we obtain

$$\lim_{\varepsilon\to 0} \frac{C}{2\varepsilon} \int_{\mathbb{R}^N} G^2_\varepsilon(x,z)dz = \frac{C}{2} \sum_p \int_{\mathbb{R}^N} \left(R^1_\varepsilon(x,z) + R^2_\varepsilon(x,z) + R^3_\varepsilon(x,z)\right) dz,$$

where

$$R^1_\varepsilon(x,z) = \frac{\partial}{\partial\varepsilon} \left[J(B^{-1}(x-\varepsilon z)z)\right] \frac{(\det B(x-\varepsilon z))'_p}{\det B^2(x-\varepsilon z)} \, z_p,$$

$$R^2_\varepsilon(x,z) = J(B^{-1}(x-\varepsilon z)z) \frac{\partial}{\partial\varepsilon} \left[\frac{(\det B(x-\varepsilon z))'_p}{\det B(x-\varepsilon z)}\right] \det B^{-1}(x-\varepsilon z) z_p,$$

and

$$R^3_\varepsilon(x,z) = J(B^{-1}(x-\varepsilon z)z) \frac{(\det B(x-\varepsilon z))'_p}{\det B(x-\varepsilon z)} \frac{\partial}{\partial\varepsilon} \left[\det B^{-1}(x-\varepsilon z)\right] z_p.$$

Therefore, the Part 2 will be divided into three steps:

Part 2.a: Compute

$$\lim_{\varepsilon\to 0} \frac{C}{2} \sum_p \int_{\mathbb{R}^N} R^1_\varepsilon(x,z)dz.$$

By identity (10) and letting ε to 0, we get

$$= -\frac{C}{2} \det B^{-2}(x) \sum_{j,k,m,p} (b^{-1}_{jk})'_m(x) (\det B(x))'_p \int_{\mathbb{R}^N} J'_j(B^{-1}(x)z) z_k z_m z_p dz.$$

Which coincides with expression (20). Hence,

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{C}{2} \sum_p \int_{\mathbb{R}^N} R^1_\varepsilon(x,z)dz \\
&= - \sum_{j,k,m,p,q,r,s} b^{-1}_{qr}(x)(b_{rq})'_p(x) b^{-1}_{jk}(x)(b_{kj})'_m(x) b_{ms}(x) b_{ps}(x) \\
&- \sum_{j,k,m,p,q,r} b^{-1}_{qr}(x)(b_{rq})'_p(x) \left\{(b_{jk})'_j(x) b_{pk}(x) + (b_{pk})'_j(x) b_{jk}(x)\right\}.
\end{aligned} \tag{25}$$

Part 2.b: Compute

$$\lim_{\varepsilon\to 0} \frac{C}{2} \sum_p \int_{\mathbb{R}^N} R^2_\varepsilon(x,z)dz.$$

If we compute de derivative of (22), we obtain that

$$\frac{\partial}{\partial\varepsilon} \left[\frac{(\det B(x-\varepsilon z))'_p}{\det B(x-\varepsilon z)}\right] = \frac{\partial}{\partial\varepsilon} \sum_{j,k} b^{-1}_{jk}(x-\varepsilon z)(b_{kj})'_p(x-\varepsilon z)$$

$$= - \sum_{j,k,m} \left\{(b^{-1}_{jk})'_m(x-\varepsilon z) z_m (b_{kj})'_p(x-\varepsilon z) + b^{-1}_{jk}(x-\varepsilon z)(b_{kj})''_{pm}(x-\varepsilon z) z_m\right\}.$$

Therefore, replacing the above expression, letting $\varepsilon \to 0$ and change variables as $w = B^{-1}(x)z$, Part 2.b reads as follows

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{C}{2}\sum_p \int_{\mathbb{R}^N} R^2_\varepsilon(x,z)dz \\
&= -\sum_{j,k,m,p,q} \Big\{ (b^{-1}_{jk})'_m(x)(b_{kj})'_p(x) b_{pq}(x) b_{mq}(x) \\
&\qquad + b^{-1}_{jk}(x)(b_{kj})''_{pm}(x) b_{pq}(x) b_{mq}(x)\Big\}.
\end{aligned} \tag{26}$$

Part 2.c: Compute

$$\lim_{\varepsilon\to 0} \frac{C}{2}\sum_p \int_{\mathbb{R}^N} R^3_\varepsilon(x,z)dz.$$

Using again the equalities (22) and (14), letting $\varepsilon \to 0$ and change variables $w = B^{-1}(x)z$, we get

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{C}{2}\sum_p \int_{\mathbb{R}^N} R^3_\varepsilon(x,z)dz \\
&= \sum_{j,k,m,p,q,r,s} b^{-1}_{jk}(x)(b_{kj})'_m b^{-1}_{qr}(x)(b_{rq})'_p(x) b_{ms}(x) b_{ps}(x).
\end{aligned} \tag{27}$$

Note that this expression cancels with the first part of (25) from Part 2.a.

Summarizing, we conclude Part 2 of the proof as follows:

$$\begin{aligned}
&\lim_{\varepsilon\to 0} \frac{C}{2\varepsilon}\int_{\mathbb{R}^N} G^2_\varepsilon(x,z)dz = \lim_{\varepsilon\to 0} \frac{C}{2}\sum_p \int_{\mathbb{R}^N} \big(R^1_\varepsilon(x,z) + R^2_\varepsilon(x,z) + R^3_\varepsilon(x,z)\big)\, dz \\
&= -\sum_{j,k,m,p,q,r} b^{-1}_{qr}(x)(b_{rq})'_p(x)\left\{(b_{jk})'_j(x) b_{pk}(x) + (b_{pk})'_j(x) b_{jk}(x)\right\} \\
&\quad - \sum_{j,k,m,p,q} \Big\{ (b^{-1}_{jk})'_m(x)(b_{kj})'_p(x) b_{pq}(x) b_{mq}(x) \\
&\qquad + b^{-1}_{jk}(x)(b_{kj})''_{pm}(x) b_{pq}(x) b_{mq}(x)\Big\}.
\end{aligned} \tag{28}$$

Finally, taking into account that the first sum of (28) is cancelled with the 2^{nd} and 3^{rd} term of (24) and the second sum of (28) is cancelled with the last two terms of (24). We have, adding Part 1 and Part 2 in (16), that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \mathcal{L}^2_\varepsilon(u(x,t)) = \Bigg\{ &\sum_{j,k,m,p,q,r,s} (b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x) b_{qs}(x) b_{rj}(x) b_{ks}(x) b_{mp}(x) \\
&+ \sum_{j,k,m,p,q,r,s} (b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x) b_{qs}(x) b_{rp}(x) b_{ks}(x) b_{mj}(x) \\
&+ \sum_{j,k,m,p,q} (b_{kq})''_{mp}(x) b_{mj}(x) b^{-1}_{jk}(x) b_{pq}(x) \\
&+ \sum_{j,k,m,p,q} (b_{kq})''_{mp}(x) b_{pj}(x) b^{-1}_{jk}(x) b_{mq}(x) \Bigg\} u(x,t).
\end{aligned}$$

Now, applying property *(1)* from Lemma 3.2, each sum satisfies

$$
\begin{aligned}
&\sum_{j,k,m,p,q,r,s} (b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)b_{qs}(x)b_{rj}(x)b_{ks}(x)b_{mp}(x)\\
&\qquad = \sum_{j,k,m,p,q,r,s} b^{-1}_{pq}(x)b^{-1}_{jk}(x)(b_{qs})'_r(x)b_{rj}(x)(b_{ks})'_m(x)b_{mp}(x)\\
&\qquad = \sum_{k,q,s}(b_{qs})'_k(x)(b_{ks})'_q(x) = \sum_{i,j,k}(b_{ik})'_j(x)(b_{jk})'_i(x),
\end{aligned}
\tag{29}
$$

replacing, in the last equality, indexes $\{q,k,s\}$ by $\{i,j,k\}$ respectively. We have

$$
\begin{aligned}
&\sum_{j,k,m,p,q,r,s} (b^{-1}_{pq})'_r(x)(b^{-1}_{jk})'_m(x)b_{qs}(x)b_{rp}(x)b_{ks}(x)b_{mj}(x)\\
&\qquad = \sum_{j,k,m,p,q,r,s} b^{-1}_{pq}(x)b^{-1}_{jk}(x)(b_{rp})'_r(x)b_{qs}(x)(b_{mj})'_m(x)b_{ks}(x)\\
&\qquad = \sum_{m,p,r}(b_{rp})'_r(x)(b_{mp})'_m(x) = \sum_{i,j,k}(b_{ik})'_i(x)(b_{jk})'_j(x),
\end{aligned}
\tag{30}
$$

replacing, in the last equality, indexes $\{r,m,p\}$ by $\{i,j,k\}$ respectively.

Now,

$$
\begin{aligned}
\sum_{j,k,m,p,q} (b_{kq})''_{mp}(x)b_{mj}(x)b^{-1}_{jk}(x)b_{pq}(x) &= \sum_{k,p,q}(b_{kq})''_{kp}(x)b_{pq}(x)\\
&= \sum_{i,j,k}(b_{ik})''_{ij}(x)b_{jk}(x),
\end{aligned}
\tag{31}
$$

replacing, in the last equality, indexes $\{k,p,q\}$ by $\{i,j,k\}$ respectively.

Also, we have

$$
\begin{aligned}
\sum_{j,k,m,p,q} (b_{kq})''_{mp}(x)b_{pj}(x)b^{-1}_{jk}(x)b_{mq}(x) &= \sum_{k,m,q}(b_{kq})''_{mk}(x)b_{mq}(x)\\
&= \sum_{i,j,k}(b_{jk})''_{ij}(x)b_{ik}(x),
\end{aligned}
\tag{32}
$$

replacing, in the last equality, indexes $\{m,k,q\}$ by $\{i,j,k\}$ respectively.

Summarizing, from (29), (30), (31) and (32), we conclude that

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\mathcal{L}^2_\varepsilon(u(x,t)) &= \sum_{i,j,k}\big\{(b_{ik})'_j(x)(b_{jk})'_i(x) + (b_{ik})'_i(x)(b_{jk})'_j(x)\\
&\quad +(b_{ik})''_{ij}(x)b_{jk}(x) + (b_{jk})''_{ij}(x)b_{ik}(x)\big\}\,u(x,t) = \sum_{i,j}\frac{\partial^2 a_{ij}(x)}{\partial x_i\partial x_j}u(x,t),
\end{aligned}
$$

and the Proposition gets proved. □

Proposition 3.5. *Let u be a $\mathcal{C}^{2+\alpha,1+\alpha/2}\left(\mathbb{R}^N\times[0,T]\right)$ function and let $\mathcal{L}_\varepsilon$ be the operator defined as*

$$
\mathcal{L}_\varepsilon(u(x,t)) = \frac{C}{\varepsilon^2}\left\{\int_{\mathbb{R}^N} J_\varepsilon\left(B^{-1}(y)(x-y)\right)\det B^{-1}(y)u(y,t)dy - u(x,t)\right\}. \tag{33}
$$

Then,

$$
\sup_{t\in[0,T]}\left\|\mathcal{L}_\varepsilon(u(x,t)) - \sum_{i,j}\frac{\partial^2}{\partial x_i\partial x_j}\left(a_{ij}(x)u(x,t)\right)\right\|_{L^\infty}\to 0 \qquad \textit{as } \varepsilon\to 0.
$$

Proof. Thanks to Propostion 3.3 and Proposition 3.4 we obtain that

$$\sup_{t\in[0,T]} \left\| \mathcal{L}_\varepsilon(u(x,t)) - \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(a_{ij}(x)u(x,t)\right) \right\|_{L^\infty}$$
$$\leq \sup_{t\in[0,T]} \| \left(\mathcal{L}^1_\varepsilon - \Lambda\right)(u(x,t))\|_{L^\infty} + \sup_{t\in[0,T]} \| \left(\mathcal{L}^2_\varepsilon - \Gamma\right)(u(x,t))\|_{L^\infty} \to 0,$$

as $\varepsilon \to 0$. □

We are now ready to prove our main result.

Proof of Theorem 1.1. We will denote by $w^\varepsilon = u^\varepsilon - v$. Note that w^ε satisfies the following equation

$$\begin{cases} w^\varepsilon_t(x,t) = \mathcal{L}_\varepsilon(w^\varepsilon(x,t)) + \tilde{F}(x,t), & x \in \mathbb{R}^N,\ t \in [0,T], \\ w^\varepsilon(x,0) = 0, & x \in \mathbb{R}^N, \end{cases} \tag{34}$$

where

$$\tilde{F}(x,t) = \mathcal{L}_\varepsilon(v(x,t)) - \sum_{i,j} \frac{\partial^2}{\partial x_i \partial x_j} \left(a_{ij}(x)v(x,t)\right).$$

In addition, thanks to Proposition 3.5, we can assert that there exists a positive function θ such that $|\tilde{F}(x,t)| \leq \theta(\varepsilon) \to 0$ as $\varepsilon \to 0$, for every $x \in \mathbb{R}^N$, $t \in [0,T]$.

Next, let us consider

$$\eta(\varepsilon) = \max\left\{ \frac{C}{\varepsilon^2} \left[\int_{\mathbb{R}^N} J_\varepsilon\left(B^{-1}(y)(x-y)\right) \det B^{-1}(y) dy - 1 \right],\, x \in \mathbb{R}^N \right\},$$

it is easy to check that $\eta(\varepsilon) < \infty$, for every $\varepsilon > 0$. Futhermore, by Proposition 3.4 and (5) we obtain

$$\eta(\varepsilon) \to \max\left\{ \sum_{i,j} \frac{\partial^2 a_{ij}(x)}{\partial x_i \partial x_j},\, x \in \mathbb{R}^N \right\} < \infty.$$

In this way, we set the following function

$$\overline{w}(x,t) = \begin{cases} \dfrac{\theta(\varepsilon)}{\eta(\varepsilon)} \left(e^{\eta(\varepsilon)t} - 1\right) + \varepsilon\, e^{\eta(\varepsilon)t}, & \text{if } \eta(\varepsilon) \neq 0, \\ \theta(\varepsilon)t + \varepsilon, & \text{if } \eta(\varepsilon) = 0, \end{cases}$$

for $x \in \mathbb{R}^N$, $t \in [0,T]$. Now, we claim that $\overline{w}$ is a supersolution of (34). Indeed, for $\eta(\varepsilon) \neq 0$

$$\overline{w}_t(x,t) = \theta(\varepsilon)e^{\eta(\varepsilon)t} + \varepsilon\,\eta(\varepsilon)\, e^{\eta(\varepsilon)t} = \eta(\varepsilon)\,\overline{w}(x,t) + \theta(\varepsilon)$$

$$\geq \mathcal{L}^2_\varepsilon(\overline{w}(x,t)) + \tilde{F}(x,t) = \mathcal{L}_\varepsilon(\overline{w}(x,t)) + \tilde{F}(x,t),$$

taking into account that $\mathcal{L}^1_\varepsilon(\overline{w}(x,t)) = 0$ in the last equality. We left to the reader to check the case $\eta(\varepsilon) = 0$. Finally, as $\overline{w}(x,0) = \varepsilon$, the claim is proved.

Similar arguments applied to the case $\underline{w}(x,t) = -\overline{w}(x,t)$ leads us to assert that $\underline{w}(x,t)$ is a subsolution of problem (34).

We conclude from the comparison principle, Theorem 2.2, that

$$\underline{w} \leq w^\varepsilon \leq \overline{w}$$

and since $\overline{w}(x,t)$, $\underline{w}(x,t) \to 0$ as $\varepsilon \to 0$ our main result gets proved. □

Remark 3.6. One can easily check that, for all test function $\varphi \in \mathcal{C}_c^2(\mathbb{R}^N)$ and $u \in L^1(\mathbb{R}^N) \cap C^{2+\alpha}(\mathbb{R}^N)$, it holds that

$$\int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u(x)\,\varphi(x)dx = \int_{\mathbb{R}^N} \sum_{j,k} \varphi''_{x_j x_k}(x)\,(B(x)B^t(x))_{(j,k)}\,u(x)\,dx + 0(\varepsilon^\alpha) \tag{35}$$

and hence, integrating by parts twice, we get

$$\int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u(x)\,\varphi(x)dx = \int_{\mathbb{R}^N} \sum_{j,k} \frac{\partial^2}{\partial x_j x_k}(a_{jk}(x)u(x))\,\varphi(x)dx + 0(\varepsilon^\alpha).$$

In fact, for $\varphi \in \mathcal{C}_c^2(\mathbb{R}^N)$ we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} \mathcal{L}_\varepsilon u(x)\,\varphi(x)dx \\
&= \int_{\mathbb{R}^N} \frac{C}{\varepsilon^2}\left\{\int J_\varepsilon(B^{-1}(y)(x-y)) \det B^{-1}(y)u(y)\varphi(x)dy - u(x)\varphi(x)\right\}dx \\
&= \int_{\mathbb{R}^N} \frac{C}{\varepsilon^2}\left(\int J_\varepsilon(B^{-1}(y)(x-y)) \det B^{-1}(y)\varphi(x)dx\right)u(y)dy - \frac{C}{\varepsilon^2}\int_{\mathbb{R}^N} u(x)\varphi(x)dx \\
&= \int_{\mathbb{R}^N} u(y)\frac{C}{\varepsilon^2}\left\{\int J_\varepsilon(B^{-1}(y)y - z)\,\varphi(B(y)z)\,dz - \varphi(y)\right\}dy \\
&= \int_{\mathbb{R}^N} u(y)\frac{C}{\varepsilon^2}\left\{\int J_\varepsilon(B^{-1}(y)y - z)\,\phi(z)\,dz - \phi(B^{-1}(y)y)\right\}dy,
\end{aligned}$$

with $\phi(z) := \varphi(B(y)z)$. Now we observe that it is well known (see [2]) that this last expression verifies

$$= \int_{\mathbb{R}^N} u(y)\Delta\phi(B^{-1}(y)y)\,dy + O(\varepsilon^\alpha).$$

Using that

$$\Delta\phi(B^{-1}(x)x) = \sum_{j,k} \varphi''_{x_j x_k}(x)\,(B(x)B^t(x))_{(j,k)}$$

we obtain (35).

Remark 3.7. Our results can be interpreted from a stochastic processes viewpoint. In fact, given the stochastic differential equation

$$d\,\mathbf{X}_t = B(\mathbf{X}_t)\,d\,\mathbf{W}_t,$$

where $\mathbf{X}_t$ is an N-dimensional random variable vector and $\mathbf{W}_t$ is an N-dimensional standard Wiener process. Our main result states that

Solutions of the rescaled nonlocal problem (6)*,* $u^\varepsilon(x,t)$*, converge uniformly to the probability density,* $u(x,t)$*, that corresponds to the process* $\mathbf{X}_t$*.*

See [12] for more details.

References

[1] P. Bates, P. Fife, X. Ren, X. Wang, Travelling waves in a convolution model for phase transitions, *Arch. Ration. Mech. Anal.* **138**, 105-136 (2997).

[2] E. Chasseigne, M. Chaves, J.D. Rossi, Asymptotic behavior for nonlocal diffusion equations, *J. Math. Pures Appl.* **86**, 271-291 (2006).

[3] C. Cortázar, J. Coville, M. Elgueta, S. Martínez, A nonlocal inhomogeneous dispersal process, *J. Differential Equations* **241**, 332-358 (2007).

[4] C. Cortázar, M. Elgueta, J. García-Melián, S. Martínez, Finite mass solutions for a nonlocal inhomogeneous dispersal equation, *Discrete Contin. Dyn. Syst.* **35** No. 4, 1409-1419 (2015).
[5] C. Cortázar, M. Elgueta, J. D. Rossi, Nonlocal Diffusion problems that approximate the heat equation with Dirichlet boundary conditions, *Israel Journal of Mathematics* **170**, 53-60 (2009).
[6] C. Cortázar, M. Elgueta, J.D. Rossi, N. Wolanski, Boundary fluxes for non-local diffusion, *J. Differential Equations* **234**, 360-390 (2007).
[7] J. Coville, On a simple criterion for the existence of a principal eigenfunction of some nonlocal operators, *J. Differential Equations* **249**, 2921-2953 (2010).
[8] L. Evans, Partial Differential Equations, *AMS*, Providence, Rhode Island (1998).
[9] M.A. Goldberg, The Derivative of a Determinant, *The American Mathematical Monthly* **79** No. 10, 1124-1126 (1972).
[10] O.A. Ladyženskaja, V.A. Solonnikov, N.N. Ural'eva, Linear and Quasilinear Equations of Parabolic Type, *AMS*, Providence, Rhode Island (1968).
[11] A. Molino and J. D. Rossi, Nonlocal diffusion problems that approximate a parabolic equation with spatial dependence. *Preprint*.
[12] H. Risken, The Fokker-Planck Equation. Methods of Solution and Applications, *Springer-Verlag*. Berlin Heidelberg New York Tokio 1984.
[13] J.-W. Sun, W.-T. Li, F.-Y. Yang, Approximate the Fokker-Planck equation by a class of nonlocal dispersal problems, *Nonlinear Analysis* **74**, 3501-3509 (2011).

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