# THE FIRST NONTRIVIAL EIGENVALUE FOR A SYSTEM OF $p-$LAPLACIANS WITH NEUMANN AND DIRICHLET BOUNDARY CONDITIONS

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ABSTRACT. We deal with the first eigenvalue for a system of two $p-$Laplacians with Dirichlet and Neumann boundary conditions. If $\Delta_p w = \operatorname{div}(|\nabla w|^{p-2} w)$ stands for the $p-$Laplacian and $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, we consider

$$\begin{cases} -\Delta_p u = \lambda \alpha |u|^{\alpha-2} u |v|^\beta & \text{in } \Omega, \\ -\Delta_q v = \lambda \beta |u|^\alpha |v|^{\beta-2} v & \text{in } \Omega, \end{cases}$$

with mixed boundary conditions

$$u = 0, \qquad |\nabla v|^{q-2} \frac{\partial v}{\partial \nu} = 0, \qquad \text{on } \partial\Omega.$$

We show that there is a first non trivial eigenvalue that can be characterized by the variational minimization problem

$$\lambda_{p,q}^{\alpha,\beta} = \min \left\{ \frac{\displaystyle\int_\Omega \frac{|\nabla u|^p}{p}\,dx + \int_\Omega \frac{|\nabla v|^q}{q}\,dx}{\displaystyle\int_\Omega |u|^\alpha |v|^\beta\,dx} : (u,v) \in \mathcal{A}_{p,q}^{\alpha,\beta} \right\},$$

where

$$\mathcal{A}_{p,q}^{\alpha,\beta} = \left\{ (u,v) \in W_0^{1,p}(\Omega) \times W^{1,q}(\Omega) \colon uv \not\equiv 0 \text{ and } \int_\Omega |u|^\alpha |v|^{\beta-2} v\,dx = 0 \right\}.$$

We also study the limit of $\lambda_{p,q}^{\alpha,\beta}$ as $p, q \to \infty$ assuming that $\frac{\alpha}{p} \to \Gamma \in (0,1)$, and $\frac{q}{p} \to Q \in (0,\infty)$ as $p, q \to \infty$. We find that this limit problem interpolates between the pure Dirichlet and Neumann cases for a single equation when we take $Q = 1$ and the limits $\Gamma \to 1$ and $\Gamma \to 0$.

*Dedicated to Juan Luis Vazquez, a great mathematician.*

## 1. Introduction

Let $\Omega$ be bounded domain in $\mathbb{R}^n$ with smooth boundary, $1 < p, q, < \infty$, and $0 < \alpha, \beta$ such that

$$\frac{\alpha}{p} + \frac{\beta}{q} = 1.$$

The aim of this work is to study the following eigenvalue problem

$$\begin{cases} -\Delta_p u = \lambda \alpha |u|^{\alpha-2} u |v|^\beta & \text{in } \Omega, \\ -\Delta_q v = \lambda \beta |u|^\alpha |v|^{\beta-2} v & \text{in } \Omega, \end{cases} \tag{1}$$

*Key words and phrases.* $p-$Laplacian, systems, eigenvalues, Neumann and Dirichlet boundary conditions.

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with mixed boundary conditions

$$u = 0, \qquad |\nabla v|^{q-2}\frac{\partial v}{\partial \nu} = 0, \qquad \text{on } \partial\Omega. \tag{2}$$

Here $\Delta_p w = \operatorname{div}(|\nabla w|^{p-2} w)$ is the usual $p-$Laplacian and $\frac{\partial}{\partial \nu}$ is the outer normal derivative.

Our first result is a variational characterization of the first non trivial eigenvalue of our problem.

**Theorem 1.1.** *Let $\beta > 1$. If $p \geq N$ or $q > N$, then the first non trivial eigenvalue is given by*

$$\lambda_{p,q}^{\alpha,\beta} := \inf \left\{ \frac{\displaystyle\int_\Omega \frac{|\nabla u|^p}{p}\,dx + \int_\Omega \frac{|\nabla v|^q}{q}\,dx}{\displaystyle\int_\Omega |u|^\alpha |v|^\beta\,dx} : (u,v) \in \mathcal{A}_{p,q}^{\alpha,\beta} \right\} \tag{3}$$

*where*

$$\mathcal{A}_{p,q}^{\alpha,\beta} := \left\{ (u,v) \in W_0^{1,p}(\Omega) \times W^{1,q}(\Omega) \colon uv \not\equiv 0 \text{ and } \int_\Omega |u|^\alpha |v|^{\beta-2} v\,dx = 0 \right\}.$$

Next we want to study the behaviour of this first non trivial eigenvalue for large values of $p$ and $q$. We look at the limit as $p, q \to \infty$ of $\lambda_{p,q}^{\alpha,\beta}$. To this end, we assume that

$$\frac{\alpha}{p} \to \Gamma \in (0,1) \qquad \text{and} \qquad \frac{q}{p} \to Q \in (0,\infty) \qquad \text{as } p, q \to \infty. \tag{A}$$

Observe that, since $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, we also get the following limit:

$$\frac{\beta}{q} \to 1 - \Gamma \quad \text{as } p, q \to \infty.$$

**Theorem 1.2.** *Under the assumption* (A)*, there exists a sequence $\{(p_n, q_n)\}_{n\in\mathbb{N}}$ with $p_n, q_n \to \infty$, such that*

$$u_n \to u_\infty, \quad v_n \to v_\infty \quad \text{uniformly in } \overline{\Omega} \text{ as } n \to \infty,$$

*where $(u_n, v_n)$ is an eigenfunction corresponding to $\lambda_1(p_n, q_n)$ normalized with $\int_\Omega |u_n|^\alpha |v_n|^\beta\,dx = 1$ for all $n \in \mathbb{N}$. Moreover,*

$$\left(\lambda_{p,q}^{\alpha,\beta}\right)^{1/p} \to \Lambda_\infty(\Gamma, Q) := \inf \left\{ \frac{\max\left\{ \|\nabla w\|_{L^\infty(\Omega)};\, \|\nabla z\|_{L^\infty(\Omega)}^Q \right\}}{\| |w|^\Gamma |z|^{(1-\Gamma)Q} \|_{L^\infty(\Omega)}} : (w,z) \in \mathcal{A}_\infty \right\}$$

*as $p, q \to \infty$. Here*

$$\mathcal{A}_\infty := \Big\{ (w,z) \in W_0^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \colon wz \not\equiv 0 \text{ and } \max_{x\in\Omega} |w|^\Gamma |z_+|^{(1-\Gamma)Q} = \max_{x\in\Omega} |w|^\Gamma |z_-|^{(1-\Gamma)Q} \Big\},$$

*where $z_+$ and $z_-$ stand for the positive and negative parts of $z$ respectively.*

*In addition, this limit $(u_\infty, v_\infty)$ is a solution to the minimization problem for $\Lambda_\infty(\Gamma, Q)$ and a viscosity solution to*

$$\begin{cases} \min\left\{ -\langle D^2 u \cdot Du, Du\rangle, |Du| - \Lambda_\infty(\Gamma,Q) u^\Gamma |v_\infty|^{(1-\Gamma)Q} \right\} = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

*and*

$$\begin{cases} \min\left\{-\langle D^2vDv, Dv\rangle, |Dv| - \Lambda_\infty(\Gamma,Q)^{1/Q} u_\infty^{\Gamma/Q}|v|^{1-\Gamma}\right\} = 0 & \text{in } \{v>0\},\\ \max\left\{-\langle D^2vDv, Dv\rangle, -|Dv| - \Lambda_\infty(\Gamma,Q)^{1/Q} u_\infty^{\Gamma/Q}|v|^{1-\Gamma}\right\} = 0 & \text{in } \{v<0\},\\ -\langle D^2vDv, Dv\rangle = 0 & \text{in } \{v=0\},\\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases}$$

In the case that $\Omega$ is a ball of radius $R$ (that is, $\Omega = B_R$), or when $\Omega$ is a rectangle (that is, $\Omega = (-R,R)\times(-L,L) \subset \mathbb{R}^2$, we assume here that $L \le R$), we can obtain an explicit value for this limit value, $\Lambda_\infty(\Gamma,Q)$.

**Theorem 1.3.**

(i) *When $\Omega$ is a ball of radius $R$ we have*

$$\Lambda_\infty(\Gamma,Q) = \left(\frac{\Gamma+Q(1-\Gamma)}{\Gamma R}\right)^{\Gamma}\left(\frac{\Gamma+Q(1-\Gamma)}{Q(1-\Gamma)R}\right)^{(1-\Gamma)Q}.$$

(ii) *When $\Omega$ is the rectangle $(-R,R)\times(-L,L)$ we get*

$$\Lambda_\infty(\Gamma,Q) = \begin{cases} \left(\dfrac{\Gamma+Q(1-\Gamma)}{\Gamma R}\right)^{\Gamma}\left(\dfrac{\Gamma+Q(1-\Gamma)}{Q(1-\Gamma)R}\right)^{(1-\Gamma)Q} & \text{if } \dfrac{\Gamma R}{Q(1-\Gamma)} \le L,\\ \dfrac{1}{(R-L)^{\Gamma}L^{1-\Gamma}}, & \text{if } \dfrac{\Gamma R}{Q(1-\Gamma)} > L. \end{cases}$$

Remark that the value $\Lambda_\infty(\Gamma,Q)$ for the ball coincides with the one for the rectangle (and does not depends on $L$) when $L$ is close to $R$; while for $L$ small the two values differ (and the latter depends on $L$ and goes to $\infty$ as $L \to 0$).

Note that for the ball, $\Omega = B_R(0)$, when $q = \alpha = p$ (hence $\beta = 0$) we have that $p\lambda_{p,p}^{p,0}$ (given by (3)) is the first eigenvalue for the Dirichlet $p-$Laplacian and for this eigenvalue, it is proved in [18] that $\left(p\lambda_{p,p}^{p,0}\right)^{1/p} \to 1/R$ as $p \to \infty$, one over the radius of the largest ball included in $\Omega$. This value corresponds to the value of $\Lambda_\infty(\Gamma,Q)$ computed in Theorem 1.3 since in this case $\Gamma = 1$ and $Q = 1$. Therefore, we can recover the well known result for a single equation with Dirichlet boundary conditions from our results. For the Neumann case we have to consider $q = \beta = p$ (and hence $\alpha = 0$). Now we have that $p\lambda_{p,p}^{0,p}$ is the first non trivial eigenvalue for the Neumann $p-$Laplacian and for this eigenvalue, it is proved in [13, 30] that $\left(p\lambda_{p,p}^{0,p}\right)^{1/p} \to 1/R$ as $p \to \infty$, that is 2 over the diameter of $\Omega$. In this case in Theorem 1.3 we have to take $\Gamma = 0$ and $Q = 1$ that gives again $\Lambda_\infty(0,1) = 1/R$. Hence, we recover again the known result for a single equation with Neumann boundary conditions. Remark that similar limits cases also hold for the case of the rectangle.

These limit behaviours hold in general. Note that if we take $Q = 1$ in the minimization problem for $\Lambda_\infty(\Gamma,Q)$ and then $\Gamma \to 1$ we get

$$\Lambda_\infty(\Gamma,1) \to \inf\left\{\frac{\max\left\{\|\nabla w\|_{L^\infty(\Omega)}; \|\nabla z\|_{L^\infty(\Omega)}\right\}}{\|w\|_{L^\infty(\Omega)}} : (w,z) \in \mathscr{B}\right\}.$$

where $\mathscr{B} := \{(w,z) \in W_0^{1,\infty}(\Omega)\times W^{1,\infty}(\Omega) \colon wz \neq 0\}$ This limit value coincides with the first eigenvalue for the Dirichlet problem for the scalar infinity Laplacian (just take $z \equiv 1$ and $w$ a first eigenfunction for the Dirichlet problem), see [18]. On the other hand when we let $\Gamma \to 0$ (keeping $Q = 1$) we obtain

$$\Lambda_\infty(\Gamma, 1) \to \inf \left\{ \frac{\max\left\{\|\nabla w\|_{L^\infty(\Omega)};\, \|\nabla z\|_{L^\infty(\Omega)}\right\}}{\|z\|_{L^\infty(\Omega)}} : (w, z) \in \mathcal{B} \right\}$$

where $\mathcal{B} := \{(w, z) \in W_0^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \colon wz \not\equiv 0 \text{ and } \max|z_+| = \max|z_-|\}$. Hence in this case we obtain the first nontrivial eigenvalue for the Neumann infinity Laplacian (in this case just take $w \equiv 0$ and $z$ a first non trivial eigenfunction for the Neumann problem), see [13, 30]. We conclude that our eigenvalue limit problem is somehow in between the Dirichlet and the Neumann cases.

Let us end the introduction giving some references and motivation for the analysis of this problem. Concerning the $p-$Laplacian and its properties we quote [5, 21, 23, 27, 31] and references therein. The limit of $p-$harmonic functions, that is, of solutions to $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$, as $p \to \infty$ has been extensively studied in the literature (see [3] and the survey [1]) and leads naturally to solutions of the infinity Laplacian, given by $-\Delta_\infty u = -\nabla u D^2 u (\nabla u)^t = 0$. Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) are related to the optimal Lipschitz extension problem (see the survey [1]) and find applications in optimal transportation, image processing and tug-of-war games (see, e.g., [8, 15, 28, 29] and the references therein). Also limits of the eigenvalue problem related to the $p$-Laplacian have been exhaustively examined (see [14, 18, 20]), and lead naturally to the infinity Laplacian eigenvalue problem $\min\{|\nabla u|(x) - \Lambda_\infty u(x),\ -\Delta_\infty u(x)\} = 0$. In fact, it is proved in [18, 20] that the limit as $p \to \infty$ exists both for the eigenfunctions, $u_p \to u_\infty$ uniformly, and for the eigenvalues $(\lambda_p)^{1/p} \to \Lambda_\infty = 1/R$, where the pair $u_\infty$, $\Lambda_\infty$ is a non trivial solution to the infinity Laplacian eigenvalue problem.

More recently, the limit problem for the fractional $p-$laplacian has been studied in [9, 11, 12, 22].

Eigenvalues for the $p-$Laplacian are related to the asymptotic behaviour of solutions to the corresponding evolutions equations, see, for example, [6, 16, 17].

Concerning eigenvalues for systems of $p-$Laplacian type there is a rich recent literature, we refer to [4, 7, 24, 26, 32] and references therein. The first case in which there is an study of the limit as $p \to \infty$ of eigenvalues for systems of $p-$Laplacians is [7] where both equations are subject to Dirichlet boundary conditions.

The paper is organized as follows: in Section 2 we collect some preliminary results; in Section 3 we deal with the first eigenvalue to our problem for fixed exponents (in this section we prove Theorem 1.1); in Section 4 we deal with the limit as $p, q \to \infty$ in a variational setting (showing the first part of Theorem 1.2); in Section 5 we compute explicitly the limit eigenvalue in the case of a ball and a rectangle (see Theorem 1.3); finally, in Section 6 we pass the the limit in the equations in the viscosity sense (finishing the proof of Theorem 1.2).

## 2. Preliminaries

We begin with some basic facts that will be needed in subsequent sections.

**Lemma 2.1.** *Let $\beta > 1$, $p \geq N$ and fix $u \in W_0^{1,p}(\Omega)$ such that $u \not\equiv 0$. Then*

$$\mathcal{A}_{p,q}^{\alpha,\beta}(u) := \left\{ w \in W^{1,q}(\Omega) \colon \int_\Omega |u|^\alpha |v|^{\beta-2} v \, dx = 0 \right\}$$

*is a closed set in* $W^{1,q}(\Omega)$.

*Proof.* Let $\{v_n\}_{n\in\mathbb{N}} \subset \mathcal{A}^{\alpha,\beta}_{p,q}(u)$ and $v \in W^{1,q}(\Omega)$ such that $v_n \to v$ strongly in $W^{1,q}(\Omega)$. Then, up to a subsequence, $|v_n|^{\beta-2}v_n \to |v|^{\beta-2}v$ strongly in $L^{\frac{q}{\beta-1}}(\Omega)$. Since $p \geq N$, by the Sobolev embedding theorem, we have that $|u|^\alpha \in L^{\frac{q}{q-\beta+1}}(\Omega)$. Therefore

$$0 = \lim_{n\to\infty} \int_\Omega |u|^\alpha |v_n|^{\beta-2} v_n \, dx = \int_\Omega |u|^\alpha |v|^{\beta-2} v \, dx$$

and hence $v \in \mathcal{A}^{\alpha,\beta}_{p,q}(u)$. □

**Lemma 2.2.** *Let* $\beta > 1$, $p \geq N$ *and* $u \in W^{1,p}_0(\Omega)$ *such that* $u \not\equiv 0$. *Then there is a positive constant* $C$ *such that*

$$\|v\|_{L^q(\Omega)} \leq C\|\nabla v\|_{L^q(\Omega)} \tag{4}$$

*for all* $v \in \mathcal{A}^{\alpha,\beta}_{p,q}(u)$.

*Proof.* We argue by contradiction. Suppose that for all $n \in \mathbb{N}$ there exists $v_n \in \mathcal{A}^{\alpha,\beta}_{p,q}(u)$ such that $\|v_n\|_{L^q(\Omega)} = 1$ and

$$\|\nabla v_n\|_{L^q(\Omega)} \leq \frac{1}{n}. \tag{5}$$

Then $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$. Thus, using the Sobolev embedding theorem, we have that there exist a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, and $v \in W^{1,q}(\Omega)$ such that

$$\begin{aligned} v_n &\rightharpoonup v \text{ weakly in } W^{1,q}(\Omega), \\ v_n &\to v \text{ strongly in } L^q(\Omega). \end{aligned}$$

Thus $\|v_n\|_{L^q(\Omega)} = 1$, and by (5), we get

$$\|\nabla v\|_{L^q(\Omega)} \leq \liminf_{n\to\infty} \|\nabla v\|_{L^q(\Omega)} \leq \lim_{n\to\infty} \frac{1}{n} = 0.$$

Then $\nabla v \equiv 0$ and hence $v$ is constant since $\Omega$ is connected. Moreover, since $v_n \rightharpoonup v$ weakly in $W^{1,q}(\Omega)$ and $\|v_n\|_{W^{1,q}(\Omega)} \to \|v\|_{W^{1,q}(\Omega)}$, we have that $v_n \to v$ strongly in $W^{1,q}(\Omega)$. By Lemma 2.1, we have that $v \in \mathcal{A}^{\alpha,\beta}_{p,q}(u)$. This is a contradiction because $v$ is a constant. □

Note that the best constant $C$ for the validity of (4) is

$$\frac{1}{C(u)} = \min\left\{ \frac{\|\nabla v\|_{L^q(\Omega)}}{\|v\|_{L^q(\Omega)}} : v \in \mathcal{A}^{\alpha,\beta}_{p,q}(u) \setminus \{0\} \right\}.$$

**Lemma 2.3.** *Let* $p \geq N$, $\beta > 1$, *and* $\{u_n\}_{n\in\mathbb{N}}$ *a bounded sequence in* $W^{1,p}_0(\Omega)$. *If*

$$\limsup_{n\to\infty} C(u_n) = \infty$$

*then, up to a subsequence,* $u_n \rightharpoonup 0$ *weakly in* $W^{1,p}(\Omega)$.

*Proof.* We first assume that $C(u_n) \to \infty$. For all $n \in \mathbb{N}$, there is $v_n \in \mathcal{A}^{\alpha,\beta}_{p,q}(u_n)$ such that $\|v_n\|_{L^q(\Omega)} = 1$ and

$$\frac{1}{C(u_n)} = \|\nabla v_n\|_{L^q(\Omega)}.$$

Then $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$. Therefore there exist a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$, and $v \in W^{1,q}(\Omega)$ such that

$$\begin{aligned} &v_{n_k} \rightharpoonup v \text{ weakly in } W^{1,q}(\Omega),\\ &v_{n_k} \to v \text{ strongly in } L^q(\Omega),\\ &|v_{n_k}|^{\beta-2}v_{n_k} \to |v|^{\beta-2}v \text{ strongly in } L^{\frac{q}{\beta-1}}(\Omega). \end{aligned} \tag{6}$$

Then $\|v\|_{L^q(\Omega)} = 1$ and

$$\|\nabla v\|_{L^q(\Omega)} \le \liminf_{k\to\infty} \|\nabla v_{n_k}\|_{L^q(\Omega)} = \lim_{k\to\infty} \frac{1}{C(u_{n_k})} = 0.$$

Therefore $v$ is a constant. Moreover, since $\|v\|_{L^q(\Omega)} = 1$, we have that $v = 1/|\Omega|^{1/q}$.

On the other hand, since $\{u_{n_k}\}_{k\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$ and $p \ge N$, there exist a subsequence, still denoted $\{u_{n_k}\}_{k\in\mathbb{N}}$, and $u \in W^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_{n_k} \rightharpoonup u \text{ weakly in } W^{1,p}(\Omega),\\ &|u_{n_k}|^{\alpha} \to |u|^{\alpha} \text{ strongly in } L^{\frac{q}{q-\beta+1}}(\Omega). \end{aligned} \tag{7}$$

Using (6) and (7), we get

$$0 = \lim_{k\to\infty} \int_\Omega |u_{n_k}|^\alpha |v_{n_k}|^{\beta-2} v_{n_k}\, dx = \int_\Omega |u|^\alpha |v|^{\beta-2} v\, dx = \frac{1}{|\Omega|^{\frac{\beta-1}{q\beta}}} \int_\Omega |u|^\alpha\, dx.$$

Therefore $u \equiv 0$. □

The proof os the next lemma is classical and therefore omitted in this paper.

**Lemma 2.4.** *If $q > N$ then there is a positive constant $C = C(q, N, \Omega)$ such that*

$$\|v\|_{L^q(\Omega)} \le C\|\nabla v\|_{L^p(\Omega)}$$

*for all $v \in \{w \in W^{1,p}(\Omega) \colon \exists x_0 \in \Omega \text{ with } w(x_0) = 0\}$.*

## 3. The first non trivial eigenvalue

A natural definition of an eigenvalue is a value $\lambda$ for which there is $(u, v) \in W_0^{1,p}(\Omega) \times W^{1,p}(\Omega) \setminus \{(0, 0)\}$ such that

$$\begin{aligned} &\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w\, dx = \lambda\alpha \int_\Omega |u|^{\alpha-2}u|v|^\beta w\, dx,\\ &\int_\Omega |\nabla v|^{q-2}\nabla v \nabla z\, dx = \lambda\beta \int_\Omega |u|^{\alpha}|v|^{\beta-2} v z\, dx, \end{aligned} \tag{8}$$

for all $(w, z) \in W_0^{1,p}(\Omega) \times W^{1,q}(\Omega)$; that is, $(u, v)$ is a nontrivial solution of (1)–(2). In this context, the pair $(u, v)$ is called an eigenfunction corresponding to $\lambda$.

Note that, if $\alpha > 1$ then $(u, v) \equiv (0, 1)$ is a solution of (1)–(2) for all $\lambda \in \mathbb{R}$, that is every $\lambda \in \mathbb{R}$ is an eigenvalue. We say that a value $\lambda$ is a non trivial eigenvalue if there is $(u, v) \in W_0^{1,p}(\Omega) \times W^{1,q}(\Omega)$ such that $uv \not\equiv 0$ in $\Omega$ and $(u, v)$ is an eigenfunction corresponding to $\lambda$.

*Remark* 3.1. If $(u, v) \in W_0^{1,p}(\Omega) \times W^{1,p}(\Omega)$ is a solution of (1)–(2) with $\lambda = 0$ then

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla w\, dx = \int_\Omega |\nabla v|^{q-2}\nabla v \nabla z\, dx = 0$$

for all $(w, z) \in W_0^{1,p}(\Omega) \times W^{1,p}(\Omega)$. Therefore $u \equiv 0$ and $v$ is constant, that is 0 is a simple eigenvalue.

If $\lambda$ is a non trivial eigenvalue then there is $(u,v) \in W_0^{1,p}(\Omega) \times W^{1,q}(\Omega)$ such that $uv \not\equiv 0$ in $\Omega$ and $(u,v)$ is a solution of (1)–(2). Then

$$\int_\Omega |\nabla u|^p \, dx = \lambda\alpha \int_\Omega |u|^\alpha |v|^\beta \, dx,$$
$$\int_\Omega |\nabla v|^q \, dx = \lambda\beta \int_\Omega |u|^\alpha |v|^\beta \, dx.$$

Therefore, using that $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, we have that

$$\lambda = \frac{\displaystyle\int_\Omega \frac{|\nabla u|^p}{p} \, dx + \int_\Omega \frac{|\nabla v|^q}{q} \, dx}{\displaystyle\int_\Omega |u|^\alpha |v|^\beta \, dx} \geq 0 \tag{9}$$

Moreover, by Remark 3.1, we have $\lambda > 0$.

On the other hand, taking $z \equiv 1$ in (8), we get

$$\int_\Omega |u|^\alpha |v|^{\beta-2} v \, dx = 0.$$

Thus, our candidate for first non trivial eigenvalue is

$$\lambda_{p,q}^{\alpha,\beta} := \inf \left\{ \frac{\displaystyle\int_\Omega \frac{|\nabla u|^p}{p} \, dx + \int_\Omega \frac{|\nabla v|^q}{q} \, dx}{\displaystyle\int_\Omega |u|^\alpha |v|^\beta \, dx} : (u,v) \in \mathcal{A}_{p,q}^{\alpha,\beta} \right\} \tag{10}$$

where

$$\mathcal{A}_{p,q}^{\alpha,\beta} := \left\{ (u,v) \in W_0^{1,p}(\Omega) \times W^{1,q}(\Omega) \colon uv \not\equiv 0 \text{ and } \int_\Omega |u|^\alpha |v|^{\beta-2} v \, dx = 0 \right\}.$$

3.1. **Scaling invariance of $\lambda_1$.** If we take $(u,v) \in \mathcal{A}_{p,q}^{\alpha,\beta}$ such that

$$\int_\Omega |u|^\alpha |v|^\beta \, dx = 1 \tag{11}$$

and we scale both functions according to

$$\tilde{u} = au \qquad \tilde{v} = bv$$

we get $\int_\Omega |\tilde{u}|^\alpha |\tilde{v}|^\beta \, dx = a^\alpha b^\beta$. Then, to still have (11) we impose $a^\alpha b^\beta = 1$. On the other hand, we have

$$\int_\Omega \frac{|\nabla \tilde{u}|^p}{p} \, dx + \int_\Omega \frac{|\nabla \tilde{v}|^q}{q} \, dx = a^p \int_\Omega \frac{|\nabla u|^p}{p} \, dx + b^q \int_\Omega \frac{|\nabla v|^q}{q} \, dx := a^p A + b^q B,$$

and then we want to compute

$$\min_{a^\alpha b^\beta = 1} a^p A + b^q B.$$

This leads to (using Lagrange's multipliers) $pa^{p-1}A = \theta\alpha a^{\alpha-1} b^\beta$ and $qb^{q-1}B = \theta\beta a^\alpha b^{\beta-1}$, with $a^\alpha b^\beta = 1$. That is, $pa^pA = \theta\alpha$ and $qb^qB = \theta\beta$ and we arrive to

$$\beta p a^p A = \alpha q b^q B.$$

This computation shows that in a minimizing sequence we can assume that the terms

$$\int_\Omega \frac{|\nabla u_n|^p}{p} \, dx \qquad \text{and} \qquad \int_\Omega \frac{|\nabla v_n|^q}{q} \, dx$$

are of the same order.

3.2. **Is $\lambda_{p,q}^{\alpha,\beta}$ a non trivial eigenvalue?** We start showing that $\lambda_{p,q}^{\alpha,\beta}$ is not a non trivial eigenvalue when $\alpha = 0$ or $\beta = 0$.

Observe that if $p = q = \beta$ and $\alpha = 0$ then $\lambda_{p,p}^{0,p} \geq {\lambda_p^{\mathbf{N}}}/{p}$ where $\lambda_p^{\mathbf{N}}$ is the first non trivial eigenvalue of the Neumann $p-$Laplacian that is

$$\lambda_p^{\mathbf{N}} = \min\left\{ \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx} : v \in W^{1,p}(\Omega) \setminus \{0\}, \int_\Omega |v|^{p-2} v\,dx = 0 \right\}.$$

Moreover, if $\phi \in C_0^1(\Omega)$ and $v$ is an eigenfunction corresponding to $\lambda_p^{\mathbf{N}}$ such that $\phi v \not\equiv 0$ then $(\varepsilon\phi, v) \in \mathcal{A}_{p,p}^{0,p}$ for all $\varepsilon > 0$. Then

$$\frac{\lambda_p^{\mathbf{N}}}{p} \leq \lambda_{p,p}^{0,p} \leq \varepsilon^p \frac{\displaystyle\int_\Omega |\nabla \phi|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx} + \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx} = \varepsilon^p \frac{\displaystyle\int_\Omega |\nabla \phi|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx} + \frac{\lambda_p^{\mathbf{N}}}{p} \quad \forall \varepsilon > 0.$$

Therefore, passing to the limit as $\varepsilon \to 0$ we have that $\lambda_{p,p}^{0,p} = {\lambda_p^{\mathbf{N}}}/{p}$

We claim that $\lambda_{p,p}^{0,p}$ is not a non trivial eigenvalue. Suppose, contrary to our claim, that $\lambda_{p,p}^{0,p}$ is a non trivial eigenvalue. Then there exists $(u, v) \in \mathcal{A}_{p,p}^{0,p}$ such that

$$\lambda_{p,p}^{0,p} = \frac{\displaystyle\int_\Omega |\nabla u|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx} + \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx} > \frac{\displaystyle\int_\Omega |\nabla v|^p\,dx}{\displaystyle\int_\Omega |v|^p\,dx}$$

since $u \neq 0$. Therefore ${\lambda_p^{\mathbf{N}}}/{p} = \lambda_{p,p}^{0,p} > {\lambda_p^{\mathbf{N}}}/{p}$, a contradiction that implies that $\lambda_{p,p}^{0,p}$ is not a non trivial eigenvalue.

Similarly, if $p = q = \alpha$ and $\beta = 0$ then $\lambda_{p,p}^{p,0} = {\lambda_p^{\mathbf{D}}}/{p}$ is not a non trivial eigenvalue. Here $\lambda_p^{\mathbf{D}}$ is the first eigenvalue of the Dirichlet $p-$Laplacian, that is,

$$\lambda_p^{\mathbf{D}} = \min\left\{ \frac{\displaystyle\int_\Omega |\nabla u|^p\,dx}{\displaystyle\int_\Omega |u|^p\,dx} : v \in W_0^{1,p}(\Omega) \setminus \{0\} \right\}.$$

Now we show that if $\beta > 1$ and $p \geq N$ or $q > N$ then $\lambda_{p,q}^{\alpha,\beta}$ is the first non trivial eigenvalue.

*Proof of Theorem 1.1.* By (9) and (10), we only need to prove that $\lambda_{p,q}^{\alpha,\beta}$ is a non trivial eigenvalue. Let $\{(u_n, v_n)\}_{n\in\mathbb{N}} \subset W_0^{1,p}(\Omega) \times W^{1,q}(\Omega)$ such that

$$\int_\Omega |u_n|^\alpha |v_n|^{\beta-2} v_n\,dx = 0, \tag{12}$$

$$\int_\Omega |u_n|^\alpha |v_n|^\beta\,dx = 1, \tag{13}$$

and

$$\lambda_{p,q}^{\alpha,\beta} = \lim_{n\to\infty} \int_\Omega \frac{|\nabla u_n|^p}{p}\,dx + \int_\Omega \frac{|\nabla v_n|^q}{q}\,dx. \tag{14}$$

Then, using the Poincare inequality, we have that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(\Omega)$.

We now split the rest of the proof into 2 cases.

*Case* $p \geq N$. By the Sobolev embedding theorem, there exist a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, and $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{aligned} &u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega),\\ &|u_n|^\alpha \to |u|^\alpha \text{ strongly in } L^{\frac{p}{\alpha}}(\Omega),\\ &|u_n|^\alpha \to |u|^\alpha \text{ strongly in } L^{\frac{q}{q-\beta+1}}(\Omega). \end{aligned} \tag{15}$$

On the other hand, by (14) and Lemma 2.2, we have that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$. Hence, by the Sobolev embedding theorem, there exist a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, and $v \in W^{1,q}(\Omega)$ such that

$$\begin{aligned} &v_n \rightharpoonup v \text{ weakly in } W^{1,q}(\Omega),\\ &|v_n|^\beta \to |v|^\beta \text{ strongly in } L^{\frac{q}{\beta}}(\Omega).\\ &|v_n|^{\beta-2}v_n \to |v|^{\beta-2}v \text{ strongly in } L^{\frac{q}{\beta-1}}(\Omega). \end{aligned} \tag{16}$$

By (14), (15), and (16), we have that

$$\lambda_{p,q}^{\alpha,\beta} \geq \int_\Omega \frac{|\nabla u|^p}{p}\,dx + \int_\Omega \frac{|\nabla v|^q}{q}\,dx. \tag{17}$$

On the other hand, by (12), (13), (15), and (16), we get

$$\int_\Omega |u|^\alpha |v|^{\beta-2} v\,dx = 0, \text{ and } \int_\Omega |u|^\alpha |v|^\beta\,dx = 1.$$

Then $(u, v) \in \mathcal{A}_{p,q}^{\alpha,\beta}$, and by (17) and (3) we have that

$$\lambda_{p,q}^{\alpha,\beta} = \int_\Omega \frac{|\nabla u|^p}{p}\,dx + \int_\Omega \frac{|\nabla v|^q}{q}\,dx,$$

that is $(u, v)$ is a minimizer of (3). Therefore $(u, v)$ is an eigenfunction corresponding to $\lambda_{p,q}^{\alpha,\beta}$.

*Case* $q > N$. By the Sobolev embedding theorem, there exist a subsequence, still denoted by $\{u_n\}_{n\in\mathbb{N}}$, and $u \in W_0^{1,p}(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u \text{ weakly in } W_0^{1,p}(\Omega),\\ |u_n|^\alpha \to |u|^\alpha \text{ strongly in } L^{\frac{p}{\alpha}}(\Omega). \end{gathered} \tag{18}$$

On the other hand, by (14) and Lemma 2.4, we have that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,q}(\Omega)$. Hence, by the Sobolev embedding theorem, there exist a subsequence, still denoted by $\{v_n\}_{n\in\mathbb{N}}$, and $v \in W^{1,q}(\Omega)$ such that

$$\begin{aligned} &v_n \rightharpoonup v \text{ weakly in } W^{1,q}(\Omega),\\ &v_n \to v \text{ strongly in } C(\overline{\Omega}). \end{aligned} \tag{19}$$

By (14), (18), and (19), we have that

$$\lambda_{p,q}^{\alpha,\beta} \geq \int_\Omega \frac{|\nabla u|^p}{p}\,dx + \int_\Omega \frac{|\nabla v|^q}{q}\,dx. \tag{20}$$

On the other hand, by (12), (13), (18), and (19), we get

$$\int_\Omega |u|^\alpha |v|^{\beta-2} v\,dx = 0 \qquad \text{and} \qquad \int_\Omega |u|^\alpha |v|^\beta\,dx = 1,$$

since $\beta > 1$. Then $(u,v) \in \mathcal{A}_{p,q}^{\alpha,\beta}$, and by (20) and (3) we have that

$$\lambda_{p,q}^{\alpha,\beta} = \int_\Omega \frac{|\nabla u|^p}{p}\,dx + \int_\Omega \frac{|\nabla v|^q}{q}\,dx,$$

which concludes the proof. □

*Remark* 3.2. Note that, if $(u,v)$ is a minimizer of (3) then so is $(|u|,v)$, that is if $(u,v)$ is a solution of (1)–(2) with $\lambda = \lambda_{p,q}^{\alpha,\beta}$ then we can assume that $u \geq 0$. Moreover, due to the results in [31] we get $u > 0$ in $\Omega$.

## 4. The limit as $p,q \to \infty$

From now on, to simplify the notation, we write $\lambda_{p,q}$ instead of $\lambda_{p,q}^{\alpha,\beta}$ and by $(u_{p,q}, v_{p,q})$ we denote an eigenfunction corresponding to $\lambda = \lambda_{p,q}^{\alpha,\beta}$ normalized with $\int_\Omega |u_{p,q}|^\alpha |v_{p,q}|^\beta\,dx = 1$.

Recall that we have assumed that

$$\frac{\alpha}{p} \to \Gamma \in (0,1), \quad \text{and} \quad \frac{q}{p} \to Q \in (0,\infty) \quad \text{as } p,q \to \infty.$$

In addition, since $\frac{\alpha}{p} + \frac{\beta}{q} = 1$, we get

$$\frac{\beta}{q} \to 1 - \Gamma \quad \text{as } p,q \to \infty.$$

Now, we deal with the limit as $p,q \to \infty$ in a variational setting (showing the first part of Theorem 1.2).

**Lemma 4.1.** *Under the assumption* (A)*, there exists a sequence* $\{(p_n,q_n)\}_{n\in\mathbb{N}}$ *such that* $p_n, q_n \to \infty$,

$$u_n \to u_\infty, \quad v_n \to v_\infty \quad \text{uniformly in } \overline{\Omega} \text{ as } n \to \infty,$$

*where* $(u_n,v_n)$ *is an eigenfunction corresponding to* $\lambda_1(p_n,q_n)$ *for all* $n \in \mathbb{N}$*. Moreover,*

$$\left(\lambda_{p,q}^{\alpha,\beta}\right)^{1/p} \to \Lambda_\infty(\Gamma,Q) \coloneqq \inf\left\{ \frac{\max\left\{\|\nabla w\|_{L^\infty(\Omega)};\ \|\nabla z\|_{L^\infty(\Omega)}^Q\right\}}{\||w|^\Gamma |z|^{(1-\Gamma)Q}\|_{L^\infty(\Omega)}} : (w,z) \in \mathcal{A}_\infty \right\}$$

*as* $p,q \to \infty$ *and* $(u_\infty,v_\infty)$ *is a minimizer of* $\Lambda(\Gamma,Q)$*.*

*Proof.* We first look for a uniform bound for $(\lambda_{p,q})^{1/p}$. To this end, let us consider a non-negative Lipschitz function $w \in W^{1,\infty}(\Omega)$ that vanishes on $\partial\Omega$.

Once this functions is fixed we choose $z \in W^{1,\infty}(\Omega)$ a Lipschitz function and after that we choose $K = K(p,q)$ such that

$$\int_\Omega |w|^\alpha |(z-K)|^{\beta-2}(z-K)\,dx = 0.$$

Note that $K(p,q)$ is bounded, in fact, we have $\inf\{z(x)\colon x \in \Omega\} \leq K(p,q) \leq \sup\{z(x)\colon x \in \Omega\}$. We normalize according to

$$\int_\Omega |w|^\alpha |(z-K)|^\beta\,dx = 1.$$

Hence, using the pair $(w, z-K)$ as test in (13) we get

$$\lambda_{p,q} \leq \int_\Omega \frac{|\nabla w|^p}{p}\,dx + \int_\Omega \frac{|\nabla z|^q}{q}\,dx.$$

Therefore

$$\limsup_{p,q\to\infty}(\lambda_{p,q})^{1/p} \le \limsup_{p\to\infty}\left\{\frac{1}{p}\|\nabla z\|^p_{L^p(\Omega)} + \frac{1}{q}\|\nabla w\|^q_{L^q(\Omega)}\right\}^{1/p} \tag{21}$$
$$= \max\left\{\|\nabla w\|_{L^\infty(\Omega)}; \|\nabla z\|^Q_{L^\infty(\Omega)}\right\} \le C.$$

Therefore, there is a constant, $C$, independent of $p$ and $q$ such that, for $p$ and $q$ large,

$$(\lambda_{p,q})^{1/p} \le C.$$

Let $(u_{p,q}, v_{p,q})$ be a minimizer for $\lambda_{p,q}$ normalized by $\int_\Omega |u_{p,q}|^\alpha |v_{p,q}|^\beta \, dx = 1$. Then, we have that

$$\frac{1}{p}\int_\Omega |\nabla u_{p,q}|^p + \frac{1}{q}\int_\Omega |\nabla v_{p,q}|^q = \lambda_{p,q},$$

from which we deduce using (21) that

$$\limsup_{p,q\to\infty}\|\nabla u_{p,q}\|_{L^p(\Omega)} \le \limsup_{p,q\to\infty}(p\lambda_{p,q})^{1/p} = \limsup_{p,q\to\infty}(\lambda_{p,q})^{1/p} \le C,$$

$$\limsup_{p,q\to\infty}\|\nabla v_{p,q}\|_{L^q(\Omega)} \le \limsup_{p,q\to\infty}(q\lambda_{p,q})^{1/q} = \limsup_{p,q\to\infty}\left[(\lambda_{p,q})^{1/p}\right]^{p/q} \tag{22}$$
$$= \left[\limsup_{p,q\to\infty}(\lambda_{p,q})^{1/p}\right]^{1/Q} \le C.$$

Now, we argue as follows: We fix $r \in (N, \infty)$. Using Holder's inequality, we obtain for $p, q > r$ large enough that

$$\left(\int_\Omega |\nabla u_{p,q}|^r\right)^{1/r} \le \left(\int_\Omega |\nabla u_{p,q}|^p\right)^{1/p} |\Omega|^{\frac{1}{r}-\frac{1}{p}} \le C. \tag{23}$$

Analogously, we have

$$\left(\int_\Omega |\nabla v_{p,q}|^r\right)^{1/r} \le \left(\int_\Omega |\nabla v_{p,q}|^q\right)^{1/q} |\Omega|^{\frac{1}{r}-\frac{1}{q}} \le C.$$

Hence, extracting a subsequence $\{(p_n, q_n)\}_{n\in\mathbb{N}}$ $p_n, q_n \to \infty$ if necessary, we have that

$$u_n = u_{p_n,q_n} \rightharpoonup u_\infty \quad \text{and} \quad v_n = v_{p_n,q_n} \rightharpoonup v_\infty$$

weakly in $W^{1,r}(\Omega)$ for any $N < r < \infty$ and uniformly in $\overline{\Omega}$.

From (22) and (23), we obtain that this weak limit verifies

$$\left(\int_\Omega |\nabla u_\infty|^r\right)^{1/r} \le |\Omega|^{1/r} \limsup_{p,q\to\infty}(\lambda_{p,q})^{1/p}.$$

As we can assume that the above inequality holds for every $r > N$ (using a diagonal argument), we get that $u_\infty \in W_0^{1,\infty}(\Omega)$ and moreover, taking the limit as $r \to \infty$, we obtain

$$|\nabla u_\infty(x)| \le \liminf_{p,q\to\infty}(p\lambda_{p,q})^{1/p} = \liminf_{p,q\to\infty}(\lambda_{p,q})^{1/p}, \qquad \text{a.e. } x \in \Omega.$$

Analogously, we obtain that the function $v_\infty$ verifies that $v_\infty \in W^{1,\infty}(\Omega)$ and

$$\begin{aligned}|\nabla v_\infty(x)| &\le \liminf_{p,q\to\infty} (q\lambda_{p,q})^{1/q} = \liminf_{p,q\to\infty} (\lambda_{p,q})^{1/q} = \liminf_{p,q\to\infty} (\lambda_{p,q})^{1/q} \\ &= \liminf_{p,q\to\infty} \left[(\lambda_{p,q})^{1/p}\right]^{p/q} = \left[\liminf_{p,q\to\infty} (\lambda_{p,q})^{1/p}\right]^{1/Q} \qquad \text{a.e. } x \in \Omega,\end{aligned}$$

Then

$$|\nabla v_\infty(x)|^Q \le \liminf_{p,q\to\infty} (\lambda_{p,q})^{1/p} \qquad \text{a.e. } x \in \Omega,$$

From the uniform convergence and the normalization condition, we obtain that

$$\| |u_\infty|^\Gamma |v_\infty|^{(1-\Gamma)Q} \|_{L^\infty(\Omega)} = 1.$$

and from

$$\int_\Omega |u_{p,q}|^\alpha |v_{p,q}|^{\beta-2} v_{p,q}\, dx = 0.$$

we get

$$\max_{x\in\Omega} |u_\infty(x)|^\Gamma |(v_\infty(x))_+|^{(1-\Gamma)Q} = \max_{x\in\Omega} |u_\infty(x)|^\Gamma |(v_\infty(x))_-|^{(1-\Gamma)Q}.$$

Therefore, $(u_\infty, v_\infty) \in \mathcal{A}_\infty$ and we get

$$\Lambda_\infty(\Gamma, Q) \le \frac{\max\left\{\|\nabla u_\infty\|_{L^\infty(\Omega)};\ \|\nabla v_\infty\|^Q_{L^\infty(\Omega)}\right\}}{\| |u_\infty|^\Gamma |v_\infty|^{(1-\Gamma)Q} \|_{L^\infty(\Omega)}} \le \liminf_{p,q\to\infty} (\lambda_{p,q})^{1/p}. \tag{24}$$

Now, we note that since $K(p,q)$ is bounded, there is a sequence $\{(p_n, q_n)\}$ such that

$$p_n, q_n \to \infty \qquad \text{and} \qquad K(p_n, q_n) \to k$$

as $n \to \infty$. From (21), we get

$$\limsup_{p,q\to\infty} (\lambda_{p,q})^{1/p} \le \frac{\max\left\{\|\nabla w\|_{L^\infty(\Omega)};\ \|\nabla(z-k)\|^Q_{L^\infty(\Omega)}\right\}}{\| |w|^\Gamma |(z-k)|^{(1-\Gamma)Q} \|_{L^\infty(\Omega)}} \tag{25}$$

for every pair $(w, z-k)$ with

$$\max_{x\in\Omega} |w(x)|^\Gamma |(z(x)-k)_+|^{(1-\Gamma)Q} = \max_{x\in\Omega} |w(x)|^\Gamma |(z(x)-k)_-|^{(1-\Gamma)Q}.$$

Thus

$$\limsup_{p,q\to\infty} (\lambda_{p,q})^{1/p} \le \Lambda_\infty(\Gamma, Q).$$

Therefore, by (24) and (25) we get

$$(\lambda_{p,q})^{1/p} \to \Lambda_\infty(\Gamma, Q)$$

as $p, q \to \infty$, and $(u_\infty, v_\infty)$ is a minimizer of $\Lambda_\infty(\Gamma, Q)$. $\square$

## 5. The value of $\Lambda_\infty$ in a ball and in a rectangle.

5.1. **The case of a ball.** Now our aim is to compute the limit value $\Lambda_\infty$ in the ball of radius $R$, that we denote as $B_R$.

By symmetry reasons we have to choose $x_0 = (a, 0, \dots, 0)$ with $0 < a < R$, the point where

$$\| |u_\infty|^\Gamma |v_\infty|^{(1-\Gamma)Q} \|_{L^\infty(B_R)} = |u_\infty|^\Gamma |v_\infty|^{(1-\Gamma)Q}(x_0) = 1.$$

Note that we can choose $v_\infty$ to be symmetric (odd in the $x_1$-direction), that is, $v_\infty(x_1, x_2, \dots, x_N) = -v_\infty(-x_1, x_2, \dots, x_N)$.

Now we are lead to compute:

$$\max\left\{ \|\nabla u_\infty\|_{L^\infty(B_R)};\ \|\nabla v_\infty\|^Q_{L^\infty(B_R)} \right\}.$$

Observe that the best choice that we can make is to take $u_\infty$ as the cone

$$u_\infty(x) = k_1(R - |x|).$$

Then we have

$$\|\nabla u_\infty\|_{L^\infty(B_R)} = k_1 \qquad \text{and} \qquad u_\infty(x_0) = k_1(R - a).$$

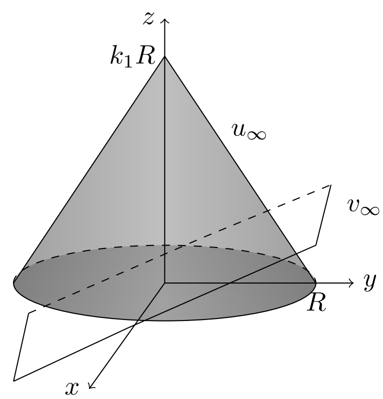
Concerning $v_\infty$ we can choose a plane

$$v_\infty(x) = k_2\langle x, e_1\rangle.$$

Then we have

$$\|\nabla v_\infty\|_{L^\infty(B_R)} = k_2 \qquad \text{and} \qquad v_\infty(x_0) = k_2 a.$$

These functions $u_\infty$ and $v_\infty$ are depicted in the following figure.




Now we have to compute

$$\min_{k_1,k_2,a} \max\left\{ k_1;\ k_2^Q \right\}$$

with the restriction

$$\begin{aligned}\max_{x\in B_R} |u_\infty|^\Gamma |v_\infty|^{(1-\Gamma)Q} &= \max_{0\le s\le R} (k_1(R-s))^\Gamma (k_2 s)^{(1-\Gamma)Q} \\ &= k_1^\Gamma k_2^{(1-\Gamma)Q} (R-a)^\Gamma a^{(1-\Gamma)Q} = 1.\end{aligned}$$

Then we have to compute

$$\max_{0\le s\le R} (R-s)^\Gamma s^{(1-\Gamma)Q}.$$

We have that this maximum is attained at a point $a$ that satisfies

$$\Gamma a = Q(1-\Gamma)(R-a),$$

hence, $a$ is given by

$$a = \frac{Q(1-\Gamma)R}{\Gamma + Q(1-\Gamma)}.$$

Therefore, the restriction is given by

$$k_1^\Gamma k_2^{(1-\Gamma)Q} \left( \frac{\Gamma R}{\Gamma + Q(1-\Gamma)} \right)^\Gamma \left( \frac{Q(1-\Gamma)R}{\Gamma + Q(1-\Gamma)} \right)^{(1-\Gamma)Q} = 1.$$

This gives

$$k_1 = \Theta k_2^{\frac{\Gamma-1}{\Gamma}Q}$$

with

$$\Theta = \frac{\Gamma + Q(1-\Gamma)}{\Gamma R} \left( \frac{\Gamma + Q(1-\Gamma)}{Q(1-\Gamma)R} \right)^{\frac{(1-\Gamma)Q}{\Gamma}}.$$

Finally we arrive to

$$\min_{k_2} \max \left\{ \Theta k_2^{\frac{\Gamma-1}{\Gamma}Q}; \, k_2^Q \right\}.$$

We must have

$$\Theta k_2^{\frac{\Gamma-1}{\Gamma}Q} = k_2^Q$$

and hence

$$k_2 = \Theta^{\frac{\Gamma}{Q}}.$$

We conclude that the optimal value for $\Lambda_\infty$ is given by

$$\Lambda_\infty(\Gamma, Q) = \Theta^\Gamma = \left( \frac{\Gamma + Q(1-\Gamma)}{\Gamma R} \right)^\Gamma \left( \frac{\Gamma + Q(1-\Gamma)}{Q(1-\Gamma)R} \right)^{(1-\Gamma)Q}.$$

5.2. **The case of a rectangle.** Now we want to compute $\Lambda_\infty(\Gamma, Q)$ when $\Omega$ is the rectangle $(-R, R) \times (-L, L) \subset \mathbb{R}^2$. Without loss of generality, we assume that $L \le R$.

Here, as for the case of the ball, we rely on symmetry. We look for a point $x_0 = (a, 0)$ with $L \le a < R$, where

$$\| |u_\infty|^\Gamma |v_\infty|^{(1-\Gamma)Q} \|_{L^\infty(\Omega)} = |u_\infty|^\Gamma |v_\infty|^{(1-\Gamma)Q}(x_0) = 1.$$

Note that we can choose $v_\infty$ to be symmetric (odd in the $x$-direction), that is, $v_\infty(x, y) = -v_\infty(-x, y)$.

Observe that the best choice that we can make is to take $u_\infty$ as the cone

$$u_\infty(x) = k_1(\rho - |(x, y) - (a, 0)|)_+,$$

with $\rho = R - a \le \min\{L, R - L\}$. Then we have

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} = k_1 \qquad \text{and} \qquad u_\infty(x_0) = k_1 \rho.$$

Concerning $v_\infty$, as before, we can choose a plane

$$v_\infty(x) = k_2 x.$$

Then we have

$$\|\nabla v_\infty\|_{L^\infty(B_R)} = k_2 \qquad \text{and} \qquad v_\infty(x_0) = k_2 a.$$

Now we have to compute

$$\min_{k_1,k_2,a} \max \left\{ k_1;\, k_2^Q \right\}$$

with the restriction

$$\begin{aligned}\max_{x\in\Omega} |u_\infty|^\Gamma |v_\infty|^{(1-\Gamma)Q} &= \max_{a\le s\le R} (k_1(R-s))^\Gamma (k_2 s)^{(1-\Gamma)Q} \\ &= k_1^\Gamma k_2^{(1-\Gamma)Q} (R-a)^\Gamma a^{(1-\Gamma)Q} = 1.\end{aligned}$$

Then we have to compute

$$\max_{a\le s\le R} (R-s)^\Gamma s^{(1-\Gamma)Q}. \tag{26}$$

When $\rho < L$, this maximum is attained at a point $a$ that is given by

$$\Gamma a = Q(1-\Gamma)(\rho + a - a),$$

that is,

$$a = Q\frac{(1-\Gamma)}{\Gamma}\rho.$$

Hence, with similar computations as the ones that we did for the ball we obtain that

$$\Lambda_\infty(\Gamma,Q) = \left(\frac{\Gamma + Q(1-\Gamma)}{\Gamma R}\right)^\Gamma \left(\frac{\Gamma + Q(1-\Gamma)}{Q(1-\Gamma)R}\right)^{(1-\Gamma)Q}, \qquad \text{if } \frac{\Gamma R}{Q(1-\Gamma)} \le L.$$

Observe that, in this case, $\Lambda_\infty(\Gamma,Q)$ coincides with the eigenvalue that we found in the case of the ball.

When $\rho = L$, (26) is attained at a point $a$ that is given by $a = R - L$ then

$$\Lambda_\infty(\Gamma,Q) = \frac{1}{(R-L)^\Gamma L^{1-\Gamma}}, \qquad \text{if } \frac{\Gamma R}{Q(1-\Gamma)} > L.$$

Note that computing the value of $\Lambda_\infty(\Gamma,Q)$ for a general domain $\Omega$ is not straightforward.

## 6. Viscosity solutions

In order to identify the limit PDE problem satisfied by any limit $(u_\infty, v_\infty)$, we introduce the definition of viscosity solutions. Since we deal with different boundary conditions for the components $u_{p,q}$ (Dirichlet) and $v_{p,q}$ (Neumann) we split the passage to the limit into two parts. Also remark that $u_\infty$ is non-negative in $\overline{\Omega}$ but $v_\infty$ changes sign. This is reflected in the fact that they are solutions to quite different equations. First, we deal with the equation and boundary condition verified by $u_\infty$ and next we deal with $v_\infty$.

6.1. **Passing to the limit in $u_{p,q}$.** Assuming that $u_{p,q}$ is smooth enough, we can rewrite the first equation in (1) as

$$-|\nabla u_{p,q}|^{p-4}\left(|\nabla u_{p,q}|^2 \Delta u_{p,q} + (p-2)\Delta_\infty u_{p,q}\right) = \alpha\lambda_{p,q} u_{p,q}^{\alpha-1} v_{p,q}^\beta. \tag{27}$$

Recall that $-\Delta_\infty u = -\nabla u D^2 u (\nabla u)^t$. This equation is non-linear, elliptic (degenerate) but not in divergence form, thus it makes sense to consider viscosity sub-solutions and super-solutions of it. Let $x, y \in \mathbb{R}$, $z \in \mathbb{R}^N$, and $S$ a real symmetric matrix. We consider the following function

$$H_p(x,y,z,S) = -|z|^{p-4}\left(|z|^2 \mathrm{trace}(S) + (p-2)\langle S\cdot z, z\rangle\right) - \alpha\lambda_{p,q}|y|^{\alpha-2} y v_{p,q}(x)^\beta.$$

Observe that $H_p$ is elliptic in the sense that $H_p(x,y,z,S) \geq H_p(x,y,z,S')$ if $S \leq S'$ in the sense of bilinear forms, and also that (27) can be written as $H_p(x,u_{p,q},\nabla u_{p,q},D^2u_{p,q}) = 0$. We are thus interested in viscosity super and sub solutions of the partial differential equation

$$\begin{cases} H_p(x,u,\nabla u,D^2u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{28}$$

**Definition 6.1.** An upper semi-continuous function $u$ defined in $\Omega$ is a *viscosity sub-solution* of (28) if, $u|_{\partial\Omega} \leq 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$ and $u(x) < \phi(x)$, if $x \neq x_0$, then

$$H_p(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \leq 0.$$

**Definition 6.2.** A lower semi-continuous function $u$ defined in $\Omega$ is a *viscosity super-solution* of (28) if, $u|_{\partial\Omega} \geq 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u(x_0) = \phi(x_0)$ and $u(x) > \phi(x)$, if $x \neq x_0$, then

$$H_p(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

We observe that in both of the above definitions the second condition is required just in a neighbourhood of $x_0$ and the strict inequality can be relaxed. We refer to [10] for more details about general theory of viscosity solutions, and to [19] for viscosity solutions related to the $\infty-$Laplacian and the $p-$Laplacian operators. The following result can be shown as in [25, Proposition 2.4], therefore we omit the proof here.

**Lemma 6.3.** *A continuous weak solution to the equation*

$$\begin{cases} -\Delta_p u = \lambda\alpha|u|^{\alpha-2}uv_{p,q}^{\beta} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

*is a viscosity solution to* (28)*.*

Now, we have all the ingredients to compute the limit of (28) as $p \to \infty$ in the viscosity sense, that is, to identify the limit equation verified by any uniform limit of $u_{p,q}$, $u_\infty$. For $x, y \in \mathbb{R}$, $z \in \mathbb{R}^N$ and $S$ a symmetric real matrix, we define the limit operator $H_\infty$ by

$$H_\infty(x,y,z,S) = \min\{-\langle S\cdot z, z\rangle, |z| - \Lambda_\infty(\Gamma,Q)|y|^{\Gamma-2}y|v_\infty|^{(1-\Gamma)Q}(x)\}.$$

Note that $H_\infty(x,u,\nabla u,D^2u) = 0$ is the limit equation that we are looking for.

**Theorem 6.4.** *A function $u_\infty$ obtained as a limit of a subsequence of $\{u_{p,q}\}$ is a viscosity solution to the problem*

$$\begin{cases} H_\infty(x,u,\nabla u,D^2u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{29}$$

*with $H_\infty$ defined in* (6.1)*, and $v_\infty$ a uniform limit of $v_{p,q}$.*

*Proof.* In this proof we use ideas from [7]. We consider a subsequence $\{(p_n,q_n)\}_{n\in\mathbb{N}}$ such that $p_n, q_n \to \infty$

$$\lim_{n\to\infty} u_{p_n,q_n} = u_\infty, \qquad \lim_{n\to\infty} v_{p_n,q_n} = v_\infty$$

uniformly in $\Omega$ and $(\lambda_{p_n,q_n})^{1/p_n} \to \Lambda_\infty(\Gamma, Q)$. In what follows we omit the subscript $n$ and denote as $u_{p,q}$, $v_{p,q}$ and $\lambda_{p,q}$ such subsequences for simplicity.

We first check that $u_\infty$ is a super-solution of (29).To this end, we consider a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) > \phi(x)$ for every $x \in B(x_0, R)$, $x \neq x_0$, with $R > 0$ fixed and verifying that $B(x_0, 2R) \subset \Omega$. We must show that

$$H_\infty(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \geq 0. \tag{30}$$

Let $x_{p,q}$ be a minimum point of $u_{p,q} - \phi$ in $\bar{B}(x_0, R)$. Since $u_{p,q} \to u_\infty$ uniformly in $\bar{B}(x_0, R)$, up to a subsequence $x_{p,q} \to x_0$.

In view of Lemma 6.3, $u_{p,q}$ is a viscosity super-solution of (28), then

$$\begin{aligned} &- |\nabla\phi(x_{p,q})|^{p-4}\Big(|\nabla\phi(x_{p,q})|^2\Delta\phi(x_{p,q}) + (p-2)\Delta_\infty\phi(x_{p,q})\Big) \\ &\qquad \geq \alpha\lambda_{p,q}|\phi(x_{p,q})|^{\alpha-2}\phi(x_p)|v_{p,q}|^\beta(x_{p,q}). \end{aligned} \tag{31}$$

Assume that $\phi(x_0) = u_\infty(x_0) > 0$ and $|v_\infty|(x_0) > 0$. Then for $p, q$ large, $\phi(x_{p,q}) > 0$ and $|v_{p,q}|(x_{p,q}) > 0$ so that the right hand side of (31) is positive. It follows that $|\nabla\phi(x_{p,q})| > 0$ and then we get

$$\begin{aligned} &-\left(\frac{|\nabla\phi(x_{p,q})|^2\Delta\phi(x_{p,q})}{(p-2)} + \Delta_\infty\phi(x_{p,q})\right) \\ &\geq \left(\frac{\alpha^{\frac{1}{p}}}{(p-2)^{\frac{1}{p}}}(\lambda_{p,q})^{\frac{1}{p}}|\phi(x_{p,q})|^{\frac{\alpha-2}{p}}\phi^{\frac{1}{p}}(x_{p,q})|v_{p,q}|^{\frac{\beta}{p}}(x_{p,q})|\nabla\phi(x_{p,q})|^{-1+\frac{4}{p}}\right)^p. \end{aligned} \tag{32}$$

Note that we have

$$\lim_{p,q\to\infty} -\left(\frac{|\nabla\phi(x_{p,q})|^2\Delta\phi(x_{p,q})}{(p-2)} + \Delta_\infty\phi(x_{p,q})\right) = -\Delta_\infty\phi(x_0) < \infty.$$

Hence

$$\limsup_{p,q\to\infty} \frac{\alpha^{\frac{1}{p}}}{(p-2)^{\frac{1}{p}}}(\lambda_{p,q})^{\frac{1}{p}}\phi^{\frac{\alpha-1}{p}}(x_{p,q})|v_{p,q}|^{\frac{\beta}{p}}(x_p)|\nabla\phi(x_{p,q})|^{-1+\frac{4}{p}} \leq 1.$$

Recalling that by assumption $\frac{\alpha}{p} \to \Gamma$ and $\frac{q}{p} \to Q$ as $p, q \to \infty$, we obtain

$$\Lambda_\infty(\Gamma, Q)\phi^\Gamma(x_0)|v_\infty|^{(1-\Gamma)Q}(x_0) \leq |\nabla\phi(x_0)| \tag{33}$$

and

$$-\Delta_\infty\phi(x_0) \geq 0, \tag{34}$$

which is (30).

Assume now that either $\phi(x_0) = u_\infty(x_0) = 0$ or $v_\infty(x_0) = 0$. In particular, (33) holds. Note first that if $\nabla\phi(x_0) = 0$ then $\Delta_\infty\phi(x_0) = 0$ by definition so that (34) holds. We now assume that $|\nabla\phi(x_0)| > 0$ and write (32). The parenthesis in the right hand side goes to 0 as $p, q \to \infty$ so that the right hand side goes to 0 and (34) follows.

To complete the proof it just remains to see that $u_\infty$ is a viscosity sub-solution. Let us consider a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) < \phi(x)$ for every $x$ in a neighbourhood of $x_0$. We want to show that

$$H_\infty(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \leq 0.$$

We first observe that if $\nabla\phi(x_0) = 0$ the previous inequality trivially holds. Hence, let us assume that $\nabla\phi(x_0) \neq 0$. Now, we argue as follows: assuming that

$$|\nabla\phi(x_0)| - \Lambda_\infty(\Gamma, Q)\phi^\Gamma(x_0)|v_\infty|^{(1-\Gamma)Q}(x_0) > 0, \tag{35}$$

we will show that

$$-\Delta_\infty\phi(x_0) \leq 0. \tag{36}$$

As before, using that $u_{p,q}$ is a viscosity sub-solution of (28), we get a sequence of points $x_{p,q} \to x_0$ such that

$$\begin{aligned} & -\left(\frac{|\nabla\phi|^2\Delta\phi(x_{p,q})}{(p-2)} + \Delta_\infty\phi(x_{p,q})\right) \\ & \quad \leq \left(\frac{\alpha^{1/p}}{(p-2)}(\lambda_{p,q})^{1/p}|\phi|^{\alpha/p}(x_{p,q})|v_{p,q}|^{\beta/p}(x_{p,q})|\nabla\phi(x_{p,q})|^{-1+4/p}\right)^p. \end{aligned} \tag{37}$$

Using (35) we get

$$\limsup_{p,q\to\infty}\left(\frac{\alpha^{1/p}}{(p-2)}(\lambda_{p,q})^{1/p}|\phi|^{\alpha/p}(x_{p,q})|v_{p,q}|^{\beta/p}(x_{p,q})|\nabla\phi(x_{p,q})|^{-1+4/p}\right)^p = 0.$$

Hence, we conclude (36) taking limits in (37) and we obtain that

$$\min\{-\Delta_\infty\phi(x_0),\, |\nabla\phi(x_0)| - \Lambda_\infty(\Gamma, Q)\phi^\Gamma(x_0)|v_\infty|^{(1-\Gamma)Q}(x_0)\} \leq 0.$$

The fact that $u_\infty = 0$ on $\partial\Omega$ is immediate from the uniform convergence of $u_{p,q}$ since $u_{p,q} = 0$ on $\partial\Omega$. □

6.2. **Passing to the limit in $v_{p,q}$.** Let

$$F_q(x, y, z, S) = -|z|^{q-4}\left(|z|^2\mathrm{trace}(S) + (q-2)\langle S\cdot z, z\rangle\right) - \beta\lambda_{p,q}|u_{p,q}|^\alpha|y|^{\beta-2}y.$$

Now we deal with viscosity super and subsolutions of the partial differential equation

$$\begin{cases} F_q(x, v, \nabla v, D^2v) = 0 & \text{in } \Omega, \\ \dfrac{\partial v}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{38}$$

Here, we have to pay special attention to the fact that $v_{p,q}$ changes sign and to the boundary condition ${\partial v_{p,q}}/{\partial\nu} = 0$ on $\partial\Omega$. To this end, following [2], we introduce the following definition of viscosity solution for the boundary value problem

$$\begin{cases} F_q(x, \nabla u, D^2u) = 0 & \text{in } \Omega, \\ B(x, u, \nabla u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{39}$$

where $B(x, u, z) = \langle z, \nu(x)\rangle$.

**Definition 6.5.** A lower semi-continuous function $u$ is a viscosity super-solution if for every $\phi \in C^2(\overline{\Omega})$ such that $u - \phi$ has a strict minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality

$$\max\{B(x_0, \phi(x_0), \nabla\phi(x_0)),\ F_q(x_0, \nabla\phi(x_0), D^2\phi(x_0))\} \geq 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F_q(x_0, \nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

**Definition 6.6.** An upper semi-continuous function $u$ is a sub-solution if for every $\phi \in C^2(\overline{\Omega})$ such that $u-\phi$ has a strict maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0)=\phi(x_0)$ we have: If $x_0 \in \partial\Omega$ the inequality

$$\min\{F_q(x_0,\nabla\phi(x_0),D^2\phi(x_0)),B(x_0,\phi(x_0),\nabla\phi(x_0))\} \le 0$$

holds, and if $x_0 \in \Omega$ then we require

$$F_q(x_0,\nabla\phi(x_0),D^2\phi(x_0)) \le 0.$$

As before, we have that any continuous weak solution of the second equation in (1) is a viscosity solution of (39). This fact can be proved as in [14, 15, 30].

We can now pass to the limit $p,q\to\infty$ to obtain the equation satisfied by $v_\infty$.

**Theorem 6.7.** *A function $v_\infty$ obtained as a limit of a subsequence of $\{v_{p,q}\}$ is a viscosity solution of the equation*

$$\begin{cases} F_\infty(x,v,\nabla v,D^2v)=0 & \text{in } \Omega,\\ \dfrac{\partial v}{\partial \nu}=0 & \text{on } \partial\Omega. \end{cases} \tag{40}$$

*with $F_\infty$ defined by*

$$F_\infty(v,z,S)=\begin{cases} \min\{-\langle S\cdot z,z\rangle, |z|-\Lambda_\infty(\Gamma,Q)^{1/Q}|u_\infty|^{\Gamma/Q}|v|^{1-\Gamma}\} & \text{in } \{v>0\},\\ \max\{-\langle S\cdot z,z\rangle, -|z|-\Lambda_\infty(\Gamma,Q)^{1/Q}|u_\infty|^{\Gamma/Q}|v|^{1-\Gamma}\} & \text{in } \{v<0\},\\ -\langle S\cdot z,z\rangle & \text{in } \{v=0\}. \end{cases}$$

*Proof.* We prove that $v_\infty$ is a super-solution of (40). The proof of the fact that it is a sub-solution is similar. Fix some point $x_0\in\overline{\Omega}$ and a smooth function $\phi$ such that $v_\infty-\phi$ has a strict minimum at $x_0$ with $v_\infty(x_0)=\phi(x_0)$. Since $v_{p,q}\to v_\infty$ uniformly there exist $x_{p,q}\in \operatorname{argmax}\{v_{p,q}-\phi\}$ such that $x_{p,q}\to x_0$ as $p,q\to\infty$.

Assume first that $x_0\in\Omega$, so that $x_{p,q}\in\Omega$ for $p,q$ large. If $\nabla\phi(x_0)=0$ then we have $\Delta_\infty\phi(x_0)=0$. We assume now that $\nabla\phi(x_0)\neq 0$. As $u_{p,q}$ is a viscosity solution of (38), we have

$$F_q(x_p,v_{p,q}(x_{p,q}),\nabla\phi(x_{p,q}),D^2\phi(x_p))\ge 0.$$

Dividing this inequality by $(q-2)|\nabla\phi(x_{p,q})|^{q-4}$ we obtain

$$\Delta_\infty\phi(x_0)+o(1)\ge v_{p,q}|\nabla\phi|^2(x_{p,q})\left(\frac{\lambda_{p,q}^{\frac{1}{q-2}}|u_{p,q}(x_{p,q})|^{\frac{\alpha}{q}}|v_{p,q}(x_{p,q})|^{\frac{\beta}{q}-1}}{|\nabla\phi(x_{p,q})|(q-2)^{\frac{1}{q-2}}}\right)^{q-2}. \tag{41}$$

If $v_\infty(x_0)>0$, then, recalling that $(\lambda_{p,q})^{\frac{1}{q-2}}\to(\Lambda_\infty(\Gamma,Q))^{1/Q}$, it follows that we must have $\frac{\Lambda_\infty(\Gamma,Q)^{1/Q}|u_\infty(x_0)|^{\Gamma/Q}|v_\infty(x_0)|^{-\Gamma}}{|\nabla\phi(x_0)|}\le 1$. Going back to (41) we also get $\Delta_\infty\phi(x_0)\ge 0$.

If $v_\infty(x_0)<0$ then we rewrite the equation as

$$-|\nabla\phi(x_{p,q})|^{-3}\left(\frac{(q-2)^{\frac{1}{q-1}}|\nabla\phi(x_{p,q})|}{\lambda_{p,q}^{\frac{1}{q-1}}|u_{p,q}(x_{p,q})|^{\frac{\alpha}{q}}|v_{p,q}(x_{p,q})|^{\frac{\beta}{q}}}\right)^{q-1}(\Delta_\infty\phi(x_0)+o(1))\le 1.$$

If $\frac{\Lambda_\infty(\Gamma,Q)^{1/Q}|u_\infty(x_0)|^{\Gamma/Q}|v_\infty(x_0)|^{-\Gamma}}{|\nabla\phi(x_0)|}>1$ then we must have $\Delta_\infty\phi(x_0)\ge 0$. Otherwise we have $\frac{\Lambda_\infty(\Gamma,Q)^{1/Q}|u_\infty(x_0)|^{\Gamma/Q}|v_\infty(x_0)|^{-\Gamma}}{|\nabla\phi(x_0)|}\le 1$.

If $v_\infty(x_0)=0$, then $v_{p,q}(x_{p,q})\to 0$ so that $|v_{p,q}(x_{p,q})|^{q-2}v_{p,q}(x_{p,q})\le v_{p,q}(x_{p,q})\to 0$. It then follows that

$$|\nabla\phi(x_{p,q})|^{q-2}\Delta\phi(x_{p,q})+(q-2)|\nabla\phi(x_{p,q})|^{q-4}\Delta_\infty\phi(x_{p,q})\ge o(1).$$

Dividing this inequality by $(q-2)|\nabla\phi(x_{p,q})|^{q-4}$ and letting $p,q\to\infty$ we obtain $\Delta_\infty\phi(x_0)\ge 0$.

Assume now that $x_0\in\partial\Omega$. We have to prove that

$$\max\left\{F_\infty(x_0,\nabla\phi(x_0),D^2\phi(x_0)),\frac{\partial\phi}{\partial\nu}(x_0)\right\}\ge 0.$$

If $x_{p,q}\in\Omega$ for some subsequence then we can proceed as before to get

$$F_\infty(x_0,\nabla\phi(x_0),D^2\phi(x_0))\ge 0.$$

Assume that $x_{p,q}\in\partial\Omega$ for every $p,q$ large. If $\nabla\phi(x_0)=0$ then $\partial\phi(x_0)/\partial\nu=0$. Then we need to deal with $\nabla\phi(x_0)\neq 0$. We have

$$\max\left\{F_p(x_{p,q},\nabla\phi(x_{p,q}),D^2\phi(x_{p,q})),\frac{\partial\phi}{\partial\nu}(x_{p,q})\right\}\ge 0.$$

If $F_p(x_{p,q},\nabla\phi(x_{p,q}),D^2\phi(x_p))\ge 0$ holds for a subsequence we are done as before. Otherwise

$$\frac{\partial\phi}{\partial\nu}(x_{p,q})\ge 0\qquad\text{for } p,q \text{ large}$$

so that $\partial\phi/\partial\nu(x_0)=\lim_{p,q\to\infty}\partial\phi/\partial\nu(x_{p,q})\ge 0$. □

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