A CONVEX-CONCAVE ELLIPTIC PROBLEM WITH A PARAMETER ON THE BOUNDARY CONDITION

JORGE GARCÍA-MELIÁN, JULIO D. ROSSI AND JOSÉ C. SABINA DE LIS

Abstract. In this paper we study existence and multiplicity of nonnegative solutions to

$$\begin{cases} \Delta u = u^p + u^q & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega. \end{cases}$$

Here Ω is a smooth bounded domain of $\mathbb{R}^N$, ν stands for the outward unit normal and p, q are in the convex-concave case, that is $0 < q < 1 < p$. We prove that there exists $\Lambda^* > 0$ such that there are no nonnegative solutions for $\lambda < \Lambda^*$, and there is a maximal nonnegative solution for $\lambda \geq \Lambda^*$. If λ is large enough, then there exist at least two nonnegative solutions. We also study the asymptotic behavior of solutions when $\lambda \to \infty$ and the occurrence of dead cores. In the particular case where Ω is the unit ball of $\mathbb{R}^N$ we show exact multiplicity of radial nonnegative solutions when λ is large enough, and also the existence of nonradial nonnegative solutions.

1. Introduction

In the well-known paper [1], the following elliptic problem

$$\begin{cases} -\Delta u = u^p + \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

was considered. Here $\lambda > 0$ is a parameter and the main point is that the nonlinearity in the equation is a combination of a convex term and a concave term, that is, the exponents verify

$$0 < q < 1 < p. \tag{1.2}$$

Among other results, it was shown in [1] that there exists a value $\Lambda > 0$ such that problem (1.1) does not have positive solutions when $\lambda > \Lambda$, while it has at least a positive solution for $\lambda = \Lambda$ and at least two positive solutions if $0 < \lambda < \Lambda$ (provided in addition that p is subcritical). These results have been subsequently generalized to deal with more general operators (cf. [11], [12] for the p-Laplacian or [8] for fully nonlinear operators) or boundary conditions (see [9] for mixed-type boundary conditions and [13] for a nonlinear boundary condition).

The purpose of this paper is to consider a related problem where the presence of a convex and a concave term allows to have multiplicity of nonnegative nontrivial solutions. The problem we deal with is the following:

$$\begin{cases} \Delta u = u^p + u^q & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a smooth $C^{2,\alpha}$ domain or $\mathbb{R}^N$, ν stands for the outward pointing unit normal and p and q verify (1.2).

Observe that an important feature in (1.3) is the presence of the parameter λ in the boundary condition. Problems with bifurcation parameters in the boundary conditions of the form (1.3) appear in a natural way when one considers the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ (see for instance [10] and the survey [27]), but in spite of this they are not very frequent in the literature. We quote the works [15], [16], [18] by the authors, which deal with different elliptic problems with the same boundary condition and [4], [5], [6], where the boundary conditions are more general.

By a nonnegative solution to (1.3) we mean a nonnegative nontrivial weak solution in $H^1(\Omega)$, that is, a function $u \in H^1(\Omega)$, $u \gneq 0$, verifying

$$-\int_\Omega \nabla u \nabla \varphi + \lambda \int_{\partial\Omega} u\varphi = \int_\Omega u^p \varphi + \int_\Omega u^q \varphi,$$

for every $\varphi \in H^1(\Omega)$, where the last two integrals are assumed to be finite. However, let us mention that weak solutions are indeed classical (they are bounded thanks for instance to Lemma 5 in [17] and hence in $C^{2,\gamma}(\overline{\Omega})$ by Lemma 7 in [16], where $\gamma = \min\{\alpha, q\}$). Recall that the functional

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{\lambda}{2}\int_{\partial\Omega} |u|^2 + \frac{1}{p+1}\int_\Omega |u|^{p+1} + \frac{1}{q+1}\int_\Omega |u|^{q+1} \tag{1.4}$$

defines a natural energy functional for (1.3).

We come next to the statement of our results. Let us begin with the questions of existence and nonexistence of nonnegative nontrivial solutions.

Theorem 1. *Assume $\Omega \subset \mathbb{R}^N$ is a $C^{2,\alpha}$ bounded domain and p and q verify* (1.2). *Then there exists $\Lambda^* > 0$ such that the following properties hold.*

(a) *There are no nonnegative nontrivial solutions to* (1.3) *for $\lambda < \Lambda^*$.*
(b) *For every $\lambda \geq \Lambda^*$, there exists a maximal nonnegative nontrivial solution $u = u_\lambda$ to* (1.3).
(c) *There exists $\Lambda^{**} \geq \Lambda^*$ such that* (1.3) *admits a second nontrivial nonnegative solution $u = v_\lambda$ for $\lambda > \Lambda^{**}$. Moreover, v_λ has nonnegative energy for large λ, i.e. $E(v_\lambda) \geq 0$ for $\lambda \geq \lambda_0$.*

In the literature on convex-concave problems, one actually has $\Lambda^* = \Lambda^{**}$. We remark that in those cases the second solution can be obtained directly by means of the mountain pass theorem because the first solution can actually be obtained as a local minimum of the functional (1.4). To prove that this actually happens an important tool is the strong maximum principle: if $\underline{u}$, $\overline{u}$ are respectively sub and supersolution with $\underline{u} \leq \overline{u}$ in Ω, then $\underline{u} < \overline{u}$ in $\overline{\Omega}$ unless $\underline{u} \equiv \overline{u}$ and both are solutions. In our present situation, this property does not hold, due to the possible appearance of dead cores in the solutions (see below). Thus in general it is difficult to prove that $\Lambda^* = \Lambda^{**}$. This can be shown in the particular case where the maximal solution $u = u_{\Lambda^*}$ to (1.3) corresponding to $\lambda = \Lambda^*$ is strictly positive on $\partial\Omega$, (this occurs, for instance, when Ω is a ball, cf. Theorem 4 below and Section 5).

Our next concern is the asymptotic behavior of solutions to (1.3) as $\lambda \to +\infty$. We will show that there are essentially two behaviors: the maximal solution increases and converges to a boundary blow-up solution, while all bounded solutions tend to zero at a precise rate.

Theorem 2. *Under the assumptions of Theorem 1, the following properties on the asymptotic behavior of* (1.3) *as* $\lambda \to \infty$ *hold true.*

(a) *The maximal solution* u_λ *to* (1.3) *verifies* $u_\lambda \to U$ *in* $C^2_{loc}(\Omega)$*, where* U *is the unique nonnegative solution to*

$$\begin{cases} \Delta U = U^p + U^q & \text{in } \Omega, \\ U = \infty & \text{on } \partial\Omega. \end{cases} \tag{1.5}$$

(b) *If* $\{v_\lambda\}_{\lambda \geq \lambda_0}$ *is a family of solutions to* (1.3) *such that* $\sup_\Omega v_\lambda \leq C$ *for* $\lambda \geq \lambda_0$ *then there exist positive constants* C_1*,* C_2*, such that*

$$C_1 \lambda^{-\frac{2}{1-q}} \leq \max_\Omega v_\lambda \leq C_2 \lambda^{-\frac{2}{1-q}}, \tag{1.6}$$

for $\lambda \geq \lambda_0$*.*

Remarks 1.

a) Theorem 2 elucidates the behavior for large λ of two specific kinds of solutions. Namely, those which became *uniformly* unbounded on $\partial\Omega$ as $\lambda \to \infty$ (the family of maximal solutions u_λ) and those solutions to (1.3) that remain uniformly bounded on $\partial\Omega$ for large λ (notice that solutions to (1.3) are subharmonic). Indeed, in the case where Ω is a ball and solutions are radially symmetric the full asymptotic behavior of (1.3) is the one described in Theorem 2. However, in the general case, asymptotic responses that are a combination of the ones in cases a) and b) are possible (see Section 6).

b) An optimum version of estimate (1.6) on the asymptotic amplitude of nonnegative solutions that stay bounded as $\lambda \to \infty$ can be obtained in the case of radially symmetric solutions (Theorem 4 (b) below).

As we have already mentioned, one of the difficulties that appear when dealing with nonnegative solutions u to problem (1.3) is the possible appearance of dead cores, that is, the set $\mathcal{O} = \{x \in \Omega : \ u(x) = 0\}$ could be nonempty. We next state some conditions which ensure that nonnegative solutions do or do not have dead cores.

Theorem 3. *Assume that the hypotheses of Theorem 1 hold. Then,*

(a) *If* $\{v_\lambda\}_{\lambda \geq \lambda_0}$ *is a family of nonnegative solutions with nonnegative energy, that is* $E(v_\lambda) \geq 0$*, then* v_λ *exhibits a dead core for large enough* λ*. Moreover,* $v_\lambda(x) = 0$ *for all* $x \in \Omega$ *satisfying*

$$d(x)^{\frac{2}{1-q}+N-1} \geq \frac{C}{\lambda}, \tag{1.7}$$

with $d(x) = dist\ (x, \partial\Omega)$ *and* $C > 0$ *a constant that does not depend on* λ*. In particular, the family* $\{v_\lambda\}$ *obtained in Theorem 1 (c) exhibits a dead core for large* λ*.*

(b) *If* $\{v_\lambda\}_{\lambda \geq \lambda_0}$ *is a family of nonnegative solutions with* $\sup_\Omega v_\lambda \leq C$*, then* v_λ *has a dead core for large enough* λ*. In this case,* $v_\lambda(x) = 0$ *when*

$$d(x) \geq \frac{C}{\lambda} \tag{1.8}$$

for large λ*, where* $C > 0$ *does not depend on* λ*.*

(c) *There exists* $R_0 > 0$ *such that if* Ω *contains a ball* B_R *with radius* $R \geq R_0$ *then every nonnegative solution to* (1.3) *has a dead core inside* Ω.

(d) *If* Ω *lies between parallel hyperplanes* π_1, π_2 *with* $\operatorname{dist}(\pi_1, \pi_2) < 2R_0$, R_0 *as in* (c)*, then the maximal solution* u_λ *is strictly positive when* λ *is large enough.*

Remark 2. Estimates (1.7) and (1.8) implies that in cases a) and b), dead cores $\mathcal{O}_\lambda = \{v_\lambda(x) = 0\}$ progressively fill Ω as $\lambda \to \infty$. A more precise version of estimate (1.8) can be obtained when (1.3) is radially symmetric (Theorem 4 (b)).

Finally, we will concentrate in the particular case where Ω is a ball in $\mathbb{R}^N$. In this case, we have more precise information, particularly when dealing with radial solutions. We will show exact multiplicity of radial nonnegative solutions to (1.3) when λ is large enough, and also the existence of nonradial nonnegative solutions, in the spirit of [16].

Theorem 4. *Under the same conditions as in Theorem 1, assume in addition that* $\Omega = B_R$*, the open ball in* $\mathbb{R}^N$ *with radius* R *and center* $x = 0$*. Then,*

(a) *There exists* $\Lambda^* > 0$ *such that problem* (1.3) *has no nonnegative nontrivial solutions for* $\lambda < \Lambda^*$*, while it has at least a nonnegative nontrivial* radial *solution when* $\lambda = \Lambda^*$ *and at least two nonnegative nontrivial radial solutions if* $\lambda > \Lambda^*$*, one of them being the maximal solution* u_λ.

(b) *There exists* $\lambda_0 > 0$ *such that problem* (1.3) *has* exactly *two nonnegative nontrivial radial solutions* u_λ *and* z_λ *when* $\lambda > \lambda_0$*, whose asymptotic behavior is given by parts* (a) *and* (b) *in Theorem 2, respectively. Moreover,*

$$z_\lambda(R) = \sup_{B_R} v_\lambda \sim \left(\frac{2}{q+1}\right)^{\frac{1}{1-q}} \lambda^{-\frac{2}{1-q}} \tag{1.9}$$

as $\lambda \to \infty$ *and*

$$\rho(\lambda) \sim \frac{2}{(1-q)\lambda}, \tag{1.10}$$

as $\lambda \to \infty$ *where* $\rho(\lambda)$ *stands for the distance from the dead core* $\{z_\lambda = 0\}$ *to the boundary* $|x| = R$ *of* B_R.

(c) *There exists* $\lambda_0' > 0$ *such that problem* (1.3) *admits a nonradial nonnegative solution* v_λ *when* $\lambda > \lambda_0'$.

The rest of the paper is organized as follows: in Section 2 we prove Theorem 1. In Section 3 we study the asymptotic behavior of solutions when $\lambda \to \infty$, while dead core formation is analyzed in Section 4. Section 5 is devoted to the problem in a ball of $\mathbb{R}^N$. Finally, Section 6 collects some results on multiplicity of solutions.

2. Existence and nonexistence of solutions

In this section we prove Theorem 1, which will be split into a series of lemmas. We begin by part (a), that is, nonnegative solutions do not exist when λ is small. We are always assuming that Ω is a smooth bounded domain and that p, q verify (1.2).

Lemma 5. *There exists $\Lambda > 0$ such that there are no nonnegative nontrivial solutions to* (1.3) *for $\lambda < \Lambda$.*

Proof. Assume that $u \in H^1(\Omega)$ is a nonnegative nontrivial weak solution, then we have

$$\lambda \int_{\partial\Omega} u^2 - \int_\Omega |\nabla u|^2 = \int_\Omega (u^{p+1} + u^{q+1}).$$

Note that this implies that $u \not\equiv 0$ on $\partial\Omega$. Using that p and q verify (1.2), we observe that $s^{p+1} + s^{q+1} \geq \max\{s^{p+1}, s^{q+1}\} \geq s^2$ for $s \geq 0$ and hence we get

$$\lambda \int_{\partial\Omega} u^2 \geq \int_\Omega |\nabla u|^2 + \int_\Omega u^2.$$

Now we use the continuity of the Sobolev trace embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ to obtain the existence of a positive constant Λ such that

$$\lambda \int_{\partial\Omega} u^2 \geq \Lambda \int_{\partial\Omega} u^2.$$

Therefore we conclude that $\lambda \geq \Lambda$ since $u \not\equiv 0$ on $\partial\Omega$. □

Now we recall that, since $p > 1$, Theorem 1 in [15] provides with a unique solution to the problem

$$\begin{cases} \Delta u = u^p & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

for every $\lambda > 0$, which will be denoted by U_λ (it is also worthy of mention that $U_\lambda > 0$ in $\overline{\Omega}$, and is increasing and continuous in λ). This solution provides with an upper bound for nonnegative nontrivial solutions to (1.3). Indeed, it is easy to show that if u is a nonnegative nontrivial solution to (1.3), then it is a subsolution to (2.1). Since MU_λ is a supersolution for large M, we obtain the following result:

Lemma 6. *Let u be a nonnegative nontrivial solution to* (1.3)*. Then*

$$u \leq U_\lambda \quad \textit{in } \Omega.$$

Next let us face the question of existence of solutions. We begin by part (b) of Theorem 1.

Lemma 7. *There exists $\Lambda^* > 0$ such that* (1.3) *has a maximal nonnegative nontrivial solution for $\lambda \geq \Lambda^*$, while no nonnegative nontrivial solutions exist when $\lambda < \Lambda^*$.*

Proof. Let us first prove that nonnegative nontrivial solutions exist for large λ. To this end, consider the problem

$$\begin{cases} \Delta u = 2u^q & \text{in } \Omega, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

It follows from Theorem 1 in [16] that there exists a family of nonnegative nontrivial solutions $\{V_\lambda\}_{\lambda>0}$ to (2.2) verifying $V_\lambda \to 0$ uniformly in $\overline{\Omega}$ as

$\lambda \to \infty$. Let Λ_1 be such that $V_\lambda \leq 1$ in Ω for $\lambda > \Lambda_1$. Then V_λ is a subsolution to (1.3), since

$$\begin{cases} \Delta V_\lambda = 2V_\lambda^q \geq V_\lambda^p + V_\lambda^q & \text{in } \Omega, \\ \dfrac{\partial V_\lambda}{\partial \nu} = \lambda V_\lambda & \text{on } \partial\Omega. \end{cases}$$

On the other hand, the unique positive solution U_λ to (2.1) is a supersolution to (1.3). Since U_λ stays bounded away from zero while $V_\lambda \to 0$ uniformly in $\overline{\Omega}$ as $\lambda \to \infty$, we also have $V_\lambda \leq U_\lambda$ when λ is large enough, so that a nonnegative nontrivial solution to (1.3) exists when λ is large enough.

Therefore, we may define Λ^* as the infimum of those λ_0 for which (1.3) has a nonnegative nontrivial solution for every $\lambda > \lambda_0$. Thanks to Lemma 5 we have $\Lambda^* > 0$. Let us next show that solutions do exist for every $\lambda > \Lambda^*$. To this aim we momentarily change notation to make explicit the dependence of (1.3) on λ and denote it as $(1.3)_\lambda$.

Take $\lambda > \Lambda^*$. Thanks to the definition of Λ^*, there exists a value μ with $\Lambda^* < \mu < \lambda$ such that $(1.3)_\mu$ admits a nontrivial nonnegative solution, which will be denoted by u_μ. Observe that u_μ is a subsolution to $(1.3)_\lambda$ since $\mu < \lambda$, while, according to Lemma 2.1, $u_\mu \leq U_\mu \leq U_\lambda$. Observing that U_λ is a supersolution to $(1.3)_\lambda$, we obtain at least a nonnegative nontrivial solution to this problem, as we wanted to show.

We now observe that by Lemma 2.1, U_λ is a supersolution to $(1.3)_\lambda$ which controls every possible nonnegative nontrivial solution. Thus it is standard to obtain that $(1.3)_\lambda$ admits a maximal solution for every $\lambda > \Lambda^*$, and the maximal solution is increasing in λ. To conclude the proof, we only need to show that there exists a nontrivial nonnegative solution when $\lambda = \Lambda^*$ as well. To this end we just have to take an arbitrary sequence $\lambda_n \downarrow \Lambda^*$ and consider the function

$$u := \lim_{n\to\infty} u_{\lambda_n},$$

where u_λ is the maximal solution to $(1.3)_\lambda$ constructed above. Observe that by the monotonicity of u_λ, this limit exists pointwise. Moreover, a standard compactness argument shows that the limit also holds in $H^1(\Omega)$. Thus u will be a nonnegative solution to $(1.3)_{\Lambda^*}$, and we only have to rule out the possibility $u = 0$. Arguing by contradiction, assume that $u = 0$, and define the functions $v_n = u_{\lambda_n}/\|u_{\lambda_n}\|_{L^2(\partial\Omega)}$, which verify

$$\begin{cases} \Delta v_n = \|u_{\lambda_n}\|^{p-1}_{L^2(\partial\Omega)} v_n^p + \|u_{\lambda_n}\|^{q-1}_{L^2(\partial\Omega)} v_n^q & \text{in } \Omega, \\ \dfrac{\partial v_n}{\partial \nu} = \lambda_n v_n & \text{on } \partial\Omega. \end{cases}$$

Then, we have

$$\lambda_n - \int_\Omega |\nabla v_n|^2 = \|u_{\lambda_n}\|^{p-1}_{L^2(\partial\Omega)} \int_\Omega v_n^{p+1} + \|u_{\lambda_n}\|^{q-1}_{L^2(\partial\Omega)} \int_\Omega v_n^{q+1}.$$

This implies that v_n is bounded in $H^1(\Omega)$ and therefore we may assume that $v_n \to v$ weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, $L^{q+1}(\Omega)$ and $L^2(\partial\Omega)$. In particular, $\|v\|_{L^2(\partial\Omega)} = 1$. Moreover,

$$\lambda_n \geq \|u_{\lambda_n}\|^{q-1}_{L^2(\partial\Omega)} \int_\Omega v_n^{q+1},$$

and we obtain that $v=0$, which is a contradiction since $\|v\|_{L^2(\partial\Omega)}=1$. This concludes the proof. $\square$

To complete the proof of Theorem 1 we have to prove part (c).

Lemma 8. *There exists $\Lambda^{**}\geq\Lambda^*$ such that for every $\lambda>\Lambda^{**}$, there exist at least two nonnegative nontrivial solutions to* (1.3).

The proof of Lemma 8 is based in the mountain pass theorem (cf. [3], [22], [25]). But observe first that the natural functional associated to solutions of (1.3) is given by

$$E(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\frac{\lambda}{2}\int_{\partial\Omega}|u|^2+\frac{1}{p+1}\int_\Omega|u|^{p+1}+\frac{1}{q+1}\int_\Omega|u|^{q+1},$$

and in order to obtain a well-defined and differentiable functional in $H^1(\Omega)$ we should assume that p is subcritical. To get rid of this supplementary undesirable hypothesis, we take advantage of the fact that every solution verifies $u\leq U_\lambda$ and modify the functional accordingly. Introduce the truncated functions

$$f(x,u)=\begin{cases}U_\lambda^p & u>U_\lambda,\\ u^p & 0\leq u\leq U_\lambda,\\ 0 & u\leq 0,\end{cases}$$

and, for technical reasons, also

$$g(x,u)=\begin{cases}U_\lambda & u>U_\lambda,\\ u & 0\leq u\leq U_\lambda,\\ 0 & u\leq 0.\end{cases}$$

Let

$$F(x,u)=\int_0^u f(x,s)\,ds,\qquad G(x,u)=\int_0^u g(x,s)\,ds,$$

and consider the truncated functional

$$J(u)=\frac{1}{2}\int_\Omega|\nabla u|^2-\lambda\int_{\partial\Omega}G(x,u)+\int_\Omega F(x,u)+\frac{1}{q+1}\int_\Omega|u|^{q+1}.$$

Notice that the sublinear term $|u|^{q+1}$ has not been truncated, since it is not necessary.

It is standard that J is a C^1 functional in $H^1(\Omega)$, whose critical points are weak solutions to the problem

$$\begin{cases}\Delta u=f(x,u)+|u|^{q-1}u & \text{in }\Omega,\\ \dfrac{\partial u}{\partial\nu}=\lambda g(x,u) & \text{on }\partial\Omega.\end{cases}\tag{2.3}$$

As a first step, let us check that weak solutions to (2.3) verify $0\leq u\leq U_\lambda$ and therefore are also weak solutions to (1.3).

Lemma 9. *Let $u\in H^1(\Omega)$ be a weak solution to* (2.3). *Then*

$$0\leq u\leq U_\lambda.$$

In particular, u is a weak solution to (1.3).

Proof. Take $u_- = \min\{u, 0\}$ as a test function in the weak formulation of (2.3) to obtain

$$-\int_\Omega |\nabla u_-|^2 + \lambda \int_{\partial\Omega} g(x,u)u_- = \int_\Omega f(x,u)u_- + \int_\Omega |u_-|^{q+1}.$$

Since $f(x,u)u_- = 0$ and $g(x,u)u_- = 0$ we have

$$-\int_\Omega |\nabla u_-|^2 = \int_\Omega |u_-|^{q+1}$$

and then we conclude that $u_- = 0$, that is, $u \geq 0$. Similarly, taking $(u - U_\lambda)_+ = \max\{u - U_\lambda, 0\}$ as test function we get,

$$\begin{aligned} -\int_{u>U_\lambda} |\nabla(u-U_\lambda)|^2 &+ \lambda \int_{\partial\Omega\cap\{u>U_\lambda\}} (g(x,u) - U_\lambda)(u - U_\lambda) \\ &= \int_{u>U_\lambda} (f(x,u) - U_\lambda^p)(u - U_\lambda) + \int_{u>U_\lambda} u^q (u - U_\lambda). \end{aligned}$$

As $g(x,u) - U_\lambda = 0$ and $f(x,u) - U_\lambda^p = 0$ when $u > U_\lambda$ we obtain

$$-\int_{u>U_\lambda} |\nabla(u-U_\lambda)|^2 = \int_{u>U_\lambda} u^q(u-U_\lambda),$$

and it follows that $u \leq U_\lambda$. This concludes the proof. □

The next step is to ensure that J has the desirable compactness properties. We will use in $H^1(\Omega)$ the norm

$$\|u\|_{H^1} = \|\nabla u\|_{L^2(\Omega)^N} + \|u\|_{L^2(\partial\Omega)},$$

which is equivalent to the usual one. Then we have:

Lemma 10. *The functional J verifies the Palais-Smale condition.*

Proof. Let $u_n \in H^1(\Omega)$ be a sequence such that $J(u_n) \leq C$ and $J'(u_n) \to 0$. First, let us see that u_n is bounded in $H^1(\Omega)$. Assume, by contradiction, that $\|u_n\|_{H^1} \to \infty$. Since $f(x,u) \geq 0$ and $G(x,u) \leq u^2/2$ we obtain

$$\frac{1}{2}\int_\Omega |\nabla u_n|^2 \leq \frac{\lambda}{2}\int_{\partial\Omega} u_n^2 + C,$$

so that $\|u_n\|_{L^2(\partial\Omega)} \to \infty$. Define $v_n = u_n/\|u_n\|_{L^2(\partial\Omega)}$. Then $\|v_n\|_{H^1(\Omega)}$ is bounded and we can extract a subsequence, denoted again by v_n, such that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$ and strongly in $L^2(\partial\Omega)$ and in $L^{q+1}(\Omega)$. Observe that

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla v_n|^2 &- \frac{\lambda}{\|u_n\|^2_{L^2(\partial\Omega)}} \int_{\partial\Omega} G(x,u_n) + \frac{1}{\|u_n\|^2_{L^2(\partial\Omega)}} \int_\Omega F(x,u_n) \\ &+ \frac{\|u_n\|^{q-1}_{L^2(\partial\Omega)}}{q+1} \int_\Omega |v_n|^{q+1} \leq \frac{C}{\|u_n\|^2_{L^2(\partial\Omega)}}. \end{aligned}$$

Since

$$\frac{\lambda}{\|u_n\|^2_{L^2(\partial\Omega)}} \int_{\partial\Omega} G(x,u_n) \to 0 \quad \text{and} \quad \frac{1}{\|u_n\|^2_{L^2(\partial\Omega)}} \int_\Omega F(x,u_n) \to 0$$

we conclude that

$$\int_\Omega |v_n|^{q+1} \to 0.$$

This implies $v = 0$, which contradicts $\|v\|_{L^2(\partial\Omega)} = 1$. This contradiction proves that u_n is bounded in $H^1(\Omega)$, thus we can extract a subsequence, still denoted by u_n, such that $u_n \rightharpoonup u$ weakly in $H^1(\Omega)$, strongly in $L^2(\partial\Omega)$ and $L^{q+1}(\Omega)$ and a.e. in Ω. Since f and g are bounded, we obtain by dominated convergence

$$\int_{\partial\Omega} g(x,u_n)u \ , \ \int_{\partial\Omega} g(x,u_n)u_n \to \int_{\partial\Omega} g(x,u)u$$

and

$$\int_{\Omega} f(x,u_n)u \ , \ \int_{\Omega} f(x,u_n)u_n \to \int_{\Omega} f(x,u)u.$$

Finally, as $J'(u_n) \to 0$ we have $J'(u_n)(u-u_n) \to 0$. Since

$$\begin{aligned} J'(u_n)(u-u_n) &= \int_\Omega \nabla u_n \nabla u - \int_\Omega |\nabla u_n|^2 - \lambda \int_{\partial\Omega} g(x,u_n)u \\ &\quad +\lambda \int_{\partial\Omega} g(x,u_n)u_n + \int_\Omega f(x,u_n)u - \int_\Omega f(x,u_n)u_n \\ &\quad + \int_\Omega |u_n|^{q-1}u_n u - \int_\Omega |u_n|^{q+1} \\ &= \int_\Omega \nabla u_n \nabla u - \int_\Omega |\nabla u_n|^2 + o(1), \end{aligned}$$

it follows that

$$\int_\Omega |\nabla u_n|^2 \to \int_\Omega |\nabla u|^2.$$

Hence, we conclude that $u_n \to u$ strongly in $H^1(\Omega)$, so that J verifies the Palais-Smale condition. $\square$

Finally, we need to check that J verifies the geometric conditions of the mountain pass lemma. Let us prove first that $u = 0$ is a local minimum of J.

Lemma 11. *The functional J has a strict local minimum at $u = 0$.*

Proof. We have $J(0) = 0$. Let us assume that there exists a sequence $u_n \to 0$ with $J(u_n) \le 0$. Then, taking $v_n = u_n/\|u_n\|_{L^2(\partial\Omega)}$ (note that $u_n \not\equiv 0$ on $\partial\Omega$ since $u_n \not\equiv 0$ in Ω and $J(u_n) \le 0$), we have

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2 \le \lambda \int_{\partial\Omega} \frac{G(x,u_n)}{\|u_n\|^2_{L^2(\partial\Omega)}} \le C \int_{\partial\Omega} v_n^2 = C,$$

for some positive constant C. Hence we can extract a subsequence, still denoted by v_n, such that $v_n \rightharpoonup v$ weakly in $H^1(\Omega)$, strongly in $L^2(\partial\Omega)$ and $L^{q+1}(\Omega)$ and a.e. in Ω. In particular, we have that $\|v_n\|_{L^2(\partial\Omega)} = 1$, and then $v \neq 0$. On the other hand,

$$\frac{1}{2}\int_\Omega |\nabla v_n|^2 - \frac{\lambda}{\|u_n\|^2_{L^2(\partial\Omega)}} \int_{\partial\Omega} G(x,u_n) + \frac{\|u_n\|^{q-1}_{L^2(\partial\Omega)}}{q+1} \int_\Omega |v_n|^{q+1} \le 0,$$

which implies that $\int_\Omega |v_n|^{q+1} \to 0$, a contradiction. $\square$

We can finally proceed with the proof of Lemma 8. This will also conclude the proof of Theorem 1.

Proof of Lemma 8. Since J verifies the Palais-Smale condition we can find a second solution using the mountain pass theorem. In fact, using Lemma 11, we only need to check the existence of $u \in H^1(\Omega)$ such that $J(u) < 0$. This is easily achieved by simply taking an arbitrary fixed function $u \in H^1(\Omega)$ which does not vanish identically on $\partial\Omega$ and considering a large enough λ. Thus the mountain pass theorem implies the existence of a critical point u of J such that $J(u) > 0$. It follows that u is a nontrivial solution to (2.3) and by Lemma 9, u is a nontrivial nonnegative solution to (1.3).

Finally, notice that Theorem 3 (a), whose forthcoming proof is independent of the present one, implies that the maximal solution has negative energy for large enough λ. Indeed, it will be proven that if for some sequence $\lambda_n \to \infty$ we had $J(u_{\lambda_n}) \geq 0$ then $u_{\lambda_n} \to 0$ pointwise in Ω, which is impossible since the maximal solution is increasing in λ. In particular, we can guarantee that the nonnegative nontrivial solution just constructed does not coincide with the maximal solution.

To summarize, we have shown the existence of $\Lambda^{**} > 0$ such that problem (1.3) admits at least two nontrivial nonnegative solutions for $\lambda \geq \Lambda^{**}$. The proof is finished. □

3. Behavior as $\lambda \to \infty$

In this section we analyze the behavior for large λ of nonnegative solutions to (1.3).

Proof of Theorem 2. (a) First, notice that by comparison we have $u_\lambda \leq U$, where U is the unique nonnegative solution to (1.5). The existence of such solution is implied by the results in [24], while the uniqueness follows by Theorem 1 in [14] (see also Remark 3 below). Then it is standard to obtain that for every sequence $\lambda_n \to \infty$, there exists a subsequence such that $u_{\lambda_n} \to V$ in $C^2_{\text{loc}}(\Omega)$, where V is a solution to $\Delta V = V^p + V^q$ in Ω.

Once we show that $u_\lambda \to \infty$ on $\partial\Omega$ as $\lambda \to \infty$, it will follow that $V = U$, as we want to show. Choose $m > 0$ and let V_m be the unique solution to the Dirichlet problem

$$\begin{cases} \Delta v = v^p + v^q & \text{in } \Omega, \\ v = m & \text{on } \partial\Omega. \end{cases}$$

If we denote by $\lambda_m = \sup_{\partial\Omega} |\nabla v_m|/v_m$, it clearly follows that v_m is a subsolution to (1.3) for $\lambda \geq \lambda_m$. Since there exist arbitrarily large supersolutions, we achieve $u_\lambda \geq v_m$ if $\lambda \geq \lambda_m$. In particular, $u_\lambda \geq m$ on $\partial\Omega$ if $\lambda \geq \lambda_m$, as we wanted to see. This concludes the proof of part (a).

(b) This proof uses a standard blow-up technique, in the same spirit as Theorem 1-iii) in in [16], thus we do not provide complete details. Let $\{v_\lambda\}$ be a family of nonnegative solutions to (1.3) with

$$M_\lambda := \sup_\Omega v_\lambda \leq C.$$

Assume that for a sequence $\lambda_n \to \infty$ we have $\lambda_n^{\frac{2}{1-q}} M_{\lambda_n} \to \infty$. For simplicity, let us denote $M_n = M_{\lambda_n}$ and $v_n = v_{\lambda_n}$. Choose a point $x_n \in \partial\Omega$ such that

$v_n(x_n) = M_n$ and introduce the scaled functions

$$w_n(y) = \frac{v_n(x_n + \lambda_n^{-1} y)}{M_n},$$

which verify

$$\begin{cases} \Delta w_n = \dfrac{1}{\lambda_n^2} \left(M_n^{p-1} w_n^p + M_n^{q-1} v_n^q \right) & \text{in } \Omega_n, \\ \dfrac{\partial w_n}{\partial \nu} = w_n & \text{on } \partial\Omega_n, \end{cases}$$

where $\Omega_n = \{y \in \mathbb{R}^N : \ x_n + \lambda_n^{-1} y \in \Omega\}$. We may assume with no loss of generality that $x_n \to x_0 \in \partial\Omega$. Then, a usual straightening of $\partial\Omega$ near x_0 together with the fact that $\|w_n\|_\infty = 1$, allow us to obtain bounds to pass to the limit and obtain that, for a subsequence, $w_n \to w$ in $C(\overline{\mathbb{R}^N_+}) \cap C^2(\mathbb{R}^N_+)$, where w solves the problem

$$\begin{cases} \Delta w = 0 & \text{in } \mathbb{R}^N_+, \\ -\dfrac{\partial w}{\partial y_1} = w & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

and $\mathbb{R}^N_+ = \{y_1 > 0\}$. Moreover, $w(0) = 1$. According to the discussion in page 15 of [16], this is impossible. Hence the upper inequality in (1.6) is proved.

To show the lower inequality, we assume that there exists a sequence $\lambda_n \to \infty$ such that $\lambda_n^{\frac{2}{1-q}} M_n \to 0$, and consider

$$z_n(y) = \frac{v_n(x_n + M_n^{1/\beta} y)}{M_n},$$

where $\beta = 2/(1-q)$. Since z_n solves

$$\begin{cases} \Delta z_n = M_n^{p-q} z_n^p + z_n^q & \text{in } \Omega_n, \\ \dfrac{\partial z_n}{\partial \nu} = M_n^{1/\beta} \lambda_n z_n & \text{on } \partial\Omega_n \end{cases}$$

and verifies $\|z_n\|_\infty = 1$, we can, as before, pass to the limit to obtain that $z_n \to z$ in $C(\overline{\mathbb{R}^N_+}) \cap C^2(\mathbb{R}^N_+)$, where z is a solution to

$$\begin{cases} \Delta z = z^q & \text{in } \mathbb{R}^N_+, \\ -\dfrac{\partial z}{\partial y_1} = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

Now we observe that this is impossible, since by the strong maximum principle $z < z(0) = 1$, and Hopf's principle would imply $-\frac{\partial z}{\partial y_1}(0) > 0$. Again we have a contradiction, and therefore we conclude the existence of a constant c such that $M_\lambda \geq c\lambda^{-2/(1-q)}$ when λ is large enough. This shows the lower inequality in (1.6). □

Remark 3. It is worth mentioning that the proof of Theorem 1 of [14] should be clarified in a specific technical step. Such proof deals with uniqueness of solutions to

$$\begin{cases} \Delta u = f(u) & \text{in } \Omega, \\ u = \infty & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

under the assumption that f is a continuous function which is increasing and such that $f(t)/t^p$ is increasing for large t and some $p > 1$. At some point, the strong comparison principle is invoked and this requires f to be locally Lipschitz (this of course does not hold in our present situation). This difficulty can be overcomed as follows: assume that u is the minimal solution to (3.1) and let v be another solution. For small $\varepsilon > 0$ let $D_\varepsilon = \{x \in \Omega : (1+\varepsilon)u(x) < v(x)\}$ and choose $\eta > 0$ such that $f(t)/t$ is increasing in $t \geq \inf_{\Omega_\eta} u$, where $\Omega_\eta = \{x \in \Omega : \ d(x) := \text{dist}(x, \partial\Omega) < \eta\}$. Notice that $D_\varepsilon \cap \partial\Omega = \emptyset$, since $u/v \to 1$ as $d(x) \to 0$ (see [7]). Then in $D_{\varepsilon,\eta} = D_\varepsilon \cap \Omega_\eta$ we have $\Delta(v - (1+\varepsilon)u) \geq (f(v) - f((1+\varepsilon)u)) \geq 0$ so that

$$v - (1+\varepsilon)u \leq \max_{\partial D_{\varepsilon,\eta}} (v - (1+\varepsilon)u) \quad \text{in } D_{\varepsilon,\eta}.$$

On the other hand, $\partial D_{\varepsilon,\eta} = (\partial D_\varepsilon \cap \Omega_\eta) \cup (D_\varepsilon \cap \partial\Omega_\eta)$, and the maximum cannot be achieved on ∂D_ε, since this would imply $v - (1+\varepsilon)u \leq 0$ in $D_{\varepsilon,\eta}$, contrary to the definition of D_ε. Then

$$v - (1+\varepsilon)u \leq \max_{D_\varepsilon \cap \{d=\eta\}} (v - (1+\varepsilon)u) \quad \text{in } D_{\varepsilon,\eta}. \tag{3.2}$$

Finally, observe that D_ε is increasing as $\varepsilon \downarrow 0$, so that denoting $\widetilde{\Omega} = \cup_{\varepsilon>0} D_\varepsilon$ and letting $\varepsilon \to 0$ in (3.2) we have

$$v - u \leq \max_{\widetilde{\Omega} \cap \{d=\eta\}} (v-u) \leq \max_{\{d=\eta\}} (v-u) =: \theta \quad \text{in } \widetilde{\Omega}_\eta, \tag{3.3}$$

where $\widetilde{\Omega}_\eta = \widetilde{\Omega} \cap \Omega_\eta$. Taking into account that $u = v$ in $\Omega \setminus \widetilde{\Omega}$, as can be easily checked, we may ensure that (3.3) holds in Ω_η.

On the other hand, $\Delta(v-u) = f(v) - f(u) \geq 0$ in $\Omega \setminus \Omega_\eta$, so that $v - u \leq \theta$ in $\Omega \setminus \Omega_\eta$ by the maximum principle. Hence $v - u \leq \theta$ in Ω and the strong maximum principle implies $v - u = \theta$, which is only possible if $\theta = 0$, that is, $u = v$. This proves uniqueness.

Remark 4. Observe that if $\{v_\lambda\}$ is a family of solutions to (1.3) with $\lambda \to \infty$ such that $\inf_\Omega v_\lambda \geq c > 0$ then $\Delta v_\lambda = v_\lambda^p + v_\lambda^q \leq (1 + c^{q-p}) v_\lambda^p$ in Ω, so that the function $(1+c^{q-p})^{\frac{1}{p-1}} v_\lambda$ is a supersolution to (2.1) and uniqueness implies

$$v_\lambda \geq (1 + c^{q-p})^{-\frac{1}{p-1}} U_\lambda.$$

Then $v_\lambda \to \infty$ uniformly on $\partial\Omega$ when $\lambda \to \infty$. It follows that $v_\lambda \to U$ uniformly on compact subsets of Ω where U is the unique nonnegative solution to (1.5).

4. Positivity vs. dead cores

In this section we prove Theorem 3. That is, we consider conditions under which nonnegative solutions to (1.3) either are strictly positive or have a dead core.

Proof of Theorem 3, (a) *and* (b). (a) Let $\{v_\lambda\}_{\lambda \geq \lambda_0}$ be a family of nonnegative solutions with nonnegative energy, that is,

$$\frac{1}{2} \int_\Omega |\nabla v_\lambda|^2 - \frac{\lambda}{2} \int_{\partial\Omega} v_\lambda^2 + \frac{1}{p+1} \int_\Omega v_\lambda^{p+1} + \frac{1}{q+1} \int_\Omega v_\lambda^{q+1} \geq 0.$$

Taking v_λ as a test function in the weak formulation of (1.3) we have

$$\int_\Omega |\nabla v_\lambda|^2 - \lambda \int_{\partial\Omega} |v_\lambda|^2 + \int_\Omega v_\lambda^{p+1} + \int_\Omega v_\lambda^{q+1} = 0,$$

so that

$$\int_\Omega v_\lambda^{p+1} \le \frac{(p+1)(1-q)}{(q+1)(p-1)} \int_\Omega v_\lambda^{q+1}$$

and then

$$\|v_\lambda\|_{L^{p+1}(\Omega)} \le C\|v_\lambda\|^\theta_{L^{q+1}(\Omega)}, \tag{4.1}$$

with $\theta = (q+1)/(p+1) \in (0,1)$ and a positive constant C which is independent of λ. Hence,

$$\|v_\lambda\|^{1-\theta}_{L^{p+1}(\Omega)} \le C|\Omega|^{(p-q)/(p+1)^2}$$

and so

$$\|v_\lambda\|_{L^{p+1}(\Omega)} \le C.$$

Now, we just observe that from (1.3) we also have

$$\lambda \int_{\partial\Omega} v_\lambda = \int_\Omega v_\lambda^p + \int_\Omega v_\lambda^q \le C,$$

so that

$$\int_{\partial\Omega} v_\lambda \le \frac{C}{\lambda}. \tag{4.2}$$

Next notice that by comparison $v_\lambda \le z_\lambda$, where z_λ is the harmonic function in Ω which coincides with v_λ on $\partial\Omega$. Thanks to Green's representation formula:

$$z_\lambda(x) = \int_{\partial\Omega} v_\lambda(y) \frac{\partial G}{\partial\nu}(x,y) dS(y),$$

where $G(x,y)$ is the Green function of the domain Ω.

On the other hand

$$\left| \frac{\partial G}{\partial\nu}(x,y) \right| \le \frac{A}{|x-y|^{N-1}},$$

for $x \in \Omega$, $y \in \partial\Omega$ with $A = A(\Omega)$ a positive constant. Hence

$$v_\lambda(x) \le \frac{C}{\lambda} d(x)^{1-N}$$

with $d(x) = \text{dist}\,(x, \partial\Omega)$ and large λ. This implies that $v_\lambda \to 0$ uniformly in compacts of Ω as $\lambda \to \infty$.

Consider now the auxiliary boundary value problem

$$\begin{cases} \Delta w = w^q & \text{in } B_R, \\ w = \mu & \text{on } \partial B_R, \end{cases} \tag{4.3}$$

with $B_R = \{|x| < R\}$ and $\mu > 0$. It has a unique solution $w = w_{\mu,B_R}(r)$, $r = |x|$, such that

$$w_{\mu,B_R}(r) \le B[(r-d)^+]^\beta, \qquad 0 \le r \le R,$$

with $\beta = 2/(1-q)$, $B^{q-1} = \beta(\beta-1)$, for $\mu \le BR^\beta$ and $d = R - (\mu/B)^{1/\beta}$.

Given a small $d_0 > 0$, for every $\xi \in \Omega$ such that $d(\xi) = d_0$ we observe problem (4.3) in the ball $B_R(\xi)$ with radius $R = d_0/2$ and value

$$\mu = \frac{C}{\lambda}\left(\frac{d_0}{2}\right)^{1-N}.$$

Since μ is an upper bound for v_λ in $\{d(x) \geq d_0/2\}$ we obtain

$$v_\lambda(x) \leq w_{\mu,B_r(\xi)}(x) \qquad x \in B_R(\xi),$$

for every ξ such that $d(\xi) = d_0$. In particular,

$$v_\lambda(\xi) \leq w_{\mu,B_r(\xi)}(\xi) = 0$$

on $\{d(\xi) = d_0\}$ when

$$\mu \leq B\left(\frac{d_0}{2}\right)^{\beta}.$$

Therefore,

$$v_\lambda = 0 \qquad \text{in} \qquad d(x) \geq d_0,$$

if

$$d_0^{\beta+N-1} \geq \frac{C}{\lambda},$$

for a certain positive constant C. This proves (1.7).

(b) It follows by Theorem 2 (b) that $v_\lambda \to 0$ uniformly in Ω. To show both the existence of a dead core and estimate (1.8) we proceed as in part (a). Specifically, we take problem (4.3) in $B_R(\xi)$ with $R = d_0$ (small enough) and

$$\mu = C_2\lambda^{-\beta}$$

the upper bound of the family according Theorem 2. Then, v_λ vanishes in $\{d(x) \geq d_0\}$ provided

$$C_2\lambda^{-\beta} \leq B d_0^{\beta},$$

which is the estimate (1.8). □

Before completing the proof of Theorem 3 it is convenient to state some basic features on certain radial initial value problems which will be instrumental for both the present and the next Section. Consider first the Cauchy problem

$$\begin{cases} (r^{N-1}u')' = r^{N-1}(u^p + u^q) & r \geq 0, \\ u(0) = c, \qquad u'(0) = 0, \end{cases} \tag{4.4}$$

where $c \geq 0$ is a parameter.

Lemma 12. *For every $c \geq 0$, problem (4.4) admits a unique solution*

$$u = u(r, c),$$

which is defined for $r \in [0, \omega(c))$ with $\omega(c) < \infty$. In addition,

a) *$u(r, c)$ is increasing with respect to r in $[0, \omega(c))$ and $\lim_{r\to\omega(c)} u(r, c) = \infty$.*
b) *$u(r, c)$ is increasing and differentiable with respect to c.*
c) *The function $\omega = \omega(c)$ is continuous, decreasing and $\lim_{c\to\infty} \omega(c) = 0$.*

Remark 5. In the one-dimensional case $u = u(x, c)$ is given by the expression

$$\int_c^u \frac{ds}{\sqrt{2(H(s) - H(c))}} = x, \tag{4.5}$$

$\omega(c)$ being the value of the (finite) integral corresponding to $u = \infty$ (H is a primitive of $h = u^p + u^q$).

Another initial value problem of a different nature must be analyzed to properly understand the dead core formation in problem (1.3). Namely,

$$\begin{cases} (r^{N-1}u')' = r^{N-1}(u^p + u^q) & r \geq d, \\ u(d) = u'(d) = 0, \end{cases} \tag{4.6}$$

where $d \geq 0$ has the status of a parameter. It should be remarked that such a problem exhibits infinitely many nontrivial nonnegative solutions. However, it only admits a positive solution in $r \geq d$. This fact and related features concerning (4.6) are stated next.

Lemma 13. *Problem* (4.6) *admits for every $d \geq 0$ a unique solution $u = u(r, d)$ defined in an interval $[d, \omega_0(d))$, $d < \omega_0(d) < \infty$, when subject to the property of being positive in $r > d$. Moreover,*

a) *$u(r, d)$ is increasing in $[d, \omega_0(d))$ and $\lim_{r \to \omega_0(d)} u(r, d) = \infty$.*
b) *For $d_1 < d_2$, $u(r, d_1) > u(r, d_2)$ for $r \geq d_2$.*
c) *$u(r, d)$ is differentiable with respect to d.*
d) *The function $\omega = \omega_0(d)$ is continuous, increasing while*

$$\frac{1}{\sqrt{N}}(\omega_0(d) - d) \leq L \leq \omega_0(d) - d \tag{4.7}$$

for all $d \geq 0$ with $L = \int_0^\infty \frac{ds}{\sqrt{H(s)}}$.

Remarks 6.

a) In the one-dimensional case $u(x, d) = u_0(x - d)$, $u_0(x)$ being given by (4.5) after setting $c = 0$.
b) It should be noticed that, following the notation given in the lemmas, $u(r, c)_{|c=0} = u(r, d)_{d=0}$ and so $\omega(0) = \omega_0(0)$. For immediate use we fix the notation $R_0 = \omega(0)$ and $u_0(r) = u(r, d)_{|d=0}$.

Further auxiliary problems playing an important rôle in next section are the radial Dirichlet problem

$$\begin{cases} \Delta u = u^p + u^q & \text{in } B_R, \\ u = \mu & \text{on } \partial B_R, \end{cases} \tag{4.8}$$

for which the existence a unique nonnegative radial solution $u = \tilde{u}(r, \mu)$ for all $\mu \geq 0$ is well-known, and the associated singular version

$$\begin{cases} \Delta u = u^p + u^q & \text{in } B_R, \\ u = \infty & \text{on } \partial B_R. \end{cases} \tag{4.9}$$

Problem (4.9) admits a unique nonnegative radial solution $U = \widetilde{U}(r)$ (see the proof of Theorem 2 (a) and Remark 3).

The main features concerning the Dirichlet problem (4.8) which are relevant for our forthcoming purposes are collected in the following lemma.

Lemma 14. *For $\mu \geq 0$ let $u = \tilde{u}(r,\mu)$ be the solution to* (4.8) *and $U = \widetilde{U}(r)$ the corresponding solution to* (4.9)*. Then, the following properties hold.*

A) *Assume that $R < R_0 = \omega(0)$. Then, the large solution $\widetilde{U}$ is positive in B_R with*

$$\inf \widetilde{U} = \omega^{-1}(R).$$

In addition,

i) *For $\mu \geq \mu_0 := u_0(R)$ there exists a unique $0 \leq c(\mu) < \omega^{-1}(R)$ such that*

$$\tilde{u}(r,\mu) = u(r, c(\mu)) \qquad 0 \leq r \leq R.$$

The function $c(\mu)$ is increasing, differentiable, $c(\mu_0) = 0$ and $c \to \omega^{-1}(R)$ as $\mu \to \infty$.

ii) *If $0 \leq \mu \leq \mu_0$ then a unique $d = d(\mu)$ exists such that*

$$\tilde{u}(r,\mu) = u(r, d(\mu)) \qquad d(\mu) \leq r \leq R.$$

The function $d(\mu)$ is decreasing, differentiable with $d(0) = R$, $d(\mu_0) = 0$. In particular, $\tilde{u}(\cdot,\mu)$ has $\overline{B}_d$ as dead core.

B) *If on the contrary $R \geq R_0$ then the large solution $\widetilde{U}$ possesses*

$$\overline{B}_{\omega_0^{-1}(R)} = \{|x| \leq \omega_0^{-1}(R)\}$$

as a dead core. Moreover, each $\mu \geq 0$ has associated a unique $\omega_0^{-1}(R) < d(\mu) \leq R$ such that

$$\tilde{u}(r,\mu) = u(r, d(\mu)) \qquad d(\mu) \leq r \leq R,$$

and so $\tilde{u}(\cdot,\mu)$ has $\overline{B}_d$ as dead core. Furthermore, $d = d(\mu)$ is differentiable, decreasing, $d(0) = R$ while $d(\mu) \to \omega_0^{-1}(R)$ as $\mu \to \infty$.

C) *The distance $R-d$ of the dead core of $\tilde{u}(r,\mu)$ to the boundary ∂B_R satisfies the asymptotic estimate,*

$$R - d \sim \sqrt{\beta(\beta-1)}\,\mu^{1/\beta}, \tag{4.10}$$

as $\mu \to 0$ where $\beta = 2/(1-q)$.

Before outlining a proof of Lemmas 12, 13 and 14 let us use them to finish the proof of Theorem 3.

Proof of Theorem 3, (c) *and* (d). (c) Let $R_0 = \omega(0) = \omega_0(0)$ (see Lemmas 12 and 13). If $\Omega \supset B_R(x_0)$ for certain $R \geq R_0$ and $x_0 \in \Omega$, then any nonnegative solution u to (1.3) satisfies $u \leq \widetilde{U}$ in $B_R(x_0)$, $\widetilde{U}$ the solution to (4.9) in $B_R(x_0)$. Since $\widetilde{U}$ possesses a dead core (Lemma 14), the same happens to u.

(d) No generality is lost by assuming that π_1, π_2 coincide with $x_1 = -R$ and $x_1 = R$, respectively. Suppose that $0 < R < R_0$. Then the solution $\widetilde{U}(x_1)$ to (4.9) corresponding to $N = 1$ is positive in $(-R, R)$ (Lemma 14) and the same happens to the solution $u(x_1, c)$ to (4.4), which is defined for $x_1 \in [R, R]$ and all $0 < c < \widetilde{U}(0)$. Fix any c_0 in that range, put $u(x_1) = u(x_1, c_0)$ and define $\underline{u}(x) = u(x_1)$ for $x \in \Omega$. If

$$\lambda_0 = \sup_{0 \le x_1 \le R} \frac{u'(x_1)}{u(x_1)},$$

then $\underline{u}$ defines a positive subsolution to (1.3) for $\lambda \ge \lambda_0$. This means that the maximal solution u_λ to (1.3) is positive in Ω for all $\lambda \ge \lambda_0$ what proves (d). $\square$

Proof of Lemma 12. We only deal now with $c > 0$, the more subtle case $c = 0$ being studied in Lemma 13. By observing that any solution u initially satisfies

$$u'(r) = \int_0^r \left(\frac{s}{r}\right)^{N-1} h(u(s))\, ds, \qquad h(u) = u^p + u^q, \tag{4.11}$$

then u must be increasing wherever defined. Standard theory, see [23], then implies that a unique solution $u(r, c)$ to (4.4) exists which is defined in a maximal interval $[0, \omega(c))$, $u(r, c) \to \infty$ as $r \to \omega(c)$, while it is smooth with respect to c (a further direct argument then says that $u(r, c)$ increases with c).

Let us show that ω is finite (reference to c is now omitted). By observing that $u(s) < u(r)$ in (4.11) we find that (cf. [24])

$$u'(r) \le \frac{r}{N} h(u(r)),$$

which, together with the equation in (4.4) implies that

$$u'' \ge \frac{1}{N} h(u) \qquad \text{for} \quad 0 \le r < \omega. \tag{4.12}$$

Multiplying by u' and integrating yields

$$\frac{r}{\sqrt{N}} \le \int_c^{u(r)} \frac{ds}{\sqrt{2(H(s) - H(c))}}. \tag{4.13}$$

The finiteness of ω then follows by letting $r \to \omega$ in the previous expression.

Continuous dependence of ω on c is more delicate. First, the uniqueness of nonnegative solution to (4.9) implies that $\omega = \omega(c)$ is increasing. Since standard theory states that ω is lower semicontinuous in c ([23]) then $\omega(c) = \lim_{c' \to c+} \omega(c')$. On the other hand, $\omega(c') \to \omega(c)$ as $c' \to c-$, otherwise

$$\omega(c) < \omega_0 = \inf \omega(c_n),$$

for a certain increasing $c_n \to c$. However, $u(r, c_n) < \widetilde{U}(r)$ for $r < \omega_0$, $\widetilde{U}$ being in this case the solution to (4.9) in B_R with $R = \omega_0$. This is incompatible with the fact that $u(r, c_n)$ diverges to ∞ at $r = \omega(c)$. Thus, the continuity of $\omega = \omega(c)$ is shown.

Finally, observe that (4.13) implies that

$$\frac{\omega(c)}{\sqrt{N}} \le \int_c^\infty \frac{ds}{\sqrt{2H(s)}},$$

and hence $\omega(c) \to 0$ as $c \to \infty$. $\square$

Proof of Lemma 13. As mentioned above, problem (4.6) admits infinitely nonnegative solutions defined in $r \geq d$ (see [20]). Existence, uniqueness and estimates near $r = d$ of a *positive* local increasing solution u to (4.6) is provided by Theorem 2.3 in [20] (see also [26] for existence). Ode's standard theory then allows us to obtain a global increasing continuation $u(r,d)$ up to a maximal interval $[d, \omega_0(d))$. Now it follows from the equation that

$$u''u' \leq h(u)u',$$

for $r \in [d, \omega_0(d))$ which together with (4.12) gives

$$\frac{r-d}{\sqrt{N}} \leq \int_0^{u(r)} \frac{ds}{\sqrt{2H(s)}} \leq r - d, \qquad r \in [d, \omega_0(d)). \tag{4.14}$$

This implies both (4.7) and that $\omega_0(d) < \infty$. On the other hand that ω_0 is continuous and increasing in d is shown as in Lemma 12, while (b) is somehow standard.

The more subtle issue of the differentiability of $u(r,d)$ with respect to d is solved by Theorem 2.6 in [20]. As a consequence of it, $w(t,d) := u(t+d,d)$, $0 \leq t \leq \eta$ is differentiable with respect d when $d \mapsto w(\cdot, d)$ is regarded as a mapping with values in $C^2[0,\eta]$ and $d \geq 0$ (η can be taken not depending on d thanks to (4.7)). Since $r \mapsto (u(r,d), u'(r,d))$ takes values in $\mathbb{R}^+ \times \mathbb{R}^+$ for $r > d$ then standard results on smoothness on initial data hold from $r = d+\eta$ ahead. Thus, a continuation argument shows that $u(r,d)$ is globally smooth with respect to d. It also follows from [20] that $z(r) = (\partial u/\partial d)(r,d)$ solves the initial value problem

$$\begin{cases} z'' + \dfrac{N-1}{r} z' = h(u(r,d))z, \\ z(d) = z'(d) = 0. \end{cases}$$

The proof is concluded. $\square$

Proof of Lemma 14. Parts A) and B) are essentially a direct consequence of Lemmas 12 and 13. As for C) observe that setting $r = R$ in (4.14) gives

$$\frac{R-d}{\sqrt{N}} \leq \int_0^{\mu} \frac{ds}{\sqrt{2H(s)}} \leq R - d$$

while

$$\int_0^{\mu} \frac{ds}{\sqrt{2H(s)}} \sim \sqrt{\beta(\beta-1)}\mu^{1/\beta}$$

as $\mu \to 0$. This suggests the choice $\tau = d\mu^{-1/\beta}$, $\rho = (R-d)\mu^{-1/\beta}$ and the scaling

$$u(r) = \mu w(r\mu^{-1/\beta} - \tau),$$

which leads to the initial value problem

$$\begin{cases} w'' + \dfrac{N-1}{t+\tau} w' = w^q + \mu^{p-q} w^p \\ w(0) = w'(0) = 0, \end{cases}$$

together with $w(\rho) = 1$. It is known that, for every $\eta > 0$, the unique positive solution $w = w(t,\tau)$ to such problem converges in $C^2[0,\eta]$ as $\tau \to \infty$ to the unique positive solution to the problem

$$\begin{cases} w'' = w^q \\ w(0) = w'(0) = 0, \end{cases} \tag{4.15}$$

(see Theorem 2.5 in [20]). Such a solution is explicitly given by $w(t,\infty) = Bt^\beta$ with $B^{q-1} = \beta(\beta-1)$. On the other hand, by our previous discussion, $\rho = (R-d)\mu^{-1/\beta}$ is bounded away from zero and from infinity as $\mu \to 0$. Therefore,

$$\lim_{\mu\to\infty} \rho = \lim_{\mu\to\infty} (R-d)\mu^{-1/\beta} = B^{-1/\beta},$$

since $t = B^{-1/\beta}$ is the only point where $w(t,\infty)$ achieves the value 1. This finishes the proof. □

5. Problem (1.3) in balls

This section will be dedicated to the proof of our last result, Theorem 4. Thus we will be mainly dealing with radial solutions. Notice that for nonnegative radial solutions u we have, according to the maximum principle,

$$u(R) = \max_{B_R} u. \tag{5.1}$$

Proof of Theorem 4 (a). Since the maximal solution to (1.3) is clearly radial, only the existence of a second nonnegative radial solution remains to be proved.

The existence of a second radial nonnegative solution follows by means of the mountain pass theorem, as in Section 2, applied to the functional J defined there but in the space of radial functions in $H^1(B_R)$, which will be denoted by $H^1_r(B_R)$. We notice that if the maximal solution had nonpositive energy, we could directly apply the mountain pass theorem as in the proof of Theorem 1. However, this is not the case in general, so that we still need to prove a further geometric property of J.

Lemma 15. *Let $\lambda > \Lambda^*$. Then either problem* (1.3) *has two nonnegative nontrivial radial solutions or the maximal solution u_λ is a local minimum of the functional J in $H^1_r(B_R)$.*

Proof. We may assume that the maximal solution u_λ is the only nonnegative radial solution to (1.3). Fix λ_1 such that $\Lambda^* \le \lambda_1 < \lambda$. Let us check that the function

$$\underline{u} = u_{\lambda_1} - \varepsilon \tag{5.2}$$

is a subsolution to (2.3), where $\varepsilon > 0$ is chosen to have $u_{\lambda_1} > \varepsilon$ on ∂B_R (this is possible thanks to (5.1)). To see this, observe that $\Delta(u_{\lambda_1} - \varepsilon) = u^p_{\lambda_1} + u^q_{\lambda_1} = f(x, u_{\lambda_1}) + |u_{\lambda_1}|^{q-1}u_{\lambda_1} \ge f(x, u_{\lambda_1} - \varepsilon) + |u_{\lambda_1} - \varepsilon|^{q-1}(u_{\lambda_1} - \varepsilon)$ in B_R. Also, we have,

$$\frac{\partial(u_{\lambda_1} - \varepsilon)}{\partial\nu} = \lambda_1 u_{\lambda_1} \le \lambda(u_{\lambda_1} - \varepsilon)$$

on ∂B_R provided that $\varepsilon \le (\lambda - \lambda_1)u_{\lambda_1}/\lambda$ on ∂B_R, which is certainly possible if ε is small enough. Thus $u_{\lambda_1} - \varepsilon$ is a subsolution to (2.3). We recall that the unique positive solution to (2.1), denoted by U_λ, is a radial supersolution, verifying $\underline{u} \le U_\lambda$. We truncate again the nonlinearities $f(x,u)$ and $g(x,u)$ as follows:

$$\tilde{f}(x,u) = \begin{cases} U_\lambda^p & u > U_\lambda, \\ |u|^{p-1}u & \underline{u} \le u \le U_\lambda, \\ |\underline{u}|^{p-1}\underline{u} & u \le \underline{u}, \end{cases}$$

and

$$\tilde{g}(x,u) = \begin{cases} U_\lambda & u > U_\lambda, \\ u & \underline{u} \le u \le U_\lambda, \\ \underline{u} & u \le \underline{u}, \end{cases}$$

and define

$$\tilde{J}(u) = \frac{1}{2}\int_{B_R} |\nabla u|^2 - \lambda \int_{\partial B_R} \tilde{G}(x,u) + \int_{B_R} \tilde{F}(x,u) + \frac{1}{q+1}\int_{B_R} |u|^{q+1},$$

where $\tilde{F}$ and $\tilde{G}$ are primitives of $\tilde{f}$ and $\tilde{g}$ respectively. Since $\tilde{f}$ and $\tilde{g}$ are bounded, it follows that $\tilde{J}$ is coercive. Also, $\tilde{J}$ is weakly sequentially lower semicontinuous, so that there exists a global minimum $u \in H_r^1(B_R)$. Arguing as in Lemma 9, we can show that $\underline{u} \le u \le U_\lambda$, thus u is a weak solution to

$$\begin{cases} \Delta u = |u|^{p-1}u + |u|^{q-1}u & \text{in } B_R, \\ \dfrac{\partial u}{\partial \nu} = \lambda u & \text{on } \partial B_R. \end{cases}$$

Indeed, u is a nonnegative solution to (1.3). To see this fact, assume that $\inf_{B_R} u < 0$, then there exists a point $x_0 \in B_R$ such that $u(x_0) = \inf_{B_R} u$ (recall that $u \ge u_{\lambda_1} - \varepsilon > 0$ on ∂B_R). Then, since $\Delta u(x_0) \ge 0$, we would have $0 \le |u(x_0)|^{p-1}u(x_0) + |u(x_0)|^{q-1}u(x_0)$, which is a contradiction. Thus u is nonnegative, and by assumption, $u = u_\lambda$, so that u_λ is a global minimum of $\tilde{J}$ in $H_r^1(B_R)$. We claim that u_λ is also a local minimum of J in the $C(\overline{B_R})$ topology. Indeed, if v is such that $\|v - u_\lambda\|_\infty < \delta < \varepsilon$, then it follows that $v < U_\lambda$ for small enough δ, while $v \ge u_\lambda - \delta \ge u_{\lambda_1} - \varepsilon = \underline{u}$. A straightforward calculation gives that $\tilde{J}(v) = J(v) + C$ for some constant C, so that u_λ is a minimum of J in the ball of center u_λ and radius δ in the $C(\overline{B_R})$ topology. Then u_λ is also a local minimum of J in $H_r^1(B_R)$ (see for instance Lemma 6.4 in [13]). This finishes the proof. □

Remark 7. Observe that the radial symmetry of the solutions is only used to ensure that $u_{\lambda_1} > 0$ on ∂B_R, since u_{λ_1} assumes its maximum and is constant there. The above proof is indeed "nonradial".

We can now conclude the proof of existence of a second nonnegative radial solution as in the proof of Lemma 8, with the use of the mountain pass lemma. Observe that, when $J(u_\lambda) \le 0$, we can argue exactly as in that proof to obtain the second solution. Thus only the case when $J(u_\lambda) > 0$ needs to be considered. Assume that u_λ is the only radial nonnegative solution for some $\lambda > \Lambda^*$. Then, according to Lemma 15, u_λ is a local minimum of J. Since $J(0) = 0 < J(u_\lambda)$, we can apply again the mountain pass theorem to obtain a second nonnegative solution, which is a contradiction. This concludes the proof of existence. □

Before performing the proof of parts (b) and (c) in Theorem 4, we need some further auxiliary facts.

Lemma 16. *Let $u = \tilde{u}(r, \mu)$ be the solution to* (4.8)*. Then*

$$\frac{\partial \tilde{u}}{\partial r}(R, \mu) \sim \sqrt{\frac{2}{p+1}}\mu^{\frac{p+1}{2}}, \tag{5.3}$$

as $\mu \to \infty$.

Lemma 17. *Assume that $0 < R < R_0$, $R_0 = \omega(0)$ and for $0 \le c < \omega^{-1}(R)$ let $u = u(r, c)$ be the solution to* (4.4) *(Lemma 12). Then,*

$$\frac{u_c'(R, c)}{u_c(R, c)} \sim \theta A^{-\frac{1}{\alpha}} \mu^{\frac{1}{\alpha}} \tag{5.4}$$

as $c \to \omega^{-1}(R)$ (i. e., as $\mu \to \infty$), where subindex means partial differentiation and

$$\alpha = \frac{2}{p-1}, \qquad A^{p-1} = \alpha(\alpha+1), \qquad \theta = \frac{\sqrt{1 + 4pA^{p-1}} - 1}{2}. \tag{5.5}$$

Similarly, if $R \ge R_0$ and $u = u(r, d)$ is the solution to (4.6) *(Lemma 13) then also*

$$\frac{u_d'(R, d)}{u_d(R, d)} \sim \theta A^{-\frac{1}{\alpha}} \mu^{\frac{1}{\alpha}}, \tag{5.6}$$

as $d \to \omega_0^{-1}(R)$ (i. e., $\mu \to \infty$), subindex d meaning partial differentiation with respect to d.

To show the preceding Lemmas we requiere an additional result that we state next. Its proof is a minor modification of the one of Theorem 1.1 in [19], and will be omitted.

Lemma 18. *Let $u = \tilde{u}(u, \mu)$ be the nonnegative solution to* (4.8)*. Then, for every small $\varepsilon > 0$ and large $u_0 > 0$, there exist positive δ, M and μ_0 such that*

$$\begin{aligned} u_0 \le \frac{A - \varepsilon}{\left(R - r + \left(\frac{A}{\mu}\right)^{1/\alpha}\right)^{\alpha}} - M &\le \tilde{u}(r, \mu) \\ &\le \frac{A + \varepsilon}{\left(R - r + \left(\frac{A}{\mu}\right)^{1/\alpha}\right)^{\alpha}} + M \end{aligned} \tag{5.7}$$

for all $R - \delta < r \le R$ and $\mu \ge \mu_0$, where α and A are given in (5.5)*.*

Proof of Lemma 16. Given positive and small η, ε, there exist positive u_0, M, δ, μ_0 such that

$$u^p \le h(u) \le (1 + \eta)u^p$$

for all $u \ge u_0$ together with (see (5.7))

$$h(\tilde{u}(r, \mu)) \le \frac{(1+\eta)(A+\varepsilon)^p}{\left(R - r + \left(\frac{A}{\mu}\right)^{1/\alpha}\right)^{\alpha p}} \left\{1 + \frac{M}{A}\left(\delta + \left(\frac{A}{\mu}\right)^{1/\alpha}\right)^{\alpha}\right\}^p,$$

and

$$h(\tilde{u}(r,\mu)) \geq \frac{(A-\varepsilon)^p}{\left(R-r+\left(\frac{A}{\mu}\right)^{1/\alpha}\right)^{\alpha p}} \left\{1-\frac{M}{\mu}\right\}^p,$$

for all $R-r \leq \delta$, $\mu \geq \mu_0$. Now observe that by conveniently diminishing δ and enlarging μ_0 we achieve

$$\frac{(1-\eta)(A-\varepsilon)^p}{\left(R-r+\left(\frac{A}{\mu}\right)^{1/\alpha}\right)^{\alpha p}} \leq h(\tilde{u}(r,\mu)) \leq \frac{(1+\eta)^2(A+\varepsilon)^p}{\left(R-r+\left(\frac{A}{\mu}\right)^{1/\alpha}\right)^{\alpha p}}, \tag{5.8}$$

provided $R-r \leq \delta$ and $\mu \geq \mu_0$. Thus

$$\begin{aligned}
\tilde{u}'(R,\mu) &= \int_0^R \left(\frac{s}{R}\right)^{N-1} h(\tilde{u})\, ds \\
&= \left(\int_0^{R-\delta} + \int_{R-\delta}^R\right) \left(\frac{s}{R}\right)^{N-1} h(\tilde{u})\, ds \\
&\leq I_{R-\delta} + (1+\eta)^2(A+\varepsilon) \int_{R-\delta}^R \left(\frac{s}{R}\right)^{N-1} \frac{ds}{\left(R-s+(A/\mu)^{1/\alpha}\right)^{\alpha p}}
\end{aligned}$$

where

$$I_{R-\delta} = \int_0^{R-\delta} \left(\frac{s}{R}\right)^{N-1} h(\tilde{u})\, ds.$$

On the other hand,

$$\begin{aligned}
\int_{R-\delta}^R \left(\frac{s}{R}\right)^{N-1} \frac{ds}{\left(R-s+(A/\mu)^{1/\alpha}\right)^{\alpha p}} &\leq \mu^{p-\frac{1}{\alpha}} \int_0^{\delta\mu^{1/\alpha}} \frac{dt}{(t+A^{1/\alpha})^{\alpha p}} \\
&\leq \frac{p-1}{p+1} A^{\frac{1}{\alpha}-p} \mu^{\frac{p+1}{2}},
\end{aligned}$$

for $\mu \geq \mu_0$. Since for fixed $\delta >$ we have that $I_{R-\delta} = O(1)$ as $\mu \to \infty$ then we arrive at

$$\varlimsup_{\mu\to\infty} \mu^{-\frac{p+1}{2}} \tilde{u}'(R,\mu) \leq (1+\eta)^2 (A+\varepsilon)^p \frac{p-1}{p+1} A^{\frac{1}{\alpha}-p}.$$

Taking $\varepsilon \to 0+$, $\eta \to 0+$ we then obtain

$$\varlimsup_{\mu\to\infty} \mu^{-\frac{p+1}{2}} \tilde{u}'(R,\mu) \leq \frac{p-1}{p+1} A^{\frac{1}{\alpha}} = \sqrt{\frac{2}{p+1}}.$$

The complementary estimate

$$\varliminf_{\mu\to\infty} \mu^{-\frac{p+1}{2}} \tilde{u}'(R,\mu) \geq \sqrt{\frac{2}{p+1}}.$$

is accomplished in an entirely similar way by employing instead the lower inequality in (5.8). $\square$

Remark 8. The exact estimate (5.3) obtained in Lemma 16 can be extended to general smooth domains $\Omega \subset \mathbb{R}^N$. Specifically, if $u = \tilde{u}(x,\mu)$ is the positive solution to the Dirichlet problem (4.8) now regarded in Ω, then it can shown by the blow-up techniques used in [21] (see also the proof of Theorem 2) that

$$\frac{\partial \tilde{u}}{\partial \nu} \sim \sqrt{\frac{2}{p+1}}\mu^{\frac{p+1}{2}},$$

as $\mu \to \infty$, where ν stands for the outward unit normal at $\partial\Omega$. Moreover, the same approach permits showing in addition that

$$\frac{\partial \tilde{u}}{\partial \nu} \sim \sqrt{\frac{2}{q+1}}\mu^{\frac{q+1}{2}},$$

as $\mu \to 0+$.

Proof of Lemma 17. Let us proceed to prove estimate (5.4) (the notation $h(u) = u^p + u^q$ will be kept in what follows). According to Lemma 12 the solution $u = u(r, c)$ to (4.4) can be differentiated with respect to c and

$$v(r) = \frac{\partial u}{\partial c}(r, c)$$

solves the initial value problem

$$\begin{cases} (r^{N-1}v')' = r^{N-1}h'(\tilde{u}(r,\mu))v & 0 \le r \le R, \\ v(0) = 1, \; v'(0) = 0, \end{cases}$$

where, in view of part A) in Lemma 14 $u(r, c) = \tilde{u}(r, \mu)$ for $\mu = \mu(c) \to \infty$ as $c \to \omega^{-1}(R)$. Therefore we are having in mind that $v(r) = v(r, \mu)$ (very often, explicit reference either to c or μ will be avoided below whenever possible). In addition, it should be remarked that

$$v(r, \mu) \to V(r) \qquad \text{as} \qquad \mu \to \infty,$$

in $C^2[0, R)$, where $v = V(r)$ is the solution to

$$\begin{cases} (r^{N-1}v')' = r^{N-1}h'(\widetilde{U}(r))v & 0 \le r < R, \\ v(0) = 1, \; v'(0) = 0, \end{cases}$$

and $\widetilde{U}$ is the positive solution to (4.9).

Fix now $0 < r_0 < R$ and set $v_0 = v(r_0, \mu)$, $v_0' = v'(r_0, \mu)$. By performing the change

$$v(r) = w(t),$$

where

$$t = \begin{cases} \log\left(\dfrac{R}{r}\right) & N = 2 \\ \dfrac{1}{N-2}\left(\dfrac{1}{r^{N-2}} - \dfrac{1}{R^{N-2}}\right) & N \ge 3, \end{cases} \tag{5.9}$$

we get

$$\frac{dw}{dt} = -r^{N-1}v,$$

and so $w = w(t)$ satisfies the initial value problem

$$\begin{cases} w'' = r^{2(N-1)}h'(\tilde{u})w & 0 \le t \le t_0, \\ w(t_0) = v_0, \; w'(t_0) = -r_0^{N-1}v_0', \end{cases} \tag{5.10}$$

where the value of t_0 corresponding to r_0 will be suitably chosen in the course of the proof.

Next, given a positive and small η, there exists $u_0 > 0$ such that

$$(1-\eta)pu^{p-1} \le h'(u) \le (1+\eta)pu^{p-1}$$

for $u \ge u_0$. By proceeding in a similar way as in the proof of Lemma 16 and by employing Lemma 18 it is possible to ensure that for small $\varepsilon > 0$ there exist a small $\delta > 0$ and a large μ_0, such that

$$\frac{(1-\eta)^2 p(A-\varepsilon)^{p-1}}{\left(R-r+\left(\frac{A}{\mu}\right)^{1/\alpha}\right)^2} \le h'(\tilde{u}(r,\mu)) \le \frac{(1+\eta)^2 p(A+\varepsilon)^{p-1}}{\left(R-r+\left(\frac{A}{\mu}\right)^{1/\alpha}\right)^2}, \tag{5.11}$$

for $R-r \le \delta$ and $\mu \ge \mu_0$.

On the other hand, it follows from (5.9) that

$$t \sim \frac{1}{R^{N-1}}(R-r),$$

as $r \to R$. Thus, by reducing δ if necessary we have

$$(1-\eta)R^{N-1}t \le R-r \le (1+\eta)R^{N-1}$$

if $r_0 \le r \le R$ with $R - r_0 < \delta$.

As a consequence of the preceding assertions we achieve that $w(t)$ satisfies

$$w(t) \le w_+(t), \qquad w'(t) \le w_+'(t)$$

for $0 \le t \le t_0$, where $w = w_+(t)$ is the solution to the problem

$$\begin{cases} w'' = \dfrac{D}{(t+b)^2} w & 0 \le t \le t_0 \\ w(t_0) = v_0,\ w'(t_0) = -r_0^{N-1} v_0', \end{cases} \tag{5.12}$$

and $D = D(\varepsilon,\eta)$ and $b = b(\mu)$ are given by

$$D = \left(\frac{1+\eta}{1-\eta}\right)^2 p(A+\varepsilon)^{p-1}, \qquad b = \frac{1}{(1-\eta)R^{N-1}}\left(\frac{A}{\mu}\right)^{\frac{1}{\alpha}}.$$

We now introduce in problem (5.12) the change

$$w(t) = z(\tau), \qquad \tau = \log(t+b),$$

and so $z(\tau)$ defines the solution to

$$\begin{cases} z'' - z' = Dz & \tau^* \le \tau \le \tau_0, \\ z(\tau_0) = z_0,\ z'(\tau_0) = z_0', \end{cases} \tag{5.13}$$

where,

$$\tau^* = \log b, \qquad \tau_0 = \log(t_0+b), \qquad z_0 = v_0, \qquad z_0' = -r_0^{N-1}(t_0+b)v_0'.$$

The solution to (5.13) is explicitly given by

$$z(\tau) = \frac{1}{\theta_1+\theta_2}\left\{(z_0\theta_2 + z_0')e^{-(\theta_2+1)\tau_0}e^{\theta_1\tau} + (z_0\theta_1 - z_0')e^{(\theta_1-1)\tau_0}e^{-\theta_2\tau}\right\},$$

where

$$\theta_1 = \frac{1+\sqrt{1+4D}}{2} \qquad -\theta_2 = \frac{1-\sqrt{1+4D}}{2}.$$

Therefore, since $\tau^* \to -\infty$ as $\mu \to \infty$ we achieve

$$\lim_{\mu\to\infty} \frac{-z'(\tau^*)}{z(\tau^*)} = \theta_2.$$

On the other hand

$$v(R,\mu) = w(0) \le w_+(0) = z(\tau^*),$$

together with

$$-R^{N-1}v'(R,\mu) = w'(0) \le w_+'(0) = \frac{z'(\tau^*)}{b}.$$

Thus

$$\frac{v'(R,\mu)}{v(R,\mu)} \ge \frac{1}{R^{N-1}b}\frac{-z'(\tau^*)}{z(\tau^*)},$$

and so

$$\varliminf_{\mu\to\infty} \mu^{-\frac{1}{\alpha}}\frac{v'(R,\mu)}{v(R,\mu)} \ge (1-\eta)A^{-\frac{1}{\alpha}}\theta_2.$$

By letting $\varepsilon \to 0+$ and $\eta \to 0+$ in the precedent expression we finally arrive at

$$\varliminf_{\mu\to\infty} \mu^{-\frac{1}{\alpha}}\frac{v'(R,\mu)}{v(R,\mu)} \ge A^{-\frac{1}{\alpha}}\theta,$$

where θ is given in (5.5).

The complementary asymptotic estimate

$$\varlimsup_{\mu\to\infty} \mu^{-\frac{1}{\alpha}}\frac{v'(R,\mu)}{v(R,\mu)} \le A^{-\frac{1}{\alpha}}\theta,$$

is obtained by using in problem (5.10) the lower estimate for $h'(\tilde u)$ given in (5.11). We find in this way the lower estimate

$$w(t) \ge w_-(t), \qquad w'(t) \ge w_-'(t), \qquad 0 \le t \le t_0,$$

where $w = w_-(t)$ solves instead

$$\begin{cases} w'' = \dfrac{D_-}{(t+b_-)^2}w & 0 \le t \le t_0, \\ w(t_0) = v_0,\ w'(t_0) = -r_0^{N-1}v_0', \end{cases}$$

with,

$$D_- = \left(\frac{1-\eta}{1+\eta}\right)^2 p(A-\varepsilon)^{p-1}, \qquad b_- = \frac{1}{(1+\eta)R^{N-1}}\left(\frac{A}{\mu}\right)^{\frac{1}{\alpha}}.$$

The analysis then proceeds in the same lines as in the lower estimate. This concludes the proof of (5.4).

To show the asymptotic estimate (5.6) first observe that Lemma 14 ensure us that $\tilde u(r,\mu) = u(r,d)$ with $\mu \to \infty$ as $d \to \omega_0^{-1}(R)$. On the other hand, smoothness of $u(r,d)$ with respect to d provided in Lemma 13 implies that

$$v(r) = \frac{\partial u}{\partial d}(r,d)$$

satisfies the initial value problem

$$\begin{cases} (r^{N-1}v')' = r^{N-1}h'(\tilde u(r,\mu))v & d \le r \le R, \\ v(d) = v'(d) = 0. \end{cases}$$

Then, the argument in the preceding proof of (5.4) can be repeated to achieve (5.6). □

We now come to the conclusion of the proof of Theorem 4.

Proof of Theorem 4 (b). According to Theorem 2, there are only two possibilities for a sequence of nonnegative radial solutions u_n to (1.3) corresponding to $\lambda_n \to \infty$ (passing to a subsequence): either $u_n(1) \to \infty$ or $u_n(1) \to 0$.

Let us first prove that in the first case we necessarily have $u_n = u_{\lambda_n}$, that is, the maximal solution is the only family of nonnegative radial solutions which becomes unbounded as $\lambda \to \infty$. To this aim, we consider the solution

$$u = \tilde{u}(r, \mu)$$

to problem (4.8) and the function

$$z(\mu) := \frac{\tilde{u}'(R, \mu)}{\tilde{u}(R, \mu)} = \frac{\tilde{u}'(R, \mu)}{\mu},$$

which is C^1 in $\mu \geq 0$ (Lemmas 12, 13 and 14). We claim that the following assertions hold:

i) $\lim_{\mu\to\infty} z(\mu) = \infty$,

ii) $\dfrac{dz}{d\mu} > 0$ for $\mu \geq \mu_0$ and large μ_0.

These facts imply the desired uniqueness, for if u_{n_1} and u_{n_2} are solutions to (1.3) with corresponding maxima μ_1 and μ_2, respectively, $\mu_i \geq \mu_0$, $i = 1, 2$, then

$$z(\mu_1) = z(\mu_2) = \lambda,$$

that implies $\mu_1 = \mu_2$ and hence $u_{n_1} = u_{n_2}$.

Let us show now claims i) and ii). That $z(\mu)$ diverges as $\mu \to \infty$ is a consequence of the estimate (5.3)(see Lemma 16):

$$\tilde{u}'(R, \mu) \sim \sqrt{\frac{2}{p+1}}\, \mu^{\frac{p+1}{2}},$$

as $\mu \to \infty$.

As for ii) notice that

$$\frac{dz}{d\mu} = \frac{\mu \tilde{u}'_\mu(R, \mu) - \tilde{u}'(R, \mu)}{\mu^2}$$

(the subindex μ denotes $\frac{\partial}{\partial \mu}$). Let us see that

$$\tilde{u}'_\mu(R, \mu) = \frac{u'_c(R, c)}{u_c(R, c)}, \tag{5.14}$$

for $R < R_0$ and large μ. Indeed, (5.14) is obtained by taking into account that $\tilde{u}'(R, \mu) = u'(R, c(\mu))$ (Lemma 14), hence $\tilde{u}'_\mu(R, \mu) = u'_c(R, c(\mu))c'(\mu)$. Since $\mu = u(R, c(\mu))$, differentiating we obtain $1 = u_c(R, c(\mu))c'(\mu)$, and (5.14) follows. Similarly,

$$\tilde{u}'_\mu(R, \mu) = \frac{u'_d(R, d)}{u_d(R, d)},$$

when $R \geq R_0$ and μ is large also in virtue of Lemma 14. In either case, (5.4) and (5.6) in Lemma 17 yield the asymptotic estimate

$$\tilde{u}'_\mu(R,\mu) \sim \theta A^{-\frac{1}{\alpha}} \mu^{\frac{1}{\alpha}} \tag{5.15}$$

as $\mu \to \infty$, the values of constants α, A and θ being those given in (5.5). Since

$$\sqrt{\frac{2}{p+1}} = \sqrt{\frac{\alpha}{\alpha+1}}$$

then $\theta A^{-\frac{1}{\alpha}} > \sqrt{\frac{\alpha}{\alpha+1}}$ and so, combining (5.3) and (5.15), we have that

$$\mu \tilde{u}'_\mu(R,\mu) - \tilde{u}'(R,\mu) > 0$$

for large μ. This proves assertion ii).

Let us consider now the case of a sequence of nonnegative radial solutions $\{u_n\}$ verifying $u_n(R) \to 0$. To deal with this situation take a general family $\{\hat{u}_\lambda\}$ of radial nonnegative solutions such that $\hat{u}_\lambda(R) \to 0$ as $\lambda \to \infty$. Then $\hat{u}_\lambda$ possesses a dead core $\overline{B}_d$, $d = d_\lambda$, for large λ, while Lemma 14 implies that $d_\lambda \to R$ as $\lambda \to \infty$. Moreover,

$$0 < R - d_\lambda \leq \frac{C}{\lambda}, \tag{5.16}$$

for large λ and a certain positive constant C (see Theorem 3 (b), or alternatively, estimate (4.10) together with the fact that $\mu \leq C_2 \lambda^{-\beta}$).

On the other hand, if $u = u(r)$ is an arbitrary nonnegative solution to (1.3) with a dead core $\overline{B}_d$ then it can be written as

$$u(r) = \lambda^\beta w(t),$$

with $\beta = \frac{2}{1-q}$, $t = \lambda(r-d)$ and where $w = w(t)$ defines a positive solution in $0 < t < R\lambda - \bar{d}$, $\bar{d} = \lambda d$, to the initial value problem

$$\begin{cases} ((t+\bar{d})^{N-1} w')' = (t+\bar{d})^{N-1}(w^q + \lambda^{-\beta(p-q)} w^p) \\ w(0) = w'(0) = 0. \end{cases} \tag{5.17}$$

In addition, $w = w(t)$ satisfies the boundary condition

$$w'(R\lambda - \bar{d}) = w(R\lambda - \bar{d}). \tag{5.18}$$

Let us recall now some basic features of (5.17). For every $\eta > 0$ there exists $\lambda = \lambda(\eta)$ such that (5.17) exhibits a unique positive solution $w = w(r, \bar{d}, \lambda)$ for all $\bar{d} \geq 0$, $\lambda \geq \lambda(\eta)$ while the mapping $(\bar{d}, \lambda) \mapsto w(\cdot, \bar{d}, \lambda)$, observed as taking values in $C^2[0,\eta]$ is continuous and C^1 with respect to $\bar{d}$ (see Theorems 2.3, 2.5 and 2.6 in [20]). In addition,

$$v(r) = \frac{\partial w}{\partial \bar{d}}(r, \bar{d}, \lambda)$$

satisfies the problem

$$\begin{cases} v'' + \dfrac{N-1}{t+\bar{d}} v' = (q w^{q-1} + p\lambda^{-\beta(p-q)} w^{p-1}) v + \dfrac{N-1}{(t+\bar{d})^2} w \\ v(0) = v'(0) = 0, \end{cases}$$

and the decaying estimate

$$|v(t)| \leq C t^\beta. \tag{5.19}$$

Furthermore, for every $\eta > 0$, $w(\cdot, \bar{d}, \lambda) \to w_\infty(\cdot)$ in $C^2[0,\eta]$ as $\bar{d} \to \infty$, $\lambda \to \infty$ where $w = w_\infty(t)$ is the positive solution to problem (4.15). Recall that such solution is explicitly provided by

$$w_\infty(t) = Bt^\beta$$

with $\beta = \frac{2}{1-q}$ and $B^{q-1} = \beta(\beta-1)$ (Theorem 2.5 in [20]). Notice that in particular, w_∞ is defined in the whole of $[0,\infty)$. Similarly, for every $\eta > 0$, $v(r) = (\partial w/\partial \bar{d})(r, \bar{d}, \lambda)$ converges in $C^2[0,\eta]$ as $\bar{d} \to \infty$, $\lambda \to \infty$ to the solution of the linear problem

$$\begin{cases} v'' = q w_\infty^{q-1}(t) v, \\ v(0) = v'(0) = 0, \end{cases}$$

with satisfies in addition condition (5.19). Therefore,

$$\lim_{\bar{d},\lambda\to\infty} \frac{\partial w}{\partial \bar{d}}(r, \bar{d}, \lambda) = 0,$$

in $C^2[0,\infty)$.

Let us examine now the fulfillment of the boundary condition (5.18). It can be equivalently expressed as

$$w'(T, \bar{d}, \lambda) - w(T, \bar{d}, \lambda) = 0, \tag{5.20}$$

together with

$$T = R\lambda - \bar{d}. \tag{5.21}$$

Taking into account that $w(t,\infty,\infty) = w_\infty(t) = Bt^\beta$, equation (5.20) is uniquely solved by $T = T_0 := \beta$ as $\bar{d} = \lambda = \infty$. Since $w_\infty''(\beta) - w_\infty(\beta) = w_\infty(\beta)^q(1 - w_\infty(\beta)^{1-q}) \neq 0$, then the implicit function theorem implies that equation (5.20) is uniquely solved in the form $T = T(\bar{d}, \lambda)$, where $T : [\bar{d}_1, \infty) \times [\lambda_1, \infty) \to \mathbb{R}$ is a continuous function which is class C^1 with respect to $\bar{d}$ and satisfies

$$\lim_{\bar{d},\lambda\to\infty} T(\bar{d}, \lambda) = T_0. \tag{5.22}$$

Moreover,

$$\lim_{\bar{d},\lambda\to\infty} \frac{\partial T}{\partial \bar{d}} = 0,$$

since

$$\frac{\partial T}{\partial \bar{d}} = \frac{w_{\bar{d}}(T(\bar{d},\lambda), \bar{d}, \lambda) - w'_{\bar{d}}(T(\bar{d},\lambda), \bar{d}, \lambda)}{w''(T(\bar{d},\lambda), \bar{d}, \lambda) - w'(T(\bar{d},\lambda), \bar{d}, \lambda)},$$

where subindex $\bar{d}$ means partial differentiation with respect $\bar{d}$.

Finally, the uniqueness assertion is ensured provided $\bar{d} \geq \bar{d}_1$, $\lambda \geq \lambda_1$ and

$$|T - T_0| \leq \varepsilon, \tag{5.23}$$

for certain small positive ε.

Now, solving (5.18) amounts to solving (5.21) with T replaced by $T(\bar{d}, \lambda)$. In other words, to solve

$$R\lambda = \bar{d} + T(\bar{d}, \lambda). \tag{5.24}$$

with respect to $(\bar{d}, \lambda)$. Due to (5.22), the function $I_\lambda(\bar{d}) = \bar{d} + T(\bar{d}, \lambda)$ is one to one in $[\bar{d}_2, \infty)$ for $\lambda \geq \lambda_2$ for certain conveniently large $\bar{d}_2, \lambda_2$.

Therefore, equation (5.24) is uniquely solved by a continuous function $\bar{d} = \bar{d}(\lambda)$ provided $\bar{d}, \lambda$ are suitably large.

Let us return to our original setting and assume $\hat{u}_\lambda$ is a family of nonnegative radial solutions to (1.3) with $\sup \hat{u}_\lambda \to 0$ as $\lambda \to \infty$. Thus,

$$\hat{u}_\lambda(r) = \lambda^{-\beta} w(\lambda r - \bar{d}_\lambda, \bar{d}_\lambda, \lambda)$$

for $d_\lambda \le r \le R$, with $\bar{d}_\lambda = \lambda d_\lambda$. Moreover,

$$T_\lambda = R\lambda - \bar{d}_\lambda,$$

solves (5.20) corresponding to $\bar{d} = \bar{d}_\lambda$. Now, in view of estimate (5.16), it follows that T_λ keeps bounded as $\lambda \to \infty$. Hence, necessarily

$$\lim_{\lambda\to\infty} T_\lambda = T_0.$$

This means that solutions $(T_\lambda, \bar{d}_\lambda, \lambda)$ to (5.20) satisfy the uniqueness condition (5.23) and so

$$T_\lambda = T(\bar{d}_\lambda, \lambda)$$

for large λ. Therefore $\bar{d}_\lambda = \bar{d}(\lambda)$ and family $\hat{u}_\lambda$ coincides, for large λ, with z_λ where

$$z_\lambda(r) = \begin{cases} 0 & 0 \le r \le d(\lambda), \\ \lambda^{-\beta} w(\lambda r - \lambda d(\lambda), \lambda d(\lambda), \lambda) & d(\lambda) < r \le R, \end{cases}$$

being $d(\lambda) = \bar{d}(\lambda)/\lambda$. This shows both the announced uniqueness and the existence of the family of radial nonnegative solutions z_λ satisfying $\sup z_\lambda \to 0$ as $\lambda \to \infty$.

Observe also for its use below that the constructed solution z_λ verifies

$$\left.\begin{aligned} &\int_{B_R} z_\lambda^{q+1} \sim C\lambda^{-\beta(q+1)-1}, \\ &\int_{B_R} z_\lambda^{p+1} = o(\lambda^{-\beta(q+1)-1}), \end{aligned}\right. \qquad \text{as } \lambda \to +\infty \tag{5.25}$$

for some positive constant C, as can be easily seen from the previous discussion. □

We finally conclude the proof of Theorem 4.

Proof of Theorem 4 (c). Let us see that the solution obtained in part (a) is not radial. For this aim, choose a function $\psi \in C_0^1(B_R)$ with $\psi > 0$ in B_R, and for some $\gamma > 0$, define

$$\phi(x) = \lambda^{-\beta} \psi(\lambda\gamma(x - x_0))$$

where $x_0 \in \partial B_R$ is fixed. Let us check that $J(t\phi) < 0$ for some positive t if λ is large enough. Notice that $t\phi \le U_\lambda$ if λ is large, so that

$$\begin{aligned} J(t\phi) &= \frac{t^2}{2}\int_{B_R} |\nabla\phi|^2 - \frac{\lambda t^2}{2}\int_{\partial B_R} \phi^2 + \frac{t^{p+1}}{p+1}\int_{B_R} \phi^{p+1} + \frac{t^{q+1}}{q+1}\int_{B_R} \phi^{q+1} \\ &\sim A\gamma^{2-N} t^2 \lambda^{-2\beta-N+2} - B\gamma^{1-N} t^2 \lambda^{-2\beta-N+2} + Ct^{p+1}\gamma^{-N}\lambda^{-\beta(p+1)-N} \\ &\quad + Dt^{q+1}\gamma^{-N}\lambda^{-\beta(q+1)-N} \\ &\sim t^2\gamma^{-N}\lambda^{-2\beta-N+2}\left(A\gamma^{2-N} - B\gamma^{1-N} + Dt^{q-1}\right), \end{aligned}$$

where A, B, C and D are positive constants. If $\gamma < B/A$, we can select a value of t, which we will fix and denote by t_0, such that $J(t_0\phi) < 0$ if λ is large enough. Observe also that

$$\sup_{0\le t\le t_0} J(t\phi) \le K\lambda^{-2\beta-N+2}$$

for some positive constant K. Thus, according to Lemmas 10 and 11, we can use the mountain pass theorem to obtain a nonnegative solution $v \in H_r^1(B_R)$ to (1.3) with

$$0 < J(v) \le K\lambda^{-2\beta-N+2}. \tag{5.26}$$

This solution is different from the maximal solution u_λ since $J(u_\lambda) < 0$ for large enough λ.

Finally, for the radial solution z_λ constructed in part (b) above, we have

$$J(z_\lambda) = \left(\frac{1}{q+1} - \frac{1}{2}\right)\int_{B_R} z_\lambda^{q+1} - \left(\frac{1}{2} - \frac{1}{p+1}\right)\int_{B_R} z_\lambda^{p+1} \sim C\lambda^{-\beta(q+1)-1}$$

as $\lambda \to \infty$, thanks to (5.25). According to (5.26), the solution v just obtained cannot be radial. This finishes the proof. □

6. Some remarks on multiplicity of solutions

Theorem 1 provides two nonnegative solutions to (1.3), the maximal solution $u = u_\lambda$, which exits for all $\lambda \ge \Lambda^*$, and an extra nonnegative energy solution $u = v_\lambda$ whose existence is only ensured for large λ. As will be described next, additional solutions can be built from u_λ and v_λ when $\partial\Omega$ possesses more than a single connected component.

Assume Γ_i is a component of $\partial\Omega$ (thus Γ_i constitutes a closed submanifold of $\partial\Omega$). Take $\delta > 0$ small and λ so that $\lambda \ge C\delta^{-(\beta+N-1)}$ (see (1.7)). Then

$$v_\lambda^{(i)}(x) = \begin{cases} v_\lambda(x) & \text{dist } (x,\Gamma_i) < \delta, \\ 0 & \text{dist } (x,\Gamma_i) \ge \delta, \end{cases}$$

defines a new solution to (1.3) if $\partial\Omega$ has more than a single component.

Suppose now that Γ_i is a component of $\partial\Omega$ such that Ω is not too thin around Γ_i. More precisely, it is said that Ω has thickness greater than R_0 (the value introduced in (c), (d) of Theorem 2) near the component Γ_i if for each $x \in \Gamma_i$ there exists an inner ball $B_R(\xi) \subset \Omega$ — R and ξ depending on x— such that $\overline{B}_R(\xi) \cap \Gamma_i \supset \{x\}$ together with $R \ge R_0$ for all $x \in \Gamma_i$. Associated to such a component Γ_i a new solution to (1.3) can be obtained from the maximal solution u_λ. Namely,

$$u_\lambda^{(i)}(x) = \begin{cases} u_\lambda(x) & \text{dist } (x,\Gamma_i) \le R_0, \\ 0 & \text{dist } (x,\Gamma_i) > R_0. \end{cases}$$

If $\partial\Omega$ exhibits more than a single component, and one of them, say Γ_i, satisfies the previous thickness condition then $u_\lambda^{(i)}$ furnishes a solution to (1.3) that does not satisfy (a) nor (b) of Theorem 2 (see Remark 1 (a)). It would be interesting to ascertain the possible existence of a family of nonnegative solutions not satisfying (a) and (b) when $\partial\Omega$ is connected.

Finally observe that further solutions to (1.3) can be constructed from $u_\lambda^{(i)}$ and $v_\lambda^{(i)}$ if $\partial\Omega$ has enough components. Specifically, suppose that $\partial\Omega$ splits up in two groups $\{\Gamma_i : i = 1, \dots, M_1\}$ and $\{\Gamma'_j : j = 1, \dots, M_2\}$ so that all the members of the first one satisfy the thickness condition. Then,

$$u_\lambda^{(\bar\sigma,\bar\eta)}(x) = \sum_{i=1}^{M_1} \sigma_i w_\lambda^{(i)}(x) + \sum_{j=1}^{M_2} \eta_j v_\lambda^{(j)}(x), \tag{6.27}$$

with $\bar\sigma \in \{0,1\}^{M_1}$, $\bar\eta \in \{0,1\}^{M_2}$, $w_\lambda^{(i)} \in \{u_\lambda^{(i)}, v_\lambda^{(i)}\}$ constitutes a new family of solutions to (1.3) for each of the possible choices of $\bar\sigma$, $\bar\eta$ and $w_\lambda^{(i)}$ (of course, one of them giving the null solution should be ruled out!).

As a further remark, in the case that Ω is an annulus with radii $R_1 < R_2$ and $R_2 - R_1 > R_0$ the family (6.27) can be enlarged. Indeed choices $0, u_\lambda, z_\lambda$ and v_λ (see Theorem 4) are now possible on each component $|x| = R_i$, $i = 1, 2$, of the boundary to construct (6.27).

Acknowledgements. Supported by Spanish Ministerio de Ciencia e Innovacion under grant MTM2008-05824. J. D. Rossi is partially supported by UBA X066 and by CONICET, Argentina.

References

[1] A. Ambrosetti, H. Brezis, G. Cerami, *Combined effects of concave and convex nonlinearities in some elliptic problems*, J. Funct. Anal. **122** (1994), 519–543.

[2] A. Ambrosetti, J. García-Azorero, I. Peral, *Multiplicity results for some nonlinear elliptic equations*, J. Funct. Anal. **137** (1996), 219–242.

[3] A. Ambrosetti, P. H. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14** (1973), 349–381.

[4] D. Arcoya, J. D. Rossi. *Antimaximum principle for quasilinear problems.* Adv. Diff. Eqns. **9 (9&10)**, (2004), 1185–1200.

[5] J. M. Arrieta, R. Pardo, A. Rodríguez-Bernal, *Bifurcation and stability of equilibria with asymptotically linear boundary conditions at infinity*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), 225–252.

[6] J. M. Arrieta, R. Pardo, A. Rodríguez-Bernal, *Equilibria and global dynamics of a problem with bifurcation from infinity*, J. Diff. Eqns. **246** (2009), 2055–2080.

[7] C. Bandle, M. Marcus, *'Large' solutions of semilinear elliptic equations: Existence, uniqueness and asymptotic behaviour*, J. Anal. Math. **58** (1992), 9–24.

[8] F. Charro, E. Colorado, I. Peral, *Multiplicity of solutions to uniformly elliptic fully nonlinear equations with concave-convex right-hand side*, J. Diff. Eqns. **246** (2009), 4221–4248.

[9] E. Colorado, I. Peral, *Semilinear elliptic problems with mixed Dirichlet-Neumann boundary conditions*, J. Funct. Anal. **199** (2003), 468–507.

[10] M. Del Pino, C. Flores, *Asymptotic behavior of best constants and extremals for trace embeddings in expanding domains*, Comm. Partial Differential Equations **26** (2001), 2189–2210.

[11] J. García-Azorero, I. Peral Alonso, *Multiplicity of solutions for elliptic problems with critical exponent or with a non-symmetric term*, Trans. Amer. Math. Soc. **323** (1991), 877–895.

[12] J. García-Azorero, I. Peral Alonso, *Some results about existence of a second positive solution in a quasilinear critical problem*, Indiana Univ. Math. J. **43** (1994), 941–957.

[13] J. García-Azorero, I. Peral, J. D. Rossi, *A convex-concave problem with a nonlinear boundary condition*, J. Diff. Eqns. **198** (1) (2004), 91–128.

[14] J. García-Melián, *Uniqueness of positive solutions for a boundary blow-up problem*, J. Math. Anal. Appl. **360** (2009), 530–536.
[15] J. García-Melián, J. D. Rossi, J. Sabina, *A bifurcation problem governed by the boundary condition I*, Nonlinear Differential Equations and Applications, NoDEA **14** (5/6) (2007), 499–525.
[16] J. García-Melián, J. D. Rossi, J. Sabina, *A bifurcation problem governed by the boundary condition II*, Proc. London Math. Soc. **94** (1) (2007), 1–25.
[17] J. García-Melián, J. D. Rossi, J. Sabina, *Existence and uniqueness of positive solutions to elliptic problems with sublinear mixed boundary conditions*, Comm. Contemp. Math. **11** (2009), 585–613.
[18] J. García-Melián, J. D. Rossi, J. Sabina, *An elliptic system with bifurcation parameters on the boundary condition*, J. Diff. Eqns. **247** (2009), 779–810.
[19] J. García-Melián, J. D. Rossi, J. Sabina, *Layer profiles of solutions to elliptic problems under parameter-dependent boundary conditions*. To appear in Zeitschrift für Analysis und ihre Anwendungen (Journal for Analysis and its Applications).
[20] J. García-Melián, J. Sabina, *Uniqueness to quasilinear problems for the p-Laplacian in radially symmetric domains*, Nonlinear Anal. T.M.A. **43** (7) (2001), 803–835.
[21] J. García-Melián, J. Sabina, *Remarks on Large Solutions*, in "Spectral Theory and Nonlinear Analysis with Applications to Spatial Ecology" (S. Cano, J. López-Gómez and C. Mora editors), World Scientific, Singapore, 2005.
[22] N. Ghoussoub, D. Preiss, *A general mountain pass principle for locating and classifying critical points*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **6** (1989), 321–330.
[23] P. Hartman, "Ordinary Differential Equations", Wiley, New York, 1964.
[24] J. B. Keller, *On solutions of $\Delta u = f(u)$*, Comm. Pure Appl. Math. **10** (1957), 503–510.
[25] P. H. Rabinowitz, "Minimax methods in critical point theory with applications to differential equations", CBMS Regional Conf. Ser. in Math., no. **65**, Amer. Math. Soc., Providence, R.I., 1986.
[26] W. Reichel, W. Walter, *Radial solutions of equations and inequalities involving the p-Laplacian*, J. Inequal. Appl. **1** (1997), 47–71.
[27] J. D. Rossi, *Elliptic problems with nonlinear boundary conditions and the Sobolev trace theorem*, Chapter 5 of "Handbook of differential equations: stationary partial differential equations" vol. II, pp. 311–406, Elsevier/North-Holland, Amsterdam, 2005.

J. García-Melián and J. C. Sabina de Lis
Departamento de Análisis Matemático, Universidad de La Laguna.
C/. Astrofísico Francisco Sánchez s/n, 38271 – La Laguna, SPAIN
and
Instituto Universitario de Estudios Avanzados (IUdEA) en Física Atomica, Molecular y Fotonica, Facultad de Física, Universidad de La Laguna
C/. Astrofísico Francisco Sánchez s/n, 38203 – La Laguna , SPAIN
E-mail address: `jjgarmel@ull.es, josabina@ull.es`

J. D. Rossi
Departamento de Análisis Matemático, Universidad de Alicante,
Ap. correos 99, 03080 Alicante, SPAIN.
On leave from
Departamento de Matemática, FCEyN UBA,
Ciudad Universitaria, Pab 1 (1428),
Buenos Aires, ARGENTINA.
E-mail address: `julio.rossi@ua.es`