THE CRITICAL HYPERBOLA FOR A HAMILTONIAN ELLIPTIC SYSTEM WITH WEIGHTS

DJAIRO G. DE FIGUEIREDO, IRENEO PERAL, AND JULIO D. ROSSI

Abstract. In this paper we look for existence results for nontrivial solutions to the system,

$$\begin{cases} -\Delta u = \dfrac{v^p}{|x|^\alpha} & \text{in } \Omega, \\ -\Delta v = \dfrac{u^q}{|x|^\beta} & \text{in } \Omega, \end{cases}$$

with Dirichlet boundary conditions, $u = v = 0$ on $\partial\Omega$ and $\alpha, \beta < N$. We find the existence of a critical hyperbola in the (p, q) plane (depending on α, β and N) below which there exists nontrivial solutions. For the proof we use a variational argument (a linking theorem).

1. Introduction.

In this paper we study the existence of nontrivial solutions of the following elliptic system,

$$\begin{cases} -\Delta u = \dfrac{v^p}{|x|^\alpha} & \text{in } \Omega, \\ -\Delta v = \dfrac{u^q}{|x|^\beta} & \text{in } \Omega, \end{cases} \tag{1.1}$$

with Dirichlet boundary conditions

$$u = v = 0 \qquad \text{on } \partial\Omega.$$

Here Ω is a bounded smooth domain in $\mathbb{R}^N$ with $0 \in \Omega$ and $s^p := \operatorname{sgn}(s)|s|^p$. We will assume that the exponents p, q are positive and that $\alpha, \beta < N$.

Our main concern in this paper is to look at the role played by the two weights when dealing with existence of solutions. We find the existence of a critical hyperbola, given by,

$$\frac{N-\alpha}{p+1} + \frac{N-\beta}{q+1} = N - 2. \tag{1.2}$$

1991 *Mathematics Subject Classification.* 35J65, 35J20, 35J55.

Key words and phrases. elliptic systems, nonlinear boundary conditions, variational problems.

I. P. partially supported by project MTM2004-02223 of MEC, Spain.

J.D.R. partially supported by Universidad de Buenos Aires (grant TX066), ANPCyT and Fundacion Antorchas.

Below this hyperbola we find existence of nontrivial solutions. Remark that when the two weights are not present, that is, for $\alpha = \beta = 0$, we recover the critical hyperbola for elliptic systems without weights that was found independently in [15] and [31] (see also [17] and [20]). Also remark that the hyperbola (1.2) is monotone with respect to α and β. The stronger the weights the smaller the hyperbola.

The main result in this paper is the following theorem.

Theorem 1.1. *Let us assume that p, q, α, β verifies*

$$\frac{N-\alpha}{p+1} + \frac{N-\beta}{q+1} > N-2, \tag{1.3}$$

$$\frac{1}{p+1} + \frac{1}{q+1} < 1 \tag{1.4}$$

and

$$q+1 < \frac{2(N-\beta)}{N-4} \quad \textit{and} \quad p+1 < \frac{2(N-\alpha)}{N-4} \qquad \textit{if } N \geq 5, \tag{1.5}$$

then there exist infinitely many strong solutions and at least one positive strong solution of (1.1).

In the case $\alpha = \beta = 0$ nonexistence on the critical hyperbola could be obtained by some Pohozaev identities, see for instance [35]. Notice that if $\alpha = \beta = 2$ and $p = q = 1$ the existence result depends on a real parameter λ. More precisely consider the linear problem

$$\begin{cases} -\Delta u = \lambda \dfrac{v}{|x|^2} & \text{in } \Omega, \\ -\Delta v = \lambda \dfrac{u}{|x|^2} & \text{in } \Omega. \end{cases} \tag{1.6}$$

By adding both equation we have that if $\lambda > (N-2)^2/4$ there is no positive distributional solution. In this sense we could conjecture that the hyperbola also is optimal in this case. Moreover, we have no sign restriction on α and β, i.e., we are able to solve systems in which one equation (or both) are of Hénon type. Notice that, if we take $p = q$, $\alpha = \beta < 0$, the hyperbola that we found gives the pioneering existence result in [30] above the critical Sobolev exponent corresponding to the weight. This Sobolev exponent is given by the corresponding Caffarelli-Kohn-Nirenberg estimate in [11]. For scalar Hénon equations and properties we refer to [6], [30], [33], [34] and the references therein.

Note that the system (1.1) has a variational structure. In fact, it can be seen as a Hamiltonian system, since, if we consider

$$H(x,u,v) = \frac{v^{p+1}}{(p+1)|x|^{\alpha}} + \frac{u^{q+1}}{(q+1)|x|^{\beta}},$$

then we have

$$H_v(x,u,v) = \frac{v^p}{|x|^\alpha} \qquad \text{and} \qquad H_u(x,u,v) = \frac{u^q}{|x|^\beta}.$$

The crucial point of our arguments is to find the proper functional setting for (1.1) that allows us to treat our problem variationally. We accomplish this by considering fractional powers of the self adjoint operator $-\Delta$ with Dirichlet boundary conditions. The main ideas are taken from [17], [20], [23] and [28]. We also use a linking theorem in a version that can be found in [23]. See also the survey [18].

We observe that the same techniques used here can be applied to deal with more general Hamiltonian systems (with adequate hypotheses on H)

$$\begin{cases} -\Delta u = H_v(x,u,v) & \text{in } \Omega, \\ -\Delta v = H_u(x,u,v), \end{cases}$$

with Dirichlet boundary conditions. For example, we can consider, for two different points, $x_1 \neq x_2 \in \Omega$,

$$\begin{cases} -\Delta u = \dfrac{v^p}{|x-x_1|^\alpha} & \text{in } \Omega, \\ -\Delta v = \dfrac{u^q}{|x-x_2|^\beta} & \text{in } \Omega. \end{cases}$$

To clarify the exposition we will state and prove our results for (1.1) and leave the details of the general case to the reader.

We end the introduction with some bibliographical discussion. Existence results for nonlinear elliptic systems have deserved a great deal of interest in recent years. For this type of results see, among others, [8], [12], [14], [17], [19], [20], [22], [24] and the survey [16]. Results of existence, non-existence and multiplicity for elliptic equations involving weights could be found, among other papers, in [1], [2], [3], [4], [5], [9], [10], [25] and [27].

The rest of the paper is organized as follows, in Section 2 we establish the functional setting in which the problem will be posed and in Section 3 we prove our main result, Theorem 1.1.

2. The functional setting

In this section we describe the functional setting that allows us to treat (1.1) variationally. The natural functional associated to (1.1) is given by

$$J(u,v) = \int_\Omega \nabla u \nabla v - \int_\Omega \left(\frac{v^{p+1}}{(p+1)|x|^\alpha} + \frac{u^{q+1}}{(q+1)|x|^\beta} \right), \tag{2.1}$$

in the natural space $H_0^1(\Omega) \times H_0^1(\Omega)$. However, in order to have a C^1 functional one has to impose conditions on p, q, α, β that are too restrictive. Therefore we use an idea from [17], [20] and [28] and use suitable fractional powers of an operator on fractional Sobolev spaces to define the functional.

Let us consider the Hilbert space $L^2(\Omega)$ and the operator

$$-\Delta : H^2(\Omega) \cap H^1_0(\Omega) \subset L^2(\Omega) \mapsto L^2(\Omega)$$

It is well known that there exists a sequence of eigenvalues of $-\Delta$, $(\lambda_n) \subset \mathbb{R}$ with eigenfunctions $\phi_n \in H^2(\Omega) \cap H^1_0(\Omega)$ such that $0 < \lambda_1 \le \lambda_2 \le ... \le \lambda_n \le ... \nearrow +\infty$.

Let us consider the fractional powers of $-\Delta$, namely for $0 < s < 1$, let

$$A^s = (-\Delta)^s,$$

that is,

$$A^s : D(A^s) \subset L^2(\Omega) \mapsto L^2(\Omega)$$

is given by

$$A^s u = \sum_{n=1}^{\infty} \lambda_n^s a_n \phi_n,$$

when u has the expansion $u = \sum a_n \phi_n$. We call

$$E^s = D(A^s) = \left\{ u \in L^2(\Omega) \ : \ \sum_{n=1}^{\infty} \lambda_n^{2s} a_n^2 < \infty \right\},$$

which is a Hilbert space with inner product, that we denote by $(\cdot,\cdot)_{E^s}$, given by

$$(u, v)_{E^s} = \langle A^s u, A^s v \rangle.$$

These spaces E^s are fractional Sobolev spaces, see [29]. In fact, we have

$$E^s \subset H^{2s}(\Omega), \qquad 0 < s \le 1.$$

By the Sobolev embedding theorem we have that

$$E^s \hookrightarrow L^r(\Omega) \qquad \text{if } \frac{1}{r} \ge \frac{1}{2} - \frac{2s}{N}$$

and the embedding is compact if we have a strict inequality. Now, we use Holder's inequality to obtain

$$\begin{aligned} \int_\Omega \frac{u^{q+1}}{|x|^\beta} &\le \left(\int_\Omega u^r \right)^{(q+1)/r} \left(\int_\Omega |x|^{-\beta r/(r-(q+1))} \right)^{(r-(q+1))/r} \\ &\le C \left(\int_\Omega u^r \right)^{(q+1)/r} \end{aligned}$$

if

$$\frac{\beta r}{r-(q+1)} < N.$$

That is

$$N(q+1) < (N-\beta) r. \tag{2.2}$$

Therefore, if

$$q+1 < \frac{2(N-\beta)}{N-4s}$$

we can choose r such that
$$\frac{1}{r} > \frac{1}{2} - \frac{2s}{N}$$
and (2.2) hold. Hence, we have the following inclusions
$$E^s \subset H^{2s}(\Omega) \hookrightarrow L^r(\Omega) \subset L^{q+1}(\Omega, |x|^{-\beta}).$$

More precisely, we have proved the following proposition,

Proposition 2.1. *Given $q > 1$, $\beta > 0$ and $s > 0$ so that*
$$q + 1 < \frac{2(N-\beta)}{N-4s}$$
the inclusion map $i : E^s \to L^{q+1}(\Omega, |x|^{\beta})$ is well defined and compact.

Let us set $E = E^s \times E^t$ where $s + t = 1$, with the norm
$$\|(u,v)\|_E^2 = \|u\|_{E^s}^2 + \|v\|_{E^t}^2.$$
Let the linear operator $L : E \to E$ be given by
$$L(u,v) = (A^{-s}A^t v, A^{-t}A^s u).$$
Next, we consider the eigenvalue problem $Lz = \lambda z$. We can rewrite this as
$$A^{-s}A^t v = \lambda u, \qquad A^{-t}A^s u = \lambda v,$$
where $z = (u,v)$. As A^s and A^t are isomorphisms, it follows that $\lambda = 1$ or $\lambda = -1$. The associated eigenvectors are
$$\text{for } \lambda = 1,\ (u, A^{-t}A^s u)\ \forall u \in E^s,$$
$$\text{for } \lambda = -1,\ (u, -A^{-t}A^s u)\ \forall u \in E^s.$$
We can define the eigenspaces
$$E^+ = \{(u, A^{-t}A^s u) \ / \ u \in E^s\},$$
$$E^- = \{(u, -A^{-t}A^s u) \ / \ u \in E^s\},$$
which give a natural splitting
$$E = E^+ \oplus E^-.$$

We can define the functional, $\mathrm{H} : E \to \mathbb{R}$ as
$$\mathrm{H}(u,v) = \int_\Omega H(x,u,v).$$

Proposition 2.2. *The functional* H *defined above is of class C^1 and its derivative is given by*
$$\mathrm{H}'(u,v)(\phi,\psi) = \int_\Omega H_u(x,u,v)\phi + \int_\Omega H_v(x,u,v)\psi.$$
Moreover, H' *is compact.*

Proof. We have

$$\int_\Omega \left| \frac{\partial H}{\partial u}(x,u,v)\phi \right| = \int_\Omega \left(\frac{|u|^q}{|x|^\beta} \right) |\phi|.$$

By Holder's inequality and Proposition 2.1 we have

$$\int_\Omega \left| \frac{\partial H}{\partial u}(x,u,v)\phi \right| \le C \left(\|u\|_{E^s}^q \right) \|\phi\|_{E^s}.$$

In a similar way we obtain the analogous inequality for H_v.

Thus H$'$ is well defined and bounded in E. Next, a standard argument gives that H is Fréchet differentiable with H$'$ continuous. The fact that H$'$ is compact comes from Proposition 2.1 (see [32] for the details). □

We consider the functional

$$I : E \mapsto \mathbb{R}$$

given by

$$\begin{aligned} I(u,v) &= \int_\Omega A^s u, A^t v - \int_\Omega \left(\frac{v^{p+1}}{(p+1)|x|^\alpha} + \frac{u^{q+1}}{(q+1)|x|^\beta} \right) \\ &= \int_\Omega A^s u, A^t v - \int_\Omega H(x,u,v). \end{aligned} \tag{2.3}$$

Where we have selected $s, t > 0$ such that

$$s + t = 1, \qquad q + 1 < \frac{2(N-\beta)}{N-4s} \quad \text{and} \quad p + 1 < \frac{2(N-\alpha)}{N-4t}.$$

Remark 2.1. *Note that this selection of s, t is possible since we are below the critical hyperbola* (1.2)*. Moreover, there exists a positive selection of s and t thanks to our last assumption on the exponents in Theorem 1.1,* (1.5)*.*

Let us now give the definition of weak solution of (1.1).

Definition 2.1. *We say that $z = (u,v) \in E = E^s \times E^t$ is an $(s,t)-$weak solution of* (1.1) *if z is a critical point of I. In other words, for every $(\phi, \psi) \in E$ we have*

$$\int_\Omega A^s u, A^t \psi + \int_\Omega A^s \phi, A^t v - \int_\Omega \frac{v^p}{|x|^\alpha}\psi - \int_\Omega \frac{u^q}{|x|^\beta}\phi = 0. \tag{2.4}$$

Now, we prove a theorem that provides regularity of (s,t)-weak solutions. In fact, every (s,t)-weak solution has two derivatives in some $L^r(\Omega)$ and hence it is a strong solution.

Theorem 2.1. *If $(u,v) \in E^s \times E^t$ is an (s,t)-weak solution of* (1.1) *then $u \in W^{2,a}(\Omega)$, $v \in W^{2,b}(\Omega)$ for every*

$$1 < a < \frac{2N}{p(N-4t)+2\alpha} \qquad \text{and} \qquad 1 < b < \frac{2N}{q(N-4s)+2\beta}.$$

Hence (u,v) is in fact a strong solution of (1.1)*.*

Proof. Let us first consider $\psi = 0$ in (2.4), then

$$\int_\Omega A^s\phi, A^t v - \int_\Omega \frac{u^q}{|x|^\beta}\phi = 0, \tag{2.5}$$

for all $\phi \in E^s$. If we take $\phi \in H^2(\Omega) \cap H_0^1(\Omega)$, we have

$$\int_\Omega A^s\phi, A^t v = -\int_\Omega \Delta\phi\, v. \tag{2.6}$$

On the other hand, since $u \in E^s$, by our previous calculations, we have

$$\frac{u^q}{|x|^\beta} \in L^b(\Omega) \qquad \text{if} \quad b < \frac{2N}{q(N-4s)+2\beta}.$$

Then, from basic elliptic theory (see [26]), there exists a function $w \in W^{2,b}(\Omega)$ such that

$$\begin{cases} -\Delta w = \dfrac{u^q}{|x|^\beta} & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega. \end{cases}$$

Now, integration by parts gives

$$0 = -\int_\Omega \Delta w\phi - \int_\Omega \frac{u^q}{|x|^\beta}\phi = -\int_\Omega w\Delta\phi - \int_\Omega \frac{u^q}{|x|^\beta}\phi. \tag{2.7}$$

Combining (2.5),(2.6) and (2.7), we obtain

$$\int_\Omega (v-w)\Delta\phi = 0,$$

with $v = w = 0$ on $\partial\Omega$. From where it follows that $v = w$ and hence v belongs to $W^{2,b}(\Omega)$. We argue similarly for u. $\square$

Remark 2.2. *Since we have*

$$q+1 < \frac{2(N-\beta)}{N-4s} \qquad \textit{and} \qquad p+1 < \frac{2(N-\alpha)}{N-4t}$$

we get

$$1 < \frac{2N}{p(N-4t)+2\alpha} \qquad \textit{and} \qquad 1 < \frac{2N}{q(N-4s)+2\beta}.$$

3. Proof of Theorem 1.1

First, we prove that there exist infinitely many solutions to (1.1). To this end, we present an abstract theorem from critical point theory from [23] (see also [7]) that provides us with infinitely many critical points of a functional. Next, we prove that this abstract result can be applied to our functional setting stated in the previous section.

Let E be a Hilbert space with inner product $(\cdot,\cdot)_E$. Assume that E has a splitting $E = X \oplus Y$ where X and Y are both infinite dimensional subspaces. Assume there exists a sequence of finite dimensional subspaces $X_n \subset X$, $Y_n \subset Y$, $E_n = X_n \oplus Y_n$ such that $\overline{\cup_{n=1}^\infty E_n} = E$. Let $T : E \to E$ be a linear bounded invertible operator.

Let $I \in C^1(E, \mathbb{R})$. Instead of the usual Palais-Smale condition we will require that the functional I satisfies the so-called $(PS)^*$ conditions with respect to E_n, i.e. any sequence $z_k \in E_{n_k}$ with $n_k \to \infty$ as $k \to \infty$, satisfying $I|'_{E_{n_k}}(z_k) \to 0$ and $I(z_k) \to c$ has a subsequence that converges in E.

Then we define the basic sets over which the linking process will take place. For $\rho > 0$ we define

$$S = S_\rho = \{y \in Y |\ \|y\|_E = \rho\}$$

and for some fixed $y_1 \in Y$ with $\|y_1\|_E = 1$ and subspaces $\mathcal{X}_1$ and $\mathcal{X}_2$, we consider

$$X \oplus \text{span}\{y_1\} = \mathcal{X}_1 \oplus \mathcal{X}_2.$$

Without loss of generality we may assume that $y_1 \in \mathcal{X}_2$. Next, we define for $M, \sigma > 0$

$$D = D_{M,\sigma} = \{x_1 + x_2 \in \mathcal{X}_1 \oplus \mathcal{X}_2 |\ \|x_1\|_E \le M,\ \|x_2\|_E \le \sigma\}.$$

Now we can state our abstract critical point result whose proof can be found in [23].

Theorem 3.1. *Let $I \in C^1(E, \mathbb{R})$ be an even functional satisfying the $(PS)^*$ condition with respect to E_n. Assume that $T : E_n \to E_n$, for n large. Let $\rho > 0$ and $\sigma > 0$ be such that $\sigma \|Ty_1\|_E > \rho$. Assume that there are constants $\alpha \le \beta$ such that*

$$\inf_{S \cap E_n} I \ge \alpha, \qquad \sup_{T(\partial D \cap E_n)} I < \alpha \quad \textit{and} \quad \sup_{T(D \cap E_n)} I \le \beta$$

for all n large. Then I has a critical value $c \in [\alpha, \beta]$.

Next, we show how the functional setting introduced in Section 2 can be used to apply Theorem 3.1.

Let ϕ_n be the eigenfunctions of $-\Delta$. Let

$$E_n = \text{span}\{\phi_1, ..., \phi_n\} \times \text{span}\{\phi_1, ..., \phi_n\}.$$

It is easy to see that $\overline{\cup_{n=1}^\infty E_n} = E$. Next, we prove that I satisfies the $(PS)^*$ condition with respect to the family E_n.

Lemma 3.1. *The functional I satisfies the $(PS)^*$ condition with respect to E_n.*

Proof. Let $(z_k)_{k \ge 1} = (u_k, v_k)_{k \ge 1} \subset E_{n_k}$ be a sequence such that

$$I(z_k) \to c \quad \text{and} \quad I' |_{E_{n_k}} (z_k) \to 0. \tag{3.1}$$

Let us first prove that (3.1) implies that (z_k) is bounded in E. From (3.1) it follows that there exists a sequence $\varepsilon_k \to 0$ such that

$$|I'(z_k) w| \le \varepsilon_k \|w\|_E, \tag{3.2}$$

for all $w \in E_{n_k}$. Let us take

$$w_k = ((w_k)_1, (w_k)_2) = \frac{(q+1)(p+1)}{p+q+2}\left(\frac{1}{q+1}u_k, \frac{1}{p+1}v_k\right).$$

Now, using (3.1) and (3.2), for k large

$$\begin{aligned} &c + 1 + \varepsilon_k \|w_k\|_E \geq I(z_k) - I'(z_k)w_k \\ &= \int_\Omega A^s u_k A^t v_k - \int_\Omega H(x, u_k, v_k) - \int_\Omega A^s u_k, A^t (w_k)_2 \\ &\quad - \int_\Omega A^s (w_k)_1, A^t v_k + \int_\Omega H_u(x, u_k, v_k)(w_k)_1 + \int_\Omega H_v(x, u_k, v_k)(w_k)_2 \\ &= -\frac{(1 - p\,q)}{p+q+2}\int_\Omega H(x, u_k, v_k). \end{aligned}$$

Now, by (1.4) we get $p\,q > 1$ and hence we obtain

$$C(1 + \|z_k\|_E) \geq \int_\Omega H(x, u_k, v_k).$$

Therefore,

$$\int_\Omega \frac{|u_k|^{q+1}}{|x|^\beta} + \frac{|v_k|^{p+1}}{|x|^\alpha} \leq C(1 + \|u_k\|_{E^s} + \|v_k\|_{E^t}). \tag{3.3}$$

Next we consider $w = (\phi, 0)$ with $\phi \in E^s_{n_k}$. From (3.2) we have

$$\int_\Omega A^s\phi, A^t v_k \leq \int_\Omega \frac{|u_k|^q}{|x|^\beta}|\phi| + \varepsilon_k \|\phi\|_{E^s}.$$

Now, using Holder's inequality,

$$\int_\Omega \frac{|u_k|^q}{|x|^\beta}|\phi| \leq \|u_k\|^q_{L^{q+1}(\Omega,|x|^{-\beta})}\|\phi\|_{L^{q+1}(\Omega,|x|^{-\beta})}.$$

So, by Proposition 2.1, we get that

$$|\langle A^s\phi, A^t v_k\rangle| \leq C\|\phi\|_{E^s}\left(\|u_k\|^q_{L^{q+1}(\Omega,|x|^{-\beta})} + 1\right).$$

By duality (A^s is an isometry between E^s and L^2) we get

$$\|v_k\|_{E^t} \leq C\left(\|u_k\|^q_{L^{q+1}(\Omega,|x|^{-\beta})} + 1\right). \tag{3.4}$$

Analogously, we obtain

$$\|u_k\|_{E^s} \leq C\left(\|v_k\|^p_{L^{p+1}(\Omega,|x|^{-\alpha})} + 1\right). \tag{3.5}$$

Now combining (3.3), (3.4) and (3.5), we obtain

$$\|u_k\|_{E^s} + \|v_k\|_{E^t} \leq C\left(\|u_k\|^{q/(q+1)}_{E^s} + \|v_k\|^{p/(p+1)}_{E^t} + 1\right).$$

Since all the involved exponents are less than one, we conclude that z_k in bounded.

Now, by compactness and the invertibility of L we can extract a subsequence of z_k that converges in E. Indeed, we can take a subsequence z_{k_j}

that converges weakly in E, as H' is compact, it follows that $\mathrm{H}'(z_{k_j})$ converges strongly in E. Hence, using the fact that $I'(z_{k_j}) \to 0$ strongly and the invertibility of L, the result follows. $\square$

Now we define the splitting of E_n. Fix $k \in \mathbb{N}$ and for $n \geq k$ let

$$X_n = \left(E_1^- \oplus \cdots \oplus E_n^-\right) \oplus \left(E_1^+ \oplus \cdots \oplus E_{k-1}^+\right) \quad \text{and} \quad Y_n = \left(E_k^+ \oplus \cdots \oplus E_n^+\right), \tag{3.6}$$

where $E_j^+ = \operatorname{span}\{(\phi_j, A^{-t}A^s\phi_j)\}$ and $E_j^- = \operatorname{span}\{(\phi_j, -A^{-t}A^s\phi_j)\}$. We have $E_n = X_n \oplus Y_n$.

Lemma 3.2. *There exist $\alpha_k > 0$ and $\rho_k > 0$ independent of n such that for all $n \geq k$*

$$\inf_{z \in S_{\rho_k} \cap Y_n} I(z) \geq \alpha_k$$

where $S_{\rho_k} = \{y \in E^+ \mid \|y\| = \rho_k\}$. Moreover, $\alpha_k \to \infty$ as $k \to \infty$.

Proof. We first recall that by Proposition 2.1, E^s is embedded in $L^\gamma(\Omega, |x|^{-\varrho})$ for any γ such that

$$\gamma \leq \frac{2(N-\varrho)}{N-4s}.$$

Hence, there exists $a = a(\gamma)$ such that

$$\|u\|_{L^\gamma(\Omega,|x|^{-\varrho})} \leq a\|u\|_{E^s} \qquad \text{for all } u \in E^s.$$

Also for $z \in E_k^+ \oplus \cdots \oplus E_j^+ \oplus \cdots$ we have

$$\|z\|_E \geq \lambda_k^{\min\{s,t\}} \|z\|_{L^2(\Omega)}$$

with $\lambda_k \to \infty$ as $k \to \infty$.

Now consider $z = (u, v) \in Y_n$. For a constant a independent of n, we observe that there exists $\kappa > 0$ such that

$$\|u\|^{q+1}_{L^{q+1}(\Omega,|x|^{-\beta})} \leq \|u\|^{2/\kappa}_{L^2(\Omega)} \|u\|^{q+1-2/\kappa}_{L^\gamma(\Omega,|x|^{-\varrho})} \leq \frac{a}{\lambda_k^{\min\{s,t\}(2/\kappa)}} \|u\|_E^{q+1}$$

Analogously, we obtain

$$\|v\|^{p+1}_{L^{p+1}(\Omega,|x|^{-\alpha})} \leq \frac{a}{\lambda_k^{\min\{s,t\}(2/\theta)}} \|v\|_E^{p+1}$$

for some $\theta > 0$. Then for $z = (u, v)$ we have

$$I(z) \geq \|z\|_E^2 - C\left(\frac{a}{\lambda_k^{\min\{s,t\}\min\{2/\kappa,\ 2/\theta\}}} \max\{\|z\|_E^{q+1},\ \|z\|_E^{p+1}\} + 1\right)$$

Then we choose

$$\rho_k^{\max\{p+1,q+1\}} = \lambda_k^{\min\{s,t\}\min\{2/\kappa,\ 2/\theta\}}$$

and observe that $\rho_k \to \infty$ as $k \to \infty$.

Therefore, for $z \in S_{\rho_k} \cap Y_n$ we find that

$$I(z) \geq \rho_k^2 - C. \tag{3.7}$$

Defining α_k as the right hand side of (3.7) and noting that both ρ_k and α_k are independent of $n \geq k$ we complete the proof of the Lemma. $\square$

Next we define, for $z = (u, v) \in E$

$$T_\sigma(z) = (\sigma^{\mu-1}u, \sigma^{\nu-1}v) \tag{3.8}$$

where μ and ν are such that

$$\mu + \nu < \min\{\mu(p+1), \nu(q+1)\}.$$

This choice of μ and ν is possible since, by (1.4), we have $p\,q > 1$.

Lemma 3.3. *There exist $B_k > 0$, σ_k and $M_k > 0$ independent of n such that for all $n \geq k$ they satisfy $\sigma_k > \rho_k$,*

$$\sup_{T_{\sigma_k}(\partial D \cap E_n)} I \leq 0 \qquad \textit{and} \qquad \sup_{T_{\sigma_k}(D \cap E_n)} I \leq B_k$$

where

$$D = \{z \in E^- \oplus E_1^+ \oplus \cdots \oplus E_k^+ \mid \|z^-\| \leq M_k,\ \|z^+\| \leq \sigma_k\}.$$

Proof. Let us consider $z = T_\sigma(u, v)$ with $(u, v) \in D$. Then we can write

$$z = (\sigma^{\mu-1}u^+, \sigma^{\nu-1}v^+) + (\sigma^{\mu-1}u^-, \sigma^{\nu-1}v^-).$$

Using the definition of the spaces E^+ and E^- we have

$$\int_\Omega A^s u, A^t v = \sigma^{\mu+\nu-2}(\|z^+\|^2 - \|z^-\|^2).$$

On the other hand, we have

$$\int_\Omega H(x, z) = \int_\Omega \sigma^{(q+1)(\mu-1)} \frac{|u^+ + u^-|^{q+1}}{|x|^\beta} + \sigma^{(p+1)(\nu-1)} \frac{|v^+ + v^-|^{p+1}}{|x|^\alpha}.$$

The functions u^+ and u^- can be written as

$$u^+ = \sum_{i=1}^k \theta_i \phi_i \qquad \text{and} \qquad u^- = \sum_{i=1}^k \gamma_i \phi_i + \tilde{u}^-,$$

where $\tilde{u}^-$ is orthogonal to ϕ_i, $i = 1, ..., k$ in $L^2(\Omega)$. Using Holder's inequality we get

$$\begin{aligned} \sum_{i=1}^k \lambda_i^{s-t}(\theta_i^2 + \theta_i \gamma_i) &= \langle u^+ + u^-, A^{s-t}u^+ \rangle \\ &\leq \|u^+ + u^-\|_{L^{q+1}(\Omega, |x|^{-\beta})} \|A^{s-t}u^+\|_{L^{a'}(\Omega, |x|^\theta)}. \end{aligned}$$

Then there exists a constant C_k such that

$$\sum_{i=1}^k \lambda_i^{s-t}(\theta_i^2 + \theta_i \gamma_i) \leq C_k \|u^+ + u^-\|_{L^{q+1}(\Omega, |x|^{-\beta})} \|u^+\|_{L^2(\Omega)}. \tag{3.9}$$

In a similar way, using that $v^+ = A^{s-t}u^+$ and $v^- = -A^{s-t}u^-$ we have that there exists a constant C_k such that

$$\sum_{i=1}^{k} \lambda_i^{s-t}(\theta_i^2 - \theta_i\gamma_i) \le C_k \|v^+ + v^-\|_{L^{p+1}(\Omega,|x|^{-\alpha})} \|v^+\|_{L^2(\Omega)}. \tag{3.10}$$

Depending on the sign of $\sum_{i=1}^k \lambda_i^{s-t}\theta_i\gamma_i$ we use (3.9) or (3.10) to conclude that

$$\|u^+\|_{L^2(\Omega)} \le C_k \|u^+ + u^-\|_{L^{q+1}(\Omega,|x|^{-\beta})}$$

or

$$\|u^+\|_{L^2(\Omega)} \le C_k \|v^+ + v^-\|_{L^{p+1}(\Omega,|x|^{-\alpha})}.$$

Hence,

$$I(z) \le \sigma^{\mu+\nu-2}(\|z^+\|^2 - \|z^-\|^2) - C_k\sigma^{(q+1)(\mu-1)}\|u^+\|_{L^2(\Omega)}^{q+1}$$

or

$$I(z) \le \sigma^{\mu+\nu-2}(\|z^+\|^2 - \|z^-\|^2) - C_k\sigma^{(p+1)(\nu-1)}\|u^+\|_{L^2(\Omega)}^{p+1}.$$

Thus we may choose $\|z^+\|_E = \sigma_k$ large enough in order to obtain $\sigma_k > \rho_k$ and, by the condition on μ and ν, $I(z) \le 0$.

Taking $\|z^+\| \le \sigma_k$ and $\|z^-\| = M_k$, we get

$$I(z) \le \sigma_k^{\mu+\nu-2}\left(\sigma_k^2 - M_k^2\right)$$

and then choosing M_k large enough we find that

$$I(z) \le 0.$$

In this way we have finished with the proof of the first part of Lemma 3.3.

Next we choose B_k large so that the second inequality holds. □

Proof of Theorem 1.1. Existence of infinitely many solutions.

For $k \ge 1$, Lemmas 3.2 and 3.3 allow us to use Theorem 3.1. As a consequence the functional I has a critical value $c_k \in [\alpha_k, B_k]$. Since $\alpha_k \to \infty$ we get infinitely many critical values of I. Therefore we have infinitely many solutions of (1.1). □

Now we turn our attention to the existence of a positive solution to (1.1). We use ideas from [17] under the functional setting of Section 2. We start by redefining the Hamiltonian. Let us define $\tilde{H} : \partial\Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\tilde{H}(x,u,v) = \begin{cases} H(x,u,v) & \text{if } u,v \ge 0, \\ H(x,0,v) & \text{if } u \le 0,\ v \ge 0, \\ H(x,u,0) & \text{if } u \ge 0,\ v \le 0, \\ 0 & \text{if } u,v \le 0. \end{cases} \tag{3.11}$$

We observe that if (u,v) is a nontrivial strong solution of

$$\begin{cases} -\Delta u = \tilde{H}_v(x,u,v) & \text{in } \Omega, \\ -\Delta v = \tilde{H}_u(x,u,v), \end{cases} \tag{3.12}$$

with Dirichlet boundary conditions, then by the maximum principle we have that u and v are strictly positive in $\overline{\Omega}$. Hence (u, v) is a positive strong solution of (1.1).

To find a nontrivial solution of (3.12) we want to apply the results of Section 3. By our assumptions, the new Hamiltonian $\tilde{H}$ is regular.

We have to adapt the proof of Theorem 1.1 to the functional I with the Hamiltonian H replaced by $\tilde{H}$. We observe that the proof of the Palais-Smale condition and the geometric conditions follows as before with some minor modifications, see [17] for the details.

Proof of Theorem 1.1. Existence of a positive strong solution.

As a consequence of the previous results the modified functional I (with the modified Hamiltonian $\tilde{H}$ instead of H) has a critical value $c \neq 0$. Hence, by the maximum principle, we obtain a positive strong solution of (1.1). □

References

[1] B. Abdellaoui and I. Peral. *Some results for semilinear elliptic equations with critical potential*, Proc. of the Royal Soc. of Edinburgh, 132A (2002), 1–24.

[2] B. Abdellaoui and I. Peral. *Existence and nonexistence results for quasilinear elliptic equations involving the p-Laplacian with a critical potential*, Annali di Matematica Pura e Applicata, 182, (2003), 247–270.

[3] B. Abdellaoui and I. Peral. *Nonexistence results for quasilinear elliptic equations related to Caffarelli-Kohn-Nirenberg inequalities*, Communications in Pure and Applied Analysis, 2 (2003), 539–566.

[4] B. Abdellaoui, E. Colorado and I. Peral, *Some remarks on elliptic equations with singular potentials and mixed boundary conditions*, Advanced Nonlinear Studies, 4 (2004), 501–531.

[5] B. Abdellaoui, E. Colorado and I. Peral, *Effect of the boundary conditions in the behavior of the optimal constant of some Caffarelli-Kohn-Nirenberg inequalities: Applications to critical nonlinear elliptic problems.* To appear in Advances on Differential Equations.

[6] M. Badiale and E. Serra, *Multiplicity results for the supercritical Hénon equation*, Adv Nonlinear Stu. 4 (2004), 453-467.

[7] T.Bartsch and D.G. de Figueiredo, *Infinitely many solutions of Nonlinear Elliptic Systems*, Progress in Nonlinear Diff. Eq. and Their Appl., Vol. 35 (1999), 51-67, Birkhäuser Verlag Basel/ Switzerland.

[8] L. Boccardo and D.G. deFigueiredo, *Some remarks on a system of quasilinear elliptic equations*, NoDEA Nonlinear Differential Equations Appl., 9(3) (2002), 309–323.

[9] H. Brezis and X. Cabré, *Some simple nonlinear PDE's without solution,* Boll. Unione. Mat. Ital. Sez. B, 8 (1998), 223–262.

[10] H. Brezis, L. Dupaigne and A. Tesei, *On a semilinear equation with inverse-square potential,* Selecta Math. 11 (2005), 1–7.

[11] L. Caffarelli, R. Kohn, L. Nirenberg, *First order interpolation inequalities with weights*, Compositio Math., 53, (1984), 259-275.

[12] Ph. Clément, D.G. deFigueiredo and E. Mitidieri, *Positive solutions of semilinear elliptic systems*, Comm. Part. Diff. Eq., 17 (1992), 923–940.

[13] E. Colorado and I. Peral, *Eigenvalues and Bifurcation for Elliptic Equations related to Caffarelli-Kohn-Nirenberg Inequalities, with Mixed Dirichlet-Neumann Boundary Conditions*, Topol. Methods Nonlinear Anal., 23 (2004), 239–273.

[14] D.G. deFigueiredo, *Positive solutions of semilinear elliptic equations*, Springer Lecture Notes in Mathematics 957 (1982), 34–87.

[15] Ph. Clément, D. G. deFigueiredo and E. Mitidieri, Positive Solutions of Semilinear Elliptic Systems, Comm.Part.Diff.Eq.17(1992), 923-940.

[16] D.G. deFigueiredo, *Semilinear elliptic systems: a survey of superlinear problems*, Resenhas IME-USP, 2 (1996), 373–391.

[17] D.G. deFigueiredo and P.L. Felmer, *On superquadratic elliptic systems*, Trans. Amer. Math. Soc., 343 (1994), 99–116.

[18] D.G. deFigueiredo, *Semilinear Elliptic Systems* in *Nonlinear Functional Analysis and Applications to Differential Equations*, pag 122-152, edited by A. Ambrosetti, K.C. Chang, I. Ekeland World Scientific, 1998.

[19] D.G. deFigueiredo and C.A. Magalhães, *On nonquadratic hamiltonian elliptic systems*, Adv. Diff. Eq., 1(5) (1996), 881–898.

[20] D.G. deFigueiredo and B. Ruf, *Elliptic systems with nonlinearityes of arbitrary growth*, Mediterr. J. Math., 1(4), (2004), 417–431.

[21] P. Felmer, *Periodic solutions of 'superquadratic' Hamiltonian systems*, J. Differential Equations, 102 (1993), 188–207.

[22] P. Felmer, R.F. Manásevich and F. de Thélin, *Existence and uniqueness of positive solutions for certain quasilinear elliptic systems*, Comm. Par. Diff. Eq., 17 (1992), 2013–2029.

[23] P. Felmer and Z-Q. Wang, *Multiplicity for symmetric indefinite functionals: application to hamiltonian and elliptic systems*, Topol. Methods Nonlinear Anal., 12 (2) (1998), 207–226.

[24] J. Fernández Bonder and J.D. Rossi, *Existence for an elliptic system with nonlinear boundary conditions via fixed point methods*. Adv. Differential Equations. 6 (1), (2001), 1–20.

[25] J. Garcia Azorero and I. Peral, *Hardy Inequalities and some Critical Elliptic and Parabolic problems* J. Differential Equations, 144 (1998), 441–476.

[26] D. Gilbarg and N. S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer-Verlag, NY (1983).

[27] N. Ghoussoub and C. Yuan, *Multiple solutions for quasi-linear involving the critical Sobolev and Hardy exponents* Trans. Amer. Math. Soc. 352 (2000), 5703–5743.

[28] J. Hulshof and R.C.A.M. van der Vorst, *Differential systems with strongly indefinite variational structure*, J. Funct. Anal. 114, (1993), 32–58.

[29] J.L. Lions and E. Magenes, Non-homogenoeus boundary value problems and applications, Vol. I and II, Springer-Verlag, Berlin, (1972).

[30] W. M. Ni, *A nonlinear Dirichlet problem on the unit ball and its applications*, Indiana Univ. Math. Jour. 31 (1982), 801-807.

[31] L. A. Peletier and R. C. A. M. van der Vorst, Existence and Non- existence of Positive Solutions of Non-linear Elliptic Systems and the Biharmonic Equation, Diff.Int.Eq.5(1992), 747-767.

[32] P. Rabinowitz, *Minimax methods in critical point theory with applications to differential equations*, CBMS Regional Conf. Ser. in Math., no. 65, Amer. Math. Soc., Providence, R.I. (1986).

[33] E. Serra, *Non radial positive solutions for the Hénon equation with critical growth*, Calc. Var. 23 (2005), 301-326.

[34] D. Smets and M. Willem, *Partial symmetry and asymptotic behavior for some elliptic variational problems*, Calc. Var. 18 (2003), 57-75.

[35] R. van der Vorst, *Variational identities and applications to differential systems*, Arch. Rat. Mech. Anal. 116 (1991), 375-398.

IMECC, Universidade Estadual de Campinas, 13081-970, Campinas, SP, Brasil.
E-mail address: djairo@ime.unicamp.br

Departamento de Matemáticas, U. Autonoma de Madrid, 28049 Madrid, Spain.
E-mail address: ireneo.peral@uam.es

Departamento de Matemática, FCEyN, UBA (1428) Buenos Aires, Argentina.
E-mail address: jrossi@dm.uba.ar