Uniqueness and non-uniqueness for the porous medium equation with non linear boundary conditions.

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Abstract

We study the uniqueness problem for non negative solutions of $u_t = \Delta u^m$ in $\Omega \times [0,T)$, $-\frac{\partial u^m}{\partial \hat{n}}(x,t) = u^\lambda(x,t)$ on $\partial\Omega \times (0,T)$ and $u(x,0) \equiv 0$ on Ω where $m > 1$, $\lambda \geq 1$ and Ω is a bounded domain with smooth boundary in $\mathbf{R}^N$. We prove that the solution $u \equiv 0$ is unique if and only if $2\lambda \geq m+1$.

1 Introduction.

Let Ω be a bounded domain in $\mathbf{R}^N$ with smooth boundary and let $m > 1$ and $\lambda \geq 1$ be two real numbers. In this article we are concerned with

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the uniqueness problem for non-negative solutions of the following initial boundary value problem in the case of null initial data.

$$\begin{array}{ll} u_t = \Delta u^m, & \text{in } \Omega \times (0,T) \\ -\frac{\partial u^m}{\partial \hat{n}} = u^\lambda, & \text{on } \partial\Omega \times (0,T) \end{array} \tag{1.1}$$

with initial data $u(x,0) = u_0 \geq 0$ in Ω. Here and in what follows $\hat{n}$ denotes the inner unit normal to $\partial\Omega$.

As it is well known, in general problem (1.1) does not have classical solutions, but if the initial data is positive in $\overline{\Omega}$ and verifies a compatibility condition at the boundary, then there exists a classical solution (see [A]). In what follows, in the case of more general non negative initial data, we will say that a continuous function u on $\overline{\Omega} \times [0,T)$ is a solution, weak solution, of (1.1) if

$$\int_\Omega u_0 \varphi(\cdot,0) + \int_0^T \int_\Omega u \varphi_t + \int_0^T \int_\Omega u^m \Delta\varphi - \int_0^T \int_{\partial\Omega} u^\lambda \varphi - \int_0^T \int_{\partial\Omega} u^m \frac{\partial \varphi}{\partial \hat{n}} = 0 \tag{1.2}$$

for all $\varphi \in C_0^\infty(\overline{\Omega} \times [0,T))$.

This type of solutions of (1.1) can be obtained, by a standard monotonicity argument, as the limit of a decreasing sequence of classical solutions.

The problem of uniqueness and non uniqueness for different parabolic non-linear equations with non Lipschitian data has been addressed by several authors. See for example [FW] for the heat equation with a nonlinear source, [EH1] and [EH2] for semi-linear reaction diffusion systems. In [PV] a non-uniqueness result is obtained for the porous medium equation with a non Lipschitz source term in the case of the Cauchy problem in $\mathbf{R}^N$. It is well known that, in the above mentioned problem, if the source term is Lipschitz then solutions are unique.

Our main result in this note is the following

Theorem 1.1 *Let $u_0 \equiv 0$, then the solution $u \equiv 0$ to problem (1.1) is unique if and only if $2\lambda \geq m+1$.*

We want to remark that in our case the nonlinear boundary condition is given by a Lipschitz function, nevertheless the non uniqueness phenomenon appears for a certain range of the parameters m and λ. We finally observe that Theorem 1.1. is a generalization to several space variables of the results of [BCE].

The idea of the proof of Theorem 1.1 is, up to some technical arguments, the following, since there is a comparison principle for problem (1.1), the existence of a non trivial sub-solution would imply the existence of a non trivial solution via a standard monotonicity argument. On the other hand, if one could construct arbitrarily small super-solutions, defined on a fixed interval of time, then by comparison it would follow that the null solution is unique. Thus, a key step in the proof is the construction of non trivial sub-solutions, in the case $2\lambda < m+1$, and arbitrarily small super-solutions, if $m+1 \leq 2\lambda$, of problem (1.1) with $u_0 \equiv 0$. This construction is based on a device, already used in [CDE], that consists of producing sub and super solutions for the multi dimensional problem starting from solutions in the half line in $\mathbf{R}^1$, provided one has a good knowledge of the one dimensional solutions. This is achieved by making a suitable change of coordinates in a neighborhood of the boundary of Ω and then performing a rescaling of variables. The solutions for the one dimensional problem we use, are the self-similar ones that were obtained in [GP1] and [GP2].

The rest of this note is organized as follows. In the next section we describe the change of variables near the boundary and how to compute the Laplacian in these coordinates, review the self similar one dimensional solutions of [GP1] and [GP2] and, state a comparison principle. In the last section we construct the sub and super solutions and prove Theorem 1.1.

2 Preliminaries.

Let us consider the following change of variables in a neighborhood of $\partial\Omega$. Let $\bar{x}$ be a point in $\partial\Omega$. We denote by $\hat{n}(\bar{x})$ the inner unit normal to $\partial\Omega$ at the point $\bar{x}$. Since $\partial\Omega$ is smooth it is well known that there exists $\delta > 0$ such that the mapping $\varphi : \partial\Omega \times [0, \delta] \to \mathbf{R}^N$ given by $\varphi(\bar{x}, s) = \bar{x} + s\hat{n}(\bar{x})$ defines new coordinates $(\bar{x}, s)$ in a neighborhood V of $\partial\Omega$ in $\overline{\Omega}$.

A straightforward computation shows that, in these coordinates, Δ applied to a function $g(\bar{x}, s) = g(s)$, which is independent of the variable $\bar{x}$, evaluated at a point $(\bar{x}, s)$ is given by

$$\Delta g(\bar{x}, s) = \frac{\partial^2 g}{\partial s^2}(\bar{x}, s) - \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1 - H_j(\bar{x})s)} \frac{\partial g}{\partial s}(\bar{x}, s) \tag{2.1}$$

where $H_j(\bar{x})$ for $i = 1, ..., N$, denote the principal curvatures of $\partial\Omega$ at $\bar{x}$.

We review now some results about the solutions of the following O.D.E.

$$(f^m)''(\eta) + p\eta f'(\eta) = qf(\eta) \qquad \text{in } [0, +\infty) \tag{2.2}$$

where p and q are real numbers.

Solutions of (2.2) provide self-similar solutions of the porous medium equation in the half line in $\mathbf{R}^1$. Equation (2.2) has been completely studied in [GP1], [GP2] and [G]. We summarize, in the form of a lemma, the part of these results that we will need later.

Lemma 2.1 *(B. H. Gilding-L. A. Peletier.) If $p > 0$ and $q > 0$ then for any $U \geq 0$ equation (2.2) has a unique weak solution f such that f is a positive classical solution on an interval $(0, \xi_0)$, $f(0) = U$, $f(\xi_0) = 0$, $(f^m)'(\xi_0) = 0$ and $f(\eta) \equiv 0$ for $\eta \in [\xi_0, \infty)$. Moreover $(f^m)'(\eta) < 0$ and $(f^m)''(\eta) > 0$ for $\eta \in [0, \xi_0)$.*

Let us define the function

$$h(\eta) = \begin{cases} \frac{(f^m)'(\eta)}{(f^m)''(\eta)} & \text{if} \quad \eta \in [0, \xi_0) \\ 0 & \text{if} \quad \eta \in [\xi_0, \infty) \end{cases}$$

and state the following elementary lemma for future reference.

Lemma 2.2 *If $p > 0$ and $q > 0$, then the function h is continuous and hence bounded.*

Proof:

It suffices to prove that h is continuous at ξ_0. This is immediate because from the equation it follows that for $\eta \in (0, \xi_0)$ one has

$$-\frac{mf^{m-1}(\eta)}{p\eta} \leq h(\eta) \leq 0$$

and the lemma is proved. $\square$

We will say that a function $\overline{u}$ is a strict super-solution of (1.1) if $\overline{u}$ is continuous in $\overline{\Omega} \times [0,T)$ and satisfies

$$\overline{u}_t \geq \Delta \overline{u}^m, \qquad \text{in } \Omega \times (0,T) \tag{2.3}$$

in the weak sense and

$$-\frac{\partial \overline{u}^m}{\partial \hat{n}} > \overline{u}^\lambda \qquad \text{on } \partial\Omega \times (0,T). \tag{2.4}$$

Analogously we say that $\underline{u}$ is a strict sub-solution of (1.1) if in the definition for strict super-solution the inequalities are reversed.

The last ingredient we need in our proof are the following comparison principles.

Lemma 2.3 *Let u be a solution of (1.1).*

1) Let $\overline{u}$ be a strict super-solution of (1.1).

If $\overline{u}(x,0) \geq u_0(x)$ in Ω and $\overline{u}(x,0) > u_0(x)$ in $\partial\Omega$, then $\overline{u} \geq u$ in $\overline{\Omega} \times (0,T)$.

2) Let $\underline{u}$ be a strict sub-solution of (1.1).

If $\underline{u}(x,0) \leq u_0(x)$ in Ω and $\underline{u}(x,0) < u_0(x)$ in $\partial\Omega$, then $\underline{u} \leq u$ in $\overline{\Omega} \times (0,T)$.

Proof:

We sketch only the proof of 1). Suppose that the conclusion is false and set $t_0 = \max\{t \in (0,t) / \ \overline{u} \geq u \text{ in } \overline{\Omega} \times (0,T)\}$. Clearly $0 \leq t_0 < T$. We claim that there exists $x_0 \in \partial\Omega$ such that $\overline{u}(x_0,t_0) = u(x_0,t_0)$. Indeed, if not, by continuity and the standard comparison result for the Dirichlet problem, one has $\overline{u} \geq u$ in $\overline{\Omega} \times (0, t_0 + \varepsilon]$ for some $\varepsilon > 0$. This contradicts the choice of t_0 and the claim is proved. From the hypotheses on the initial condition and continuity it follows that $t_0 > 0$. Now, using the boundary conditions, one has $-\frac{\partial \overline{u}^m}{\partial \hat{n}}(x_0,t_0) > -\frac{\partial u^m}{\partial \hat{n}}(x_0,t_0)$ that also contradicts the choice of t_0 and the lemma is proved. $\square$

Lemma 2.4 *Let u_1 and u_2 be solutions of (1.1) with initial conditions satisfying the compatibility condition on the boundary. If $0 < u_1(x,0) < u_2(x,0)$ for all $x \in \overline{\Omega}$, then $u_1 \leq u_2$ in $\overline{\Omega} \times [0,T)$.*

Proof:

The proof is similar to the one of the previous lemma but, in this case one takes $t_0 = \max\{t \in (0,T) / \ u_2 \geq u_1 + k \text{ in } \overline{\Omega} \times (0,t)\}$ where $k > 0$ is such that $u_2(x,0) > u_1(x,0) + 2k$ for all $x \in \overline{\Omega}$. $\square$

3 Proof of Theorem 1.

Throughout this section we will always assume that $u_0 \equiv 0$ in (1.1).

We split the proof into two cases.

The case $2\lambda < m+1$**:**

Let f be the solution of (2.2) with

$$p = \frac{(m-\lambda)}{m+1-2\lambda}$$

and

$$q = \frac{1}{m+1-2\lambda}.$$

In this case it is easy to see, by rescaling, that it is possible to choose U such that $-(f^m)'(0) = f^\lambda(0)$. Then the function

$$v(s,t) = t^q f(\frac{s}{t^p})$$

satisfies $v_t = (v^m)_{ss}$ and $-(v^m)_s(0,t) = v^\lambda(0,t)$ in $[0,\infty) \times [0,\infty)$.

We proceed now to do the rescaling. Let ε be such that $0 < \varepsilon < 1$ and pick c such that $0 < c < \frac{m-1}{2}$. Choose T_0 such that $\xi_0 \varepsilon^{\frac{m-1}{2}}((1-\varepsilon^c)T_0)^p \leq \delta$. For points in V of coordinates $(\bar{x}, s, t)$ such that $0 \leq t \leq T_0$ and $0 \leq s \leq \xi_0 \varepsilon^{\frac{m-1}{2}}((1-\varepsilon^c)t)^p$ define

$$\underline{u}_\varepsilon(\bar{x}, s, t) = \varepsilon v(\frac{s}{\varepsilon^{(m-1)/2}}, (1-\varepsilon^c)t)$$

and, extend $\underline{u}_\varepsilon$ as zero to the whole of $\overline{\Omega} \times [0, T_0]$.

We can state now

Proposition 3.1 *There exists ε_0 such that for any ε such that $0 < \varepsilon \leq \varepsilon_0$ the function $\underline{u}_\varepsilon$ is a strict sub-solution of (1.1) in $\overline{\Omega} \times [0, T_0]$.*

Proof:

As $0 < \varepsilon < 1$ and $\lambda < \frac{m+1}{2}$ one has

$$-\frac{\partial \underline{u}_\varepsilon^m}{\partial s}(\bar{x}, 0, t) < \underline{u}_\varepsilon^\lambda(\bar{x}, 0, t). \tag{3.1}$$

and hence the boundary condition is satisfied.

We set
$$\xi = \frac{s}{\varepsilon^{(m-1)/2}((1-\varepsilon^c)t)^p}.$$
A straightforward computation shows that if $(\bar{x}, s, t)$ is such that $0 < t < T_0$ and $0 \le s \le \xi_0 \varepsilon^{\frac{m-1}{2}}((1-\varepsilon^c)t)^p$, then
$$(\underline{u}_\varepsilon)_t(\bar{x}, s, t) - \Delta \underline{u}_\varepsilon^m(\bar{x}, s, t) =$$
$$-\varepsilon^{1+c}((1-\varepsilon^c)t)^{q-1}(f^m)''(\xi)\left[1 - \varepsilon^{\frac{m-1}{2}-c}((1-\varepsilon^c)t)^p \sum_{j=1}^{N-1} \frac{H_j(\bar{x})}{(1-H_j(\bar{x})s)} h(\xi)\right].$$

Now since, by Lemma 2.2, the function h is bounded and $(f^m)''(\xi) > 0$ for $\xi \in [0, \xi_0]$, we obtain that if ε is small enough then
$$(\underline{u}_\varepsilon)_t(\bar{x}, s, t) - \Delta \underline{u}_\varepsilon^m(\bar{x}, s, t) \le 0 \tag{3.2}$$
if $0 < t < T_0$ and $0 \le s \le \xi_0 \varepsilon^{\frac{m-1}{2}}((1-\varepsilon^c)t)^p$. Finally that $\underline{u}_\varepsilon$ is a sub-solution, in the weak sense in the whole of $\overline{\Omega} \times [0, T_0)$, follows from the fact that $\underline{u}_\varepsilon$ is continuous in $\overline{\Omega} \times [0, T_0)$ and, since $(f^m)'(\xi_0) = 0$, one has $\nabla \underline{u}_\varepsilon^m = 0$ on the free boundary as it can be checked by a direct computation. The proposition is proved. $\square$

We are in a position now to give the proof of non-uniqueness in the case $2\lambda < m+1$. Pick a sequence, v_n, $n = 0, 1, 2, \ldots$, of positive classical solutions of (1.1) with compatible initial data such that $0 < v_n(x, 0) < v_j(x, 0)$ if $n > j$ and $v_n(x, 0) \to 0$ as $n \to \infty$. By Lemma 2.3 and Lemma 2.4, making T_0 smaller if necessary, we obtain that for a fixed small enough ε one has
$$\underline{u}_\varepsilon \le v_n \le v_j \le v_0$$
in $\overline{\Omega} \times [0, T_0)$ if $n > j$.

We define now $u(x, t) = \lim v_n(x, t)$ as $n \to \infty$. By the monotone convergence theorem we obtain that u satisfies (1.2) and it follows from Theorem 6.2, pag. 113, of [dB] that u is continuous in $\overline{\Omega} \times [0, T_0]$. Clearly $\underline{u}_\varepsilon \le u$ and hence u is non-trivial. This proves the theorem in the case $2\lambda < m+1$.

The case $m + 1 \le 2\lambda$:

In this case let f be the solution of (2.2) with $p = \frac{m-1}{2}$, $q = 1$ and, say, such that $f(0) = 1$.

Let $A>0$ and define for $(s,t)\in[0,\infty)\times[0,\infty)$

$$v(s,t)=A^{\frac{1}{m-1}}e^{At}f(\frac{s}{e^{(m-1)At/2}}).$$

Then v satisfies $v_t=(v^m)_{ss}$ and $-(v^m)_s(0,t)=-A^{1/2}(f^m)'(0)v^{\frac{m+1}{2}}(0,t)$. Therefore if we pick, once and for all, A such that $-A^{1/2}(f^m)'(0)>1$, then one has

$$\begin{array}{lll} v_t=(v^m)_{ss} & \text{in } [0,\infty)\times[0,\infty) & \\ -(v^m)_s(0,t)>v^{\frac{m+1}{2}}(0,t) & \text{on } [0,\infty) & (3.3)\\ v(s,0)=A^{\frac{1}{m-1}}f(s) & \text{on } [0,\infty). & \end{array}$$

We will construct now super solutions for (1.1) when $2\lambda\geq m+1$.

Let $\varepsilon>0$ and pick c such that $0<c<\frac{m-1}{2}$. Let T_0 be such that

$$\xi_0(\varepsilon e^{A(1+\varepsilon^c)T_0})^{(m-1)/2}<\delta,$$

define

$$\overline{u}_\varepsilon(\bar{x},s,t)=\varepsilon v(\frac{s}{\varepsilon^{(m-1)/2}},(1+\varepsilon^c)t)$$

for $0\leq t<T_0$ and $0\leq s<\xi_0(\varepsilon e^{A(1+\varepsilon^c)t})^{(m-1)/2}$ and, extend $\overline{u}_\varepsilon$ as zero to the whole of $\overline{\Omega}\times[0,T_0)$.

We can state now

Proposition 3.2 *There exist ε_0 and T_0 such that for any $0\leq\varepsilon\leq\varepsilon_0$ and $0<T\leq T_0$ the function $\overline{u}_\varepsilon$ is a strict super-solution of (1.1).*

Proof:

Since $2\lambda\geq m+1$ it is clear that, as long as t verifies $\varepsilon A^{\frac{1}{m-1}}e^{A(1+\varepsilon^c)t}<1$, one has

$$-\frac{\partial\overline{u}_\varepsilon^m}{\partial s}(\bar{x},0,t)>\overline{u}_\varepsilon^\lambda(\bar{x},0,t).$$

Therefore, making T_0 smaller if necessary, the boundary condition is satisfied.

Setting

$$\xi=\frac{s}{(\varepsilon e^{A(1+\varepsilon^c)t})^{(m-1)/2}},$$

the same calculation as in the previous case shows that

$$(\overline{u}_\varepsilon)_t(\bar{x},s,t)-\Delta\overline{u}_\varepsilon^m(\bar{x},s,t)=\varepsilon^{1+c}A^{\frac{m}{m-1}}\times$$

$$e^{A(1+\varepsilon^c)t}(f^m)''(\xi)\left[1+\varepsilon^{(m-1)/2}(A^{\frac{1}{m-1}}e^{A(1+\varepsilon^c)t})^{\frac{m-1}{2}}\sum_{j=1}^{N-1}\frac{H_j(\bar{x})}{(1-H_j(\bar{x})s)}h(\xi)\right]$$

whenever $0 \leq t < T_0$ and $0 \leq s < \xi_0(\varepsilon e^{A(1+\varepsilon^c)t})^{(m-1)/2}$.

Since the function h is bounded, as seen in Lemma 2.2, and $(f^m)''(\xi) \geq 0$ we obtain that if ε is small enough, then

$$(\overline{u}_\varepsilon)_t(\bar{x},s,t) - \Delta\overline{u}_\varepsilon^m(\bar{x},s,t) \geq 0 \qquad (3.4)$$

as long as $0 \leq t < T_0$ and $0 \leq s < \xi_0(\varepsilon e^{A(1+\varepsilon^c)t})^{(m-1)/2}$.

Now, $\overline{u}_\varepsilon$ is continuous on $\overline{\Omega} \times [0, T_0)$ and as before $\nabla\overline{u}_\varepsilon^m = 0$ on the free boundary. It follows that $\overline{u}_\varepsilon$ is a super-solution on the whole of $\overline{\Omega} \times [0, T_0)$. This ends the proof of the proposition. $\square$

We are ready now to give the proof of uniqueness in the case $m+1 \leq 2\lambda$. Assume for a contradiction that there exists a solution u of (1.1) with $u_0 \equiv 0$ which is not identically null. Without lost of generality we can assume that it is not identically zero in $\overline{\Omega} \times [0, T)$ for some $T < T_0$. It follows, by Lemma 2.3, that

$$u \leq \overline{u}_\varepsilon \text{ on } \overline{\Omega} \times [0, T)$$

for all $\varepsilon \leq \varepsilon_0$. This is a contradiction since $\overline{u}_\varepsilon \to 0$ uniformly in $\overline{\Omega} \times [0, T)$ as $\varepsilon \to 0$. The theorem is proved.

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