LARGE SOLUTIONS FOR A CLASS OF SEMILINEAR INTEGRO-DIFFERENTIAL EQUATIONS WITH CENSORED JUMPS.

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ABSTRACT. We study existence of large solutions, that is, solutions that verify $u(x) \to +\infty$ as $x \to \partial\Omega$, for equations like

$$-\mathcal{I}(u,x) + u(x)^p = 0, \ x \in \Omega,$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$, $p > 1$ and $\mathcal{I}$ is a nonlocal operator of the form

$$\mathcal{I}(u,x) = \text{P.V.} \int_{|z| \leq \varrho(x)} [u(x+z) - u(x)]|z|^{-(N+\alpha)} dz,$$

where $\alpha \in (0,2)$ and $\varrho : \bar{\Omega} \to \mathbb{R}$ is a function whose main particularity is that $0 < \varrho(x) \leq \text{dist}(x, \partial\Omega)$. We also obtain uniqueness of the solution in a class of large solutions whose blow-up rate depends on p, α and the rate at which ϱ shrinks near the boundary.

1. INTRODUCTION.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary. This paper is concerned with the nonlocal equation

$$-\mathcal{I}(u,x) + u^p(x) = 0, \quad x \in \Omega, \tag{1.1}$$

where $p > 1$ and $\mathcal{I}$ is a nonlocal operator with the form

$$\mathcal{I}(u,x) = \int_{|z| \leq \varrho(x)} [u(x+z) - u(x) - \mathbf{1}_B \langle Du(x), z \rangle] \nu(dz),$$

where ν is a positive, regular measure in $\mathbb{R}^N$ and $\varrho : \bar{\Omega} \to \mathbb{R}$ is a function whose main particularity is that $0 < \varrho(x) \leq \text{dist}(x, \partial\Omega)$ for each $x \in \Omega$. The precise set of assumptions on ν, ϱ and p will be given later on.

Our interest here is the study of *large solutions* also called *blow-up solutions* for (1.1), that is, solutions of (1.1) satisfying

$$u(x) \to +\infty \quad \text{as } x \to \partial\Omega. \tag{1.2}$$

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The study of large solutions for (1.1) is motivated by the wide literature concerning second-order operators, namely equations of the form

$$-\Delta u + f(u) = 0 \quad \text{in } \Omega, \tag{1.3}$$

with $f : \mathbb{R} \to \mathbb{R}$ satisfying adequate assumptions. In the seminal works of Keller [22] and Osserman [28], it is proved that the interaction between the diffusive and the reactive term in (1.3) allows the existence of large solutions to this problem, if and only if the nonlinearity f satisfies the so-called *Keller-Osserman condition* given by

$$\int_1^{+\infty} \frac{ds}{\sqrt{F(s)}} < +\infty, \qquad \text{where} \quad F(s) = \int_0^s f(t)dt.$$

In particular, when $f(t) = t^p$, the above condition is verified by a superlinear nonlinearity, that is, for $p > 1$.

After the works by Keller and Osserman, a broad variety of very interesting results concerning existence, uniqueness and asymptotic behavior near the boundary for large solutions of second-order reaction-diffusion equations have been obtained in the PDE framework using different techniques, see [3, 12, 14, 25, 26, 27, 29] for a nonexhaustive list of references. It is also worth to mention the deep connection of problems like (1.3) with stochastic superprocesses and the so-called Brownian snake, see [15, 24] and references therein. Additionally, we remark that the study of large solutions is also addressed in second-order quasilinear problems associated to coercive gradient terms in [23] and in infinity Laplacian problems in [21].

In recent years, large solutions for reaction-diffusion problems associated to anomalous diffusion have been subject of study by several authors. The typical example of this anomalous diffusive operator is the so-called fractional Laplacian of order $\alpha \in (0, 2)$, defined as

$$(-\Delta)^\alpha u(x) = C_{N,\alpha} \text{P.V.} \int_{\mathbb{R}^N} [u(x+z) - u(x)]|z|^{-(N+\alpha)} dz, \tag{1.4}$$

where P.V. stands for the Cauchy principal value and $C_{N,\alpha}$ is a well-known normalizing constant, see [16] for a very complete introduction to this operator. In this case, the prototype equation takes the form

$$(-\Delta)^\alpha u + u^p = 0 \quad \text{in } \Omega. \tag{1.5}$$

Typically, in addition to (1.5), we must prescribe the value of u in Ω^c as an exterior data in order to evaluate u in the fractional Laplacian. This is the first feature that must be taken into account in addressing problems like (1.5). For instance, in [17] the authors construct large solutions to (1.5) provided the exterior data blows-up at the boundary. This condition was afterwards relaxed in the subsequent paper [11], where in addition it is provided a uniqueness result for a class of large solutions with a controlled blow-up rate at the boundary. We also remark that the blow-up rate is naturally restricted by the request convergence of the integral defining the fractional Laplacian.

In this paper we get the existence of a large solution following the method used in some local and nonlocal problems of the references mentioned above, where a large solution is obtained as the limit of a sequence of solutions of Dirichlet problems with bounded boundary data tending to infinity. In this task, the reaction-diffusion nature of the problem and suitable comparison principles provide the required compactness to pass to the limit. The reactive term (the nonlinearity) allows us to construct a locally bounded supersolution which implies the approximating sequence is uniformly bounded in $L^\infty_{loc}(\Omega)$, meanwhile the diffusive term (the elliptic operator) allows to construct an adequate subsolution which "lifts to infinity" the approximating sequence at the boundary. This diffusive term is also the key ingredient in the application of elliptic regularity results.

At this point we should mention the main features of our problem playing a role in the application of the aforementioned method. We start remarking that $\mathcal{I}$ has a *censored or regional nature* since by its form we don't need to consider any datum outside $\overline{\Omega}$. Moreover, this can be understood as a weaker nonlocality (compared, for instance, with the fractional Laplacian (1.4)) since we just require the values of u in a neighborhood of x (depending on $\varrho(x)$) to compute $\mathcal{I}(u)$ at x. Despite this feature, the nonlocal dependence on Ω of $\mathcal{I}$ is strong enough to prevent the use of some second-order methods concerning the analysis of auxiliary problems localized in subdomains, basically because in that case we are qualitatively changing the structure of the problem. On the other hand, this type of nonlocality can be understood as a degenerate ellipticity: for integro-differential operators the ellipticity is related with the singularity at the origin of the measure ν defining $\mathcal{I}$, but as we will see below, in our case this ellipticity is also influenced by the rate at which the function ϱ shrinks near the boundary. In fact, the blow-up rate of the constructed large solution is determined both by the singularity of the measure ν and the rate at which $\varrho(x)$ shrinks as $x \to \partial\Omega$.

The paper is organized as follows: In Section 2 we introduce the precise assumptions we use to deal with this problem and state the main results. Two important estimates are provided in Section 3. These estimates are used in Section 4 to solve the Dirichlet problem with bounded datum, which is the key step to get the existence of a large solution given in Section 5. We prove uniqueness (in a certain class of large solutions) in Section 6 and we finish with a precise estimate of the blow-up rate in Section 7.

Basic Notation. For $x \in \mathbb{R}^N$ and $r > 0$, we denote $B_r(x)$ the open ball centered at x with radius r, B_r if $x = 0$, and B if additionally $r = 1$.

For $x \in \mathcal{O}$, we denote $d_{\mathcal{O}}(x) = dist(x, \partial\mathcal{O})$. In the case $\mathcal{O} = \Omega$ we simply write $d(x)$. For $\delta > 0$, we denote $A_\delta(\mathcal{O}) = \{x \in \mathcal{O} : d_{\mathcal{O}}(x) < \delta\}$, and we simplify to A_δ if $\mathcal{O} = \Omega$. By the smoothness of the boundary of the domain, we can find $\delta_0 > 0$ such that the distance function is of class C^2 in A_{δ_0}, see [18]. We can also assume that $\delta_0 < 1$.

For a set $D \subset \mathbb{R}^N$, $x \in \Omega$, $p \in \mathbb{R}^N$ and $\phi \in L^1(\Omega)$, we define

$$\mathcal{I}[D](\phi, x, p) = \int_{B_\varrho(x) \cap D} [\phi(x+z) - \phi(x) - \mathbf{1}_B \langle p, z\rangle] \nu(dz),$$

each time the integral makes sense. We also write

$$\mathcal{I}[D](\phi, x) = \mathcal{I}[D](\phi, x, D\phi(x)).$$

Since in most of the cases the arguments are carried out near the boundary, we are typically integrating on balls with a small radius and therefore we omit the $\mathbf{1}_B$ term in the integrand.

2. Assumptions and statements of the main results.

The nonlocal operator $\mathcal{I}$ that will be considered in this work has the form

$$\mathcal{I}(u, x) = \int_{B_{\varrho(x)}} [u(x+z) - u(x) - \mathbf{1}_B \langle Du(x), z\rangle] K^\alpha(z) dz, \tag{2.1}$$

where for some $\alpha \in (0, 2)$, $K^\alpha(z) := K(z)|z|^{-(N+\alpha)}$, $z \in \mathbb{R}^N \setminus \{0\}$ and the functions K, ϱ satisfy the following assumptions:

(M) $K : \mathbb{R}^N \to \mathbb{R}$ is measurable, bounded, radially symmetric function satisfying $K \geq \kappa > 0$ in $\mathbb{R}^N$, for some $\kappa > 0$.

(D) There exists $\sigma \geq 1$ and $0 < \Lambda \leq 1$ such that

$$\varrho(x) = \Lambda d^\sigma(x).$$

Note that the symmetry of K and the form of the region of integration defining (2.1) allow us to write

$$\mathcal{I}(u, x) = \text{P.V.} \int_{B_{\varrho(x)}} [u(x+z) - u(x)] K^\alpha(z) dz,$$

where P.V. stands for the Cauchy principal value. Moreover, we can drop the P.V. in the case $\alpha \in (0, 1)$, see [16]. We use this fact to avoid the compensator term $\mathbf{1}_B \langle Du(x), z\rangle$ in some situations, for a sake of simplicity.

Our first result reads as follows:

Theorem 2.1. *Let $\mathcal{I}$ as in* (2.1) *satisfying* **(M)**. *Assume* **(D)** *holds with p, σ and Λ satisfying one of the following configurations:*

(i) Linear Censorship: $\sigma = \Lambda = 1$ *and* $p > 1 + \alpha$.

(ii) Linear Strict Censorship: $\sigma = 1, \Lambda < 1$ *and* $p > 1$.

(iii) Superlinear Censorship: $1 < \sigma < 2/(2 - \alpha)$ *and* $p > 1$.

Define

$$\gamma = \alpha_\sigma/(p-1), \tag{2.2}$$

with

$$\alpha_\sigma = \sigma(\alpha - 2) + 2 > 0. \tag{2.3}$$

Then, there exists a classical solution u to problem (1.1)-(1.2)*. This solution is strictly positive in Ω and it is the minimal solution over the class of large solutions, in the sense that any solution v to* (1.1)-(1.2) *satisfies $u \le v$ in Ω. In addition, there exist constants $0 < \bar{c} < \bar{C}$ such that*

$$\bar{c}\ d^{-\gamma}(x) \le u(x) \le \bar{C}\ d^{-\gamma}(x), \quad \textit{for } x \to \partial\Omega. \tag{2.4}$$

Moreover, this solution is the unique solution in the class of large solutions $v : \Omega \to \mathbb{R}$ satisfying the boundary blow-up rate

$$0 < \liminf_{\Omega \ni x \to \partial\Omega} v(x) d^{-\gamma}(x) \le \limsup_{\Omega \ni x \to \partial\Omega} v(x) d^{-\gamma}(x) < +\infty. \tag{2.5}$$

Roughly speaking, α_σ represents the order of the operator $\mathcal{I}$ in the different configurations of the problem. Note that this order is not only influenced by the degree of the singularity of K^α, but by the speed at which the domain of integration shrinks near the boundary.

Note that in cases (ii) and (iii) of the above theorem, the domain of integration in the nonlocal operator "does not touches the boundary" when x is sufficiently close to $\partial\Omega$ and for this reason we intend them as the "*strictly censored case*". The main particularity in this situation is that the problem allows blow-up solutions which are not in $L^1(\Omega)$. This makes a qualitative difference with the remaining case (i) (which, just to differentiate it from the previous case, we refer as "*weakly censored case*"). In this case, the integrable blow-up profile at the boundary of the solution is a structural requirement to evaluate it in the nonlocal operator, see also [11, 17] for further discussions about this fact.

Our second result deals with the precise blow-up rate for a large solution to problem (1.1).

Theorem 2.2. *Let $\mathcal{I}$ defined as in* (2.1) *satisfying* **(M)** *with K continuous at the origin. Let* **(D)** *holds with Λ, σ and p satisfying one of the configurations of Theorem 2.1. Then, the blow-up solution u to problem* (1.1) *satisfies the asymptotic behavior*

$$\lim_{d(x) \to 0} u(x) d^{\gamma}(x) = \bar{C}_0, \tag{2.6}$$

where γ is given in (2.2) *and $\bar{C}_0 > 0$ is a constant depending on N, Λ, σ, α, p, b_0 and $K(0)$.*

Our results are obtained in the framework of viscosity solutions theory for nonlocal problems, see [5, 6] and references therein. See also [13] for a complete introduction of the viscosity theory for second-order operators. Thus, for a (bounded) subsolution for a problem set up in an open set $\mathcal{O}$ we mean an upper semicontinuous function on $\bar{\mathcal{O}}$ which satisfies the subsolution's viscosity inequality associated to the problem in $\mathcal{O}$. Analogous remark can be done for supersolutions and therefore for solutions. For a large solution u to a problem set up in $\mathcal{O}$ we mean a viscosity solution in $\mathcal{O}$ satisfying the blow-up condition (1.2).

We remark that the application of (nonlocal) elliptic regularity results to our problem (see [4, 9, 10]) and a standard bootstrap argument allows us to conclude that (continuous) viscosity solutions to (1.1)-(1.2) which are locally bounded in Ω are in fact classical solutions.

3. Two important estimates.

Recalling $\delta_0 > 0$ is such that the distance function $x \mapsto \mathrm{dist}(x, \partial\Omega)$ is smooth in A_{δ_0}, in this section we consider $d : \bar{\Omega} \to \mathbb{R}$ a smooth function, strictly positive in $\Omega \setminus A_{\delta_0}$ and such that $d(x) = \mathrm{dist}(x, \partial\Omega)$ on $\bar{A}_{\delta_0}$.

We start with the following estimate which is related with the existence of a blow-up supersolution, see Proposition 5.1 below.

Lemma 3.1. *Let $\beta > 0$ in the strictly censored case and $\beta \in (0,1)$ in the weakly censored case. Then there exist $C_0 > 0$ and $\bar{\delta} \in (0, \delta_0)$ such that*

$$-\mathcal{I}_\sigma(d^{-\beta}, x) \geq -C_0 d^{-\beta-\alpha_\sigma}(x), \quad \textit{for all } x \in A_{\bar{\delta}}.$$

Proof: We start with the strictly censored case. A simple Taylor expansion allows us to write

$$\mathcal{I}(d^{-\beta}, x) = \frac{1}{2} \int_{B_{\varrho(x)}} \int_0^1 \langle D^2 d^{-\beta}(x + sz)z, z\rangle K^\alpha(z) dz.$$

From the explicit expression

$$D^2 d^{-\beta}(x) = \beta(\beta+1) d^{-(\beta+2)}(x) Dd(x) \otimes Dd(x) - \beta d^{-(\beta+1)}(x) D^2 d(x),$$

using that for $s \in (0,1)$ and that for $z \in B_{\varrho(x)}$ the term $d(x + sz)$ is comparable to $d(x)$, together with the smoothness of d, we obtain the existence of a constant $C > 0$ depending only on the data such that

$$\mathcal{I}(d^{-\beta}, x) \leq C d^{-(\beta+2)}(x) \int_{B_{\varrho(x)}} |z|^2 K^\alpha(z) dz,$$

from which we obtain the result.

For the case $\Lambda = \sigma = 1$, we consider $\epsilon \in (0,1)$ and we write

$$\mathcal{I}(d^{-\gamma}, x) = \mathcal{I}[B_{(1-\epsilon)d(x)}](d^{-\beta}, x) + \mathcal{I}[B_{d(x)} \setminus B_{(1-\epsilon)d(x)}](d^{-\beta}, x).$$

For the first term in the right-hand side, performing the above analysis we get the existence of a constant $C_\epsilon > 0$ such that

$$\mathcal{I}[B_{(1-\epsilon)d(x)}](d^{-\beta}, x) \leq C_\epsilon d^{-\beta-\alpha}(x),$$

meanwhile, for the second term we can write

$$\mathcal{I}[B_{d(x)} \setminus B_{(1-\epsilon)d(x)}](d^{-\beta}, x) \leq ||K||_{L^\infty} \int_{B_{\varrho(x)}} d^{-\beta}(x+z)|z|^{-(N+\alpha)}dz,$$

and using that $d(x+z) \geq d(x) - |z|$ for all $z \in B_{\varrho(x)}$, there exists $C > 0$ depending only on the data such that

$$\mathcal{I}[B_{d(x)} \setminus B_{(1-\epsilon)d(x)}](d^{-\beta}, x) \leq C \int_{(1-\epsilon)d(x)}^{d(x)} (d(x)-r)^{-\beta} r^{-(1+\alpha)}dr,$$

from which, since $\beta \in (0,1)$, we obtain that

$$\mathcal{I}[B_{d(x)} \setminus B_{(1-\epsilon)d(x)}](d^{-\beta}, x) = o_\epsilon(1) d^{-\beta-\alpha}(x),$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$ uniformly on $x \in A_{\bar{\delta}}$. Fixing $\epsilon \in (0,1)$, we conclude the result. $\square$

The following lemma is related with the existence of a subsolution that allows us to “lift” to infinity the solution to our problem near the boundary (see Proposition 5.2 below).

Lemma 3.2. *Let $\beta \in (0,1)$. There exists $c_0 > 0$ and $\bar{\delta} \in (0,1)$ such that*

$$-\mathcal{I}(d^\beta, x) \geq c_0 d^{\beta-\alpha_\sigma}(x), \quad \textit{for all } x \in A_{\bar{\delta}}.$$

Proof: We focus our attention in the strictly censored case. As we did in Lemma 3.1, we consider $\bar{\delta} < \delta_0$ and apply a Taylor expansion with reminder to write

$$\begin{aligned}\mathcal{I}(d^\beta, x) =& \frac{\beta(\beta-1)}{2} \int_{B_{\varrho(x)}} \int_0^1 d^{\beta-2}(x+sz)|\langle Dd(x+sz), z\rangle|^2 K^\alpha(z)dz \\ &+ \frac{\beta}{2} \int_{B_{\varrho(x)}} \int_0^1 d^{\beta-1}(x+sz)\langle D^2 d(x+sz)z, z\rangle K^\alpha(z)dz \\ =&: I_1 + I_2.\end{aligned}$$

Concerning I_2, by the strict censorship of the jumps, we note that for $s \in (0,1)$ and $z \in B_{\varrho(x)}$, $d(x+sz)$ is comparable to $d(x)$. This together with the smoothness of $\partial\Omega$ allows us to write

$$I_2 \leq C d^{\beta-1}(x) \int_{B_{\varrho(x)}} |z|^{2-N-\alpha}dz \leq C d^{\beta-1+\sigma(2-\alpha)}(x) = C d^{\beta+1-\alpha_\sigma}(x),$$

for some constant $C > 0$ depending only on β and the data.

For I_1, we denote by D_x the set

$$D_x = \{z \in B_{\varrho(x)} : |\langle Dd(x), z\rangle| \geq |z|/2\}.$$

Thus, using the smoothness of the distance function, for each $z \in D_x$ and $s \in (0,1)$ we can write

$$|\langle Dd(x+sz), z\rangle| \geq |z|/2 - |s\langle D^2(\xi)z, z\rangle|,$$

for some ξ in the line joining x and $x+sz$. Thus, taking $\bar\delta$ smaller if necessary, we conclude that

$$|\langle Dd(x+sz), z\rangle| \geq |z|/4,$$

for all $s \in (0,1)$ and $z \in D_x$. Using this estimate, the strict censorship of the operator, the positivity of the integrand and that $\beta < 1$, we obtain

$$I_1 \geq c\kappa d^{\beta-2}(x) \int_{D_x} |z|^{2-N-\alpha} dz,$$

where $c > 0$ depends on β and the data. Now, since the measure of D_x is comparable to the measure of the whole ball of radius $\varrho(x)$ we get

$$I_1 \geq c d^{\beta-2+\sigma(2-\alpha)}(x) = c d^{\beta-\alpha_\sigma}(x),$$

where $c > 0$. Thus, joining the estimates for I_1 and I_2, we arrive at

$$\mathcal{I}(d^\beta, x) \geq d^{\beta-\alpha_\sigma}(x)(c - Cd(x)),$$

and we conclude the result by taking $\bar\delta$ small in terms of c and C. □

4. The Dirichlet problem with bounded boundary data.

In this section we provide comparison principles which are going to play a key role in the remaining sections of this paper. We start with the following:

Proposition 4.1. (Comparison Principle) *Let $\mathcal{I}$ as in* (2.1) *satisfying* **(M)** *and* **(D)***, and let $p > 0$. Let u, v be viscosity sub and supersolution for problem* (1.1) *respectively. If $u \leq v$ on $\partial\Omega$, then $u \leq v$ in Ω.*

The proof of this result follows the classical lines of the viscosity theory, for instance see [20] and references therein.

We need a second version of the comparison principle. For its statement, we introduce the following notation: Given $\mathcal{O} \subset \mathbb{R}^N$ a bounded domain and $\omega \subset \mathcal{O}$, we define

$$\Sigma(\omega, \mathcal{O}) = \bigcup_{x\in\omega} B_{\varrho(x)}(x), \tag{4.1}$$

where ϱ is defined as in **(D)** relative to $d_{\mathcal{O}}$. If $\mathcal{O} = \Omega$ we simply put $\Sigma(\omega)$ and if $\omega = \Omega \setminus A_a$ for some $a > 0$, we write Σ_a.

Proposition 4.2. (Annular Comparison Principle) *Let $\mathcal{I}$ defined as in* (2.1) *satisfying* **(M)** *and* **(D)***, and $a, p > 0$. Let $u, v \in L^1(\Sigma_a) \cap L^\infty_{loc}(\Sigma_a)$ be respective viscosity sub and supersolution to equation* (1.1) *in $\Omega \setminus \bar A_a$, with $u \leq v$ in $\bar A_a \cap \Sigma_a$. Then, $u \leq v$ in $\Omega \setminus \bar A_a$.*

Proof: We start considering the case $u, v \in L^\infty(\Omega)$. Arguing by contradiction, assume that

$$M := \sup_{x \in \Omega \setminus \bar{A}_a} \{u(x) - v(x)\} > 0. \tag{4.2}$$

By the assumptions on u and v we have that this supremum is achieved at some point $x_0 \in \Omega \setminus A_a$, and since $u \leq v$ in $\bar{A}_a \cap \Sigma_a$ we conclude that $x_0 \in \Omega \setminus \bar{A}_a$. For $\epsilon, \eta > 0$ we consider the function

$$(x, y) \mapsto \Phi(x, y) := u(x) - v(y) - \phi(x, y),$$

where $\phi(x, y) = |x - y|^2/\epsilon^2 + \eta|x - x_0|^2$.

Note that there exists $(\bar{x}, \bar{y}) \in (\Omega \setminus A_a) \times (\Omega \setminus A_a)$ depending on ϵ, η, maximum point of Φ in this set. Using the inequality $\Phi(\bar{x}, \bar{y}) \geq \Phi(x_0, x_0)$ and the boundedness of u and v in $\Omega \setminus A_a$, classical arguments in viscosity solution's theory drive us to the following facts

$$\bar{x}, \bar{y} \to x_0, \quad u(\bar{x}) \to u(x_0), \quad v(\bar{y}) \to v(x_0); \quad \text{as } \epsilon \to 0. \tag{4.3}$$

Without loss of generality, we may assume that $d(\bar{x}) \geq d(\bar{y})$. Since $d(x_0) > a$, by the first convergence in (4.3) we can assume $d(\bar{y}) - a \geq (d(x_0) - a)/2$ for all ϵ small.

Since for all ϵ small we have $\bar{x}, \bar{y} \in \Omega \setminus \bar{A}_a$, we can use the viscosity inequalities for u and v at $\bar{x}$ and $\bar{y}$, respectively. Hence, we denote

$$h = (d(x_0) - a)/2 > 0$$

and for all $0 < \delta \leq h$ we can write

$$-\mathcal{I}[B_\delta](\phi(\cdot, \bar{y}), \bar{x}) - \mathcal{I}[B_\delta^c](u, \bar{x}, D_x\phi(\bar{x}, \bar{y})) + u(\bar{x})^p \leq 0,$$

and

$$\mathcal{I}[B_\delta](\phi(\bar{x}, \cdot), \bar{x}) - \mathcal{I}[B_\delta^c](v, \bar{y}, -D_y\phi(\bar{x}, \bar{y})) + v(\bar{y})^p \geq 0.$$

We substract the above inequalities to obtain

$$-(\epsilon^{-2} + \eta)o_\delta(1) - I_1 - I_2 + u(\bar{x})^p - v(\bar{y})^p \leq 0, \tag{4.4}$$

where

$$I_1 = \int_{B_{\varrho(\bar{y})} \setminus B_\delta} [u(\bar{x} + z) - v(\bar{y} + z) - (u(\bar{x}) - v(\bar{y})) - 2\eta \mathbf{1}_B \langle \bar{x} - x_0, z\rangle] K^\alpha(z) dz,$$

$$I_2 = \int_{B_{\varrho(\bar{x})} \setminus B_{\varrho(\bar{y})}} [u(\bar{x} + z) - u(\bar{x})] K^\alpha(z) dz.$$

Now we deal with these nonlocal terms. For I_1, we consider the sets

$$D_{int} := \{z : \bar{x} + z, \bar{y} + z \in \Omega \setminus A_a\},$$
$$\Theta_1 := B_{\varrho(\bar{y})} \cap D_{int},$$
and
$$\Theta_2 := B_{\varrho(\bar{y})} \setminus D_{int}.$$

Note that $D_{int} \to ((\Omega \setminus A_a) - x_0)$ as $\epsilon \to 0$. Hence, we see that $B_h \subset \Theta_1$ and $\Theta_2 \subset B_h^c$ for all ϵ small, depending only on h. In particular, we have $\Theta_2 = \Theta_2 \setminus B_\delta$ for all such an ϵ.

Using that $(\bar{x}, \bar{y})$ is a maximum of Φ in $(\Omega \setminus A_a) \times (\Omega \setminus A_a)$, for all $z \in \Theta_1$ we have the inequality

$$u(\bar{x} + z) - v(\bar{y} + z) \leq u(\bar{x}) - v(\bar{y}) + \eta(|\bar{x} + z - x_0|^2 - |\bar{x} - x_0|^2).$$

On the other hand, for all ϵ small the function $K^\alpha \mathbf{1}_{\Theta_2}$ is uniformly bounded. We divide the region of integration of I_1 as follows:

$$B_{\varrho(\bar{y})} \setminus B_\delta = (\Theta_1 \setminus B_\delta) \cup \Theta_2.$$

Using the above facts together with the positivity of $u(\bar{x}) - v(\bar{y})$ and that $|\bar{x} - x_0| = o_\epsilon(1)$ for all ϵ small, we are able to write

$$I_1 \leq \eta \int_{\Theta_1 \setminus B_\delta} |z|^2 K^\alpha(z) dz + \int_{\Theta_2} [u(\bar{x} + z) - v(\bar{y} + z)] K^\alpha(z) dz + \eta o_\epsilon(1).$$

For the first integral term in the right-hand side, we use that $\alpha \in (0, 2)$ to write

$$\int_{\Theta_1 \setminus B_\delta} |z|^2 K^\alpha(z) dz \leq C\eta,$$

for some $C > 0$ depending only on the data.

For the second integral term, we use the boundedness of u and v, the upper semicontinuity of $u - v$, the first fact in (4.3) and Fatou's Lemma, together with the condition $u \leq v$ in $\bar{A}_a$ to get

$$\limsup_{\epsilon \to 0} \int_{\Theta_2} [u(\bar{x} + z) - v(\bar{y} + z)] K^\alpha(z) dz \leq 0.$$

Joining the last two estimates, we conclude that

$$I_1 \leq C\eta + o_\epsilon(1).$$

On the other hand, noting that the set $B_{\varrho(\bar{x})} \setminus B_{\varrho(\bar{y})}$ vanishes and remains uniformly away the origin as $\epsilon \to 0$, using the boundedness of u and v and applying Dominated Convergence Theorem we see that

$$|I_2| = o_\epsilon(1).$$

Thus, using the estimates for I_1, I_2 into (4.4) we conclude

$$-(\epsilon^{-2} + \eta) o_\delta(1) - O(\eta) - o_\epsilon(1) + u(\bar{x})^p - v(\bar{y})^p \leq 0,$$

from which, letting $\delta \to 0$, $\epsilon \to 0$ and $\eta \to 0$ we arrive at

$$u^p(x_0) - v^p(x_0) \leq 0,$$

which contradicts (4.2).

In the general case, for $R > 0$ and $x \in \Sigma_a$, we define the functions

$$u_R(x) = \max\{\min\{u(x), R\}, -R\},$$

and

$$v_R(x) = \max\{\min\{v(x), R\}, -R\}.$$

Since $u, v \in L^\infty_{loc}(\Sigma_a)$, we have $u = u_R, v = v_R$ in $\Omega \setminus A_{m(R)}$ with $m(R) \to 0$ as $R \to \infty$, and from this point, we consider R large in order to have $m(R) \leq a/4$. Since $u, v \in L^1(\Sigma_a)$, for all R large enough there exists a constant $c_R > 0$ with $c_R \to 0$ as $R \to \infty$, such that u_R, v_R satisfy

$$-\mathcal{I}(u_R, x) + u_R(x) \leq c_R \quad x \in \Omega \setminus A_a,$$

and

$$-\mathcal{I}(v_R, x) + v_R(x) \geq -c_R \quad x \in \Omega \setminus A_a.$$

From this point, we argue as in the previous case, fixing R large in order to have $2c_R \leq (u(x_0)^p - v(x_0)^p)/2$. This concludes the proof. □

We also require the following maximum principle, whose proof follows classical arguments

Proposition 4.3. *Let $\mathcal{I}$ defined in* (2.1) *satisfying* **(M)** *and* **(D)**. *Let $\omega \subset \Omega$ and assume a $u : \Sigma(\omega) \to \mathbb{R}$ bounded satisfies $-\mathcal{I}(u) \leq 0$ in ω. If there exists a constant C such that $u \leq C$ in $\Sigma(\omega) \setminus \omega$, then $u \leq C$ in ω.*

Remark 4.4. Assumptions **(M), (D)** can be relaxed in the above comparison results. In fact, the kernel K^α in **(M)** can be replaced by a positive, regular measure ν in $\mathbb{R}^N$ satisfying the *Lévy condition*

$$\int_{\mathbb{R}^N} \min\{1, |z|^2\} \nu(dz) < +\infty,$$

meanwhile, instead of **(D)** we can consider $\varrho \in C(\bar{\Omega})$ satisfying

$$\lambda_0 \, d^\sigma(x) \leq \varrho(x) \leq \Lambda_0 \, d(x),$$

for some $\sigma \geq 1$ and $0 < \lambda_0 \leq \Lambda_0$.

Lemma 4.5. *Let $\mathcal{I}$ as in* (2.1) *satisfying* **(M)** *and* **(D)**, *and $p > 0$. Then, if $g \in C^2(\bar{\Omega})$, there exist $\psi_1, \psi_2 \in C(\bar{\Omega})$, respective viscosity sub and supersolution to* (1.1) *with $\psi_1 = \psi_2 = g$ on $\partial\Omega$ and $\psi_1 \leq g \leq \psi_2$ in Ω.*

Proof: We look for the supersolution's case (the result for subsolutions is analogous). Consider $\beta \in (0, 1)$, and for $M > 0$ to be fixed later, we consider the function

$$v(x) = g(x) + M d^\beta(x).$$

For each $x \in \Omega$ with $d(x) \le \bar{\delta}$, using Lemma 3.2 we have

$$-\mathcal{I}(v, x) \ge M c_0 d(x)^{\beta-\alpha_\sigma} - C\|g\|_{C^2(\bar{\Omega})} d(x)^{2-\alpha_\sigma},$$

for some $C > 0$. Thus, there exist $\bar{c}_1 > 0$ and $0 < \delta_1 < \bar{\delta}$ such that

$$-\mathcal{I}(v, x) \ge M \bar{c}_1 d(x)^{\beta-\alpha_\sigma},$$

for all $d(x) \le \delta_1$.

Denote $C_g = \|g\|_{L^\infty(\bar{\Omega})}$ and consider the function

$$w(x) = C_g + M(\delta_1/2)^\beta + 1.$$

Note that for all $d(x) \le \delta_1/2$

$$v(x) \le C_g + M(\delta_1/2)^\beta,$$

concluding that for all $M > 0$ we have

$$v(x) < w(x), \quad \text{for all } d(x) \le \delta_1/2. \tag{4.5}$$

On the other hand, we can choose M large depending on δ_1, ζ, C_g and β in order to get

$$M(\delta_1^\beta - (\delta_1/2)^\beta) \ge 2C_g + 1,$$

which implies, for all $d(x) \ge \delta_1$ the inequality

$$v(x) \ge -C_g + M\delta_1^\beta > C_g + M(\delta_1/2)^\beta + 1,$$

and therefore

$$v(x) > w(x), \quad \text{for all } d(x) \ge \delta_1.$$

Now define $\psi = \min\{v, w\}$. Note $\psi \in C(\bar{\Omega})$, $\psi \ge g$ in Ω and $\psi = g$ on $\partial\Omega$. We claim this function ψ is a supersolution for (1.1). Let $x_0 \in \Omega$ and ϕ smooth such that x_0 is a minimum point of $\psi - \phi$ in Ω. Without loss of generality, we may assume $\psi(x_0) = \phi(x_0)$ and then we have $\psi(x) \ge \phi(x)$ for all $x \in \Omega$. We split the analysis in three cases:

Case 1: $d(x_0) \ge \delta_1/2$ and $\psi(x_0) = w(x_0)$. In this case, we note that for all $z \in \Omega$ we have

$$w(z) - \phi(z) \ge \psi(z) - \phi(z) \ge \psi(x_0) - \phi(x_0) = w(x_0) - \phi(x_0),$$

and therefore the function $w - \phi$ has a minimum at x_0 and in particular we see that $D\phi(x_0) = Dw(x_0) = 0$. Then, for each $\delta > 0$ we can write

$$-\mathcal{I}[B_\delta](\phi, x_0) \ge -\mathcal{I}[B_\delta](w, x_0) = 0,$$

and

$$-\mathcal{I}[B_\delta^c](\psi, x_0, D\phi(x_0)) \ge -\mathcal{I}[B_\delta^c](w, x_0) = 0,$$

concluding that

$$-\mathcal{I}[B_\delta](\phi, x_0) - \mathcal{I}[B_\delta^c](\psi, x_0, D\phi(x_0)) + \psi^p(x_0) \ge 0,$$

which is the desired viscosity inequality.

Case 2: $d(x_0) \geq \delta_1/2$ *and* $\psi(x_0) \neq w(x_0)$. In this case, by (4.5) we necessarily have $d(x_0) \leq \delta_1$ and $\phi(x_0) = v(x_0)$. Hence, using the same argument as in the previous case but with v playing the role of w, we have

$$\begin{aligned}&\mathcal{I}[B_\delta](\phi, x_0) - \mathcal{I}[B_\delta^c](\psi, x_0, D\phi(x_0)) + \psi^p(x_0)\\ &\geq \ -\mathcal{I}(v, x_0) + v^p(x_0)\\ &\geq \ M\bar{c}_1 d(x_0)^{\beta - \alpha_\sigma} + v(x_0)^p,\end{aligned}$$

and we take $0 < \beta < \min\{1, \alpha_\sigma\}$ to get a negative power in $d(x_0)$. Thus, by the boundedness of g, we conclude the supersolution's inequality by taking δ_1 smaller if it is necessary.

Case 3: $d(x_0) < \delta_1/2$. By an argument similar to the last one, we can write

$$\begin{aligned}&\mathcal{I}[B_\delta](\phi, x_0) - \mathcal{I}[B_\delta^c](\psi, x_0, D\phi(x_0)) + \psi^p(x_0)\\ &\geq M\bar{c}_1 d(x_0)^{\beta - \alpha_\sigma} + v(x_0)^p,\end{aligned}$$

and taking δ_1 smaller only on terms of C_g and p, we get the result. □

Theorem 4.6. *Let* $\mathcal{I}$ *be as in* (2.1) *satisfying* **(M)** *and* **(D)**, *let* $p > 0$ *and* $g \in C^2(\bar{\Omega})$. *Then, there exists a unique viscosity solution* $u \in C(\bar{\Omega})$ *for the equation* (1.1) *satisfying* $u = g$ *on* $\partial\Omega$.

This result follows by the application of Perron's method (see [19]), comparison principle given in Proposition 4.1 Lemma 4.5. It is also possible to show a well-posedness result for problem (1.1) with $u = g$ on $\partial\Omega$ in the case the boundary condition g is merely continuous. We refer to [20] for a proof of this kind of result.

5. Proof of Theorem 2.1: Existence.

Lemma 3.1 together with the influence of the reactive term in (1.1) allows us to contruct a blow-up supersolution to our problem.

Proposition 5.1. *Let* $\bar{\delta} > 0$ *as in Lemma 3.1. Under the assumptions of Theorem 2.1, there exist* $M, L > 0$ *such that the function*

$$\bar{w}(x) = Md^{-\gamma}(x) + L,$$

is a viscosity supersolution to the problem (1.1)–(1.2).

Proof: Note that for $x \in \Omega$ such that $d(x) < \bar{\delta}/2$, the lower bound in Lemma 3.1 we have

$$-\mathcal{I}(\bar{w}, x) + \bar{w}^p(x) \geq -C_0 M d^{-\gamma p}(x) + M^p d^{-\gamma p}(x),$$

and since $p > 1$, this last term is positive provided we take $M > 0$ large in terms of C_0. From this point, we fix such an M.

Now we deal with the supersolution's visosity inequality in $\Omega \setminus A_{\bar\delta/4}$ to conclude the proof. In the strictly censored case, for all $x \in \Omega \setminus A_{\bar\delta/4}$ the function

$$d^{-\gamma}(x+z) - d^{-\gamma}(x) - \mathbf{1}_B \langle Dd^{-\gamma}(x), z\rangle$$

is uniformly bounded (depending on $\bar\delta$) for each $z \in B_{\varrho(x)}$. This fact and the smoothness of d in $\Omega \setminus A_{\bar\delta/4}$ allows us to write for each x in this set

$$|\mathcal{I}(\bar w, x)| \le C(\bar\delta).$$

Thus, taking $L \ge C(\bar\delta)^{1/p}$, we have the result.

On the other hand, in the weakly censored case we can argue in the same way as in Lemma 3.1, writing the nonlocal operator as

$$\mathcal{I}(\bar w, x) = \mathcal{I}[B_{(1-\epsilon)d(x)}](\bar w, x) + \mathcal{I}[B_{d(x)} \setminus B_{(1-\epsilon)d(x)}](\bar w, x).$$

We use the above analysis for the first term. For the second term, we argue by integrability of $\bar w$ since $\gamma < 1$ in this case. □

We show next the existence of a subsolution to (1.1). As we will see, in this task the diffusive term plays a key role.

Proposition 5.2. *For $R, \mu > 0$ and $\beta \in (0,1)$, consider the function*

$$\underline{w}_R(x) = \mu(R - R^{\frac{\beta}{\gamma}+1} d^\beta(x))_+.$$

Then, there exists μ_0 small (depending only on the data) such that, for all $\mu \le \mu_0$ and all R large enough, $\underline{w}_R$ is a viscosity subsolution to equation (1.1).

Proof: Denoting $\delta_R = R^{-1/\gamma}$, we have $\underline{w}_R = 0$ in $\Omega \setminus A_{\delta_R}$. If $x \in \Omega \setminus \bar A_{\delta_R}$, we have $\underline{w}_R(x) = 0$ and since it is a minimum point of $\underline{w}_R$ in Ω, we clearly have $\mathcal{I}(\underline{w}_R, x) \ge 0$, concluding the subsolution's inequality for (1.1). Moreover, if $x \in \partial A_{\delta_R} \cap \Omega$, we see that there is no smooth function touching $\underline{w}_R$ from above at x, concluding at once that $\underline{w}_R$ is subsolution for (1.1) in $\Omega \setminus A_{\delta_R}$.

Now, consider R large enough in order to have $\delta_R \le \bar\delta/2$ with $\bar\delta$ given in Lemma 3.2. Thus, since

$$\underline{w}_R \ge \mu(R - R^{\frac{\beta}{\gamma}+1} d^\beta)$$

with equality when we evaluate at $x \in A_{\delta_R}$, for all such points we can write

$$\mathcal{I}(\underline{w}_R, x) \ge -\mu R^{\frac{\beta}{\gamma}+1} \mathcal{I}(d^\beta, x).$$

Then, applying Lemma 3.2 to the above inequality we arrive at

$$-\mathcal{I}(\underline{w}_R, x) \le -\mu R^{\frac{\beta}{\gamma}+1} c_0 \, d^{\beta - \alpha_\sigma}(x), \quad \text{for each } x \in A_{\delta_R},$$

and consequently, for x in this set we have

$$-\mathcal{I}(\underline{w}_R, x) + \underline{w}_R(x)^p \le -\mu c_0 R^{\frac{\gamma+\alpha_\sigma}{\gamma}} + \mu^p R^p.$$

Recalling the definition of γ in (2.2) we have $(\gamma + \alpha_\sigma)/\gamma = p$ and hence

$$-\mathcal{I}(\underline{w}_R, x) = \mu R^p(-c_0 + \mu^{p-1}).$$

Taking $\mu > 0$ small in terms of c_0 and p, we conclude the result. □

Proof of Theorem 2.1 - Existence: Take $R > 0$ and denote by u_R the unique solution for the problem (1.1) with $u_R = R$ on $\partial\Omega$, whose existence is guarranteed by Theorem 4.6. We note that, by the comparison principle given in Proposition 4.1, we have $0 \leq u_R \leq R$ on $\bar{\Omega}$ and the sequence $\{u_R\}_R$ is increasing in R.

Let $\bar{w}$ be as in Proposition 5.1 and for each $R > 0$ denote $\delta_R = (R/M)^{-1/\gamma}$. Note that for all $x \in \bar{A}_{\delta_R}$ we have $\bar{w} \geq R$. Applying the annular comparison principle (Proposition 4.2) in $\Omega \setminus \bar{A}_{\delta_R}$ we can write

$$u_R \leq \bar{w} \quad \text{in } \Omega, \quad \text{for all } R > 0. \tag{5.1}$$

In particular, this says that $\{u_R\}_R$ is bounded in $L^\infty_{loc}(\Omega)$. Thus, for each $\delta > 0$ we have the existence of a constant $C_\delta > 0$ not depending on R such that $u_R \leq C_\delta$ in $\Omega \setminus A_\delta$. Recalling the definition of Σ_δ given in (4.1), consider $\tilde{u}_R : \mathbb{R}^N \to \mathbb{R}$ the function defined as $\tilde{u}_R = u_R$ in Σ_δ and $\tilde{u}_R = 0$ in Σ_δ^c. Then, $\tilde{u}_R$ satisfies

$$-I(\tilde{u}_R, x) = -\tilde{u}_R^p(x) + \Theta_R(x) \quad x \in \Omega \setminus A_\delta,$$

where

$$I(\phi, x) := \int_{\mathbb{R}^N} [\phi(x+z) - \phi(x) - \mathbf{1}_B \langle D\phi(x), z\rangle] K^\alpha(z) dz, \tag{5.2}$$

and

$$\Theta_R(x) := \int_{B^c_{\varrho(x)}} [\tilde{u}_R(x+z) - \tilde{u}_R(x)] K^\alpha(z) dz.$$

But we see that for each $x \in \Omega \setminus A_\delta$ we have $d(x) \geq \delta$ and therefore the function $z \mapsto K^\alpha(z)\mathbf{1}_{B_{\varrho(x)}}(z)$ is bounded (depending on δ) for each $x \in \Omega \setminus A_\delta$. Then, for each $x \in \Omega \setminus A_\delta$ we see that

$$|\Theta_R(x)| \leq K_\delta\Big(1 + \int_{\Sigma_\delta \setminus B_\delta(x)} \bar{w}(z) dz\Big),$$

where $K_\delta > 0$ is a constant not depending on R. From this, we conclude that

$$|\Theta_R(x)| \leq \bar{K}_\delta,$$

for some $\bar{K}_\delta > 0$ depending on δ, but not on $R > 0$, since $\bar{w}$ is in $L^1(\Sigma_\delta)$ for all $\delta > 0$. In conclusion, we get the existence of a constant $\tilde{C}_\delta > 0$ not depending on R such that

$$|I(\tilde{u}_R)| \leq \tilde{C}_\delta \quad \text{in } \Omega \setminus A_\delta,$$

but assumption **(M)** says I defined in (5.2) is uniformly elliptic in the sense of Caffarelli and Silvestre (see [9, 10]). Hence, applying interior regularity results and recalling $\tilde{u}_R = u_R$ in Ω, we conclude $\{u_R\}$ is bounded in $C^{1,\sigma}_{loc}(\Omega)$, for some $\sigma > 0$. Thus, standard stability results for viscosity solutions allow us to take the limit as $R \to 0$ and conclude the existence of $\underline{u} \in C^{1,\sigma}_{loc}(\Omega)$, viscosity solution to problem (1.1). This solution is in fact classical, and by (5.1) we see that $\underline{u} \leq \bar{w}$ in Ω.

Now we show that $\underline{u}$ blows-up at the boundary. For this, by Proposition 5.2 and the comparison principle we have $u_R \geq \underline{w}_R$ in Ω. Then, since u_R is increasing in R, for all R large enough and $\beta \in (0,1)$ we see that

$$\underline{u}(x) \geq \underline{w}_R(x) = \mu(R - R^{\frac{\beta}{\gamma}+1} d^\beta(x))_+, \quad \text{for all } x \in \Omega. \tag{5.3}$$

Note that given $a > 0$ small enough, the function

$$\theta_a(R) = (R - R^{\frac{\beta}{\gamma}+1} a^\beta)_+$$

is strictly positive in the interval $(0, a^{-\gamma/\beta})$ and attains its maximum at

$$R(a) := ((\beta+\gamma)\gamma^{-1})^{-\frac{\gamma}{\beta}} a^{-\gamma} < a^{-\gamma/\beta},$$

for all $\beta \in (0,1)$ and all $a < a_0$ with a_0 not depending on β. Then, the maximum value of θ_a is

$$\theta_a(R(a)) = \frac{\beta}{\beta+\gamma}\Big(\frac{\beta+\gamma}{\gamma}\Big)^{-\gamma/\beta} a^{-\gamma}.$$

Thus, for each x with $d(x) \leq a_0$, inequality (5.3) drives us to

$$\underline{u}(x) \geq \mu \max\{\theta_{d(x)}(R) : R > 0\} = \frac{\mu\beta}{\beta+\gamma}\Big(\frac{\beta+\gamma}{\gamma}\Big)^{-\gamma/\beta} d^{-\gamma}(x),$$

and since we can maximize in $\beta \in (0,1)$ the expression in the right-hand side of the last inequality, we conclude the existence of $C_\gamma > 0$ such that

$$\underline{u}(x) \geq \mu C_\gamma d^{-\gamma}(x),$$

which implies the blow-up behavior of the solution and the lower bound in (2.5).

Finally, we note that if v is a large solutionn for (1.1), we can use Proposition 4.2 in the same way as we did to obtain (5.1) to get $u_R \leq v$ for all $R > 0$ large enough. Taking limit as $R \to \infty$, we conclude that $\underline{u} \leq v$, concluding $\underline{u}$ is minimal. The strict positivity of the solution can be easily obtained by contradiction, evaluating at a point $x_0 \in \Omega$ where the solution vanishes, which would coincide with a minimum of the function. □

6. Proof of Theorem 2.1: Uniqueness.

We will follow closely the ideas given in [11]. The strategy consists in considering any large solution v to (1.1) in the class (2.5) and the minimal large solution u given by the existence proof. Then, we define the open set

$$\mathcal{A} = \{x \in \Omega : v(x) > u(x)\}, \tag{6.1}$$

and we conclude by proving that this set is empty. Thus, our argument relies on obtaining a contradiction from the assumption $\mathcal{A} \neq \emptyset$.

Lemma 6.1. *Let $k > 1$ and denote $w_k = v - ku$. If*

$$\mathcal{A}_k := \{x \in \Omega : w_k > 0\} \neq \emptyset,$$

then

$$\partial\mathcal{A}_k \cap \partial\Omega \neq \emptyset.$$

Proof: If we assume $\mathcal{A}_k \subset\subset \Omega$, then w_k attains a strictly positive maximum in Ω at a point $\bar{x} \in \mathcal{A}_k$. Using this and the equation satisfied by u and v, we clearly have

$$0 \leq -\mathcal{I}(w_k, \bar{x}) = -\mathcal{I}(v, \bar{x}) + k\mathcal{I}(u, \bar{x}) = -v^p(\bar{x}) + ku^p(\bar{x}).$$

Since $\bar{x} \in \mathcal{A}_k$ we can write

$$-v^p(\bar{x}) + ku^p(\bar{x}) = -v^p(\bar{x}) + (ku(\bar{x}))^p - (k^p - k)u^p(\bar{x}) < -(k^p - k)u^p(\bar{x}),$$

from which we arrive at

$$0 \leq -k^p(1 - k^{1-p})u^p(\bar{x}).$$

Since $p, k > 1$ and u is strictly positive, we arrive to a contradiction. □

Lemma 6.2. *Consider $\tilde{V}$ the function*

$$\tilde{V}(x) = (1 - |x|^2)^3 \mathbf{1}_B.$$

Let $x_0 \in \Omega$, denote $r = \Lambda d(x_0)/2$ and consider V the function

$$V(x) = \tilde{V}((x - x_0)/r).$$

Then, there exists $\theta_0 > 0$ not depending on r nor on x_0 such that

$$-\mathcal{I}(V, x) \leq \theta_0 \ r^{-\alpha_\sigma}, \quad \textit{for all } x \in B_r(x_0).$$

Proof: Note that V is bounded and smooth. By definition, we can write

$$\begin{aligned}
& -\mathcal{I}(V, x) \\
& = \int_{B_\varrho(x)} [\tilde{V}((x - x_0)/r + z/r) - \tilde{V}((x - x_0)/r)]K^\alpha(z) \\
& = r^{-\alpha} \int_{B_{\varrho(x)/r}} [\tilde{V}((x - x_0)/r + y) - \tilde{V}((x - x_0)/r)]K(ry)|y|^{-(N+\alpha)}dy,
\end{aligned}$$

and since $\tilde{V}$ is smooth and K is uniformly bounded, we arrive at

$$-\mathcal{I}(V, x) \leq Cr^{-\alpha} \int_{B_{\varrho(x)/r}} |y|^{2-N-\alpha}dy,$$

for some C depending only on $||\tilde{V}||_{C^2}$ and K, but not on x_0, x or r. Now, since $d(x)$ is comparable to $d(x_0)$, we have that $\varrho(x)$ is comparable to $\varrho(x_0)$ and then, from the definition of r we get the existence of a constant $C > 0$ such that $\varrho(x)/r \leq Cr^{\sigma-1}$. Using this, after integration we conclude that

$$-\mathcal{I}(V, x) \leq \theta_0 \ r^{(\sigma-1)(2-\alpha)-\alpha} = \theta_0 \ r^{-\alpha_\sigma},$$

for some constant $\theta_0 > 0$ depending on the data. □

Lemma 6.3. *Let $k > 1$ and $w_k, \mathcal{A}_k$ as in Lemma 6.1. If $\mathcal{A}_k \neq \emptyset$, then*

$$\sup\{w_k : x \in \Omega\} = +\infty.$$

Proof: Assume

$$\sup\{w_k : x \in \Omega\} = \bar{M} < +\infty.$$

As in Lemma 6.1, this supremum cannot be achieved in Ω. Let $x_0 \in \mathcal{A}_k$ sufficiently close to the boundary and let $r = \Lambda d(x_0)/2$. Note that

$$-\mathcal{I}(w_k, x) \leq -k^p(1 - k^{1-p})u^p(x),$$

for each $x \in B_r(x_0) \cap \mathcal{A}_k$. Now, recalling that γ is given by (2.2), since we know that there exists a constant $c > 0$ such that $ud^\gamma \geq c$ in an Ω-neighborhood of $\partial\Omega$, we conclude that

$$-\mathcal{I}(w_k, x) \leq -c_k d^{-\gamma p}(x), \quad \text{for all } x \in B_r(x_0) \cap \mathcal{A}_k, \tag{6.2}$$

where $c_k = ck^p(1 - k^{1-p})$. On the other hand, let us consider the function

$$w(x) = 2\bar{M}V(0)^{-1}V(x), \quad x \in \Omega,$$

where V is the function defined in Lemma 6.2 associated to x_0 and r. Note that $w(x_0) = \max_\Omega\{w\} = 2\bar{M}$ and by Lemma 6.2 we have

$$-\mathcal{I}(w, x) \leq 2\bar{M}V(0)^{-1}\theta_0 r^{-\alpha_\sigma}, \quad \text{for all } x \in B_r(x_0).$$

From the fact that $0 < \alpha_\sigma < \gamma p$ and since $d(x)$ is comparable to r, taking x_0 closer to the boundary if necessary (and therefore r smaller), the last inequality together with (6.2) and the linearity of $\mathcal{I}$ allow us to write

$$-\mathcal{I}(w_k + w, x) \leq 0, \quad \text{in } B_r(x_0) \cap \mathcal{A}_k.$$

Recalling the definition of Σ in (4.1), denote

$$\tilde{\Omega} = \Sigma(B_r(x_0) \cap \mathcal{A}_k).$$

Let $x \in \tilde{\Omega} \setminus (B_r(x_0) \cap \mathcal{A}_k)$. If $x \in \mathcal{A}_k$, then $(w_k + w)(x) \leq \bar{M}$, meanwhile if $x \in (B_r(x_0) \cup \mathcal{A}_k)^c$, then $(w_k + w)(x) \leq 0$. Thus, by maximum principle given in Proposition 4.3, for all $x \in B_r(x_0) \cap \mathcal{A}_k$ we see that

$$(w_k + w)(x) \leq \max\{\bar{M}, \sup\{(w + w_k)(y) : y \in \tilde{\Omega} \cap B_r(x_0) \setminus \mathcal{A}_k\}\}.$$

At this point, we remark that the above maximum equals $\bar{M}$, otherwise, evaluating the left-hand side of the above expression at x_0 and using that $w_k \leq 0$ in $\mathcal{A}_k^c$ we can write

$$2\bar{M} = w(x_0) \leq \sup\{w(y) : y \in \tilde{\Omega} \cap B_r(x_0) \setminus \mathcal{A}_k\} - w_k(x_0) < 2\bar{M},$$

which is a contradiction. Hence, we have

$$w_k(x_0) + w(x_0) \leq \bar{M}, \tag{6.3}$$

which is again a contradiction. □

Now we are ready to provide the proof of the uniqueness part in Theorem 2.1.

Proof of Theorem 2.1 - Uniqueness: Assuming $\mathcal{A} \neq \emptyset$, there exists $k > 1$ such that $\mathcal{A}_k \neq \emptyset$. Then, by Lemma 6.3 we have the existence of $x_0 \in \mathcal{A}_k$ close to $\partial\Omega$ such that

$$w_k(x_0) = \max\{w_k(y) : y \in \Omega \setminus A_{d(x_0)}\} > 0.$$

Let $r = \Lambda d(x_0)/2$ and for $L > 0$ to be fixed, we consider the function

$$w(x) = L\, r^{-\gamma} V(x),$$

which, by Lemma 6.2, satisfies

$$-\mathcal{I}(w, x) \leq L\, \theta_0\, r^{-\gamma-\alpha_\sigma} \quad \text{in } B_r(x_0).$$

For $n \in \mathbb{N}$ define

$$M_n = \max\{w_k(y) : y \in \Omega \setminus A_{r/n}\}$$

and the function $\tilde{w}_n$ given by

$$\tilde{w}_n = (w_k + w)\mathbf{1}_{\{w_k \leq M_n\}} + M_n \mathbf{1}_{\{w_k > M_n\}}.$$

In what follows, we are always assuming $n \geq 4$ and therefore we have $B_r(x_0) \cap A_{r/n} = \emptyset$.

Claim: *Let $c_k = k^p(1 - k^{1-p})$ and consider $L = c\, c_k$ for some $c > 0$ small and independent of k or x_0. Consider $n_0 \geq 4$ in the strict censored case and $n_0 = (c\, c_k/4)^{1/(\gamma-1)}$ in the weakly censored case. Then, we have*

$$-\mathcal{I}(\tilde{w}_{n_0}, x) \leq 0 \quad \text{in } B_r(x_0) \cap \mathcal{A}_k. \tag{6.4}$$

We use this claim to conclude the uniqueness result and postpone its proof. Using the same argument as at the end of the proof of Lemma 6.3 (see the inequality (6.3)), we conclude

$$\tilde{w}_{n_0}(x_0) = w_k(x_0) + w(x_0) \leq M_{n_0}. \tag{6.5}$$

Let $x_1 \in \Omega \setminus A_{r/n_0}$ such that $w_k(x_1) = M_{n_0}$. By the asymptotic behavior of u given by (2.5), we can write

$$\bar{C}^{-1} n_0^{-\gamma} u(x_1) \leq r^{-\gamma},$$

and applying (6.5) we arrive at

$$L\bar{C}^{-1} n_0^{-\gamma} u(x_1) \leq L r^{-\gamma} = w(x_0) < w_k(x_0) + w(x_0) \leq M_{n_0} = w_k(x_1),$$

from which we obtain

$$(1 + c^*) k u(x_1) \leq v(x_1),$$

where $c^* = c^*(k) > 0$ not depending on x_0 and increasing with k by the above claim. Thus, we can repeat the argument above replacing x_0 by x_1

and k by $k_1 = k(1+c^*)$. Then, we proceed inductuively to construct a sequence (x_m) such that

$$(1+c^*)^m k u(x_m) \leq v(x_m),$$

which contradicts the fact that u and v belong to the same class (2.5).

Now we prove the claim. Note that

$$-\mathcal{I}(\tilde{w}_n, x) = -\mathcal{I}(w_k + w, x) - \mathcal{I}(\tilde{w}_n - (w_k + w), x) =: -I_1 - I_2, \tag{6.6}$$

and we should estimate each term in the right-hand side.

Concerning I_1, since $d(x)$ is comparable to r for each $x \in B_r(x_0)$, we proceed as in Lemma 6.3 to write

$$-I_1 \leq -Cc_k r^{-\gamma p} + L\theta_0 r^{-\gamma - \alpha_\sigma},$$

where $C > 0$ is an universal constant and $c_k = k^p(1 - k^{1-p})$. But we have $\gamma p = \gamma + \alpha_\sigma$ and therefore we get

$$-I_1 \leq (-Cc_k + L\theta_0) r^{-\gamma p}. \tag{6.7}$$

For I_2, we denote $\Theta_n = \{w_k > M_n\} - x$ and by definition of $\tilde{w}_n$ we have

$$0 \leq -I_2 = -\int_{B_{\varrho(x)} \cap \Theta_n} [M_n - (w_k + w)(x+z)] K^\alpha(z) dz. \tag{6.8}$$

Note that $\{w_k > M_n\} \subset A_{r/n}$. Then, we divide the analysis depending on the operator is strictly censored or not. If $\sigma = 1$ and $\Lambda < 1$, for each $x \in B_r(x_0)$ we have $d(x) \geq d(x_0)(1 - \Lambda/2)$ and then we see that

$$d(x+z) \geq d(x) - \varrho(x) = (1-\Lambda) d(x) \geq d(x_0)(1-\Lambda/2)(1-\Lambda),$$

for each $z \in B_{\varrho(x)}$. Thus, for all $n \geq \Lambda(1-\Lambda/2)(1-\Lambda)/2$ (note that this number is independent of x_0) we see that $B_{\varrho(x)} \cap \Theta = \emptyset$, concluding that $I_2 = 0$. If $\sigma > 1$, we have $d(x) \geq d(x_0)/2$ and then, for each $z \in B_{\varrho(x)}$ we can see that

$$d(x+z) \geq d(x_0)(1 - d^{\sigma-1}(x))/2 \geq d(x_0)/4,$$

if we consider x_0 closer to the boundary if necessary. Taking $n \geq 4$ we conclude $I_2 = 0$ as before.

If $\sigma = \Lambda = 1$, using that M_n and u are nonnegative, that $w \leq CLr^{-\gamma}$ for some $C > 0$ and that v is in the class (2.5), from (6.8) we obtain

$$-I_2 \leq C||K||_\infty \int_{B_{d(x)} \cap \Theta_n} \Big(\bar{C} d^{-\gamma}(x+z) + Lr^{-\gamma}\Big) |z|^{-(N+\alpha)} dz$$

where, $C > 0$ depends on the data and the constants in (2.5), and $\gamma < 1$. At this point, we use $d(x+z) \geq d(x) - |z|$ for all $z \in B_{d(x)}$. Recalling that

$d(x) \geq r$ for each $x \in B_r(x_0)$, and that $B_{d(x)} \cap \Theta_n \subset B_{d(x)} \setminus B_{d(x)-r/n}$ for $n \geq 4$, we arrive at

$$\begin{aligned} -I_2 &\leq C \int_{B_{d(x)} \setminus B_{d(x)-r/n}} \Big((d(x) - |z|)^{-\gamma} + Lr^{-\gamma} \Big) |z|^{-(N+\alpha)} dz \\ &\leq C \Big(d(x)^{-(\gamma+\alpha)} \int_{1-\frac{1}{n}}^{1} (1-s)^{-\gamma} s^{-(1+\alpha)} ds + Lr^{-\gamma} d(x)^{-\alpha} \int_{1-\frac{1}{n}}^{1} s^{-(1+\alpha)} ds \Big) \end{aligned}$$

and since $d(x) \geq r$, from the last expression we can write

$$-I_2 \leq C(n^{\gamma-1} + Ln^{-1}) r^{-(\gamma+\alpha)}$$

for some $C > 0$ not depending on x_0 or n. Using this last fact and (6.7) into (6.6) and recalling $\gamma p = \gamma + \alpha$ in the weakly censored case, we arrive at

$$-\mathcal{I}(\tilde{w}_n, x) \leq C(-c\, c_k + L + n^{\gamma-1} + Ln^{-1}) r^{-\gamma p},$$

for some $C, c > 0$ depending on the data, but not on x, x_0 or r. From this, we conclude the claim by taking $L = c\, c_k/4$; and $n \geq 4$ in the strict censored case and $n = (c\, c_k/4)^{1/(\gamma-1)}$ in the weakly censored case. □

7. Proof of Theorem 2.2.

We start with the following lemma.

Lemma 7.1. *Let $\mathcal{I}$ as in* (2.1) *satisfying* **(M)** *and denote*

$$\underline{K} = \liminf_{y \to 0} K(y) \quad \text{and} \quad \overline{K} = \limsup_{y \to 0} K(y).$$

Assume **(D)** *holds and let $\beta \in (0,1)$ in the weakly censored case (case* (i) *of Theorem 2.1) and $\beta > 0$ in the strict censored case (cases* (ii), (iii) *of Theorem 2.1). Then, there exists $\bar{\delta} \in (0, \delta_0)$ such that*

$$c\underline{K}(1 + o_{d(x)}(1)) \leq d^{\alpha_\sigma + \beta}(x) \mathcal{I}(d^{-\beta}, x) \leq c\bar{K}(1 + o_{d(x)}(1)), \quad x \in A_{\bar{\delta}},$$

where α_σ is defined in (2.3) *and $c > 0$ is a constant depending only on N, σ, Λ and α.*

Proof: We concentrate first in the strict censored case. For a set $A \subset \mathbb{R}^N$ we denote $\xi_A(x)$ the projection of x to A which satisfies $|x - \xi_A(x)| = d_A(x)$. In the case $A = \partial\Omega$, we simply write $\xi(x) = \xi_{\partial\Omega}(x)$.

From this point, we consider $x \in \Omega$ close to the boundary. After a traslation and rotation, we may assume $x = d(x) e_N$ with $e_N = (0', 1)$ and $\xi(x) = 0$ is the origin. Denote $\Omega_x = d(x)^{-1}\Omega$, $K_x(z) = K(d(x)z)$ and write

$$\begin{aligned} & d(x)^{\beta+\alpha}\, \mathcal{I}(d^{-\beta}, x) \\ &= d(x)^{\alpha} \int_{B_{\varrho(x)}} [d^{-\beta}(e_N + z/d(x)) - 1 + \beta Dd(x) \cdot (z/d(x))] K^{\alpha}(z) dz \\ &= \int_{B_{\Lambda d^{\sigma-1}(x)}} [d_{\Omega_x}^{-\beta}(e_N + y) - 1 + \beta y_N] K_x(y) |y|^{-(N+\alpha)} dy, \end{aligned}$$

where we have used that $Dd(x) = e_N$. Defining $\tilde{C}$ as

$$\tilde{C}(x) := \int_{B_{\Lambda d^{\sigma-1}(x)}} [d_{\Omega_x}^{-\beta}(e_N + y) - 1 + \beta y_N]|y|^{-(N+\alpha)} dy,$$

from the definition of $\underline{K}, \overline{K}$ we see that

$$(\underline{K} - o_{d(x)}(1))\tilde{C}(x) \leq d(x)^{\beta+\alpha}\mathcal{I}(d^{-\beta}, x) \leq (\overline{K} - o_{d(x)}(1))\tilde{C}(x),$$

and then our interest turns out to give an estimate for $\tilde{C}$. We claim the existence of constants $c, \bar{\delta} > 0$ such that

$$\tilde{C}(x) = c\ d^{-\alpha_\sigma+\alpha}(x)(1 + o_{d(x)}(1)), \quad \text{for } x \in A_{\bar{\delta}}, \tag{7.1}$$

from which we conclude the result.

We start with the estimates of $\tilde{C}$ when the boundary is flat. Denote $H_+ = \{(z', z_N) \in \mathbb{R}^N : z_N > 0\}$, assume there exists a radius $R > 0$ such that $B_R \cap \Omega \subset H_+$, $B_R \cap \partial\Omega \subset \partial H_+$, and that $\xi(y) = (y', 0)$ for all $y = (y', y_N) \in B_{d(x)}(x)$. In this case, we can write

$$\tilde{C}(x) = \int_{B_{\Lambda d^{\sigma-1}(x)}} [(1 + y_N)^{-\beta} - 1 + \beta y_N]|y|^{-(N+\alpha)} dy,$$

and since we are in the strict censored setting, we can perform a Taylor expansion in order to write

$$\begin{aligned}\tilde{C}(x) &= \int_0^1 (1-s) \int_{B_{\Lambda d^{\sigma-1}(x)}} (1 + s y_N)^{-(\beta+2)} y_N^2 |y|^{-(N+\alpha)} dy\ ds \\ &= d^{(\sigma-1)(2-\alpha)}(x)\ \Psi(d^{\sigma-1}(x)),\end{aligned}$$

where

$$\Psi(\tau) := \Lambda^{2-\alpha} \int_0^1 (1-s) \int_{B_1} (1 + s\Lambda\tau z_N)^{-(\beta+2)} z_N^2 |z|^{-(N+\alpha)} dz\ ds, \tag{7.2}$$

which is well defined and smooth in $(-\Lambda^{-1}, \Lambda^{-1})$ and does not depend on x. In case (ii) we have $\Lambda^{-1} > 1$ and therefore we evaluate directly Ψ at $\tau = 1$. Since $\sigma = 1$ we see that $\alpha_\sigma = \alpha$ and then we can write

$$\tilde{C}(x) = \Psi(1) = \Psi(1) d^{-\alpha_\sigma+\alpha}(x).$$

On the other hand, we remark that the function Ψ is bounded with bounded derivatives in $(-\Lambda^{-1}/2, \Lambda^{-1}/2)$. Hence, in case (iii) we can assume $d(x)^{\sigma-1} < \Lambda^{-1}/2$ and performing a Taylor expansion of Ψ at $\tau = 0$ we conclude that

$$\tilde{C}(x) = \Psi(0) d^{-\alpha_\sigma+\alpha}(x)(1 + o_{d(x)}(1)).$$

From these last two estimates we conclude (7.1) for the flat boundary, with $c = \Psi(1)$ in case (ii) and $c = \Psi(0)$ in case (iii).

For a general domain, by the smoothness of d_{Ω_x} (inherited from d), for each $y \in B_{\Lambda d^{\sigma-1}(x)}$ we can write

$$d_{\Omega_x}(e_N + y) = d_{\Omega_x}(e_N) + \langle Dd_{\Omega_x}(e_N), y\rangle + \frac{1}{2}\langle D^2 d_{\Omega_x}(\tilde{y})y, y\rangle,$$

for some $\tilde{y}$ on the line joining e_N and $e_N + y$, which lies inside $B_{\Lambda d^{\sigma-1}(x)}(e_N)$. We remark that $d_{\Omega_x}(e_N) = 1, Dd_{\Omega_x}(e_N) = e_N$, and since

$$d_{\Omega_x}(z) = d^{-1}(x)d(d(x)z) \quad \text{for all } z \in B_{\Lambda d^{\sigma-1}(x)}(e_N),$$

we conclude the existence of a constant $C > 0$ depending only on the smoothness of $\partial\Omega$, Λ and σ such that, for all x sufficiently close to the boundary

$$|d_{\Omega_x}^{-\beta}(e_N + y) - (1 + y_N)^{-\beta}| \leq Cd(x)|y|^2, \quad \text{for all } y \in B_{\Lambda d^{\sigma-1}(x)}.$$

From this, since $\alpha \in (0, 2)$ we see that

$$\tilde{C}(x) = d^{(\sigma-1)(2-\alpha)}(x)\ \Psi(d^{\sigma-1}(x)) + o_{d(x)}(1),$$

and using the estimates for Ψ given in the flat case, we arrive to (7.1).

Finally, for the case (i), we consider $\epsilon > 0$ and write

$$\mathcal{I}(d^{-\beta}, x) = \mathcal{I}[B_{(1-\epsilon)d(x)}](d^{-\beta}, x) + \mathcal{I}[B_{d(x)} \setminus B_{(1-\epsilon)d(x)}](d^{-\beta}, x)$$

Note that the first integral can be tackled as an operator in the case (ii). On the other hand, since $\beta \in (0, 1)$ we have

$$\mathcal{I}[B_{d(x)} \setminus B_{(1-\epsilon)d(x)}](d^{-\beta}, x) = o_\epsilon(1),$$

and we conclude the result performing the above analysis and let $\epsilon \to 0$, together with the Dominated Convergence Theorem. $\square$

Corollary 7.2. *Assume hypotheses of Lemma 7.1 hold, with K continuous at the origin. Then, there exists $C_0 > 0$ such that, for each $x \in \Omega$ near the boundary we have*

$$\mathcal{I}(d^{-\beta}, x) = C_0\ d^{-(\alpha_\sigma+\beta)}(x)(1 + o_{d(x)}(1)).$$

The constant C_0 depends only on $N, \sigma, \Lambda, \alpha$ and $K(0)$.

Proof of Theorem 2.2: Let $h > 0$ and denote

$$\Omega_h = \{x \in \mathbb{R}^N : dist(x, \partial\Omega) > -h\}.$$

Consider the problem

$$-\mathcal{I}_h(u) + u^p = 0 \quad \text{in } \Omega, \tag{7.3}$$

where $\mathcal{I}_h$ is the operator defined in (2.1) associated to d_{Ω_h}.

Let γ be as in (2.2) and $\bar{C}_0 = C_0^{1/(p-1)}$, with C_0 as in Corollary 7.2. Let $\epsilon > 0$ and for $C_\epsilon > 0$ a large constant to be fixed later, consider the functions

$$\bar{w}_{\epsilon,h}(x) = (\bar{C}_0 + \epsilon) d_{\Omega_h}^{-\gamma}(x) + C_\epsilon,$$

and

$$\underline{w}_{\epsilon,h}(x) = (\bar{C}_0 - \epsilon) d_{\Omega_h}^{-\gamma}(x) - C_\epsilon.$$

Note that by Corollary 7.2, we can take C_ϵ large enough in order to have $\underline{w}_{\epsilon,h}, \bar{w}_{\epsilon,h}$ are respective viscosity sub and supersolution to (7.3). Then, by Proposition 4.2, we can state a version for Perron's method in the same way as in [20] and conclude the existence of a (discontinuous) solution $u_{\epsilon,h}$ for the problem (7.3) in the sense that $u^*_{\epsilon,h}$ is a viscosity subsolution and $(u_{\epsilon,h})_*$ is a viscosity supersolution. In fact, $u_{\epsilon,h}$ satisfies, for all $\epsilon, h > 0$

$$\underline{w}_{\epsilon,h} \le (u_{\epsilon,h})_* \le u_{\epsilon,h} \le u^*_{\epsilon,h} \le \bar{w}_{\epsilon,h} \quad \text{in } \Omega_h \setminus \Omega. \tag{7.4}$$

Moreover, there exists a constant $\tilde{C} > 0$ large enough such that, for all $\epsilon > 0$ and each h small enough depending on ϵ, the functions

$$\tilde{w}_1(x) = 2\bar{C}_0 d_{\Omega_h}^{-\gamma}(x) + \tilde{C}, \qquad \tilde{w}_2(x) = \bar{C}_0 d_{\Omega_h}^{-\gamma}(x)/2 - \tilde{C},$$

are respective viscosity sub and supersolutions for the problem (7.3) and satisfy

$$\tilde{w}_2 \le \underline{w}_{\epsilon,h} \le u_{\epsilon,h} \le \bar{w}_{\epsilon,h} \le \tilde{w}_1 \quad \text{in } \Omega_h \setminus \Omega.$$

Thus, applying the annular comparison principle given in Proposition 4.2, the family of solutions $\{u_{\epsilon,h}\}$ is uniformly bounded in $L^\infty_{loc}(\Omega)$ for all ϵ and $h \le h_\epsilon$ with h_ϵ small in terms of ϵ. Then, half-relaxed limits method (see [7, 8]) implies that the functions $\underline{u}, \bar{u}$ defined as

$$\underline{u}(x) = \limsup_{h,\epsilon \to 0^+, y \to x} u^*_{\epsilon,h}(y); \quad \bar{u}(x) = \liminf_{h,\epsilon \to 0^+, y \to x} (u_{\epsilon,h})_*(y)$$

are sub and supersolutions to (1.1) in Ω respectively, which, by definition, satisfy $\bar{u} \le \underline{u}$. But from (7.4) we have

$$\lim_{x \to \partial\Omega} d^{-\gamma}(x)\bar{u}(x) = \lim_{x \to \partial\Omega} d^{-\gamma}(x)\underline{u}(x) = \bar{C}_0,$$

and therefore, for each $\beta \in (0,1)$, $\beta\underline{u} \le \bar{u}$ in A_δ for some $\delta = \delta(\beta) > 0$, and we clearly have

$$-\mathcal{I}(\beta\underline{u}) + \underline{u}^p \le -\mathcal{I}(\bar{u}) + \bar{u}^p \quad \text{in } \Omega \setminus A_\delta.$$

Thus, by the annular comparison principle, we arrive at $\beta\underline{u} \le \bar{u}$ in Ω and therefore, making $\beta \to 1$ we conclude that $u := \underline{u} = \bar{u}$ is a continuous viscosity solution to (1.1) (and classical, a fortiori) with the asymptotic behavior given by (2.6). □

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