Asymptotics for evolution problems with nonlocal diffusion

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Abstract. In these notes we review recent results concerning solutions to nonlocal evolution equations with different boundary conditions, Dirichlet or Neumann and even for the Cauchy problem. We deal with existence/uniqueness of solutions and their asymptotic behavior. We also review some recent results concerning limits of solutions to nonlocal equations when a rescaling parameter goes to zero. We recover in these limits some of the most frequently used diffusion models: the heat equation with Neumann or Dirichlet boundary conditions, the $p-$Laplace evolution equation with Neumann boundary conditions and a convection-diffusion equation.

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CHAPTER 1

Introduction

First, let us briefly introduce the prototype of nonlocal problem that will be considered along this work.

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial, continuous function with $\int_{\mathbb{R}^N} J(z)\,dz = 1$. Nonlocal evolution equations of the form

$$u_t(x,t) = (J * u - u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), \tag{1.1}$$

and variations of it, have been recently widely used to model diffusion processes. More precisely, as stated in [**59**], if $u(x,t)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(y-x)u(y,t)\,dy = (J*u)(x,t)$ is the rate at which individuals are arriving at position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external or internal sources, leads immediately to the fact that the density u satisfies equation (1.1). For recent references on nonlocal diffusion see, [**15**], [**16**], [**17**], [**32**], [**35**], [**59**], and references therein.

These type of problems have been used to model very different applied situations, for example in biology ([**32**], [**71**]), image processing ([**70**], [**63**]), particle systems ([**22**]), coagulation models ([**61**]), etc.

Concerning boundary conditions for nonlocal problems we consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and look at the nonlocal problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), & x \in \Omega,\ t>0, \\ u(x,t) = 0, & x \notin \Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{1.2}$$

In this model we have that diffusion takes place in the whole $\mathbb{R}^N$ but we impose that u vanishes outside Ω. This is the analogous of what is called Dirichlet boundary conditions for the heat equation. However, the boundary data is not understood in the usual sense, since we are not imposing that $u|_{\partial\Omega} = 0$.

Let us turn our attention to Neumann boundary conditions. We study

$$
(1.3)\qquad \begin{aligned} & u_t(x,t) = \int_\Omega J(x-y)(u(y,t)-u(x,t))\,dy, && x \in \Omega,\ t>0, \\ & u(x,0) = u_0(x), && x \in \Omega. \end{aligned}
$$

In this model we have that the integral terms take into account the diffusion inside Ω. In fact, as we have explained the integral $\int J(x-y)(u(y,t)-u(x,t))\,dy$ takes into account the individuals arriving or leaving position x from other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. The individuals may not enter nor leave Ω. This is the analogous of what is called homogeneous Neumann boundary conditions in the literature.

Here we review some results concerning existence and uniqueness for these models and their asymptotic behavior as $t \to \infty$. These results says that these problems are well posed in appropriate functional spaces although they do not have a smoothing property. Moreover, the asymptotic behavior for the linear nonlocal models coincide with the one that holds for the heat equation.

We will also review some recent results concerning limits of nonlocal problems when a scaling parameter (that measures the radius of influence of the nonlocal term) goes to zero. We recover in these limits some well known diffusion problems, namely, the heat equation with Neumann or Dirichlet boundary conditions, the $p-$Laplace equation with Neumann boundary conditions and a convection-diffusion equation.

The content of these notes summarizes the research of the author in the last years and is contained in [**7**], [**8**], [**9**], [**34**], [**40**], [**41**], [**42**], [**43**], [**67**], [**68**], [**76**]. We refer to these papers for extra details and further references.

There is a huge amount of papers dealing with nonlocal problems. Among them we quote [**14**], [**17**], [**38**], [**35**], [**44**], [**45**], [**47**], [**48**], [**81**] and [**87**], devoted to travelling front type solutions to the parabolic problem in $\Omega = \mathbb{R}$, and [**19**], [**36**], [**37**], [**46**], [**75**], which dealt with source term of logistic type, bistable or power-like nonlinearity. The particular instance of the parabolic problem in $\mathbb{R}^N$ is considered in [**34**], [**68**], while the "Neumann" boundary condition for the same problem is treated in [**7**], [**42**] and [**43**]. See also [**67**] for the appearance of convective terms and [**39**], [**40**] for interesting features in other related nonlocal problems. We finally mention the paper [**66**], where some logistic equations and systems of Lotka-Volterra type are studied.

There is also an increasing interest in free boundary problems and regularity issues for nonlocal problems. We refer to [**12**], [**30**], [**31**], [**82**], but we are not dealing with such issues in the present work.

The Bibliography of this work does not escape the usual rule of being incomplete. In general, we have listed those papers which are more close to the topics discussed here. But, even for those papers, the list is far from being exhaustive and we apologize for omissions.

These notes are organized as follows:

(1) in Chapter 2 we deal with the linear diffusion equation in the whole $\mathbb{R}^N$ or in a bounded domain with Dirichlet or Neumann boundary condition paying special attention to the asymptotic behavior as $t \to \infty$;
(2) in Chapter 3 we find refined asymptotics;
(3) in Chapter 4 we deal with higher order nonlocal problems;
(4) in Chapter 5 we approximate the heat equation with Neumann boundary conditions;
(5) in Chapter 6 we approximate the heat equation with Dirichlet boundary conditions;
(6) in Chapter 7 we deal with approximations of higher order problems;
(7) in Chapter 8 we present a nonlocal convection-diffusion equation;
(8) in Chapter 9 we deal with a nonlinear nonlocal Neumann problem. In this Chapter we can consider a nonlocal analogous to the porous medium equation;
(9) in Chapter 10 we face a nonlocal diffusion model analogous to the $p-$Laplacian with Neumann boundary conditions,
(10) finally, in Chapter 11 we take the limit as $p \to \infty$ in the previous model.

CHAPTER 2

The linear problem

The aim of this chapter is to study the asymptotic behavior of solutions of a nonlocal diffusion operator in the whole $\mathbb{R}^N$ or in a bounded smooth domain with Dirichlet or Neumann boundary conditions.

0.1. The Cauchy problem. We will consider the linear nonlocal diffusion problem presented in the Introduction

$$\begin{aligned} &u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), \\ &u(x,0) = u_0(x), \end{aligned} \tag{2.1}$$

We will understand a solution of (2.1) as a function

$$u \in C^0([0,+\infty); L^1(\mathbb{R}^N))$$

that verifies (2.1) in the integral sense, see Theorem 3. Our first result states that the decay rate as t goes to infinity of solutions of this nonlocal problem is determined by the behavior of the Fourier transform of J near the origin. The asymptotic decays are the same as the ones that hold for solutions of the evolution problem with right hand side given by a power of the laplacian.

In the sequel we denote by $\hat{f}$ the Fourier transform of f. Let us recall our hypotheses on J that we will assume throughout this chapter,

(H) $J \in C(\mathbb{R}^N, \mathbb{R})$ is a nonnegative, radial function with $\int_{\mathbb{R}^N} J(x)\,dx = 1$.

This means that J is a radial density probability which implies obviously that $|\hat{J}(\xi)| \le 1$ with $\hat{J}(0) = 1$, and we shall assume that $\hat{J}$ has an expansion of the form

$$\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$$

for $\xi \to 0$ ($A > 0$). Remark that in this case, (H) implies also that $0 < \alpha \le 2$ and $\alpha \ne 1$ if J has a first momentum.

The main result of this chapter reads as follows,

THEOREM 1. *Let u be a solution of* (2.1) *with $u_0, \hat{u}_0 \in L^1(\mathbb{R}^N)$. If there exist $A > 0$ and $0 < \alpha \le 2$ such that*

$$\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \qquad \xi \to 0, \tag{2.2}$$

then the asymptotic behavior of $u(x,t)$ *is given by*

$$\lim_{t\to+\infty} t^{N/\alpha} \max_x |u(x,t) - v(x,t)| = 0,$$

where v *is the solution of* $v_t(x,t) = -A(-\Delta)^{\alpha/2} v(x,t)$ *with initial condition* $v(x,0) = u_0(x)$. *Moreover, we have*

$$\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} \le C\, t^{-N/\alpha},$$

and the asymptotic profile is given by

$$\lim_{t\to+\infty} \max_y \left| t^{N/\alpha} u(yt^{1/\alpha}, t) - \|u_0\|_{L^1} G_A(y) \right| = 0,$$

where $G_A(y)$ *satisfies* $\hat{G}_A(\xi) = e^{-A|\xi|^\alpha}$.

In the special case $\alpha = 2$, the decay rate is $t^{-N/2}$ and the asymptotic profile is a gaussian $G_A(y) = (4\pi A)^{N/2} \exp(-A|y|^2/4)$ with $A \cdot \mathrm{Id} = -(1/2) D^2 \hat{J}(0)$. Note that in this case (that occurs, for example, when J is compactly supported) the asymptotic behavior is the same as the one for solutions of the heat equation and, as happens for the heat equation, the asymptotic profile is a gaussian.

The decay in L^∞ of the solutions together with the conservation of mass give the decay of the L^p-norms by interpolation. As a consequence of the previous theorem, we find that this decay is analogous to the decay of the evolution given by the fractional laplacian, that is,

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \le C\, t^{-\frac{N}{\alpha}\left(1-\frac{1}{p}\right)},$$

see Corollary 11. We refer to [**38**] for the decay of the L^p-norms for the fractional laplacian, see also [**33**], [**52**] and [**54**] for finer decay estimates of L^p-norms for solutions of the heat equation.

We shall make an extensive use of the Fourier transform in order to obtain explicit solutions in frequency formulation. Let us recall that if $f \in L^1(\mathbb{R}^N)$ then $\hat{f}$ and $\check{f}$ are bounded and continuous, where $\hat{f}$ is the Fourier transform of f and $\check{f}$ its inverse Fourier transform. Moreover,

$$\lim_{|\xi|\to\infty} \hat{f}(\xi) = 0 \qquad \text{and} \qquad \lim_{|x|\to\infty} \check{f}(x) = 0.$$

We begin by collecting some properties of the function J.

Lemma 2. *Let* J *satisfy hypotheses* (H). *Then,*

i) $|\hat{J}(\xi)| \le 1$, $\hat{J}(0) = 1$.
ii) *If* $\int_{\mathbb{R}^N} J(x)|x|\,dx < +\infty$ *then*

$$\left(\nabla_\xi \hat{J}\right)_i (0) = -i \int_{\mathbb{R}^N} x_i J(x)\,dx = 0$$

and if $\int_{\mathbb{R}^N} J(x)|x|^2\,dx < +\infty$ *then*

$$\left(D^2\hat{J}\right)_{ij}(0) = -\int_{\mathbb{R}^N} x_i x_j J(x)\,dx,$$

therefore $\left(D^2\hat{J}\right)_{ij}(0) = 0$ *when* $i \neq j$ *and* $\left(D^2\hat{J}\right)_{ii}(0) \neq 0$. *Hence the Hessian matrix of* $\hat{J}$ *at the origin is given by*

$$D^2\hat{J}(0) = -\left(\frac{1}{N}\int_{\mathbb{R}^N} |x|^2 J(x)\,dx\right)\cdot \mathrm{Id}.$$

iii) *If* $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|)^\alpha$ *then necessarily* $\alpha \in (0,2]$, *and if* J *has a first momentum, then* $\alpha \neq 1$. *Finally, if* $\alpha = 2$, *then*

$$A\cdot \mathrm{Id} = -(1/2)\big(D^2\hat{J}\big)_{ij}(0).$$

Proof. Points i) and ii) are rather straightforward (recall that J is radially symmetric). Now we turn to iii). Let us recall a well-known probability lemma that says that if $\hat{J}$ has an expansion of the form,

$$\hat{J}(\xi) = 1 + i\langle a,\xi\rangle - \frac{1}{2}\langle \xi, B\xi\rangle + o(|\xi|^2),$$

then J has a second momentum and we have

$$a_i = \int x_i J(x)dx, \qquad B_{ij} = \int x_i x_j J(x)dx < \infty.$$

Thus if iii) holds for some $\alpha > 2$, it would turn out that the second moment of J is null, which would imply that $J \equiv 0$, a contradiction. Finally, when $\alpha = 2$, then clearly $B_{ij} = -\big(D^2\hat{J}\big)_{ij}(0)$ hence the result since by symmetry, the Hessian is diagonal. □

Now, we prove existence and uniqueness of solutions using the Fourier transform.

Theorem 3. *Let* $u_0 \in L^1(\mathbb{R}^N)$ *such that* $\hat{u_0} \in L^1(\mathbb{R}^N)$. *There exists a unique solution* $u \in C^0([0,\infty); L^1(\mathbb{R}^N))$ *of* (2.1), *and it is given by*

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\hat{u_0}(\xi).$$

Proof. We have

$$u_t(x,t) = J * u - u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t).$$

Applying the Fourier transform to this equation we obtain

$$\hat{u}_t(\xi,t) = \hat{u}(\xi,t)(\hat{J}(\xi)-1).$$

Hence,

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\hat{u_0}(\xi).$$

Since $\hat{u_0} \in L^1(\mathbb{R}^N)$ and $e^{(\hat{J}(\xi)-1)t}$ is continuous and bounded, the result follows by taking the inverse of the Fourier transform. $\square$

REMARK 4. One can also understand solutions of (2.1) directly in Fourier variables. This concept of solution is equivalent to the integral one in the original variables under our hypotheses on the initial condition.

Now we prove a lemma concerning the fundamental solution of (2.1).

LEMMA 5. *Let $J \in \mathcal{S}(\mathbb{R}^N)$, the space of rapidly decreasing functions. The fundamental solution of (2.1), that is the solution of (2.1) with initial condition $u_0 = \delta_0$, can be decomposed as*

$$w(x,t) = e^{-t}\delta_0(x) + v(x,t), \tag{2.3}$$

with $v(x,t)$ smooth. Moreover, if u is a solution of (2.1) it can be written as

$$u(x,t) = (w * u_0)(x,t) = \int_{\mathbb{R}^N} w(x-z,t)u_0(z)\,dz.$$

PROOF. By the previous result we have

$$\hat{w}_t(\xi,t) = \hat{w}(\xi,t)(\hat{J}(\xi)-1).$$

Hence, as the initial datum verifies $\hat{u_0} = \hat{\delta}_0 = 1$,

$$\hat{w}(\xi,t) = e^{(\hat{J}(\xi)-1)t} = e^{-t} + e^{-t}(e^{\hat{J}(\xi)t} - 1).$$

The first part of the lemma follows applying the inverse Fourier transform in $\mathcal{S}(\mathbb{R}^N)$.

To finish the proof we just observe that $w * u_0$ is a solution of (2.1) (just use Fubini's theorem) with $(w * u_0)(x,0) = u_0(x)$. $\square$

REMARK 6. The above proof together with the fact that $\hat{J}(\xi) \to 0$ (since $J \in L^1(\mathbb{R}^N)$) shows that if $\hat{J} \in L^1(\mathbb{R}^N)$ then the same decomposition (2.3) holds and the result also applies.

Next, we prove the first part of our main result.

THEOREM 7. *Let u be a solution of (2.1) with $u_0, \hat{u_0} \in L^1(\mathbb{R}^N)$. If*

$$\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha), \qquad \xi \to 0,$$

the asymptotic behavior of $u(x,t)$ is given by

$$\lim_{t\to+\infty} t^{N/\alpha} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = -A(-\Delta)^{\alpha/2}v(x,t)$ with initial condition $v(x,0) = u_0(x)$.

PROOF. As in the proof of the previous lemma we have

$$\hat{u}_t(\xi,t) = \hat{u}(\xi,t)(\hat{J}(\xi)-1).$$

Hence

$$\hat{u}(\xi,t) = e^{(\hat{J}(\xi)-1)t}\hat{u}_0(\xi).$$

On the other hand, let $v(x,t)$ be a solution of

$$v_t(x,t) = -A(-\Delta)^{\alpha/2}v(x,t),$$

with the same initial datum $v(x,0) = u_0(x)$. Solutions of this equation are understood in the sense that

$$\hat{v}(\xi,t) = e^{-A|\xi|^\alpha t}\hat{u}_0(\xi).$$

Hence in Fourier variables,

$$\begin{aligned} \int_{\mathbb{R}^N} |\hat{u}-\hat{v}|(\xi,t)\,d\xi &= \int_{\mathbb{R}^N} \left| \left(e^{t(\hat{J}(\xi)-1)} - e^{-A|\xi|^\alpha t}\right)\hat{u}_0(\xi)\right| d\xi \\ &\le \int_{|\xi|\ge r(t)} \left| \left(e^{t(\hat{J}(\xi)-1)} - e^{-A|\xi|^\alpha t}\right)\hat{u}_0(\xi)\right| d\xi \\ &\quad + \int_{|\xi|< r(t)} \left| \left(e^{t(\hat{J}(\xi)-1)} - e^{-A|\xi|^\alpha t}\right)\hat{u}_0(\xi)\right| d\xi = I + II. \end{aligned}$$

To get a bound for I we proceed as follows, we decompose it in two parts,

$$I \le \int_{|\xi|\ge r(t)} \left|e^{-A|\xi|^\alpha t}\hat{u}_0(\xi)\right| d\xi + \int_{|\xi|\ge r(t)} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| d\xi = I_1 + I_2.$$

First, we deal with I_1. We have,

$$t^{N/\alpha}\int_{|\xi|>r(t)} e^{-A|\xi|^\alpha t}|\hat{u}_0(\xi)|d\xi \le \|\hat{u}_0\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta|>r(t)t^{1/\alpha}} e^{-A|\eta|^\alpha} \to 0,$$

as $t\to\infty$ if we impose that

$$r(t)t^{1/\alpha} \to \infty \qquad \text{as } t\to\infty. \tag{2.4}$$

Now, remark that from our hypotheses on J we have that $\hat{J}$ verifies

$$\hat{J}(\xi) \le 1 - A|\xi|^\alpha + |\xi|^\alpha h(\xi),$$

where h is bounded and $h(\xi)\to 0$ as $\xi\to 0$. Hence there exists $D>0$ such that

$$\hat{J}(\xi) \le 1 - D|\xi|^\alpha, \qquad \text{for } |\xi|\le a,$$

and $\delta>0$ such that

$$\hat{J}(\xi) \le 1-\delta, \qquad \text{for } |\xi|\ge a.$$

Therefore, I_2 can be bounded by

$$\int_{|\xi|\ge r(t)} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| \, d\xi \le \int_{a\ge|\xi|\ge r(t)} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| \, d\xi$$
$$+ \int_{|\xi|\ge a} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| \, d\xi \le \int_{a\ge|\xi|\ge r(t)} \left|e^{t(\hat{J}(\xi)-1)}\hat{u}_0(\xi)\right| \, d\xi + Ce^{-\delta t}.$$

Using this bound and changing variables, $\eta = \xi t^{1/\alpha}$,

$$t^{N/\alpha} I_2 \le C \int_{at^{1/\alpha}\ge|\eta|\ge t^{1/\alpha}r(t)} \left|e^{-D|\eta|^\alpha}\hat{u}_0(\eta t^{-1/\alpha})\right| \, d\eta + t^{N/\alpha}Ce^{-\delta t}$$
$$\le C \int_{|\eta|\ge t^{1/\alpha}r(t)} e^{-D|\eta|^\alpha} \, d\eta + t^{N/\alpha}Ce^{-\delta t},$$

and then

$$t^{N/\alpha} I_2 \to 0, \qquad \text{as } t \to \infty,$$

if (2.4) holds.

Now we estimate II as follows,

$$t^{N/\alpha} \int_{|\xi|<r(t)} |e^{(\hat{J}(\xi)-1+A|\xi|^\alpha)t} - 1| \, e^{-A|\xi|^\alpha t} \, |\hat{u}_0(\xi)| \, d\xi$$
$$\le Ct^{N/\alpha} \int_{|\xi|<r(t)} t|\xi|^\alpha h(\xi) e^{-A|\xi|^\alpha t} \, d\xi,$$

provided we impose

$$t\,(r(t))^\alpha h(r(t)) \to 0 \qquad \text{as } t \to \infty. \tag{2.5}$$

In this case, we have

$$t^{N/\alpha} II \le C \int_{|\eta|<r(t)t^{1/\alpha}} |\eta|^\alpha h(\eta/t^{1/\alpha}) e^{-A|\eta|^\alpha} d\eta,$$

and we use dominated convergence, $h(\eta/t^{1/\alpha}) \to 0$ as $t \to \infty$ while the integrand is dominated by $\|h\|_\infty |\eta|^\alpha \exp(-c|\eta|^\alpha)$, which belongs to $L^1(\mathbb{R}^N)$.

This shows that

$$t^{N/\alpha}(I + II) \to 0 \qquad \text{as } t \to \infty, \tag{2.6}$$

provided we can find a $r(t) \to 0$ as $t \to \infty$ which fulfills both conditions (2.4) and (2.5). This is done in Lemma 8, which is postponed just after the end of the present proof. To conclude, we only have to observe that from (2.6) we obtain

$$t^{N/\alpha} \max_x |u(x,t) - v(x,t)| \le t^{N/\alpha} \int_{\mathbb{R}^N} |\hat{u} - \hat{v}|(\xi, t) \, d\xi \to 0, \qquad t \to \infty,$$

which ends the proof of the theorem. □

The following Lemma shows that there exists a function $r(t)$ satisfying (2.4) and (2.5), as required in the proof of the previous theorem.

LEMMA 8. *Given a function $h \in C(\mathbb{R}, \mathbb{R})$ such that $h(\rho) \to 0$ as $\rho \to 0$ with $h(\rho) > 0$ for small ρ, there exists a function r with $r(t) \to 0$ as $t \to \infty$ which satisfies*

$$\lim_{t\to\infty} r(t)t^{1/\alpha} = \infty$$

and

$$\lim_{t\to\infty} t(r(t))^{\alpha} h(r(t)) = 0.$$

PROOF. For fixed t large enough, we choose $r(t)$ as a small solution of

$$r(h(r))^{1/(2\alpha)} = t^{-1/\alpha}. \tag{2.7}$$

This equation defines a function $r = r(t)$ which, by continuity arguments, goes to zero as t goes to infinity. Indeed, if there exists $t_n \to \infty$ with no solution of (2.7) for $r \in (0, \delta)$ then $h(r) \equiv 0$ in $(0, \delta)$ a contradiction. □

REMARK 9. In the case when $h(t) = t^s$ with $s > 0$, we can look for a function h of power-type, $r(t) = t^{\beta}$ with $\beta < 0$ and the two conditions read as follows:

$$\beta + 1/\alpha > 0, \quad 1 + \beta\alpha + s\beta < 0. \tag{2.8}$$

This implies that $\beta \in (-1/\alpha, -1/(\alpha + s))$ which is of course always possible.

As a consequence of Theorem 7, we obtain the following corollary which completes the results gathered in the main theorem.

COROLLARY 10. *If $\hat{J}(\xi) = 1 - A|\xi|^{\alpha} + o(|\xi|^{\alpha})$, $\xi \to 0$, $0 < \alpha \le 2$, the asymptotic behavior of solutions of* (2.1) *is given by*

$$\|u(\cdot, t)\|_{L^{\infty}(\mathbb{R}^N)} \le \frac{C}{t^{N/\alpha}}.$$

Moreover, the asymptotic profile is given by

$$\lim_{t\to+\infty} \max_{y} \left| t^{N/\alpha} u(yt^{1/\alpha}, t) - \|u_0\|_{L^1} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G}_A(\xi) = e^{-A|\xi|^{\alpha}}$.

PROOF. From Theorem 7 we obtain that the asymptotic behavior is the same as the one for solutions of the evolution given by the fractional laplacian.

It is easy to check that this asymptotic behavior is exactly the one described in the statement of the corollary. Indeed, in Fourier variables we have for $t \to \infty$

$$\hat{v}(t^{-1/\alpha}\eta, t) = e^{-A|\eta|^{\alpha}} \hat{u_0}(\eta t^{-1/\alpha}) \longrightarrow e^{-A|\eta|^{\alpha}} \hat{u_0}(0) = e^{-A|\eta|^{\alpha}} \|u_0\|_{L^1(\mathbb{R}^N)}.$$

Therefore

$$\lim_{t\to+\infty} \max_{y} \left| t^{N/\alpha} v(yt^{1/\alpha}, t) - \|u_0\|_{L^1} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G_A}(\xi) = e^{-A|\xi|^\alpha}$. □

Now we find the decay rate in L^p of solutions of (2.1).

COROLLARY 11. *Let* $1 < p < \infty$. *If* $\hat{J}(\xi) = 1 - A|\xi|^\alpha + o(|\xi|^\alpha)$, $\xi \to 0$, $0 < \alpha \le 2$, *then, the decay of the* L^p*-norm of the solution of* (2.1) *is given by*

$$\|u(\cdot,t)\|_{L^p(\mathbb{R}^N)} \le Ct^{-\frac{N}{\alpha}\left(1-\frac{1}{p}\right)}.$$

PROOF. By interpolation, see [**27**], we have

$$\|u\|_{L^p(\mathbb{R}^N)} \le \|u\|_{L^1(\mathbb{R}^N)}^{\frac{1}{p}} \|u\|_{L^\infty(\mathbb{R}^N)}^{1-\frac{1}{p}}.$$

As (2.1) preserves the L^1 norm, the result follows from the previous results that give the decay in L^∞ of the solutions. □

0.2. The Dirichlet problem. Next we consider a bounded smooth domain $\Omega \subset \mathbb{R}^N$ and impose boundary conditions to our model. From now on we assume that J is continuous.

Consider the nonlocal problem

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), & x \in \Omega,\ t>0, \\ u(x,t) = 0, & x \notin \Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{2.9}$$

In this model we have that diffusion takes place in the whole $\mathbb{R}^N$ but we impose that u vanishes outside Ω. This is the analogous of what is called Dirichlet boundary conditions for the heat equation. However, the boundary data is not understood in the usual sense, see Remark 17. As for the Cauchy problem we understand solutions in an integral sense, see Theorem 14.

In this case we find an exponential decay given by the first eigenvalue of an associated problem and the asymptotic behavior of solutions is described by the unique (up to a constant) associated eigenfunction. Let $\lambda_1 = \lambda_1(\Omega)$ be given by

$$\lambda_1 = \inf_{u \in L^2(\Omega)} \frac{\displaystyle\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)(u(x)-u(y))^2\,dx\,dy}{\displaystyle\int_\Omega (u(x))^2\,dx} \tag{2.10}$$

and ϕ_1 an associated eigenfunction (a function where the infimum is attained).

THEOREM 12. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u of (2.9) such that $u \in C([0,\infty);L^1(\Omega))$. Moreover, if $u_0 \in L^2(\Omega)$, solutions decay to zero as $t \to \infty$ with an exponential rate*

$$\|u(\cdot,t)\|_{L^2(\Omega)} \le \|u_0\|_{L^2(\Omega)} e^{-\lambda_1 t}. \tag{2.11}$$

If u_0 is continuous, positive and bounded then there exist positive constants C and C^* such that

$$\|u(\cdot,t)\|_{L^\infty(\Omega)} \le C\, e^{-\lambda_1 t} \tag{2.12}$$

and

$$\lim_{t\to\infty} \max_x \left| e^{\lambda_1 t} u(x,t) - C^* \phi_1(x) \right| = 0. \tag{2.13}$$

A solution of the Dirichlet problem is defined as follows: $u \in C([0,\infty);L^1(\Omega))$ satisfying

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t), & x \in \Omega,\ t>0, \\ u(x,t) = 0, & x \notin \Omega,\ t>0, \\ u(x,0) = u_0(x) & x \in \Omega. \end{cases}$$

Before studying the asymptotic behavior, we shall first derive existence and uniqueness of solutions, which is a consequence of Banach's fixed point theorem.

Fix $t_0 > 0$ and consider the Banach space

$$X_{t_0} = \left\{ w \in C([0,t_0];L^1(\Omega)) \right\}$$

with the norm

$$|||w||| = \max_{0\le t\le t_0} \|w(\cdot,t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $\mathcal{T} : X_{t_0} \to X_{t_0}$ defined by

$$\mathcal{T}_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_{\mathbb{R}^N} J\left(x-y\right)\left(w(y,s) - w(x,s)\right) dy\,ds,$$
$$\mathcal{T}_{w_0}(w)(x,t) = 0, \qquad x \notin \Omega.$$

LEMMA 13. *Let $w_0, z_0 \in L^1(\Omega)$ and $w, z \in X_{t_0}$, then there exists a constant C depending on J and Ω such that*

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le Ct_0|||w-z||| + ||w_0 - z_0||_{L^1(\Omega)}.$$

PROOF. We have

$$\begin{aligned} \int_\Omega |\mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t)|\,dx &\le \int_\Omega |w_0 - z_0|(x)\,dx \\ + \int_\Omega \Bigg| \int_0^t \int_{\mathbb{R}^N} J\left(x-y\right) \Big[& (w(y,s) - z(y,s)) \\ & -(w(x,s) - z(x,s)) \Big]\,dy\,ds \Bigg|\,dx. \end{aligned}$$

Hence, taking into account that w and z vanish outside Ω,

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le ||w_0 - z_0||_{L^1(\Omega)} + Ct_0|||w - z|||,$$

as we wanted to prove. □

THEOREM 14. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u, such that $u \in C([0,\infty); L^1(\Omega))$.*

PROOF. We check first that $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}. Taking z_0, $z \equiv 0$ in Lemma 13 we get that $\mathcal{T}(w) \in C([0,t_0]; L^1(\Omega))$.

Choose t_0 such that $Ct_0 < 1$. Now taking $z_0 \equiv w_0 \equiv u_0$ in Lemma 13 we get that $\mathcal{T}_{u_0}$ is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial data $u(x,t_0) \in L^1(\Omega)$ and obtain a solution up to $[0, 2t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$. □

Next we look for steady states of (2.9).

PROPOSITION 15. *$u \equiv 0$ is the unique stationary solution of (2.9).*

PROOF. Let u be a stationary solution of (2.9). Then

$$0 = \int_{\mathbb{R}^N} J(x-y)(u(y) - u(x))\,dy, \qquad x \in \Omega,$$

and $u(x) = 0$ for $x \notin \Omega$. Hence, using that $\int J = 1$ we obtain that for every $x \in \mathbb{R}^N$ it holds,

$$u(x) = \int_{\mathbb{R}^N} J(x-y)u(y)\,dy.$$

This equation, together with $u(x) = 0$ for $x \notin \Omega$, implies that $u \equiv 0$. □

Now, let us analyze the asymptotic behavior of the solutions. As there exists a unique stationary solution, it is expected that solutions converge to zero as $t \to \infty$. Our main concern will be the rate of convergence.

First, let us look the eigenvalue given by (2.10), that is we look for the first eigenvalue of

$$u(x) - \int_{\mathbb{R}^N} J(x-y)u(y)\,dy = \lambda_1 u(x). \tag{2.14}$$

This is equivalent to,

$$(1-\lambda_1)u(x) = \int_{\mathbb{R}^N} J(x-y)u(y)\,dy. \tag{2.15}$$

Let $T : L^2(\Omega) \to L^2(\Omega)$ be the operator given by

$$T(u)(x) := \int_{\mathbb{R}^N} J(x-y)u(y)\,dy.$$

In this definition we have extended by zero a function in $L^2(\Omega)$ to the whole $\mathbb{R}^N$. Hence we are looking for the largest eigenvalue of T. Since T is compact this eigenvalue is attained at some function $\phi_1(x)$ that turns out to be an eigenfunction for our original problem (2.14).

By taking $|\phi_1|$ instead of ϕ_1 in (2.10) we may assume that $\phi_1 \geq 0$ in Ω. Indeed, one simply has to use the fact that $(a-b)^2 \geq (|a|-|b|)^2$.

Next, we analyze some properties of the eigenvalue problem (2.14).

PROPOSITION 16. *Let λ_1 the first eigenvalue of* (2.14) *and denote by $\phi_1(x)$ a corresponding non-negative eigenfunction. Then $\phi_1(x)$ is strictly positive in Ω and λ_1 is a positive simple eigenvalue with $\lambda_1 < 1$.*

PROOF. In what follows, we denote by $\bar{\phi}_1$ the natural continuous extension of ϕ_1 to $\bar{\Omega}$. We begin with the positivity of the eigenfunction ϕ_1. Assume for contradiction that the set $\mathbf{B} = \{x \in \Omega : \phi_1(x) = 0\}$ is non-void. Then, from the continuity of ϕ_1 in Ω, we have that $\mathbf{B}$ is closed. We next prove that $\mathbf{B}$ is also open, and hence, since Ω is connected, standard topological arguments allows to conclude that $\Omega \equiv \mathbf{B}$ yielding to a contradiction. Consider $x_0 \in \mathbf{B}$. Since $\phi_1 \geq 0$, we obtain from (2.15) that $\overline{\Omega \cap B_1(x_0)} \in \mathbf{B}$. Hence $\mathbf{B}$ is open and the result follows. Analogous arguments apply to prove that $\bar{\phi}_1$ is positive in $\bar{\Omega}$.

Assume now for contradiction that $\lambda_1 \leq 0$ and denote by M^* the maximum of $\bar{\phi}_1$ in $\bar{\Omega}$ and by x^* a point where such maximum is attained. Assume for the moment that $x^* \in \Omega$. From Proposition 15, one can choose x^* in such a way that $\phi_1(x) \neq M^*$ in $\Omega \cap B_1(x^*)$. By using (2.15) we obtain that,

$$M^* \leq (1-\lambda_1)\phi_1(x^*) = \int_{\mathbb{R}^N} J(x^*-y)\phi_1(y) < M^*$$

and a contradiction follows. If $x^* \in \partial\Omega$, we obtain a similar contradiction after substituting and passing to the limit in (2.15) on a sequence $\{x_n\} \in \Omega$, $x_n \to x^*$ as $n \to \infty$. To obtain the upper bound, assume that $\lambda_1 \geq 1$. Then, from (2.15) we obtain for every $x \in \Omega$ that

$$0 \geq (1-\lambda_1)\phi_1(x^*) = \int_{\mathbb{R}^N} J(x^*-y)\phi_1(y)$$

a contradiction with the positivity of ϕ_1.

Finally, to prove that λ_1 is a simple eigenvalue, let $\phi_1 \neq \phi_2$ be two different eigenfunctions associated to λ_1 and define

$$C^* = \inf\{C > 0 : \bar{\phi}_2(x) \leq C\bar{\phi}_1(x),\ x \in \bar{\Omega}\}.$$

The regularity of the eigenfunctions and the previous analysis shows that C^* is nontrivial and bounded. Moreover from its definition, there must exists $x^* \in \bar{\Omega}$ such that $\bar{\phi}_2(x^*) = C^*\bar{\phi}_1(x^*)$. Define $\phi(x) = C^*\phi_1(x) - \phi_2(x)$. From the linearity of (2.14), we have that ϕ is a non-negative eigenfunction associated to λ_1 with $\bar{\phi}(x^*) = 0$. From he positivity of the eigenfunctions stated above, it must be $\phi \equiv 0$. Therefore, $\phi_2(x) = C^*\phi_1(x)$ and the result follows. This completes the proof. ☐

REMARK 17. Note that the first eigenfunction ϕ_1 is strictly positive in Ω (with positive continuous extension to $\bar{\Omega}$) and vanishes outside Ω. Therefore a discontinuity occurs on $\partial\Omega$ and the boundary value is not taken in the usual "classical" sense.

PROOF OF THEOREM 12. Using the symmetry of J, we have

$$\begin{aligned} \frac{\partial}{\partial t}\left(\frac{1}{2}\int_\Omega u^2(x,t)\,dx\right) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)[u(y,t)-u(x,t)]u(x,t)\,dy\,dx \\ &= -\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)[u(y,t)-u(x,t)]^2\,dy\,dx. \end{aligned}$$

From the definition of λ_1, (2.10), we get

$$\frac{\partial}{\partial t}\int_\Omega u^2(x,t)\,dx \le -2\lambda_1\int_\Omega u^2(x,t)\,dx.$$

Therefore

$$\int_\Omega u^2(x,t)\,dx \le e^{-2\lambda_1 t}\int_\Omega u_0^2(x)\,dx$$

and we have obtained (2.11).

We now establish the decay rate and the convergence stated in (2.12) and (2.13) respectively. Consider a nontrivial and non-negative continuous initial data $u_0(x)$ and let $u(x,t)$ be the corresponding solution to (1.1). We first note that $u(x,t)$ is a continuous function satisfying $u(x,t)>0$ for every $x\in\Omega$ and $t>0$, and the same holds for $\bar{u}(x,t)$, the unique natural continuous extension of $u(x,t)$ to $\overline{\Omega}$. This instantaneous positivity can be obtained by using analogous topological arguments to those in Proposition 16.

In order to deal with the asymptotic analysis, is more convenient to introduce the rescaled function $v(x,t)=e^{\lambda_1 t}u(x,t)$. By substituting in (1.1), we find that the function $v(x,t)$ satisfies

$$v_t(x,t) = \int_{\mathbb{R}^N} J(x-y)v(y,t)\,dy - (1-\lambda_1)v(x,t). \tag{2.16}$$

On the other hand, we have that $C\phi_1(x)$ is a solution of (2.16) for every $C\in\mathbb{R}$ and moreover, it follows from the eigenfunction analysis above, that the set of stationary solutions of (2.16) is given by $\mathbf{S}^* = \{C\phi_1,\, C\in\mathbb{R}\}$.

Define now for every $t>0$, the function

$$C^*(t) = \inf\{C>0 : v(x,t)\le C\phi_1(x),\, x\in\Omega\}.$$

By definition and by using the linearity of equation (2.16), we have that $C^*(t)$ is a non-increasing function. In fact, this is a consequence of the comparison principle applied to the solutions $C^*(t_1)\phi_1(x)$ and $v(x,t)$ for t larger than any fixed $t_1>0$. It implies that $C^*(t_1)\phi_1(x)\ge v(x,t)$ for every $t\ge t_1$, and therefore, $C^*(t_1)\ge C^*(t)$ for every $t\ge t_1$. In an analogous way, one can see that the function

$$C_*(t) = \sup\{C>0 : v(x,t)\ge C\phi_1(x),\, x\in\Omega\},$$

is non-decreasing. These properties imply that both limits exist,

$$\lim_{t\to\infty} C^*(t) = K^* \quad \text{and} \quad \lim_{t\to\infty} C_*(t) = K_*,$$

and also provides the compactness of the orbits necessary in order passing to the limit (after subsequences if needed) to obtain that $v(\cdot, t+t_n) \to w(\cdot, t)$ as $t_n \to \infty$ uniformly on compact subsets in $\Omega \times \mathbb{R}_+$ and that $w(x,t)$ is a continuous function which satisfies (2.16). We also have for every $g \in \omega(u_0)$ there holds,

$$K_*\phi_1(x) \le g(x) \le K^*\phi_1(x).$$

Moreover, $C^*(t)$ plays a role of a Lyapunov function and this fact allows to conclude that $\omega(u_0) \subset \mathbf{S}^*$ and the uniqueness of the convergence profile. In more detail, assume that $g \in \omega(u_0)$ does not belong to $\mathbf{S}^*$ and consider $w(x,t)$ the solution of (2.16) with initial data $g(x)$ and define

$$C^*(w)(t) = \inf\{C > 0 : w(x,t) \le C\phi_1(x),\, x \in \Omega\}.$$

It is clear that $W(x,t) = K^*\phi_1(x) - w(x,t)$ is a non-negative continuous solution of (2.16) and it becomes strictly positive for every $t > 0$. This implies that there exists $t^* > 0$ such that $C^*(w)(t^*) < K^*$ and by the convergence, the same holds before passing to the limit. Hence, $C^*(t^* + t_j) < K^*$ if j is large enough and a contradiction with the properties of $C^*(t)$ follows. The same arguments allow to establish the uniqueness of the convergence profile. □

0.3. The Neumann problem. Let us turn our attention to Neumann boundary conditions. We study

$$(2.17) \qquad \begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)(u(y,t) - u(x,t))\, dy, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases}$$

Again solutions are to be understood in an integral sense, see Theorem 20. In this model we have that the integral terms take into account the diffusion inside Ω. In fact, as we have explained the integral $\int J(x-y)(u(y,t) - u(x,t))\, dy$ takes into account the individuals arriving or leaving position x from other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. The individuals may not enter nor leave Ω. This is the analogous of what is called homogeneous Neumann boundary conditions in the literature.

Again in this case we find that the asymptotic behavior is given by an exponential decay determined by an eigenvalue problem. Let β_1 be given by

$$(2.18) \qquad \beta_1 = \inf_{u \in L^2(\Omega), \int_\Omega u = 0} \frac{\dfrac{1}{2}\displaystyle\int_\Omega \int_\Omega J(x-y)(u(y) - u(x))^2\, dy\, dx}{\displaystyle\int_\Omega (u(x))^2\, dx}.$$

Concerning the asymptotic behavior of solutions of (2.17) our last result reads as follows:

THEOREM 18. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u of* (2.17) *such that $u \in C([0,\infty); L^1(\Omega))$. This solution preserves the total mass in Ω*

$$\int_\Omega u(y,t)\,dy = \int_\Omega u_0(y)\,dy.$$

Moreover, let $\varphi = \frac{1}{|\Omega|}\int_\Omega u_0$, then the asymptotic behavior of solutions of (2.17) *is described as follows: if $u_0 \in L^2(\Omega)$,*

$$\|u(\cdot,t) - \varphi\|_{L^2(\Omega)} \le e^{-\beta_1 t}\|u_0 - \varphi\|_{L^2(\Omega)}, \tag{2.19}$$

and if u_0 is continuous and bounded there exist a positive constant C such that

$$\|u(\cdot,t) - \varphi\|_{L^\infty(\Omega)} \le Ce^{-\beta_1 t}. \tag{2.20}$$

Solutions of the Neumann problem are functions $u \in C([0,\infty); L^1(\Omega))$ which satisfy

$$\begin{aligned} & u_t(x,t) = \int_\Omega J(x-y)(u(y,t) - u(x,t))\,dy, && x \in \Omega,\ t > 0, \\ & u(x,0) = u_0(x) && x \in \Omega. \end{aligned}$$

As in the previous chapter, see also [**42**], existence and uniqueness will be a consequence of Banach's fixed point theorem. The main arguments are basically the same but we repeat them here to make this chapter self-contained.

Fix $t_0 > 0$ and consider the Banach space

$$X_{t_0} = C([0,t_0]; L^1(\Omega))$$

with the norm

$$|||w||| = \max_{0 \le t \le t_0} \|w(\cdot,t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $\mathcal{T} : X_{t_0} \to X_{t_0}$ defined by

$$\mathcal{T}_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_\Omega J(x-y)\,(w(y,s) - w(x,s))\,dy\,ds. \tag{2.21}$$

The following lemma is the main ingredient in the proof of existence.

LEMMA 19. *Let $w_0, z_0 \in L^1(\Omega)$ and $w, z \in X_{t_0}$, then there exists a constant C depending only on Ω and J such that*

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le Ct_0|||w - z||| + \|w_0 - z_0\|_{L^1(\Omega)}.$$

Proof. We have

$$\int_\Omega |\mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t)|\, dx \le \int_\Omega |w_0 - z_0|(x)\, dx$$
$$+ \int_\Omega \left| \int_0^t \int_\Omega J(x-y) \Big[(w(y,s) - z(y,s)) - (w(x,s) - z(x,s)) \Big]\, dy\, ds \right| dx.$$

Hence

$$\int_\Omega |\mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t)|\, dx \le \|w_0 - z_0\|_{L^1(\Omega)}$$
$$+ \int_0^t \int_\Omega |(w(y,s) - z(y,s))|\, dy + \int_0^t \int_\Omega |(w(x,s) - z(x,s))|\, dx.$$

Therefore, we obtain,

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le C t_0 |||w - z||| + \|w_0 - z_0\|_{L^1(\Omega)},$$

as we wanted to prove. □

Theorem 20. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u of* (2.17) *such that $u \in C([0,\infty); L^1(\Omega))$. Moreover, the total mass in Ω verifies,*

$$\int_\Omega u(y,t)\, dy = \int_\Omega u_0(y)\, dy. \tag{2.22}$$

Proof. We check first that $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}. From (2.21) we see that for $0 < t_1 < t_2 \le t_0$,

$$\|\mathcal{T}_{u_0}(w)(t_2) - \mathcal{T}_{u_0}(w)(t_1)\|_{L^1(\Omega)} \le 2 \int_{t_1}^{t_2} \int_\Omega |w(y,s)|\, dx\, dy\, ds.$$

On the other hand, again from (2.21)

$$\|\mathcal{T}_{u_0}(w)(t) - w_0\|_{L^1(\Omega)} \le Ct|||w|||.$$

These two estimates give that $\mathcal{T}_{u_0}(w) \in C([0,t_0]; L^1(\Omega))$. Hence $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}.

Choose t_0 such that $Ct_0 < 1$. Now taking $z_0 \equiv w_0 \equiv u_0$, in Lemma 19 we get that $\mathcal{T}_{u_0}$ is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial data $u(x,t_0) \in L^1(\Omega)$ and obtain a solution up to $[0, 2t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$.

We finally prove that if u is the solution, then the integral in Ω of u satisfies (2.22). Since

$$u(x,t) - u_0(x) = \int_0^t \int_\Omega J(x-y)\,(u(y,s) - u(x,s))\, dy\, ds.$$

We can integrate in x and apply Fubini's theorem to obtain

$$\int_\Omega u(x,t)\,dx - \int_\Omega u_0(x)\,dx = 0$$

and the theorem is proved. $\square$

Now we study the asymptotic behavior as $t \to \infty$. We start by analyzing the corresponding stationary problem so we consider the equation

$$0 = \int_\Omega J(x-y)(\varphi(y) - \varphi(x))\,dy. \tag{2.23}$$

The only solutions are constants. In fact, in particular, (2.23) implies that φ is a continuous function. Set

$$K = \max_{x \in \overline{\Omega}} \varphi(x)$$

and consider the set

$$\mathcal{A} = \{x \in \overline{\Omega} \mid \varphi(x) = K\}.$$

The set $\mathcal{A}$ is clearly closed and non empty. We claim that it is also open in $\overline{\Omega}$. Let $x_0 \in \mathcal{A}$. We have then

$$\varphi(x_0) = \Big(\int_\Omega J(x_0-y)\,dy\Big)^{-1} \int_\Omega J(x_0-y)\varphi(y)\,dy,$$

and $\varphi(y) \le \varphi(x_0)$ this implies $\varphi(y) = \varphi(x_0)$ for all $y \in \Omega \cap B(x_0,d)$, and hence $\mathcal{A}$ is open as claimed. Consequently, as Ω is connected, $\mathcal{A} = \overline{\Omega}$ and φ is constant.

We have proved the following proposition:

Proposition 21. *Every stationary solution of* (2.17) *is constant in* Ω.

Now we prove the exponential rate of convergence to steady states of solutions in L^2. Let us take β_1 as

$$\beta_1 = \inf_{u \in L^2(\Omega), \int_\Omega u = 0} \frac{\dfrac{1}{2}\displaystyle\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2\,dy\,dx}{\displaystyle\int_\Omega (u(x))^2\,dx}. \tag{2.24}$$

It is clear that $\beta_1 \ge 0$. Let us prove that β_1 is in fact strictly positive. To this end we consider the subspace of $L^2(\Omega)$ given by the orthogonal to the constants, $H = \langle cts \rangle^\perp$ and the symmetric (self-adjoint) operator $T : H \mapsto H$ given by

$$T(u) = \int_\Omega J(x-y)(u(x)-u(y))\,dy = -\int_\Omega J(x-y)u(y)\,dy + A(x)u(x).$$

Note that T is the sum of an inversible operator and a compact operator. Since T is symmetric we have that its spectrum verifies $\sigma(T) \subset [m, M]$, where

$$m = \inf_{u \in H,\, \|u\|_{L^2(\Omega)}=1} \langle Tu, u \rangle$$

and

$$M = \sup_{u\in H,\, \|u\|_{L^2(\Omega)}=1} \langle Tu, u\rangle,$$

see [**27**]. Remark that

$$\begin{aligned} m &= \inf_{u\in H,\, \|u\|_{L^2(\Omega)}=1} \langle Tu, u\rangle \\ &= \inf_{u\in H,\, \|u\|_{L^2(\Omega)}=1} \int_\Omega \int_\Omega J(x-y)(u(x)-u(y))\, dy\, u(x)\, dx \\ &= \beta_1. \end{aligned}$$

Then $m \geq 0$. Now we just observe that

$$m > 0.$$

In fact, if not, as $m \in \sigma(T)$ (see [**27**]), we have that $T : H \mapsto H$ is not inversible. Using Fredholm's alternative this implies that there exists a nontrivial $u \in H$ such that $T(u) = 0$, but then u must be constant in Ω. This is a contradiction with the fact that H is orthogonal to the constants.

To study the asymptotic behavior of the solutions we need an upper estimate on β_1.

LEMMA 22. *Let β_1 be given by* (2.24) *then*

$$\beta_1 \leq \min_{x\in\overline{\Omega}} \int_\Omega J(x-y)\, dy. \tag{2.25}$$

PROOF. Let

$$A(x) = \int_\Omega J(x-y)\, dy.$$

Since $\overline{\Omega}$ is compact and A is continuous there exists a point $x_0 \in \overline{\Omega}$ such that

$$A(x_0) = \min_{x\in\overline{\Omega}} A(x).$$

For every ε small let us choose two disjoint balls of radius ε contained in Ω, $B(x_1,\varepsilon)$ and $B(x_2,\varepsilon)$ in such a way that $x_i \to x_0$ as $\varepsilon \to 0$. We use

$$u_\varepsilon(x) = \chi_{B(x_1,\varepsilon)}(x) - \chi_{B(x_2,\varepsilon)}(x)$$

as a test function in the definition of β_1, (2.24). Then we get that for every ε small it holds

$$\begin{aligned}\beta_1 \leq &\ \frac{\displaystyle\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u_\varepsilon(y)-u_\varepsilon(x))^2\,dy\,dx}{\displaystyle\int_\Omega (u_\varepsilon(x))^2\,dx}\\ &= \frac{\displaystyle\int_\Omega A(x)u_\varepsilon^2(x)\,dx - \int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)\,u_\varepsilon(x)\,dy\,dx}{\displaystyle\int_\Omega (u_\varepsilon(x))^2\,dx}\\ &= \frac{\displaystyle\int_\Omega A(x)u_\varepsilon^2(x)\,dx - \int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)\,u_\varepsilon(x)\,dy\,dx}{2|B(0,\varepsilon)|}.\end{aligned}$$

Using the continuity of A and the explicit form of u_ε we obtain

$$\lim_{\varepsilon\to 0}\frac{\displaystyle\int_\Omega A(x)u_\varepsilon^2(x)\,dx}{2|B(0,\varepsilon)|} = A(x_0)$$

and

$$\lim_{\varepsilon\to 0}\frac{\displaystyle\int_\Omega\int_\Omega J(x-y)u_\varepsilon(y)\,u_\varepsilon(x)\,dy\,dx}{2|B(0,\varepsilon)|} = 0.$$

Therefore, (2.25) follows. □

Now let us prove the exponential convergence of $u(x,t)$ to the mean value of the initial datum.

THEOREM 23. *For every $u_0 \in L^2(\Omega)$ the solution $u(x,t)$ of* (2.17) *satisfies*

$$\|u(\cdot,t)-\varphi\|_{L^2(\Omega)} \leq e^{-\beta_1 t}\|u_0-\varphi\|_{L^2(\Omega)}. \tag{2.26}$$

Moreover, if u_0 is continuous and bounded, there exists a positive constant $C>0$ such that,

$$\|u(\cdot,t)-\varphi\|_{L^\infty(\Omega)} \leq Ce^{-\beta_1 t}. \tag{2.27}$$

Here β_1 is given by (2.24).

PROOF. Let

$$H(t) = \frac{1}{2}\int_\Omega (u(x,t)-\varphi)^2\,dx.$$

Differentiating with respect to t and using (2.24) and the conservation of the total mass, we obtain

$$H'(t) = -\frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u(y,t)-u(x,t))^2\,dy\,dx \leq -\beta_1\int_\Omega (u(x,t)-\varphi)^2\,dx.$$

Hence

$$H'(t) \le -2\beta_1 H(t).$$

Therefore, integrating we obtain,

$$H(t) \le e^{-2\beta_1 t} H(0), \tag{2.28}$$

and (2.26) follows.

In order to prove (2.27) let $w(x,t)$ denote the difference

$$w(x,t) = u(x,t) - \varphi.$$

We seek for an exponential estimate in L^∞ of the decay of $w(x,t)$. The linearity of the equation implies that $w(x,t)$ is a solution of (2.17) and satisfies

$$w(x,t) = e^{-A(x)t} w_0(x) + e^{-A(x)t} \int_0^t e^{A(x)s} \int_\Omega J(x-y) w(y,s)\, dy\, ds.$$

Recall that $A(x) = \int_\Omega J(x-y)dx$. By using (2.26) and the Holder inequality it follows that

$$|w(x,t)| \le e^{-A(x)t} w_0(x) + C e^{-A(x)t} \int_0^t e^{A(x)s - \beta_1 s}\, ds.$$

Integrating this inequality, we obtain that the solution $w(x,t)$ decays to zero exponentially fast and moreover, it implies (2.27) thanks to Lemma 22. □

CHAPTER 3

Refined asymptotics

In this chapter we find refined asymptotic expansions for solutions to the nonlocal evolution equation

$$\begin{cases} u_t(x,t) = J * u - u(x,t), & t > 0,\ x \in \mathbb{R}^d, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{3.1}$$

where $J : \mathbb{R}^d \to \mathbb{R}$ verifies $\int_{\mathbb{R}^d} J(x)dx = 1$.

For the heat equation a precise asymptotic expansion in terms of the fundamental solution and its derivatives was found in [**52**]. In fact, if G_t denotes the fundamental solution of the heat equation, namely, $G_t(x) = (4\pi t)^{-d/2}e^{-|x|^2/(4t)}$, under adequate assumptions on the initial condition, we have,

$$\Big\| u(x,t) - \sum_{|\alpha|\le k} \frac{(-1)^{|\alpha|}}{\alpha!} \Big(\int_{\mathbb{R}^d} u_0(x)x^\alpha \Big) \partial^\alpha G_t \Big\|_{L^q(\mathbb{R}^d)} \le Ct^{-A} \tag{3.2}$$

with $A = (\frac{d}{2})(\frac{(k+1)}{d} + (1 - \frac{1}{q}))$. As pointed out by the authors in [**52**], the same asymptotic expansion can be done in a more general setting, dealing with the equations $u_t = -(-\Delta)^{\frac{s}{2}}u$, $s > 0$.

Now we need to introduce some notation. We will say that $f \sim g$ as $\xi \sim 0$ if $|f(\xi) - g(\xi)| = o(g(\xi))$ when $\xi \to 0$ and, to simplify, $f \le g$ if there exists a constant c independent of the relevant quantities such that $f \le cg$. In the sequel we denote by $\widehat{J}$ the Fourier transform of J.

Our main objective here is to study if an expansion analogous to (3.2) holds for the non-local problem (3.1). Concerning the first term, in [**34**] it is proved that if J verifies $\widehat{J}(\xi) - 1 \sim -|\xi|^s$ as $\xi \sim 0$, then the asymptotic behavior of the solution to (3.1), $u(x,t)$, is given by

$$\lim_{t\to+\infty} t^{\frac{d}{s}} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = -(-\Delta)^{\frac{s}{2}}v(x,t)$ with initial condition $v(x,0) = u_0(x)$. As a consequence, the decay rate is given by $\|u(\cdot,t)\|_{L^\infty(\mathbb{R}^d)} \le C\,t^{-\frac{d}{s}}$ and the asymptotic profile is as follows,

$$\lim_{t\to+\infty} \Big\| t^{\frac{d}{s}} u(yt^{\frac{1}{s}}, t) - \Big(\int_{\mathbb{R}^d} u_0 \Big) G^s(y) \Big\|_{L^\infty(\mathbb{R}^d)} = 0,$$

where $G^s(y)$ satisfies $\widehat{G^s}(\xi) = e^{-|\xi|^s}$.

Here we find a complete expansion for $u(x,t)$, a solution to (3.1), in terms of the derivatives of the regular part of the fundamental solution, K_t. As we have mentioned, the fundamental solution $w(x,t)$ of problem (3.1) satisfies

$$w(x,t) = e^{-t}\delta_0(x) + K_t(x),$$

where the function K_t (the regular part of the fundamental solution) is given by

$$\widehat{K_t}(\xi) = e^{-t}(e^{t\widehat{J}(\xi)} - 1).$$

In contrast with the previous analysis done in [**34**] where the long time behavior is studied in the $L^\infty(\mathbb{R}^d)$-norm, here we also consider $L^q(\mathbb{R}^d)$ norms for $q \geq 1$. The cases $2 \leq q \leq \infty$ are derived from Hausdorff-Young's inequality and Plancherel's identity. The other cases $1 \leq q < 2$ are more tricky. They are reduced to L^2-estimates on the momenta of $\partial^\alpha K_t$ and therefore more restrictive assumptions on J have to be imposed.

THEOREM 24. *Let s and m be positive and such that*

$$\widehat{J}(\xi) - 1 \sim -|\xi|^s, \qquad \xi \sim 0 \tag{3.3}$$

and

$$|\widehat{J}(\xi)| \leq \frac{1}{|\xi|^m}, \qquad |\xi| \to \infty. \tag{3.4}$$

Then for any $2 \leq q \leq \infty$ and $k + 1 < m - d$ there exists a constant

$$C = C(q,k)\||x|^{k+1}u_0\|_{L^1(\mathbb{R}^d)}$$

such that

$$\Big\| u(x,t) - \sum_{|\alpha| \leq k} \frac{(-1)^{|\alpha|}}{\alpha!}\Big(\int_{\mathbb{R}^d} u_0(x)x^\alpha\Big)\partial^\alpha K_t\Big\|_{L^q(\mathbb{R}^d)} \leq Ct^{-A} \tag{3.5}$$

for all $u_0 \in L^1(\mathbb{R}^d, 1 + |x|^{k+1})$. Here $A = \frac{(k+1)}{s} + \frac{d}{s}(1 - \frac{1}{q})$.

REMARK 25. The condition $k + 1 < m - d$ guarantees that all the partial derivatives $\partial^\alpha K_t$ of order $|\alpha| = k + 1$ make sense. In addition if $\widehat{J}$ decays at infinity faster than any polynomial,

$$\forall\, m > 0,\ \exists\, c(m) \text{ such that } |\widehat{J}(\xi)| \leq \frac{c(m)}{|\xi|^m}, \quad |\xi| \to \infty, \tag{3.6}$$

then the expansion (3.5) holds for all k.

To deal with L^q-norms for $1 \leq q < 2$ we have to impose more restrictive assumptions.

THEOREM 26. *Assume that J satisfies* (3.3) *with $[s] > d/2$ and that for any $m \geq 0$ and α there exists $c(m,\alpha)$ such that*

$$|\partial^\alpha \widehat{J}(\xi)| \leq \frac{c(m,\alpha)}{|\xi|^m}, \qquad |\xi| \to \infty. \tag{3.7}$$

Then for any $1 \leq q < 2$, the asymptotic expansion (3.5) *holds.*

Note that, when J has an expansion of the form $\widehat{J}(\xi) - 1 \sim -|\xi|^2$ as $\xi \sim 0$ (this happens for example if J is compactly supported), then the decay rate in L^∞ of the solutions to the non-local problem (3.1) and the heat equation coincide (in both cases they decay as $t^{-\frac{d}{2}}$). Moreover, the first order term also coincide (in both cases it is a Gaussian). See [**34**] and Theorem 24.

0.4. Estimates on K_t. To prove our result we need some estimates on the kernel K_t. In this subsection we obtain the long time behavior of the kernel K_t and its derivatives.

The behavior of $L^q(\mathbb{R}^d)$-norms with $2 \leq q \leq \infty$ follows by Hausdorff-Young's inequality in the case $q = \infty$ and Plancherel's identity for $q = 2$. However the case $1 \leq q < 2$ is more tricky. In order to evaluate the $L^1(\mathbb{R}^d)$-norm of the kernel K_t we use the following inequality

$$\|f\|_{L^1(\mathbb{R}^d)} \leq \|f\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{2n}} \||x|^n f\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2n}}, \tag{3.8}$$

which holds for $n > d/2$ and which is frequently attributed to Carlson (see for instance [**13**]). The use of the above inequality with $f = K_t$ imposes that $|x|^n \partial^\alpha K_t$ belongs to $L^2(\mathbb{R}^d)$. To guarantee that property and to obtain the decay rate for the $L^2(\mathbb{R}^d)$-norm of $|x|^n \partial^\alpha K_t$ we will impose in Lemma 28 additional hypotheses.

The following lemma gives us the decay rate of the $L^q(\mathbb{R}^d)$-norms of the kernel K_t and its derivatives for $2 \leq q \leq \infty$.

LEMMA 27. *Let $2 \leq q \leq \infty$ and J satisfying* (3.3) *and* (3.4)*. Then for all indexes α such that $|\alpha| < m - d$ there exists a constant $c(q,\alpha)$ such that*

$$\|\partial^\alpha K_t\|_{L^q(\mathbb{R}^d)} \leq c(q,\alpha)\, t^{-\frac{d}{s}(1-\frac{1}{q})-\frac{|\alpha|}{s}}$$

holds for sufficiently large t.

Moreover, if J satisfies (3.6) *then the same result holds with no restriction on α.*

PROOF OF LEMMA 27. We consider the cases $q = 2$ and $q = \infty$. The other cases follow by interpolation. We denote by *e.s.* the exponentially small terms.

First, let us consider the case $q = \infty$. Using the definition of K_t, $\widehat{K_t}(\xi) = e^{-t}(e^{t\widehat{J}(\xi)} - 1)$, we get, for any $x \in \mathbb{R}^d$,

$$|\partial^\alpha K_t(x)| \leq e^{-t} \int_{\mathbb{R}^d} |\xi|^{|\alpha|} |e^{t\widehat{J}(\xi)} - 1|\, d\xi.$$

Using that $|e^y - 1| \le 2|y|$ for $|y|$ small, say $|y| \le c_0$, we obtain that

$$|e^{t\widehat{J}(\xi)} - 1| \le 2t|\widehat{J}(\xi)| \le \frac{2t}{|\xi|^m}$$

for all $|\xi| \ge h(t) = (c_0 t)^{\frac{1}{m}}$. Then

$$e^{-t}\int_{|\xi|\ge h(t)} |\xi|^{|\alpha|}|e^{t\widehat{J}(\xi)} - 1|\, d\xi \le te^{-t}\int_{|\xi|\ge h(t)} \frac{|\xi|^{|\alpha|}}{|\xi|^m}\, d\xi \le te^{-t}c(m-|\alpha|)$$

provided that $|\alpha| < m - d$.

Is easy to see that if (3.6) holds no restriction on the indexes α has to be assumed.

It remains to estimate

$$e^{-t}\int_{|\xi|\le h(t)} |\xi|^{|\alpha|}|e^{t\widehat{J}(\xi)} - 1| d\xi.$$

We observe that the term $e^{-t}\int_{|\xi|\le h(t)} |\xi|^{|\alpha|} d\xi$ is exponentially small, so we concentrate on

$$I(t) = e^{-t}\int_{|\xi|\le h(t)} |e^{t\widehat{J}(\xi)}||\xi|^{|\alpha|} d\xi.$$

Now, let us choose $R > 0$ such that

$$|\widehat{J}(\xi)| \le 1 - \frac{|\xi|^s}{2} \text{ for all } |\xi| \le R. \tag{3.9}$$

Once R is fixed, there exists $\delta > 0$ with

$$|\widehat{J}(\xi)| \le 1 - \delta \text{ for all } |\xi| \ge R. \tag{3.10}$$

Then

$$\begin{aligned}
|I(t)| &\le e^{-t}\int_{|\xi|\le R} |e^{t\widehat{J}(\xi)}||\xi|^{|\alpha|} d\xi + e^{-t}\int_{R\le|\xi|\le h(t)} |e^{t\widehat{J}(\xi)}||\xi|^{|\alpha|} d\xi \\
&\le \int_{|\xi|\le R} e^{t(|\widehat{J}(\xi)|-1)}|\xi|^{|\alpha|} d\xi + e^{-t\delta}\int_{R\le|\xi|\le h(t)} |\xi|^{|\alpha|} d\xi \\
&\le \int_{|\xi|\le R} e^{-\frac{t|\xi|^s}{2}}|\xi|^{|\alpha|} + e.s. \\
&= t^{-\frac{|\alpha|}{s}-\frac{d}{s}}\int_{|\eta|\le Rt^{\frac{1}{s}}} e^{-\frac{|\eta|^s}{2}}|\eta|^{|\alpha|} + e.s. \le t^{-\frac{|\alpha|}{s}-\frac{d}{s}}.
\end{aligned}$$

Now, for $q = 2$, by Plancherel's identity we have

$$\|\partial^\alpha K_t\|^2_{L^2(\mathbb{R}^d)} \le e^{-2t}\int_{\mathbb{R}^d} |e^{t\widehat{J}(\xi)} - 1|^2|\xi|^{2|\alpha|} d\xi.$$

Putting out the exponentially small terms, it remains to estimate

$$\int_{|\xi|\leq R} |e^{t(\widehat{J}(\xi)-1)}|^2 |\xi|^{2|\alpha|} d\xi,$$

where R is given by (3.9). The behavior of $\widehat{J}$ near zero gives

$$\int_{|\xi|\leq R} |e^{t(\widehat{J}(\xi)-1)}|^2 |\xi|^{2|\alpha|} d\xi \leq \int_{|\xi|\leq R} e^{-t|\xi|^s} |\xi|^{2|\alpha|} d\xi \leq t^{-\frac{d}{s}-\frac{2|\alpha|}{s}},$$

which finishes the proof. □

Once the case $2 \leq q \leq \infty$ has been analyzed the next step is to obtain similar decay rates for the L^q-norms with $1 \leq q < 2$. These estimates follow from an L^1-estimate and interpolation.

LEMMA 28. *Assume that J verifies* (3.7) *and*

$$\widehat{J}(\xi) - 1 \sim -|\xi|^s, \qquad \xi \sim 0$$

with $[s] > d/2$. Then for any index $\alpha = (\alpha_1, \ldots, \alpha_d)$

$$\|\partial^\alpha K_t\|_{L^1(\mathbb{R}^d)} \leq t^{-\frac{|\alpha|}{s}}. \tag{3.11}$$

Moreover, for $1 < q < 2$ we have

$$\|\partial^\alpha K_t\|_{L^q(\mathbb{R}^d)} \leq t^{-\frac{d}{s}(1-\frac{1}{q})-\frac{|\alpha|}{s}}$$

for large t.

REMARK 29. There is no restriction on s if J is such that

$$|\partial^\alpha \widehat{J}(\xi)| \leq \min\{|\xi|^{s-|\alpha|}, 1\}, \quad |\xi| \leq 1.$$

This happens if s is a positive integer and $\widehat{J}(\xi) = 1 - |\xi|^s$ in a neighborhood of the origin.

REMARK 30. The case $\alpha = (0, \ldots, 0)$ can be easily treated when J is nonnegative. As a consequence of the mass conservation (just integrate the equation and use Fubini's theorem, see [**34**]),

$$\int_{\mathbb{R}^d} w(x,t) = 1,$$

we obtain

$$\int_{\mathbb{R}^d} |K_t| \leq 1$$

and therefore (3.11) follows with $\alpha = (0, \ldots, 0)$.

REMARK 31. The condition (3.7) imposed on J is satisfied, for example, for any smooth, compactly supported function J.

Proof of Lemma 28. The estimates for $1 < q < 2$ follow from the cases $q = 1$ and $q = 2$ (see Lemma 27) using interpolation.

To deal with $q = 1$, we use inequality (3.8) with $f = \partial^\alpha K_t$ and n such that $[s] \geq n > d/2$. We get

$$\|\partial^\alpha K_t\|_{L^1(\mathbb{R}^d)} \leq \|\partial^\alpha K_t\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{2n}} \||x|^n \partial^\alpha K_t\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2n}}.$$

The condition $n \leq [s]$ guarantees that $\partial^n_{\xi_j} \widehat{J}$ makes sense near $\xi = 0$ and thus the derivatives $\partial^n_{\xi_j} \widehat{K_t}$, $j = 1, \dots, d$, exist. Observe that the moment of order n of K_t imposes the existence of the partial derivatives $\partial^n_{\xi_j} \widehat{K_t}$, $j = 1, \dots, d$.

In view of Lemma 27 we obtain

$$\|\partial^\alpha K_t\|_{L^1(\mathbb{R}^d)} \leq t^{-\left(\frac{d}{2s}+\frac{|\alpha|}{s}\right)\left(1-\frac{d}{2n}\right)} \||x|^n \partial^\alpha K_t\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2n}}.$$

Thus it is sufficient to prove that

$$\||x|^n \partial^\alpha K_t\|_{L^2(\mathbb{R}^d)} \leq t^{\frac{n}{s}-\frac{d}{2s}-\frac{|\alpha|}{s}}$$

for all sufficiently large t. Observe that by Plancherel's theorem

$$\begin{aligned} \int_{\mathbb{R}^d} |x|^{2n} |\partial^\alpha K_t(x)|^2 dx &\leq c(n) \int_{\mathbb{R}^d} (x_1^{2n} + \cdots + x_d^{2n}) |\partial^\alpha K_t(x)|^2 dx \\ &= c(n) \sum_{j=1}^d \int_{\mathbb{R}^d} |\partial^n_{\xi_j} (\xi^\alpha \widehat{K_t})|^2 d\xi \end{aligned}$$

where $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_d^{\alpha_d}$. Therefore, it remains to prove that for any $j = 1, \dots, d$, it holds

$$\int_{\mathbb{R}^d} |\partial^n_{\xi_j} (\xi^\alpha \widehat{K_t})|^2 d\xi \leq t^{\frac{2n}{s}-\frac{d}{s}-\frac{2|\alpha|}{s}}, \qquad \text{for } t \text{ large.} \tag{3.12}$$

We analyze the case $j = 1$, the others follow by the same arguments. Leibnitz's rule gives

$$\partial^n_{\xi_1} (\xi^\alpha \widehat{K_t})(\xi) = \xi_2^{\alpha_2} \dots \xi_d^{\alpha_d} \sum_{k=0}^n \binom{n}{k} \partial^k_{\xi_1} (\xi_1^{\alpha_1}) \partial^{n-k}_{\xi_1} (\widehat{K_t})(\xi)$$

and guarantees that

$$\begin{aligned} |\partial^n_{\xi_1} (\xi^\alpha \widehat{K_t})(\xi)|^2 &\leq \xi_2^{2\alpha_2} \dots \xi_d^{2\alpha_d} \sum_{k=0}^n |\partial^k_{\xi_1} (\xi_1^{\alpha_1})|^2 |\partial^{n-k}_{\xi_1} \widehat{K_t}(\xi)|^2 \\ &\leq \xi_2^{2\alpha_2} \dots \xi_d^{2\alpha_d} \sum_{k=0}^{\min\{n,\alpha_1\}} \xi_1^{2(\alpha_1-k)} |\partial^{n-k}_{\xi_1} \widehat{K_t}(\xi)|^2. \end{aligned}$$

The last inequality reduces (3.12) to the following one:

$$\int_{\mathbb{R}^d} \xi_1^{2(\alpha_1-k)} \xi_2^{2\alpha_2} \dots \xi_d^{2\alpha_d} |\partial^{n-k}_{\xi_1} \widehat{K_t}(\xi)|^2 d\xi \leq t^{\frac{2n}{s}-\frac{d}{s}-\frac{2|\alpha|}{s}}$$

for all $0 \le k \le \min\{\alpha_1, n\}$. Using the elementary inequality (it follows from the convexity of the log function)

$$\xi_1^{2(\alpha_1-k)}\xi_2^{2\alpha_2}\dots\xi_d^{2\alpha_d} \le (\xi_1^2+\dots+\xi_d^2)^{\alpha_1-k+\alpha_2+\dots+\alpha_d} = |\xi|^{2(|\alpha|-k)}$$

it remains to prove that for any r nonnegative and any m such that $n-\min\{\alpha_1,n\} \le m \le n$ the following inequality is valid,

$$I(r,m,t) = \int_{\mathbb{R}^d} |\xi|^{2r}|\partial^m_{\xi_1}\widehat{K_t}|^2 d\xi \le t^{-\frac{d}{s}+\frac{2}{s}(m-r)}. \tag{3.13}$$

First we analyze the case $m=0$. In this case

$$I(r,0,t) = \int_{\mathbb{R}^d} |\xi|^{2r}|e^{t(\widehat{J}(\xi)-1)} - e^{-t}|^2 d\xi$$

and in view of Lemma 27 we obtain the desired decay property.

Observe that under hypothesis (3.7) no restriction on r is needed.

In what follow we analyze the case $m \ge 1$. First we recall the following elementary identity

$$\partial^m_{\xi_1}(e^g) = e^g \sum_{i_1+2i_2+\ldots+mi_m=m} a_{i_1,\ldots,i_m}(\partial^1_{\xi_1}g)^{i_1}(\partial^2_{\xi_1}g)^{i_2}\ldots(\partial^m_{\xi_1}g)^{i_m}$$

where $a_{i_1,\ldots,i_m}$ are universal constants independent of g. Tacking into account that $\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)} - e^{-t}$ we obtain for any $m \ge 1$ that

$$\partial^m_{\xi_1}\widehat{K_t}(\xi) \;=\; e^{t(\widehat{J}(\xi)-1)} \sum_{i_1+2i_2+\ldots+mi_m=m} a_{i_1,\ldots,i_m} t^{i_1+\dots+i_m}\prod_{j=1}^m[\partial^j_{\xi_1}\widehat{J}(\xi)]^{i_j}$$

and hence

$$|\partial^m_{\xi_1}\widehat{K_t}(\xi)|^2 \;\le\; e^{2t|\widehat{J}(\xi)-1|} \sum_{i_1+2i_2+\ldots+mi_m=m} t^{2(i_1+\dots+i_m)}\prod_{j=1}^m[\partial^j_{\xi_1}\widehat{J}(\xi)]^{2i_j}.$$

Using that all the partial derivatives of $\widehat{J}$ decay, as $|\xi|\to\infty$, faster than any polinomial in $|\xi|$, we obtain that

$$\int_{|\xi|>R} |\xi|^{2r}|\partial^m_{\xi_1}\widehat{K_t}(\xi)|^2 d\xi \le e^{-\delta t}t^{2m}$$

where R and δ are chosen as in (3.9) and (3.10). Tacking into account that $n \le [s]$ and that $|\widehat{J}(\xi)-1+|\xi|^s| \le o(|\xi|^s)$ as $|\xi|\to 0$ we obtain

$$|\partial^j_{\xi_1}\widehat{J}(\xi)| \le |\xi|^{s-j},\ j=1,\ldots,n$$

for all $|\xi| \leq R$. Then for any $m \leq n$ and for all $|\xi| \leq R$ the following holds

$$\begin{aligned}|\partial_{\xi_1}^m \widehat{K_t}(\xi)|^2 &\leq e^{-t|\xi|^s} \sum_{i_1+2i_2+\ldots+mi_m=m} t^{2(i_1+\cdots+i_m)} \prod_{j=1}^{m} |\xi|^{2(s-j)i_j} \\ &\leq e^{-t|\xi|^s} \sum_{i_1+2i_2+\ldots+mi_m=m} t^{2(i_1+\cdots+i_m)} |\xi|^{\sum_{j=1}^m 2(s-j)i_j}.\end{aligned}$$

Using that for any $l \geq 0$

$$\int_{\mathbb{R}^d} e^{-t|\xi|^s} |\xi|^l d\xi \leq t^{-\frac{d}{s}-\frac{l}{s}},$$

we obtain

$$\int_{|\xi|\leq R} |\xi|^{2r} |\partial_{\xi_1}^m K_t(\xi)|^2 d\xi \leq t^{-\frac{d}{s}} \sum_{i_1+2i_2+\cdots+mi_m=m} t^{2p(i_1,\ldots,i_d)-\frac{2r}{s}}$$

where

$$\begin{aligned}p(i_1,\ldots,i_m) &= (i_1+\cdots+i_m) - \frac{1}{s}\sum_{j=1}^{m}(s-j)i_j \\ &= \frac{1}{s}\sum_{j=1}^{m} j\, i_j = \frac{m}{s}.\end{aligned}$$

This ends the proof. □

Now we are ready to prove Theorems 24 and 26.

Proof of Theorems 24 and 26. Following [**52**] we obtain that the initial condition $u_0 \in L^1(\mathbb{R}^d, 1+|x|^{k+1})$ has the following decomposition

$$u_0 = \sum_{|\alpha|\leq k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0 x^\alpha \, dx \right) D^\alpha \delta_0 + \sum_{|\alpha|=k+1} D^\alpha F_\alpha$$

where

$$\|F_\alpha\|_{L^1(\mathbb{R}^d)} \leq \|u_0\|_{L^1(\mathbb{R}^d,\, |x|^{k+1})}$$

for all multi-indexes α with $|\alpha| = k+1$.

In view of (2.3) the solution u of (3.1) satisfies

$$u(x,t) = e^{-t} u_0(x) + (K_t * u_0)(x).$$

The first term being exponentially small it suffices to analyze the long time behavior of $K_t * u_0$. Using the above decomposition, Lemma 27 and Lemma 28 we get

$$\begin{aligned}\Big\|K_t * u_0 - \sum_{|\alpha|\le k} \frac{(-1)^{|\alpha|}}{\alpha!}\Big(\int u_0(x)x^\alpha dx\Big)\partial^\alpha K_t\Big\|_{L^q(\mathbb{R}^d)} \le \\ \le \sum_{|\alpha|=k+1} \|\partial^\alpha K_t * F_\alpha\|_{L^q(\mathbb{R}^d)} \\ \le \sum_{|\alpha|=k+1} \|\partial^\alpha K_t\|_{L^q(\mathbb{R}^d)}\|F_\alpha\|_{L^1(\mathbb{R}^d)} \\ \le t^{-\frac{d}{s}(1-\frac{1}{q})}t^{-\frac{(k+1)}{s}}\|u_0\|_{L^1(\mathbb{R}^d,\,|x|^{k+1})}.\end{aligned}$$

This ends the proof. □

0.5. Asymptotics for the higher order terms. Our next aim is to study if the higher order terms of the asymptotic expansion that we have found in Theorem 24 have some relation with the corresponding ones for the heat equation. Our next results say that the difference between them is of lower order. Again we have to distinguish between $2 \le q \le \infty$ and $1 \le q < 2$.

THEOREM 32. *Let J as in Theorem 24 and assume in addition that there exists $r > 0$ such that*

$$\widehat{J}(\xi) - (1 - |\xi|^s) \sim B|\xi|^{s+r}, \qquad \xi \sim 0, \tag{3.14}$$

for some real number B. Then for any $2 \le q \le \infty$ and $|\alpha| \le m - d$ there exists a positive constant $C = C(q, d, s, r)$ such that the following holds

$$\|\partial^\alpha K_t - \partial^\alpha G_t^s\|_{L^q(\mathbb{R}^d)} \le Ct^{-\frac{d}{s}(1-\frac{1}{q})}t^{-\frac{|\alpha|+r}{s}}, \tag{3.15}$$

where G_t^s is defined by its Fourier transform $\widehat{G_t^s}(\xi) = \exp(-t|\xi|^s)$.

THEOREM 33. *Let J be as in the above Theorem with $[s] > d/2$. We also assume that all the derivatives of $\widehat{J}$ decay at infinity faster as any polinomial:*

$$|\partial^\alpha \widehat{J}(\xi)| \le \frac{c(m, \alpha)}{|\xi|^m}, \qquad \xi \to \infty.$$

Then for any $1 \le q < 2$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_d)$, (3.15) holds.

Note that these results do not imply that the asymptotic expansion

$$\sum_{|\alpha|\le k} \frac{(-1)^{|\alpha|}}{\alpha!}\left(\int u_0(x)x^\alpha\right)\partial^\alpha K_t$$

coincides with the expansion that holds for the equation $u_t = -(-\Delta)^{\frac{s}{2}} u$:

$$\sum_{|\alpha| \le k} \frac{(-1)^{|\alpha|}}{\alpha!} \left(\int u_0(x) x^\alpha \right) \partial^\alpha G_t^s.$$

They only say that the corresponding terms agree up to a better order. When J is compactly supported or rapidly decaying at infinity, then $s = 2$ and we obtain an expansion analogous to the one that holds for the heat equation.

PROOF OF THEOREM 32. CASE $2 \le q \le \infty$. Recall that we have defined G_t^s by its Fourier transform $\widehat{G_t^s} = \exp(-t|\xi|^s)$.

We consider the case $q = \infty$, the case $q = 2$ can be handled similarly and the rest of the cases, $2 < q < \infty$, follow again by interpolation.

Writing each of the two terms in Fourier variables we obtain

$$\|\partial^\alpha K_t - \partial^\alpha G_t^s\|_{L^\infty(\mathbb{R}^d)} \le \int_{\mathbb{R}^d} |\xi|^{|\alpha|} \left| e^{-t}(e^{t\widehat{J}(\xi)} - 1) - e^{-t|\xi|^s} \right| d\xi.$$

Let us choose a positive R such that

$$\left| \widehat{J}(\xi) - 1 + |\xi|^s \right| \le C|\xi|^{r+s}, \qquad \text{for } |\xi| \le R,$$

satisfying (3.10) for some $\delta > 0$. For $|\xi| \ge R$ all the terms are exponentially small as $t \to \infty$. Thus the behavior of the difference $\partial^\alpha K_t - \partial^\alpha G_t$ is given by the following integral:

$$I(t) = \int_{|\xi| \le R} |\xi|^{|\alpha|} \left| e^{t(\widehat{J}(\xi) - 1)} - e^{-t|\xi|^s} \right| d\xi.$$

In view of the elementary inequality $|e^y - 1| \le c(R)|y|$ for all $|y| \le R$ we obtain that

$$\begin{aligned} I(t) &= \int_{|\xi| \le R} |\xi|^{|\alpha|} e^{-t|\xi|^s} \left| e^{t(\widehat{J}(\xi) - 1 + |\xi|^s)} - 1 \right| d\xi \\ &\le \int_{|\xi| \le R} |\xi|^{|\alpha|} e^{-t|\xi|^s} \left| t(\widehat{J}(\xi) - 1 + |\xi|^s) \right| d\xi \\ &\le t \int_{|\xi| \le R} |\xi|^{|\alpha|} e^{-t|\xi|^s} |\xi|^{s+r} d\xi \\ &\le t^{-\frac{d}{s} - \frac{r}{s} - \frac{|\alpha|}{s}}. \end{aligned}$$

This finishes the proof. □

PROOF OF THEOREM 33. CASE $1 \le q < 2$. Using the same ideas as in the proof of Lemma 28 it remains to prove that for some $d/2 < n \le [s]$ the following holds

$$\| |x|^n (\partial^\alpha K_t - \partial^\alpha G_t^s) \|_{L^2(\mathbb{R}^d)} \le t^{-\frac{d}{2s} + \frac{n - (|\alpha| + r)}{s}}.$$

Applying Plancherel's identity the proof of the last inequality is reduced to the proof of the following one

$$\int_{\mathbb{R}^d} |\partial^n_{\xi_j}[\xi^\alpha(\widehat{K_t} - \widehat{G^s_t})]|^2 d\xi \le t^{-\frac{d}{2s}+\frac{n-(|\alpha|+r)}{s}}, \quad j = 1, \dots d,$$

provided that all the above terms make sense. This means that all the partial derivatives $\partial^k_{\xi_j}\widehat{K_t}$ and $\partial^k_{\xi_j}\widehat{G^s_t}$, $j = 1, \dots, d$, $k = 0, \dots, n$ have to be defined. And thus we need $n \le [s]$.

We consider the case $j = 1$ the other cases being similar. Applying again Leibnitz's rule we get

$$\begin{aligned} |\partial^n_{\xi_1}[\xi^\alpha(\widehat{K_t} - \widehat{G^s_t})]|^2 &\le \xi_2^{2\alpha_2}\dots\xi_d^{2\alpha_d}\sum_{k=0}^{\min\{n,\alpha_1\}} \xi_1^{2(\alpha_1-k)}|\partial^{n-k}_{\xi_1}(\widehat{K_t} - \widehat{G^s_t})|^2 \\ &\le \sum_{k=0}^{\min\{n,\alpha_1\}} |\xi|^{2(|\alpha|-k)}|\partial^{n-k}_{\xi_1}(\widehat{K_t} - \widehat{G^s_t})|^2. \end{aligned}$$

In the following we prove that

$$\int_{\mathbb{R}^d} |\xi|^{2m_1}|\partial^m_{\xi_1}(\widehat{K_t} - \widehat{G^s_t})|^2 d\xi \le t^{-\frac{d}{s}+\frac{2(m-m_1-r)}{s}}$$

for all $|\alpha| - \min\{n, \alpha_1\} \le m_1 \le |\alpha|$ and $n - \min\{n, \alpha_1\} \le m \le n$.

Using that the integral outside of a ball of radius R decay exponentially, it remains to analyze the decay of the following integral

$$\int_{|\xi|\le R} |\xi|^{2m_1}|\partial^m_{\xi_1}(\widehat{K_t} - \widehat{G^s_t})|^2 d\xi$$

where R is as in the proof of Theorem 32. Using the definition of $\widehat{K_t}$ and G^s_t we obtain that

$$\partial^m_{\xi_1}\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)} \sum_{i_1+2i_2+\ldots+mi_m=m} a_{i_1,\ldots,i_m} t^{i_1+\cdots+i_m} \prod_{j=1}^m [\partial^j_{\xi_1}\widehat{J}(\xi)]^{i_j}$$

and

$$\partial^m_{\xi_1}\widehat{G^s_t}(\xi) = e^{tp_s(\xi)} \sum_{i_1+2i_2+\ldots+mi_m=m} a_{i_1,\ldots,i_m} t^{i_1+\cdots+i_m} \prod_{j=1}^m [\partial^j_{\xi_1}p_s(\xi)]^{i_j}$$

where $p_s(\xi) = -|\xi|^s$. Then

$$|\partial^m_{\xi_1}\widehat{K_t}(\xi) - \partial^m_{\xi_1}\widehat{G^s_t}(\xi)|^2 \le I_1(\xi, t) + I_2(\xi, t)$$

where

$$I_1(\xi, t) = |e^{t(\widehat{J}(\xi)-1)} - e^{tp_s(\xi)}|^2 \sum_{i_1+2i_2+\ldots+mi_m=m} t^{2(i_1+\cdots+i_m)} \prod_{j=1}^m |\partial^j_{\xi_1}p_s(\xi)|^{2i_j}$$

and

$$I_2(\xi,t)=e^{2tp_s(\xi)}\sum_{i_1+2i_2+\ldots+mi_m=m} t^{2(i_1+\cdots+i_m)}\times \\ \times\left|\prod_{j=1}^m[\partial^j_{\xi_1}\widehat{J}(\xi)]^{i_j}-\prod_{j=1}^m[\partial^j_{\xi_1}p_s(\xi)]^{i_j}\right|^2.$$

First, let us analyze $I_1(\xi,t)$.

Tacking into account that $|\partial^j_{\xi_1}p_s(\xi)|\le|\xi|^{s-j}$ for all $j\le m\le[s]$, $|\xi|\le R$, and that

$$\begin{aligned}|e^{t(\widehat{J}(\xi)-1)}-e^{tp_s(\xi)}|^2 &= e^{-2t|\xi|^s}\left|e^{t(\widehat{J}(\xi)-1+|\xi|^s)}-1\right|^2\\ &\le e^{-2t|\xi|^s}\left|t(\widehat{J}(\xi)-1+|\xi|^s)\right|^2\\ &\le t^2e^{-2t|\xi|^s}|\xi|^{2(r+s)}\end{aligned}$$

the same arguments as in the proof of Lemma 28 give us the right decay.

It remains to analyze $I_2(\xi,t)$. We make use of the following elementary inequality

$$\left|\prod_{j=1}^m a_j-\prod_{j=1}^m b_j\right|\le\sum_{j=1}^m|b_1\ldots b_{j-1}||a_j-b_j||a_{j+1}\ldots a_m|.$$

Then by Cauchy's inequality we also have

$$\left|\prod_{j=1}^m a_j-\prod_{j=1}^m b_j\right|^2\le\sum_{j=1}^m b_1^2\ldots b_{j-1}^2(a_j-b_j)^2a_{j+1}^2\ldots a_m^2.$$

Applying the last inequality with $a_j=\partial^j_{\xi_1}\widehat{J}(\xi)$ and $b_j=\partial^j_{\xi_1}p_s(\xi)$ we obtain

$$I_2(\xi,t)\le e^{2tp_s(\xi)}\sum_{i_1+2i_2+\ldots+mi_m=m}t^{2(i_1+\cdots+i_m)}g(i_1,\ldots,i_m,\xi)$$

where

$$g(\mathbf{i},\xi)=\sum_{j=1}^m\prod_{k=1}^{j-1}|\partial^k_{\xi_1}p_s(\xi)|^{2i_k}([\partial^k_{\xi_1}\widehat{J}(\xi)]^{i_k}-[\partial^k_{\xi_1}p_s(\xi)]^{i_k})^2\prod_{k=j+1}^n[\partial^k_{\xi_1}\widehat{J}(\xi)]^{2i_k}$$

and $\mathbf{i}=(i_1,\ldots,i_m)$.

Choosing eventually a smaller R we can guarantee that for $|\xi|\le R$ and $k\le[s]$ the following inequalities hold:

$$|\partial^k_{\xi_1}\widehat{J}(\xi)-\partial^k_{\xi_1}p_s(\xi)|\le|\xi|^{s+r-k},$$

$$|\partial^k_{\xi_1}\widehat{J}(\xi)|\le|\xi|^{s-k},$$

$$|\partial^k_{\xi_1}p_s(\xi)|\le|\xi|^{s-k}.$$

Hence, we get

$$\begin{aligned}|[\partial_{\xi_1}^k \widehat{J}(\xi)]^{i_k} - [\partial_{\xi_1}^k p_s(\xi)]^{i_k}| &\le |\partial_{\xi_1}^k \widehat{J}(\xi) - \partial_{\xi_1}^k p_s(\xi)| \times \\ &\times \sum_{l=0}^{i_k-1} [\partial_{\xi_1}^k \widehat{J}(\xi)]^l [\partial_{\xi_1}^k p_s(\xi)]^{i_k-l-1} \\ &\le |\xi|^{s+r-k} |\xi|^{(i_k-1)(s-k)} = |\xi|^r |\xi|^{i_k(s-k)}.\end{aligned}$$

This yields to the following estimate on the function $g(i_1, \dots, i_m, \xi)$:

$$g(i_1, \dots, i_m, \xi) \le |\xi|^{2r} |\xi|^{2\sum_{j=1}^m i_k(s-k)},$$

and consequently

$$\begin{aligned}\int_{\mathbb{R}^d} I_2(t,\xi) d\xi \le \int_{\mathbb{R}^d} & e^{-2t|\xi|^s} \times \\ & \times \sum_{i_1+2i_2+\ldots+mi_m=m} t^{2(i_1+\cdots+i_m)} |\xi|^{2r+2\sum_{j=1}^m i_k(s-k)} d\xi.\end{aligned}$$

Making a change of variable and using similar arguments as in the proof of Lemma 28 we obtain the desired result. □

0.6. A different approach. In this final subsection we obtain the first two terms in the asymptotic expansion of the solution under less restrictive hypotheses on J.

THEOREM 34. *Let $u_0 \in L^1(\mathbb{R}^d)$ with $\widehat{u_0} \in L^1(\mathbb{R}^d)$ and $s < l$ be two positive numbers such that*

$$\widehat{J}(\xi) - (1 - |\xi|^s) \sim B|\xi|^l, \quad \xi \sim 0,$$

for some real number B.

Then for any $2 \le q \le \infty$

$$\lim_{t\to\infty} t^{\frac{d}{s}(1-\frac{1}{q})+\frac{l-s}{s}} \|u(t) - v(t) - Bt[(-\Delta)^{\frac{l}{2}} v](t)\|_{L^q(\mathbb{R}^d)} \to 0, \tag{3.16}$$

where v is the solution to $v_t = -(-\Delta)^{\frac{s}{2}} v$ with $v(x,0) = u_0(x)$.

Moreover

$$\lim_{t\to\infty} \left\| t^{\frac{d}{s}+\frac{l}{s}-1} \left(u(yt^{\frac{1}{s}}, t) - v(yt^{\frac{1}{s}}, t) \right) - Bh(y) \left(\int_{\mathbb{R}^d} u_0 \right) \right\|_{L^\infty(\mathbb{R}^d)} = 0, \tag{3.17}$$

where h is given by $\widehat{h}(\xi) = e^{-|\xi|^s} |\xi|^l$.

Let us point out that the asymptotic expansion given by (3.5) involves K_t (and its derivatives) which is not explicit. On the other hand, the two-term asymptotic expansion (3.16) involves G_t^s, a well known explicit kernel (v is just the convolution of G_t^s and u_0). However, our ideas and methods allow us to find only two terms in the latter expansion. The case $1 \le q < 2$ in (3.16) can be also treated, but additional hypothesis on J have to be imposed.

Proof of Theorem 34. The method that we use here is just to estimate the difference

$$\|u(t) - v(t) - Bt(-\Delta)^{\frac{l}{2}}v(t)\|_{L^q(\mathbb{R}^d)}$$

using Fourier variables.

As before, it is enough to consider the cases $q = 2$ and $q = \infty$. We analyze the case $q = \infty$, the case $q = 2$ follows in the same manner by applying Plancherel's identity.

By Hausdorff-Young's inequality we get

$$\begin{aligned}
&\|u(t) - v(t) - tB(-\Delta)^{\frac{l}{2}}v(t)\|_{L^\infty(\mathbb{R}^d)} \\
&\leq \int_{\mathbb{R}^d} \left|\widehat{u}(t,\xi) - \widehat{v}(t,\xi) - tB\widehat{(-\Delta)^{\frac{l}{2}}}v(t,\xi)\right| d\xi \\
&= \int_{\mathbb{R}^d} \left|e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s}(1 + tB|\xi|^l)||\widehat{u_0}(\xi)\right| d\xi.
\end{aligned}$$

As before, let us choose $R > 0$ such that

$$|\widehat{J}(\xi)| \leq 1 - \frac{|\xi|^s}{2}, \quad |\xi| \leq R.$$

Then there exists $\delta > 0$ such that

$$|\widehat{J}(\xi)| \leq 1 - \delta, \quad |\xi| \geq R.$$

Hence

$$\int_{|\xi|\geq R} |e^{t(\widehat{J}(\xi)-1)}||\widehat{u_0}(\xi)|d\xi \leq e^{-\delta t}\|\widehat{u_0}\|_{L^1(\mathbb{R}^d)}$$

and

$$\begin{aligned}
\int_{t^{-\frac{1}{l}}\leq|\xi|\leq R} |e^{t(\widehat{J}(\xi)-1)}||\widehat{u_0}(\xi)|d\xi &\leq \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^d)} \int_{t^{-\frac{1}{l}}\leq|\xi|\leq R} e^{-t|\xi|^s/2} \\
&\leq t^{-\frac{d}{s}} \int_{|\xi|\geq t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\xi|^s/2} d\xi \leq t^{-\frac{d}{s}} e^{-t^{1-\frac{s}{l}}/4}.
\end{aligned}$$

Also

$$\begin{aligned}
\int_{|\xi|\geq t^{-\frac{1}{l}}} e^{-t|\xi|^s}(1 + tB|\xi|^l)|\widehat{u_0}(\xi)|d\xi &\leq \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^d)} \int_{|\xi|\geq t^{-\frac{1}{l}}} e^{-t|\xi|^s} t|\xi|^l \, d\xi \\
&\leq t^{1-\frac{d}{s}-\frac{l}{s}} \int_{|\eta|\geq t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\eta|^s}|\eta|^l \, d\xi \\
&\leq t^{1-\frac{d}{s}-\frac{l}{s}} e^{-t^{1-\frac{s}{l}}/2} \int_{|\eta|\geq t^{\frac{1}{s}-\frac{1}{l}}} e^{-|\eta|^s/2}|\eta|^l \, d\xi.
\end{aligned}$$

Therefore, we have to analyze

$$I(t) = \int_{|\xi|\leq t^{-\frac{1}{l}}} |e^{t(\widehat{J}(\xi)-1)} - e^{-t|\xi|^s}(1 + tB|\xi|^l)||\widehat{u_0}(\xi)| \, d\xi.$$

We write $\widehat{J}(\xi) = 1 - |\xi|^s + B|\xi|^l + |\xi|^l f(\xi)$ where $f(\xi) \to 0$ as $|\xi| \to 0$. Thus

$$I(t) \leq I_1(t) + I_2(t)$$

where

$$I_1(t) = \int_{|\xi| \leq t^{-\frac{1}{l}}} e^{-t|\xi|^s} |e^{Bt|\xi|^l + t|\xi|^l f(\xi)} - (1 + Bt|\xi|^l + t|\xi|^l f(\xi))||\widehat{u_0}(\xi)|\, d\xi$$

and

$$I_2(t) = \int_{|\xi| \leq t^{-\frac{1}{l}}} e^{-t|\xi|^s} t|\xi|^l f(\xi)|\widehat{u_0}(\xi)|\, d\xi.$$

For I_1 we have

$$\begin{aligned} I_1(t) &\leq \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^d)} \int_{|\xi| \leq t^{-\frac{1}{l}}} e^{-t|\xi|^s} (t|\xi|^l + t|\xi|^l |f(\xi)|)^2\, d\xi \\ &\leq \int_{|\xi| \leq t^{-\frac{1}{l}}} e^{-t|\xi|^s} t^2 |\xi|^{2l}\, d\xi \leq t^{-\frac{d}{s} + 2 - \frac{2l}{s}} \end{aligned}$$

and then

$$t^{\frac{d}{s} + \frac{l}{s} - 1} I_1(t) \leq t^{1 - \frac{l}{s}} \to 0, \quad t \to \infty.$$

It remains to prove that

$$t^{\frac{d}{s} + \frac{l}{s} - 1} I_2(t) \to 0, \quad t \to \infty.$$

Making a change of variable we obtain

$$t^{\frac{d}{s} - 1 + \frac{l}{s}} I_2(t) \leq \|\widehat{u_0}\|_{L^\infty(\mathbb{R}^d)} \int_{|\xi| \leq t^{\frac{1}{s} - \frac{1}{l}}} e^{-|\xi|^s} |\xi|^l f(\xi t^{-\frac{1}{s}})\, d\xi.$$

The integrand is dominated by $\|f\|_{L^\infty(\mathbb{R}^d)} |\xi|^l \exp(-|\xi|^s)$, which belongs to $L^1(\mathbb{R}^d)$. Hence, as $f(\xi/t^{\frac{1}{s}}) \to 0$ when $t \to \infty$, this shows that

$$t^{\frac{d}{s} + \frac{l}{s} - 1} I_2(t) \to 0,$$

and finishes the proof of (3.16).

Thanks to (3.16), the proof of (3.17) is reduced to show that

$$\lim_{t \to \infty} \left\| t^{\frac{d}{s} + \frac{l}{s}} [(-\Delta)^{\frac{l}{2}} v](y t^{\frac{1}{s}}, t) - h(y) \left(\int_{\mathbb{R}^d} u_0 \right) \right\|_{L^\infty(\mathbb{R}^d)} = 0.$$

For any $y \in \mathbb{R}^d$ by making a change of variables we obtain

$$I(y,t) = t^{\frac{d}{s} + \frac{l}{s}} [(-\Delta)^{\frac{l}{2}} v](y t^{\frac{1}{s}}, t) = \int_{\mathbb{R}^d} e^{-|\xi|^s} |\xi|^l e^{iy\xi} \widehat{u_0}(\xi / t^{\frac{1}{s}}).$$

Thus, using the dominated convergence theorem we obtain

$$\left\| I(y,t) - h(y) \int_{\mathbb{R}^d} u_0 \right\|_{L^\infty(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} e^{-|\xi|^s} |\xi|^l |\widehat{u_0}(\xi/t^{\frac{1}{s}}) - \widehat{u_0}(0)|\, d\xi \to 0$$

as $t \to \infty$. □

CHAPTER 4

Higher order problems

Our main concern in this chapter is the study of the asymptotic behavior of solutions of a nonlocal diffusion operator of higher order in the whole $\mathbb{R}^N$, $N \geq 1$.

Let us consider the following nonlocal evolution problem:

$$
(4.1) \qquad \begin{cases} u_t(x,t) = & (-1)^{n-1}\,(J * Id - 1)^n\,(u(x,t)) \\ \qquad\quad = & (-1)^{n-1}\left(\displaystyle\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(J*)^k(u)\right)(x,t), \\ u(x,0) = & u_0(x), \end{cases}
$$

for $x \in \mathbb{R}^N$ and $t > 0$.

Note that in our problem (4.1) we just have the iteration k-times of the nonlocal operator $J * u - u$ as right hand side of the equation. This can be seen as a nonlocal generalization of higher order equations of the form

$$
(4.2) \qquad v_t(x,t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x,t),
$$

with A and α are positive constants specified later in this section. Note that when $\alpha = 2$ (4.2) is just $v_t(x,t) = -A^n(-\Delta)^n v(x,t)$. Nonlocal higher order problems have been, for instance, proposed as models for periodic phase separation. Here the nonlocal character of the problem is associated with long-range interactions of "particles" in the system. An example is the nonlocal Cahn-Hilliard equation (cf. e.g. [**64**], [**73**], [**74**]).

Here we propose (4.1) as a model for higher order nonlocal evolution. For this model, we first prove existence and uniqueness of a solution, but our main aim is to study the asymptotic behaviour as $t \to \infty$ of solutions to (4.1).

For a function f we denote by $\hat{f}$ the Fourier transform of f and by $\check{f}$ the inverse Fourier transform of f. Our hypotheses on the convolution kernel J that we will assume throughout this chapter are:

The kernel $J \in C(\mathbb{R}^N, \mathbb{R})$ is a nonnegative, radial function with total mass equals one, $\int_{\mathbb{R}^N} J(x)\,dx = 1$. This means that J is a radial density probability which implies that its Fourier transform verifies $|\hat{J}(\xi)| \leq 1$ with $\hat{J}(0) = 1$. Moreover, we assume that

$$
(4.3) \qquad \hat{J}(\xi) = 1 - A\,|\xi|^{\alpha} + o(|\xi|^{\alpha}) \quad \textit{for } \xi \to 0,
$$

for some $A > 0$ and $\alpha > 0$.

Under these conditions on J we have the following results.

0.7. Existence and uniqueness. First, we show existence and uniqueness of a solution

THEOREM 35. *Let* $u_0 \in L^1(\mathbb{R}^N)$ *such that* $\hat{u_0} \in L^1(\mathbb{R}^N)$. *There exists a unique solution* $u \in C^0([0,\infty); L^1(\mathbb{R})^N)$ *of* (4.1) *that, in Fourier variables, is given by the explicit formula,*

$$\hat{u}(\xi,t) = e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\hat{u_0}(\xi).$$

PROOF. We have

$$u_t(x,t) = (-1)^{n-1}\left(\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(J*)^k(u)\right)(x,t).$$

Applying the Fourier transform to this equation we obtain

$$\begin{aligned}\hat{u}_t(\xi,t) = &\ (-1)^{n-1}\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(\hat{J}(\xi))^k\hat{u}(\xi,t)\\ = &\ (-1)^{n-1}(\hat{J}(\xi)-1)^n\hat{u}(\xi,t).\end{aligned}$$

Hence

$$\hat{u}(\xi,t) = e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\hat{u_0}(\xi).$$

Since $\hat{u_0}(\xi) \in L^1(\mathbb{R}^N)$ and $e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}$ is continuous and bounded, $\hat{u}(\cdot,t) \in L^1(\mathbb{R}^N)$ and the result follows by taking the inverse Fourier transform. □

Now we prove a lemma concerning the fundamental solution of (4.1).

LEMMA 36. *The fundamental solution* w *of* (4.1), *that is the solution of the equation with initial condition* $u_0 = \delta_0$, *can be decomposed as*

$$w(x,t) = e^{-t}\delta_0(x) + v(x,t), \tag{4.4}$$

with $v(x,t)$ *smooth. Moreover, if* u *is a solution of* (4.1) *it can be written as*

$$u(x,t) = (w * u_0)(x,t) = \int_{\mathbb{R}^N} w(x-z,t)u_0(z)\,dz.$$

PROOF. By the previous result we have

$$\hat{w}_t(\xi,t) = (-1)^{n-1}(\hat{J}(\xi)-1)^n\hat{w}(\xi,t).$$

Hence, as the initial datum verifies $\hat{w_0} = \hat{\delta_0} = 1$, we get

$$\hat{w}(\xi,t) = e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} = e^{-t} + e^{-t}\left(e^{[(-1)^{n-1}(\hat{J}(\xi)-1)^n+1]t} - 1\right).$$

The first part of the lemma follows applying the inverse Fourier transform.

To finish the proof we just observe that $w * u_0$ is a solution of (4.1) with $(w*u_0)(x,0) = u_0(x)$. □

0.8. Asymptotic behavior. Proof of Theorem 37. Next, we deal with the asymptotic behavior as $t \to \infty$.

THEOREM 37. *Let u be a solution of (4.1) with $u_0, \hat{u_0} \in L^1(\mathbb{R}^N)$. Then the asymptotic behavior of $u(x,t)$ is given by*

$$\lim_{t\to+\infty} t^{\frac{N}{\alpha n}} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x,t)$ with initial condition $v(x,0) = u_0(x)$ and A and α as in (4.3). Moreover, we have that there exists a constant $C > 0$ such that

$$\|u(.,t)\|_{L^\infty(\mathbb{R}^N)} \le C\, t^{-\frac{N}{\alpha n}}$$

and the asymptotic profile is given by

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{\alpha n}} u(y t^{\frac{1}{\alpha n}}, t) - \|u_0\|_{L^1(\mathbb{R}^N)}\, G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G}_A(\xi) = e^{-A^n|\xi|^{\alpha n}}$.

Now we prove the first part of Theorem 37.

THEOREM 38. *Let u be a solution of (4.1) with u_0, $\hat{u_0} \in L^1(\mathbb{R}^N)$. Then, the asymptotic behavior of $u(x,t)$ is given by*

$$\lim_{t\to+\infty} t^{\frac{N}{\alpha n}} \max_x |u(x,t) - v(x,t)| = 0,$$

where v is the solution of $v_t(x,t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x,t)$, with initial condition $v(x,0) = u_0(x)$.

PROOF. As in the previous section, we have in Fourier variables,

$$\hat{u}_t(\xi,t) = (-1)^{n-1}(\hat{J}(\xi) - 1)^n \hat{u}(\xi,t).$$

Hence

$$\hat{u}(\xi,t) = e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\hat{u_0}(\xi).$$

On the other hand, let $v(x,t)$ be a solution of $v_t(x,t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x,t)$, with the same initial datum $v(x,0) = u_0(x)$. Solutions of this equation are understood in the sense that

$$\hat{v}(\xi,t) = e^{-A^n|\xi|^{\alpha n} t}\hat{u_0}(\xi).$$

Hence in Fourier variables

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\hat{u} - \hat{v}|\,(\xi,t)\, d\xi = \int_{\mathbb{R}^N} \left|(e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} - e^{-A^n|\xi|^{\alpha n} t})\hat{u_0}(\xi)\right|\, d\xi \\
&\le \int_{|\xi|\ge r(t)} \left|\left(e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} - e^{-A^n|\xi|^{\alpha n} t}\right)\hat{u_0}(\xi)\right|\, d\xi \\
&\qquad + \int_{|\xi|< r(t)} \left|\left(e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t} - e^{-A^n|\xi|^{\alpha n} t}\right)\hat{u_0}(\xi)\right|\, d\xi \\
&= I + II,
\end{aligned}$$

where I and II denote the first and the second integral respectively. To get a bound for I we decompose it in two parts,

$$\begin{aligned} I &\leq \int_{|\xi|\geq r(t)} \left| e^{-A^n|\xi|^{\alpha n}t}\hat{u_0}(\xi)\right| \, d\xi + \int_{|\xi|\geq r(t)} \left| e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\hat{u_0}(\xi)\right| \, d\xi \\ &= I_1 + I_2. \end{aligned}$$

First we consider I_1. Setting $\eta = \xi t^{1/(\alpha n)}$ and writing I_1 in the new variable η we get,

$$I_1 \leq \|\hat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta|\geq r(t)t^{\frac{1}{\alpha n}}} e^{-A^n|\eta|^{\alpha n}} t^{-\frac{N}{\alpha n}} \, d\eta,$$

and hence

$$t^{\frac{N}{\alpha n}} I_1 \leq \|\hat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta|\geq r(t)t^{\frac{1}{\alpha n}}} e^{-A^n|\eta|^{\alpha n}} \, d\eta \overset{t\to\infty}{\longrightarrow} 0$$

if we impose that

$$r(t)t^{\frac{1}{\alpha n}} \overset{t\to\infty}{\longrightarrow} \infty. \tag{4.5}$$

To deal with I_2 we have to use different arguments for n even and n odd. Let us begin with the easier case of an even n.

- *n even* - Using our hypotheses on J we get

$$I_2 \leq Ce^{-t},$$

with $r(t) \overset{t\to\infty}{\longrightarrow} 0$ and therefore

$$t^{\frac{N}{\alpha n}} I_2 \leq Ce^{-t} t^{\frac{N}{\alpha n}} \overset{t\to\infty}{\longrightarrow} 0.$$

Now consider the case when n is odd.

- *n odd* - From our hypotheses on J we have that $\hat{J}$ verifies

$$\hat{J}(\xi) \leq 1 - A|\xi|^\alpha + |\xi|^\alpha h(\xi),$$

where h is bounded and $h(\xi) \to 0$ as $\xi \to 0$. Hence there exists $D > 0$ and a constant a such that

$$\hat{J}(\xi) \leq 1 - D|\xi|^\alpha, \quad \text{for } |\xi| \leq a.$$

Moreover, because $\left|\hat{J}(\xi)\right| \leq 1$ and J is a radial function, there exists a $\delta > 0$ such that

$$\hat{J}(\xi) \leq 1 - \delta, \quad \text{for } |\xi| \geq a.$$

Therefore I_2 can be bounded by

$$\begin{aligned} I_2 &\leq \int_{a\geq|\xi|\geq r(t)} \left|e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\hat{u_0}(\xi)\right| \, d\xi \\ &\quad + \int_{|\xi|\geq a} \left|e^{(-1)^{n-1}(\hat{J}(\xi)-1)^n t}\hat{u_0}(\xi)\right| \, d\xi \\ &\leq \|\hat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{a\geq|\xi|\geq r(t)} e^{-D^n|\xi|^{\alpha n}t} \, d\xi + Ce^{-\delta^n t}. \end{aligned}$$

Changing variables as before, $\eta = \xi t^{1/(\alpha n)}$, we get

$$\begin{aligned} t^{\frac{N}{\alpha n}} I_2 &\leq \|\hat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{at^{\frac{1}{\alpha n}}\geq|\eta|\geq r(t)t^{\frac{1}{\alpha n}}} e^{-D^n|\eta|^{\alpha n}} \, d\eta + Ct^{\frac{N}{\alpha n}} e^{-\delta^n t} \\ &\leq \|\hat{u_0}\|_{L^\infty(\mathbb{R}^N)} \int_{|\eta|\geq r(t)t^{\frac{1}{\alpha n}}} e^{-D^n|\eta|^{\alpha n}} \, d\eta + Ct^{\frac{N}{\alpha n}} e^{-\delta^n t} \to 0, \end{aligned}$$

as $t \to \infty$ if (4.5) holds.

It remains only to estimate II. We proceed as follows

$$II = \int_{|\xi|<r(t)} e^{-A^n|\xi|^{\alpha n}t} \left|e^{t[(-1)^{n-1}(\hat{J}(\xi)-1)^n + A^n|\xi|^{\alpha n}]} - 1\right| |\hat{u_0}(\xi)| \, d\xi.$$

Applying the binomial formula and taking into account the two different cases when n is even and odd we can conclude

$$t^{\frac{N}{\alpha n}} II \leq Ct^{\frac{N}{\alpha n}} \int_{|\xi|<r(t)} e^{-A^n|\xi|^{\alpha n}t} t(|\xi|^{\alpha n} h(\xi) + K(|\xi|^{\alpha k} h(\xi)^k)) \, d\xi,$$

where $K(|\xi|^{\alpha k} h(\xi)^k)$ is a polynomial in $|\xi|^\alpha$ and $h(\xi)$ with $0 < k \leq n$ and provided we impose

$$t(r(t))^{\alpha n} h(r(t)) \to 0 \quad \text{as } t \to \infty. \tag{4.6}$$

In this case we have

$$\begin{aligned} t^{\frac{N}{\alpha n}} II \leq C \int_{|\eta|<r(t)t^{\frac{1}{\alpha n}}} & e^{-A^n|\eta|^{\alpha n}} (|\eta|^{\alpha n} h(\eta/t^{1/(\alpha n)}) \\ & + K(|\eta|^{\alpha k} h(\eta/t^{1/(\alpha n)})^k) \frac{1}{t^{(\alpha k)/(\alpha n)}}) \, d\eta. \end{aligned}$$

To show the convergence of II to zero we use dominated convergence. Because of our assumption on h we know $h(\eta/t^{1/(\alpha n)}) \to 0$ as $t \to \infty$ (note that clearly also $h(\eta/t^{1/(\alpha n)})^k$ converges to zero for every $k > 0$). Further the integrand is dominated by

$$\|h\|_{L^\infty(\mathbb{R}^N)} e^{-A^n|\eta|^{\alpha n}} |\eta|^{\alpha n},$$

which belongs to $L^1(\mathbb{R}^N)$.

Combining this with our previous results we have that

$$t^{\frac{N}{\alpha n}} \int_{\mathbb{R}^N} |\hat{u} - \hat{v}| (\xi, t)\, d\xi \le t^{\frac{N}{\alpha n}} (I + II) \to 0 \quad \text{as } t \to \infty, \tag{4.7}$$

provided we can find a $r(t) \to 0$ as $t \to \infty$ which fulfills both conditions (4.5) and (4.6). This is done in Lemma 39, which is postponed just after the present proof. To conclude we only have to observe that from the convergence of the Fourier transforms $\hat{u}(\cdot, t) - \hat{v}(\cdot, t) \to 0$ in L^1 the convergence of $u - v$ in L^∞ follows. Indeed, from (4.7) we obtain

$$t^{\frac{N}{\alpha n}} \max_x |u(x, t) - v(x, t)| \le t^{\frac{N}{\alpha n}} \int_{\mathbb{R}^N} |\hat{u} - \hat{v}| (\xi, t)\, d\xi \to 0, \quad t \to \infty,$$

which ends the proof of the theorem. □

The following Lemma shows that there exists a function $r(t)$ satisfying (4.5) and (4.6), as required in the proof of the previous theorem.

Lemma 39. *Given a function $h \in C(\mathbb{R}, \mathbb{R})$ such that $h(\rho) \to 0$ as $\rho \to 0$ with $h(\rho) > 0$ for small ρ, there exists a function r with $r(t) \to 0$ as $t \to \infty$ which satisfies*

$$\lim_{t \to \infty} r(t) t^{\frac{1}{\alpha n}} = \infty$$

and

$$\lim_{t \to \infty} t (r(t))^{\alpha n} h(r(t)) = 0.$$

Proof. For fixed t large enough, we choose $r(t)$ as a solution of

$$r (h(r))^{\frac{1}{2\alpha n}} = t^{-\frac{1}{\alpha n}}. \tag{4.8}$$

This equation defines a function $r = r(t)$ which, by continuity arguments goes to zero as t tends to infinity, satisfying also the additional asymptotic conditions in the lemma. Indeed, if there exists $t_n \to \infty$ with no solution of (4.8) for $r \in (0, \delta)$ then $h(r) \equiv 0$ in $(0, \delta)$, which is a contradiction to our assumption that $h(r) > 0$ for r small. □

As a consequence of Theorem 38, we obtain the following corollary which completes the results gathered in Theorem 37 in the Introduction.

Corollary 40. *The asymptotic behavior of solutions of (4.1) is given by*

$$\|u(., t)\|_{L^\infty(\mathbb{R}^N)} \le \frac{C}{t^{\frac{N}{\alpha n}}}.$$

Moreover, the asymptotic profile is given by

$$\lim_{t \to +\infty} \max_y \left| t^{\frac{N}{\alpha n}} u(y t^{\frac{1}{\alpha n}}, t) - \|u_0\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G_A}(\xi) = e^{-A^n |\xi|^{\alpha n}}$.

Proof. From Theorem 38 we obtain that the asymptotic behavior is the same as the one for solutions of the evolution given by a power n of the fractional Laplacian. It is easy to check that the asymptotic behavior is in fact the one described in the statement of the corollary. In Fourier variables we have

$$\begin{aligned} \lim_{t\to\infty} \hat{v}(\eta t^{-\frac{1}{\alpha n}}, t) = & \lim_{t\to\infty} e^{-A^n|\eta|^{\alpha n}} \hat{u_0}(\eta t^{-\frac{1}{\alpha n}}) \\ = & \; e^{-A^n|\eta|^{\alpha n}} \hat{u_0}(0) \\ = & \; e^{-A^n|\eta|^{\alpha n}} \left\| u_0 \right\|_{L^1(\mathbb{R}^N)} . \end{aligned}$$

Therefore

$$\lim_{t\to+\infty} \max_y \left| t^{\frac{N}{\alpha n}} v(yt^{\frac{1}{\alpha n}}, t) - \left\| u_0 \right\|_{L^1(\mathbb{R}^N)} G_A(y) \right| = 0,$$

where $G_A(y)$ satisfies $\hat{G_A}(\xi) = e^{-A^n|\xi|^{\alpha n}}$. □

With similar arguments as in the proof of Theorem 38 one can prove that also the asymptotic behavior of the derivatives of solutions u of (4.1) is the same as the one for derivatives of solutions v of the evolution of a power n of the fractional Laplacian, assuming sufficient regularity of the solutions u of (4.1).

Theorem 41. *Let u be a solution of (4.1) with $u_0 \in W^{k,1}(\mathbb{R}^N)$, $k \leq \alpha n$ and $\hat{u_0} \in L^1(\mathbb{R}^N)$. Then, the asymptotic behavior of $D^k u(x,t)$ is given by*

$$\lim_{t\to+\infty} t^{\frac{N+k}{\alpha n}} \max_x \left| D^k u(x,t) - D^k v(x,t) \right| = 0,$$

where v is the solution of $v_t(x,t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x,t)$ with initial condition $v(x,0) = u_0(x)$.

Proof. We begin again by transforming our problem for u and v in a problem for the corresponding Fourier transforms $\hat{u}$ and $\hat{v}$. For this we consider

$$\begin{aligned} \max_x \left| D^k u(x,t) - D^k v(x,t) \right| &= \max_\xi \left| (\widehat{D^k u(\xi,t)})^\vee - (\widehat{D^k v(\xi,t)})^\vee \right| \\ &\leq \int_{\mathbb{R}^N} \left| \widehat{D^k u(\xi,t)} - \widehat{D^k v(\xi,t)} \right| \, d\xi = \int_{\mathbb{R}^N} |\xi|^k \left| \hat{u}(\xi,t) - \hat{v}(\xi,t) \right| \, d\xi. \end{aligned}$$

Showing $\int_{\mathbb{R}^N} |\xi|^k |\hat{u}(\xi,t) - \hat{v}(\xi,t)| \, d\xi \to 0$ as $t \to \infty$ works analogue to the proof of Theorem 38. The additional term $|\xi|^k$ is always dominated by the exponential terms. □

CHAPTER 5

A linear Neumann problem with a rescale of the kernel

The purpose of this chapter is to show that the solutions of the usual Neumann boundary value problem for the heat equation can be approximated by solutions of a sequence of nonlocal "Neumann" boundary value problems.

Given a bounded, connected and smooth domain Ω, one of the most common boundary conditions that has been imposed in the literature to the heat equation, $u_t = \Delta u$, is the *Neumann boundary condition*, $\partial u/\partial\eta(x,t) = g(x,t)$, $x \in \partial\Omega$, which leads to the following classical problem,

$$
\begin{cases} u_t - \Delta u = 0 & \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \eta} = g & \text{on} \quad \partial\Omega \times (0,T), \\ u(x,0) = u_0(x) & \text{in} \quad \Omega. \end{cases} \tag{5.1}
$$

Now we propose a nonlocal "Neumann" boundary value problem, namely

$$
u_t(x,t) = \int_\Omega J(x-y)\big(u(y,t) - u(x,t)\big)\,dy + \int_{\mathbb{R}^N\setminus\Omega} G(x,x-y)g(y,t)\,dy, \tag{5.2}
$$

where $G(x,\xi)$ is smooth and compactly supported in ξ uniformly in x.

Recall that in the previous chapter we have considered homogeneous boundary data, that is, $g \equiv 0$.

In this model the first integral takes into account the diffusion inside Ω. In fact, as we have explained, the integral $\int J(x-y)(u(y,t)-u(x,t))\,dy$ takes into account the individuals arriving or leaving position x from or to other places. Since we are integrating in Ω, we are imposing that diffusion takes place only in Ω. The last term takes into account the prescribed flux of individuals that enter or leave the domain.

The nonlocal Neumann model (5.2) and the Neumann problem for the heat equation (5.1) share many properties. For example, a comparison principle holds for both equations when G is nonnegative and the asymptotic behavior of their solutions as $t \to \infty$ is similar, see [**42**].

Existence and uniqueness of solutions of (5.2) with general G is proved by a fixed point argument. Also, we prove a comparison principle when $G \geq 0$.

0.9. Existence and uniqueness. In this chapter we deal with existence and uniqueness of solutions of (5.2). Our result is valid in a general L^1 setting.

THEOREM 42. *Let Ω be a bounded domain. Let $J \in L^1(\mathbb{R}^N)$ and $G \in L^\infty(\Omega \times \mathbb{R}^N)$. For every $u_0 \in L^1(\Omega)$ and $g \in L^\infty_{loc}([0,\infty); L^1(\mathbb{R}^N \setminus \Omega))$ there exists a unique solution u of (5.2) such that $u \in C([0,\infty); L^1(\Omega))$ and $u(x,0) = u_0(x)$.*

As in [**40**] and [**42**], existence and uniqueness will be a consequence of Banach's fixed point theorem. We follow closely the ideas of those works in our proof, so we will only outline the main arguments. Fix $t_0 > 0$ and consider the Banach space

$$X_{t_0} = C([0,t_0]; L^1(\Omega))$$

with the norm

$$|||w||| = \max_{0 \le t \le t_0} \|w(\cdot,t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $T_{u_0,g} : X_{t_0} \to X_{t_0}$ defined by

$$\begin{aligned} T_{u_0,g}(w)(x,t) = u_0(x) \quad &+ \int_0^t \int_\Omega J(x-y)\,(w(y,s) - w(x,s))\,dy\,ds \\ &+ \int_0^t \int_{\mathbb{R}^N \setminus \Omega} G(x,x-y) g(y,t)\,dy\,ds. \end{aligned} \tag{5.3}$$

The following lemma is the main ingredient in the proof of existence.

LEMMA 43. *Let J and G as in Theorem 42. Let $g,\ h \in L^\infty((0,t_0); L^1(\mathbb{R}^N \setminus \Omega))$ and $u_0,\ v_0 \in L^1(\Omega)$. There exists a constant C depending only on Ω, J and G such that for $w, z \in X_{t_0}$,*

$$|||T_{u_0,g}(w) - T_{v_0,h}(z)||| \le \|u_0 - v_0\|_{L^1} + Ct_0 \left(|||w - z||| + \|g - h\|_{L^\infty((0,t_0);L^1(\mathbb{R}^N \setminus \Omega))} \right). \tag{5.4}$$

PROOF. We have

$$\begin{aligned} \int_\Omega |T_{u_0,g}(w)(x,t) - T_{v_0,h}(z)(x,t)|\,dx &\le \int_\Omega |u_0(x) - v_0(x)|\,dx \\ &+ \int_\Omega \left| \int_0^t \int_\Omega J(x-y) \Big[(w(y,s) - z(y,s)) - (w(x,s) - z(x,s)) \Big] dy\,ds \right| dx \\ &+ \int_\Omega \int_0^t \int_{\mathbb{R}^N \setminus \Omega} |G(x,x-y)||g(y,s) - h(y,s)|\,dy\,ds\,dx. \end{aligned}$$

Therefore, we obtain (5.4). □

PROOF OF THEOREM 42. Let $T = T_{u_0,g}$. We check first that T maps X_{t_0} into X_{t_0}. From (5.3) we see that for $0 \le t_1 < t_2 \le t_0$,

$$\|T(w)(t_2) - T(w)(t_1)\|_{L^1(\Omega)} \le A \int_{t_1}^{t_2} \int_\Omega |w(y,s)|\,dy\,ds + B \int_{t_1}^{t_2} \int_{\mathbb{R}^N \setminus \Omega} |g(y,s)|\,dy\,ds.$$

On the other hand, again from (5.3)

$$\|T(w)(t) - u_0\|_{L^1(\Omega)} \le Ct\big\{|||w||| + \|g\|_{L^\infty((0,t_0);L^1(\mathbb{R}^N\setminus\Omega))}\big\}.$$

These two estimates give that $T(w) \in C([0,t_0]; L^1(\Omega))$. Hence T maps X_{t_0} into X_{t_0}.

Choose t_0 such that $Ct_0 < 1$. From Lemma 43 we get that T is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial datum $u(x,t_0) \in L^1(\Omega)$ and obtain a solution in $[0, 2\,t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$. □

Our next aim is to prove a comparison principle for (5.2) when $J,\ G \ge 0$. To this end we define what we understand by sub and supersolutions.

DEFINITION 44. *A function $u \in C([0,T); L^1((\Omega))$ is a supersolution of* (5.2) *if $u(x,0) \ge u_0(x)$ and*

$$u_t(x,t) \ge \int_\Omega J(x-y)\big(u(y,t)-u(x,t)\big)\,dy + \int_{\mathbb{R}^N\setminus\Omega} G(x,x-y)g(y,t)\,dy.$$

Subsolutions are defined analogously by reversing the inequalities.

LEMMA 45. *Let $J,\ G \ge 0$, $u_0 \ge 0$ and $g \ge 0$. If $u \in C(\overline{\Omega}\times[0,T])$ is a supersolution to* (5.2)*, then $u \ge 0$.*

PROOF. Assume that $u(x,t)$ is negative somewhere. Let $v(x,t) = u(x,t) + \varepsilon t$ with ε so small such that v is still negative somewhere. Then, if we take (x_0,t_0) a point where v attains its negative minimum, there holds that $t_0 > 0$ and

$$\begin{aligned} v_t(x_0,t_0) = u_t(x_0,t_0) + \varepsilon &> \int_\Omega J(x-y)(u(y,t_0) - u(x_0,t_0))\,dy \\ &= \int_\Omega J(x-y)(v(y,t_0) - v(x_0,t_0))\,dy \ge 0 \end{aligned}$$

which is a contradiction. Thus, $u \ge 0$. □

COROLLARY 46. *Let $J,\ G \ge 0$ and bounded. Let u_0 and v_0 in $L^1(\Omega)$ with $u_0 \ge v_0$ and $g,\ h \in L^\infty((0,T); L^1(\mathbb{R}^N\setminus\Omega))$ with $g \ge h$. Let u be a solution of* (5.2) *with initial condition u_0 and flux g and v be a solution of* (5.2) *with initial condition v_0 and flux h. Then,*

$$u \ge v \qquad a.e.$$

PROOF. Let $w = u - v$. Then, w is a supersolution with initial datum $u_0 - v_0 \ge 0$ and boundary datum $g - h \ge 0$. Using the continuity of solutions with respect to the initial and Neumann data (Lemma 43) and the fact that $J \in L^\infty(\mathbb{R}^N)$, $G \in L^\infty(\Omega\times\mathbb{R}^N)$ we may assume that $u,\ v \in C(\overline{\Omega}\times[0,T])$. By Lemma 45 we obtain that $w = u - v \ge 0$. So the corollary is proved. □

Corollary 47. *Let J, $G \geq 0$ and bounded. Let $u \in C(\overline{\Omega} \times [0,T])$ (resp. v) be a supersolution (resp. subsolution) of* (5.2). *Then, $u \geq v$.*

Proof. It follows the lines of the proof of the previous corollary. □

0.10. Rescaling the kernel. Our main goal now is to show that the Neumann problem for the heat equation (5.1), can be approximated by suitable nonlocal Neumann problems like (5.2) when we rescale them appropriately.

More precisely, for given J and G we consider the rescaled kernels

$$J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right), \qquad G_\varepsilon(x,\xi) = C_1 \frac{1}{\varepsilon^N} G\left(x, \frac{\xi}{\varepsilon}\right) \tag{5.5}$$

with

$$C_1^{-1} = \frac{1}{2} \int_{B(0,d)} J(z) z_N^2 \, dz,$$

which is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it. Then, we consider the solution $u^\varepsilon(x,t)$ to

$$\begin{cases} u_t^\varepsilon(x,t) &= \displaystyle\frac{1}{\varepsilon^2} \int_\Omega J_\varepsilon(x-y)(u^\varepsilon(y,t) - u^\varepsilon(x,t))\, dy \\ & \qquad\qquad \displaystyle+ \frac{1}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} G_\varepsilon(x, x-y) g(y,t)\, dy, \\ u^\varepsilon(x,0) &= u_0(x). \end{cases} \tag{5.6}$$

We will show that

$$u^\varepsilon \to u$$

in different topologies according to two different choices of the kernel G.

Let us give an heuristic idea in one space dimension, with $\Omega = (0,1)$, of why the scaling involved in (5.5) is the correct one. We assume that

$$\int_1^\infty G(1, 1-y)\, dy = -\int_{-\infty}^0 G(0,-y)\, dy = \int_0^1 J(y)\, y\, dy$$

and, as stated above, $G(x,\cdot)$ has compact support independent of x. In this case (5.6) reads

$$\begin{aligned} u_t(x,t) = \ & \frac{1}{\varepsilon^2} \int_0^1 J_\varepsilon(x-y) \big(u(y,t) - u(x,t)\big) dy + \frac{1}{\varepsilon} \int_{-\infty}^0 G_\varepsilon(x, x-y)\, g(y,t)\, dy \\ & + \frac{1}{\varepsilon} \int_1^{+\infty} G_\varepsilon(x, x-y)\, g(y,t)\, dy := \mathcal{A}_\varepsilon u(x,t). \end{aligned}$$

If $x \in (0,1)$ a Taylor expansion gives that for any fixed smooth u and ε small enough, the right hand side $\mathcal{A}_\varepsilon u$ in (5.6) becomes

$$\mathcal{A}_\varepsilon u(x) = \frac{1}{\varepsilon^2} \int_0^1 J_\varepsilon \left(x-y\right) \left(u(y)-u(x)\right) dy \approx u_{xx}(x)$$

and if $x = 0$ and ε small,

$$\mathcal{A}_\varepsilon u(0) = \frac{1}{\varepsilon^2} \int_0^1 J_\varepsilon \left(-y\right) \left(u(y)-u(0)\right) dy + \frac{1}{\varepsilon} \int_{-\infty}^0 G_\varepsilon \left(0,-y\right) g(y)\, dy \approx \frac{C_2}{\varepsilon}(u_x(0)-g(0)).$$

Analogously, $\mathcal{A}_\varepsilon u(1) \approx (C_2/\varepsilon)(-u_x(1)+g(1))$. However, the proofs of our results are much more involved than simple Taylor expansions due to the fact that for each $\varepsilon > 0$ there are points $x \in \Omega$ for which the ball in which integration takes place, $B(x, d\varepsilon)$, is not contained in Ω. Moreover, when working in several space dimensions, one has to take into account the geometry of the domain.

0.11. Uniform convergence in the case $g \equiv 0$. Our first result deals with homogeneous boundary conditions, this is, $g \equiv 0$.

THEOREM 48. *Assume $g \equiv 0$. Let Ω be a bounded $C^{2+\alpha}$ domain for some $0 < \alpha < 1$. Let $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ be the solution to* (5.1) *and let u^ε be the solution to* (5.6) *with J_ε as above. Then,*

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^\infty(\Omega)} \to 0$$

as $\varepsilon \to 0$.

Note that this result holds for every G since $g \equiv 0$, and that the assumed regularity in u is guaranteed if $u_0 \in C^{2+\alpha}(\overline{\Omega})$ and $\partial u_0/\partial\eta = 0$. See, for instance, [**62**].

We will prove Theorem 48 by constructing adequate super and subsolutions and then using comparison arguments to get bounds for the difference $u^\varepsilon - u$.

We set $w^\varepsilon = u^\varepsilon - u$ and let $\tilde{u}$ be a $C^{2+\alpha,1+\alpha/2}$ extension of u to $\mathbb{R}^N \times [0,T]$. We define

$$L_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_\Omega J_\varepsilon(x-y)\big(v(y,t)-v(x,t)\big)dy$$

and

$$\tilde{L}_\varepsilon(v) = \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N} J_\varepsilon(x-y)\big(v(y,t)-v(x,t)\big)dy.$$

Then

$$\begin{aligned} w_t^\varepsilon &= L_\varepsilon(u^\varepsilon) - \Delta u + \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} G_\varepsilon(x,x-y)g(y,t)\,dy \\ &= L_\varepsilon(w^\varepsilon) + \tilde{L}_\varepsilon(\tilde{u}) - \Delta u + \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} G_\varepsilon(x,x-y)g(y,t)\,dy \\ &\qquad - \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(y,t)-\tilde{u}(x,t)\big)\,dy. \end{aligned}$$

Or

$$w_t^\varepsilon - L_\varepsilon(w^\varepsilon) = F_\varepsilon(x,t),$$

where, noting that $\Delta u = \Delta\tilde{u}$ in Ω,

$$\begin{aligned} F_\varepsilon(x,t) = \tilde{L}_\varepsilon(\tilde{u}) - \Delta\tilde{u} &+ \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} G_\varepsilon(x,x-y)g(y,t)\,dy \\ &- \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(y,t)-\tilde{u}(x,t)\big)\,dy. \end{aligned}$$

Our main task in order to prove the uniform convergence result is to get bounds on F_ε.

First, we observe that it is well known that by the choice of C_1, the fact that J is radially symmetric and $\tilde{u} \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N\times[0,T])$, we have that

$$\sup_{t\in[0,T]} \|\tilde{L}_\varepsilon(\tilde{u}) - \Delta\tilde{u}\|_{L^\infty(\Omega)} = O(\varepsilon^\alpha)\,. \tag{5.7}$$

In fact,

$$\frac{C_1}{\varepsilon^{N+2}}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right)\left(\tilde{u}(y,t)-\tilde{u}(x,t)\right)\,dy - \Delta\tilde{u}(x,t)$$

becomes, under the change variables $z=(x-y)/\varepsilon$,

$$\frac{C_1}{\varepsilon^2}\int_{\mathbb{R}^N} J\left(z\right)\left(\tilde{u}(x-\varepsilon z,t)-\tilde{u}(x,t)\right)dy - \Delta\tilde{u}(x,t)$$

and hence (5.7) follows by a simple Taylor expansion.

Next, we will estimate the last integral in F_ε. We remark that the next lemma is valid for any smooth function, not only for a solution to the heat equation.

LEMMA 49. *If θ is a $C^{2+\alpha,1+\alpha/2}$ function on $\mathbb{R}^N\times[0,T]$ and $\dfrac{\partial\theta}{\partial\eta} = h$ on $\partial\Omega$, then for $x\in\Omega_\varepsilon = \{z\in\Omega\,|\,\mathrm{dist}\,(z,\partial\Omega)<d\varepsilon\}$ and ε small,*

$$\begin{aligned} \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\theta(y,t)-\theta(x,t)\big)\,dy &= \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}h(\bar{x},t)\,dy \\ + \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{|\beta|=2}\frac{D^\beta\theta}{2}(\bar{x},t)&\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big]dy + O(\varepsilon^\alpha), \end{aligned}$$

where $\bar{x}$ is the orthogonal projection of x on the boundary of Ω so that $\|\bar{x}-y\| \le 2d\varepsilon$.

PROOF. Since $\theta \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$ we have

$$\begin{aligned}\theta(y,t)-\theta(x,t) &= \theta(y,t)-\theta(\bar{x},t)-\big(\theta(x,t)-\theta(\bar{x},t)\big)\\ &= \nabla\theta(\bar{x},t)\cdot(y-x)+\sum_{|\beta|=2}\frac{D^\beta\theta}{2}(\bar{x},t)\big[(y-\bar{x})^\beta-(x-\bar{x})^\beta\big]\\ &\quad + O(||\bar{x}-x||^{2+\alpha})+O(||\bar{x}-y||^{2+\alpha}).\end{aligned}$$

Therefore,

$$\begin{aligned}\frac{1}{\varepsilon^2}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\theta(y,t)-\theta(x,t)\big)\,dy &= \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\nabla\theta(\bar{x},t)\cdot\frac{(y-x)}{\varepsilon}\,dy\\ + \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{|\beta|=2}\frac{D^\beta\theta}{2}(\bar{x},t)\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta-\big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big]\,dy + O(\varepsilon^\alpha).\end{aligned}$$

Fix $x \in \Omega_\varepsilon$. Let us take a new coordinate system such that $\eta(\bar{x}) = e_N$. Since $\dfrac{\partial\theta}{\partial\eta} = h$ on $\partial\Omega$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\nabla\theta(\bar{x},t)\cdot\frac{(y-x)}{\varepsilon}\,dy\\ &= \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}h(\bar{x},t)\,dy + \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\,dy.\end{aligned}$$

We will estimate this last integral. Since Ω is a $C^{2+\alpha}$ domain we can chose vectors e_1, e_2, ..., e_{N-1} so that there exists $\kappa > 0$ and constants $f_i(\bar{x})$ such that

$$B_{2d\varepsilon}(\bar{x}) \cap \left\{ y_N - \big(\bar{x}_N + \sum_{i=1}^{N-1} f_i(\bar{x})(y_i-x_i)^2\big) > \kappa\varepsilon^{2+\alpha} \right\} \subset \mathbb{R}^N \setminus \Omega,$$

$$B_{2d\varepsilon}(\bar{x}) \cap \left\{ y_N - \big(\bar{x}_N + \sum_{i=1}^{N-1} f_i(\bar{x})(y_i-x_i)^2\big) < -\kappa\varepsilon^{2+\alpha} \right\} \subset \Omega.$$

Therefore

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\Big)\,dy\\
&\quad=\int_{(\mathbb{R}^N\setminus\Omega)\cap\ \left|y_N-\left(\bar{x}_N+\sum_{i=1}^{N-1}f_i(\bar{x})(y_i-x_i)^2\right)\right|\le\kappa\varepsilon^{2+\alpha}} J_\varepsilon(x-y)\Big(\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\Big)\,dy\\
&\qquad+\int_{y_N-\left(\bar{x}_N+\sum_{i=1}^{N-1}f_i(\bar{x})(y_i-x_i)^2\right)>\kappa\varepsilon^{2+\alpha}} J_\varepsilon(x-y)\Big(\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\Big)\,dy\\
&\quad=I_1+I_2.
\end{aligned}$$

If we take $z=(y-x)/\varepsilon$ as a new variable, recalling that $\bar{x}_N-x_N=\varepsilon s$, we obtain

$$|I_1|\le C_1\sum_{i=1}^{N-1}|\theta_{x_i}(\bar{x},t)|\int_{\left|z_N-\left(s+\varepsilon\sum_{i=1}^{N-1}f_i(\bar{x})(z_i)^2\right)\right|\le\kappa\varepsilon^{1+\alpha}} J(z)|z_i|\,dz\le C\,\kappa\,\varepsilon^{1+\alpha}.$$

On the other hand,

$$I_2=C_1\sum_{i=1}^{N-1}\theta_{x_i}(\bar{x},t)\int_{z_N-\left(s+\varepsilon\sum_{i=1}^{N-1}f_i(\bar{x})(z_i)^2\right)>\kappa\varepsilon^{1+\alpha}} J(z)\,z_i\,dz.$$

Fix $1\le i\le N-1$. Then, since J is radially symmetric, $J(z)\,z_i$ is an odd function of the variable z_i and, since the set $\Big\{z_N-\big(s+\varepsilon\sum_{i=1}^{N-1}f_i(\bar{x})(z_i)^2\big)>\kappa\varepsilon^{1+\alpha}\Big\}$ is symmetric in that variable we get

$$I_2=0.$$

Collecting the previous estimates the lemma is proved. □

We will also need the following inequality.

Lemma 50. *There exist $K>0$ and $\bar{\varepsilon}>0$ such that, for $\varepsilon<\bar{\varepsilon}$,*

$$\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}\,dy\ge K\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy. \tag{5.8}$$

Proof. Let us put the origin at the point $\bar{x}$ and take a coordinate system such that $\eta(\bar{x})=e_N$. Then, $x=(0,-\mu)$ with $0<\mu<d\varepsilon$. Then, arguing as before,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}\,dy=\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\,dy\\
&=\int_{\{y_N>\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\,dy+\int_{\mathbb{R}^N\setminus\Omega\cap\{|y_N|<\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\,dy\\
&\ge\int_{\{y_N>\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\,dy-C\varepsilon.
\end{aligned}$$

Fix c_1 small such that

$$\frac{1}{2}\int_{\{z_N>0\}} J(z)\, z_N\, dz \geq 2c_1 \int_{\{0<z_N<2c_1\}} J(z)\, dz.$$

We divide our arguments into two cases according to whether $\mu \leq c_1\varepsilon$ or $\mu > c_1\varepsilon$.

Case I Assume $\mu \leq c_1\varepsilon$. In this case we have,

$$\begin{aligned} &\int_{\{y_N>\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\, dy = C_1 \int_{\{z_N>\kappa\varepsilon+\frac{\mu}{\varepsilon}\}} J(z)\, z_N\, dz \\ &= C_1 \left(\int_{\{z_N>0\}} J(z)\, z_N\, dz - \int_{\{0<z_N<\kappa\varepsilon+\frac{\mu}{\varepsilon}\}} J(z)\, z_N\, dz \right) \\ &\geq C_1 \left(\int_{\{z_N>0\}} J(z)\, z_N\, dz - 2c_1 \int_{\{0<z_N<2c_1\}} J(z)\, dz \right) \geq \frac{C_1}{2} \int_{\{z_N>0\}} J(z)\, z_N\, dz. \end{aligned} \tag{5.9}$$

Then,

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy - K \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy \\ &\geq C_1 \left(\frac{1}{2}\int_{\{z_N>0\}} J(z)\, z_N\, dz - K \right) - C\varepsilon \geq 0, \end{aligned}$$

if ε is small enough and

$$K < \frac{1}{4}\int_{\{z_N>0\}} J(z)\, z_N\, dz.$$

Case II Assume that $\mu \geq c_1\varepsilon$. For y in $\mathbb{R}^N \setminus \Omega \cap B(\bar{x}, d\varepsilon)$ we have

$$\frac{y_N}{\varepsilon} \geq -\kappa\varepsilon.$$

Then,

$$\begin{aligned} &\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\frac{y_N+\mu}{\varepsilon}\, dy - K \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy \\ &\geq (c_1-\kappa\varepsilon) \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy - K \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy \\ &= \big(c_1-\kappa\varepsilon-K\big) \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy \geq 0, \end{aligned}$$

if ε is small and

$$K < \frac{c_1}{2}.$$

This ends the proof of (5.8). □

We now prove Theorem 48.

Proof of Theorem 48. We will use a comparison argument. First, let us look for a supersolution. Let us pick an auxiliary function v as a solution to

$$\begin{cases} v_t - \Delta v = h(x,t) & \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial v}{\partial \eta} = g_1(x,t) & \text{on} \quad \partial\Omega \times (0,T), \\ v(x,0) = v_1(x) & \text{in} \quad \Omega. \end{cases}$$

for some smooth functions $h(x,t) \geq 1$, $g_1(x,t) \geq 1$ and $v_1(x) \geq 0$ such that the resulting v has an extension $\tilde{v}$ that belongs to $C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$, and let M be an upper bound for v in $\bar{\Omega} \times [0,T]$. Then,

$$v_t = L_\varepsilon v + (\Delta v - \tilde{L}_\varepsilon \tilde{v}) + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)(\tilde{v}(y,t) - \tilde{v}(x,t))\, dy + h(x,t).$$

Since $\Delta v = \Delta \tilde{v}$ in Ω, we have that v is a solution to

$$\begin{cases} v_t - L_\varepsilon v = H(x,t,\varepsilon) & \text{in} \quad \Omega \times (0,T), \\ v(x,0) = v_1(x) & \text{in} \quad \Omega, \end{cases}$$

where by (5.7), Lemma 49 and the fact that $h \geq 1$,

$$\begin{aligned} H(x,t,\varepsilon) =& (\Delta \tilde{v} - \tilde{L}_\varepsilon \tilde{v}) + \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)(\tilde{v}(y,t) - \tilde{v}(x,t))\, dy + h(x,t) \\ \geq& \Big(\frac{1}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon} g_1(\bar{x},t)\, dy \\ &+ \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \sum_{|\beta|=2} \frac{D^\beta \tilde{v}}{2}(\bar{x},t) \Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta \Big]\, dy \Big) + 1 - C\varepsilon^\alpha \\ \geq& \Big(\frac{g_1(\bar{x},t)}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy - D_1 \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)\, dy \Big) + \frac{1}{2} \end{aligned}$$

for some constant D_1 if ε is small so that $C\varepsilon^\alpha \leq 1/2$.

Now, observe that Lemma 50 implies that for every constant $C_0 > 0$ there exists ε_0 such that,

$$\frac{1}{\varepsilon} \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y) \eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy - C_0 \int_{\mathbb{R}^N \setminus \Omega} J_\varepsilon(x-y)\, dy \geq 0,$$

if $\varepsilon < \varepsilon_0$.

Now, since $g = 0$, by (5.7) and Lemma 49 we obtain

$$|F_\varepsilon| \le C\varepsilon^\alpha + \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y) \sum_{|\beta|=2} \frac{D^\beta \tilde{u}}{2}(\bar{x},t)\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big]dy$$
$$\le C\varepsilon^\alpha + C_2 \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy.$$

Given $\delta > 0$, let $v_\delta = \delta v$. Then v_δ verifies

$$\begin{cases} (v_\delta)_t - L_\varepsilon v_\delta = \delta H(x,t,\varepsilon) & \text{in} \quad \Omega \times (0,T), \\ v_\delta(x,0) = \delta v_1(x) & \text{in} \quad \Omega. \end{cases}$$

By our previous estimates, there exists $\varepsilon_0 = \varepsilon_0(\delta)$ such that for $\varepsilon \le \varepsilon_0$,

$$|F_\varepsilon| \le \delta H(x,t,\varepsilon).$$

So, by the comparison principle for any $\varepsilon \le \varepsilon_0$ it holds that

$$-M\delta \le -v_\delta \le w_\varepsilon \le v_\delta \le M\delta.$$

Therefore, for every $\delta > 0$,

$$-M\delta \le \liminf_{\varepsilon\to 0} w_\varepsilon \le \limsup_{\varepsilon \to 0} w_\varepsilon \le M\delta.$$

and the theorem is proved. □

0.12. The non-homogeneous case. $g \neq 0$**.** Now we will make explicit the functions G we will deal with in the case $g \neq 0$.

To define the first one let us introduce some notation. As before, let Ω be a bounded $C^{2+\alpha}$ domain. For $x \in \Omega_\varepsilon := \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) < d\varepsilon\}$ and ε small enough we write $x = \bar{x} - s\,d\,\eta(\bar{x})$ where $\bar{x}$ is the orthogonal projection of x on $\partial\Omega$, $0 < s < \varepsilon$ and $\eta(\bar{x})$ is the unit exterior normal to Ω at $\bar{x}$. Under these assumptions we define

$$G_1(x,\xi) = -J(\xi)\,\eta(\bar{x})\cdot\xi \quad \text{for } x \in \Omega_\varepsilon. \tag{5.10}$$

Notice that the last integral in (5.6) only involves points $x \in \Omega_\varepsilon$ since when $y \notin \Omega$, $x - y \in supp\, J_\varepsilon$ implies that $x \in \Omega_\varepsilon$. Hence the above definition makes sense for ε small.

For this choice of the kernel, $G = G_1$, we have the following result.

THEOREM 51. *Let Ω be a bounded $C^{2+\alpha}$ domain, $g \in C^{1+\alpha,(1+\alpha)/2}(\overline{(\mathbb{R}^N \setminus \Omega)} \times [0,T])$, $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega} \times [0,T])$ the solution to* (5.1)*, for some $0 < \alpha < 1$. Let J as before and $G(x,\xi) = G_1(x,\xi)$, where G_1 is defined by* (5.10)*. Let u^ε be the solution to* (5.6)*. Then,*

$$\sup_{t\in[0,T]} \|u^\varepsilon(\cdot,t) - u(\cdot,t)\|_{L^1(\Omega)} \to 0$$

as $\varepsilon \to 0$.

Observe that G_1 may fail to be nonnegative and hence a comparison principle may not hold. However, in this case our proof of convergence to the solution of the heat equation does not rely on comparison arguments for (5.2). If we want a nonnegative kernel G, in order to have a comparison principle, we can modify $(G_1)_\varepsilon$ by taking

$$(\tilde{G}_1)_\varepsilon(x,\xi) = (G_1)_\varepsilon(x,\xi) + \kappa\varepsilon J_\varepsilon(\xi) = \frac{1}{\varepsilon} J_\varepsilon(\xi)\left(-\eta(\bar{x})\cdot\xi + \kappa\varepsilon^2\right)$$

instead.

Note that for $x \in \overline{\Omega}$ and $y \in \mathbb{R}^N \setminus \Omega$, $(\tilde{G}_1)_\varepsilon(x, x-y) = \frac{1}{\varepsilon} J_\varepsilon(x-y)\left(-\eta(\bar{x})\cdot(x-y) + \kappa\varepsilon^2\right)$ is nonnegative for ε small if we choose the constant κ as a bound for the curvature of $\partial\Omega$, since $|x-y| \le d\,\varepsilon$. As will be seen in Remark 53, Theorem 51 remains valid with $(G_1)_\varepsilon$ replaced by $(\tilde{G}_1)_\varepsilon$.

0.13. Convergence in L^1 in the case $G = G_1$. Using the previous notations, first we prove that F_ε goes to zero as ε goes to zero.

LEMMA 52. *If $G = G_1$ then*

$$F_\varepsilon(x,t) \to 0 \quad \text{in} \quad L^\infty\big([0,T]; L^1(\Omega)\big)$$

as $\varepsilon \to 0$.

PROOF. As $G = G_1 = -J(\xi)\,\eta(\bar{x})\cdot\xi$, for $x \in \Omega_\varepsilon$, by (5.7) and Lemma 49,

$$\begin{aligned} F_\varepsilon(x,t) =& \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}\big(g(y,t)-g(\bar{x},t)\big)\,dy \\ & - \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{|\beta|=2}\frac{D^\beta\tilde{u}}{2}(\bar{x},t)\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big]\,dy + O(\varepsilon^\alpha). \end{aligned}$$

As g is smooth, we have that F_ε is bounded in Ω_ε. Recalling the fact that $|\Omega_\varepsilon| = O(\varepsilon)$ and $F_\varepsilon(x,t) = O(\varepsilon^\alpha)$ on $\Omega \setminus \Omega_\varepsilon$ we get the convergence result. ☐

We are now ready to prove Theorem 51.

PROOF OF THEOREM 51. In the case $G = G_1$ we have proven in Lemma 52 that $F_\varepsilon \to 0$ in $L^1(\Omega\times[0,T])$. On the other hand, we have that $w^\varepsilon = u^\varepsilon - u$ is a solution to

$$\begin{aligned} w_t - L_\varepsilon(w) &= F_\varepsilon \\ w(x,0) &= 0. \end{aligned}$$

Let z^ε be a solution to

$$\begin{aligned} z_t - L_\varepsilon(z) &= |F_\varepsilon| \\ z(x,0) &= 0. \end{aligned}$$

Then $-z^\varepsilon$ is a solution to

$$\begin{aligned} z_t - L_\varepsilon(z) &= -|F_\varepsilon| \\ z(x,0) &= 0. \end{aligned}$$

By comparison we have that

$$-z^\varepsilon \le w^\varepsilon \le z^\varepsilon \quad \text{and} \quad z^\varepsilon \ge 0.$$

Integrating the equation for z^ε we get

$$\|z^\varepsilon(\cdot,t)\|_{L^1(\Omega)} = \int_\Omega z^\varepsilon(x,t)\,dx = \int_\Omega \int_0^t |F_\varepsilon(x,s)|\,ds\,dx\,.$$

Applying Lemma 52 we get

$$\sup_{t\in[0,T]} \|z^\varepsilon(\cdot,t)\|_{L^1(\Omega)} \to 0$$

as $\varepsilon \to 0$. So the theorem is proved. □

REMARK 53. *Notice that if we consider a kernel G which is a modification of G_1 of the form*

$$G_\varepsilon(x,\xi) = (G_1)_\varepsilon(x,\xi) + A(x,\xi,\varepsilon)$$

with

$$\int_{\mathbb{R}^N\setminus\Omega} |A(x,x-y,\varepsilon)|\,dy \to 0$$

in $L^1(\Omega)$ as $\varepsilon \to 0$, then the conclusion of Theorem 51 is still valid. In particular, we can take $A(x,\xi,\varepsilon) = \kappa\varepsilon J_\varepsilon(\xi)$.

0.14. The case $G = C\,J$. Finally, the other "Neumann" kernel we propose is just a scalar multiple of J, that is,

$$G(x,\xi) = G_2(x,\xi) = C_2 J(\xi),$$

where C_2 is such that

$$\int_0^d \int_{\{z_N>s\}} J(z)\big(C_2 - z_N\big)\,dz\,ds = 0. \tag{5.11}$$

This choice of G is natural since we are considering a flux with a jumping probability that is a scalar multiple of the same jumping probability that moves things in the interior of the domain, J.

Several properties of solutions to (5.2) have been recently investigated in [**42**] in the case $G = G_2$ for different choices of g.

For the case of G_2 we can still prove convergence but in a weaker sense.

THEOREM 54. *Let Ω be a bounded $C^{2+\alpha}$ domain, $g \in C^{1+\alpha,(1+\alpha)/2}(\overline{(\mathbb{R}^N\setminus\Omega)} \times [0,T])$, $u \in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ the solution to (5.1), for some $0<\alpha<1$. Let J as before and $G(x,\xi) = G_2(x,\xi) = C_2 J(\xi)$, where C_2 is defined by (5.11). Let u^ε be the solution to (5.6). Then, for each $t \in [0,T]$*

$$u_\varepsilon(x,t) \rightharpoonup u(x,t) \quad * -\textit{weakly in } L^\infty(\Omega)$$

as $\varepsilon \to 0$.

0.15. Weak convergence in L^1 in the case $G = G_2$. First, we prove that in this case F_ε goes to zero as measures.

LEMMA 55. *If $G = G_2$ then there exists a constant C independent of ε such that*

$$\int_0^T \int_\Omega |F_\varepsilon(x,s)|\, dx\, ds \le C.$$

Moreover,

$$F_\varepsilon(x,t) \rightharpoonup 0 \qquad \text{as measures}$$

as $\varepsilon \to 0$. That is, for any continuous function θ, it holds that

$$\int_0^T \int_\Omega F_\varepsilon(x,t)\theta(x,t)\, dx\, dt \to 0$$

as $\varepsilon \to 0$.

PROOF. As $G = G_2 = C_2 J(\xi)$ and g and $\tilde{u}$ are smooth, taking again the coordinate system of Lemma 49, we obtain

$$\begin{aligned} F_\varepsilon(x,t) =& \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(C_2 g(y,t) - \frac{y_N - x_N}{\varepsilon} g(\bar{x},t)\Big) \\ &- \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1} \tilde{u}_{x_i}(\bar{x},t)\frac{(y_i - x_i)}{\varepsilon}\, dy \\ &- \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{|\beta|=2}\frac{D^\beta \tilde{u}(\bar{x},t)}{2}\Big[\big(\frac{(y-\bar{x})}{\varepsilon}\big)^\beta - \big(\frac{(x-\bar{x})}{\varepsilon}\big)^\beta\Big] dy + O(\varepsilon^\alpha) \\ =& \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(C_2 g(\bar{x},t) - \frac{y_N - x_N}{\varepsilon} g(\bar{x},t)\Big) \\ &- \frac{1}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1} \tilde{u}_{x_i}(\bar{x},t)\frac{(y_i - x_i)}{\varepsilon}\, dy + O(1)\chi_{\Omega_\varepsilon} + O(\varepsilon^\alpha). \end{aligned}$$

Let

$$\begin{aligned} B_\varepsilon(x,t) := &\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(C_2 g(\bar{x},t) - \frac{y_N - x_N}{\varepsilon} g(\bar{x},t)\Big) \\ &- \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1} \tilde{u}_{x_i}(\bar{x},t)\frac{(y_i - x_i)}{\varepsilon}\, dy. \end{aligned}$$

Proceeding in a similar way as in the proof of Lemma 49 we get for ε small,

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big(C_2 g(\bar{x},t) - \frac{y_N - x_N}{\varepsilon} g(\bar{x},t)\Big) \\
&= g(\bar{x},t)\int_{(\mathbb{R}^N\setminus\Omega)\cap\{|y_N-\bar{x}_N|\le\kappa\varepsilon^2\}} J_\varepsilon(x-y)\left(C_2 - \frac{(y_N-x_N)}{\varepsilon}\right) dy \\
&\quad + g(\bar{x},t)\int_{(\mathbb{R}^N\setminus\Omega)\cap\{y_N-\bar{x}_N>0\}} J_\varepsilon(x-y)\left(C_2 - \frac{(y_N-x_N)}{\varepsilon}\right) dy \\
&\quad - g(\bar{x},t)\int_{(\mathbb{R}^N\setminus\Omega)\cap\{0<y_N-\bar{x}_N<\kappa\varepsilon^2\}} J_\varepsilon(x-y)\left(C_2 - \frac{(y_N-x_N)}{\varepsilon}\right) dy \\
&= C_1 g(\bar{x},t)\int_{\{z_N>s\}} J(z)(C_2 - z_N)\,dz + O(\varepsilon)\chi_{\Omega_\varepsilon}.
\end{aligned}$$

And

$$\begin{aligned}
&\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\frac{(y_i-x_i)}{\varepsilon}\,dy \\
&= \sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\int_{\{|y_N-\bar{x}_N|\le\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{(y_i-x_i)}{\varepsilon}dy \\
&\quad + \sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\int_{\{y_N-\bar{x}_N>\kappa\varepsilon^2\}} J_\varepsilon(x-y)\frac{(y_i-x_i)}{\varepsilon}dy \\
&= C_1\sum_{i=1}^{N-1}\tilde{u}_{x_i}(\bar{x},t)\int_{\{z_N-s>\kappa\varepsilon\}} J(z)z_i dz + O(\varepsilon)\chi_{\Omega_\varepsilon} \\
&= I_2 + O(\varepsilon)\chi_{\Omega_\varepsilon}.
\end{aligned}$$

As in Lemma 49 we have $I_2 = 0$. Therefore,

$$B_\varepsilon(x,t) = C_1 g(\bar{x},t)\int_{\{z_N>s\}} J(z)\big(C_2 - z_N\big)\,dz + O(\varepsilon)\chi_{\Omega_\varepsilon}.$$

Now, we observe that B_ε is bounded and supported in Ω_ε. Hence

$$\int_0^t\int_\Omega |F_\varepsilon(x,\tau)|\,dx\,d\tau \le \frac{1}{\varepsilon}\int_0^t\int_{\Omega_\varepsilon}|B_\varepsilon(x,\tau)|\,dx\,d\tau + Ct|\Omega_\varepsilon| + Ct|\Omega|\varepsilon^\alpha \le C.$$

This proves the first assertion of the lemma.

Now, let us write for a point $x\in\Omega_\varepsilon$

$$x = \bar{x} - \mu\eta(\bar{x}) \qquad \text{with } 0<\mu<d\varepsilon.$$

For ε small and $0<\mu<d\varepsilon$, let dS_μ be the area element of $\{x\in\Omega\,|\,\mathrm{dist}\,(x,\partial\Omega)=\mu\}$. Then, $dS_\mu = dS + O(\varepsilon)$, where dS is the area element of $\partial\Omega$.

So that, taking now $\mu = s\varepsilon$ we get for any continuous test function θ,

$$\begin{aligned}\frac{1}{\varepsilon}\int_0^T\int_{\Omega_\varepsilon} & B_\varepsilon(x,t)\theta(\bar{x},t)\,dx\,dt \\ &= O(\varepsilon) + C_1\int_0^T\int_{\partial\Omega} g(\bar{x},t)\theta(\bar{x},t)\int_0^d\int_{\{z_N>s\}} J(z)\big(C_2 - z_N\big)\,dz\,ds\,dS\,dt \\ &= O(\varepsilon)\to 0 \quad \text{as } \varepsilon\to 0,\end{aligned}$$

since we have chosen C_2 so that

$$\int_0^d\int_{\{z_N>s\}} J(z)\big(C_2 - z_N\big)\,dz\,ds = 0.$$

Now, with all these estimates, we go back to F_ε. We have

$$F_\varepsilon(x,t) = \frac{1}{\varepsilon}B_\varepsilon(x,t) + O(1)\chi_{\Omega_\varepsilon} + O(\varepsilon^\alpha).$$

Thus, we obtain

$$\int_0^T\int_{\Omega_\varepsilon} F_\varepsilon(x,t)\theta(\bar{x},t)\,dx\,dt \to 0 \qquad \text{as } \varepsilon\to 0.$$

Now, if $\sigma(r)$ is the modulus of continuity of θ,

$$\begin{aligned}\int_0^T\int_{\Omega_\varepsilon} F_\varepsilon(x,t)\theta(x,t)\,dx\,dt &= \int_0^T\int_{\Omega_\varepsilon} F_\varepsilon(x,t)\theta(\bar{x},t)\,dx\,dt \\ &\quad + \int_0^T\int_{\Omega_\varepsilon} F_\varepsilon(x,t)\big(\theta(x,t)-\theta(\bar{x},t)\big)\,dx\,dt \\ &\le \int_0^T\int_{\Omega_\varepsilon} F_\varepsilon(x,t)\theta(\bar{x},t)\,dx\,dt + C\sigma(\varepsilon)\int_0^T\int_{\Omega_\varepsilon} |F_\varepsilon(x,t)|\,dx\,dt \to 0 \qquad \text{as } \varepsilon\to 0.\end{aligned}$$

Finally, the observation that $F_\varepsilon = O(\varepsilon^\alpha)$ in $\Omega\setminus\Omega_\varepsilon$ gives

$$\int_0^T\int_{\Omega\setminus\Omega_\varepsilon} F_\varepsilon(x,t)\theta(x,t)\,dx\,dt \to 0 \qquad \text{as } \varepsilon\to 0$$

and this ends the proof. □

Now we prove that u^ε is uniformly bounded when $G = G_2$.

Lemma 56. *Let $G = G_2$. There exists a constant C independent of ε such that*

$$\|u^\varepsilon\|_{L^\infty(\overline{\Omega}\times[0,T])} \le C.$$

PROOF. Again we will use a comparison argument. Let us look for a supersolution. Pick an auxiliary function v as a solution to

$$\begin{cases} v_t - \Delta v = h(x,t) & \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial v}{\partial \eta} = g_1(x,t) & \text{on} \quad \partial\Omega \times (0,T), \\ v(x,0) = v_1(x) & \text{in} \quad \Omega. \end{cases} \tag{5.12}$$

for some smooth functions $h(x,t) \geq 1$, $v_1(x) \geq u_0(x)$ and

$$g_1(x,t) \geq \frac{2}{K}(C_2+1) \max_{\partial\Omega\times[0,T]} |g(x,t)| + 1 \qquad (K \text{ as in (5.8)})$$

such that the resulting v has an extension $\tilde{v}$ that belongs to $C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$ and let M be an upper bound for v in $\bar{\Omega} \times [0,T]$. As before v is a solution to

$$\begin{cases} v_t - L_\varepsilon v = H(x,t,\varepsilon) & \text{in} \quad \Omega \times (0,T), \\ v(x,0) = v_1(x) & \text{in} \quad \Omega, \end{cases}$$

where H verifies

$$H(x,t,\varepsilon) \geq \Big(\frac{g_1(\bar{x},t)}{\varepsilon} \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\eta(\bar{x}) \cdot \frac{(y-x)}{\varepsilon}\, dy - D_1 \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy\Big) + \frac{1}{2}.$$

So that, by Lemma 50,

$$H(x,t,\varepsilon) \geq \big(\frac{g_1(\bar{x},t)\, K}{\varepsilon} - D_1\big) \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\, dy + \frac{1}{2}$$

for $\varepsilon < \bar{\varepsilon}$.

Let us recall that

$$\begin{aligned} F_\varepsilon(x,t) = \tilde{L}_\varepsilon(\tilde{u}) - \Delta\tilde{u} + \frac{C_2}{\varepsilon} \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y) g(y,t)\, dy \\ - \frac{1}{\varepsilon^2} \int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\big(\tilde{u}(y,t) - \tilde{u}(x,t)\big)\, dy. \end{aligned}$$

Then, proceeding once again as in Lemma 49 we have,

$$
\begin{aligned}
|F_\varepsilon(x,t)| &\le \frac{|g(\bar{x},t)|\,C_2}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy + \frac{|g(\bar{x},t)|}{\varepsilon}\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\Big|\eta(\bar{x})\cdot\frac{(y-x)}{\varepsilon}\Big|\,dy \\
&\quad + C\varepsilon^\alpha + C\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy \\
&\le \Big[\frac{(C_2+1)}{\varepsilon}\max_{\partial\Omega\times[0,T]}|g(x,t)| + C\Big]\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy + C\varepsilon^\alpha \\
&\le \Big(\frac{g_1(\bar{x},t)\,K}{2\varepsilon} + C\Big)\int_{\mathbb{R}^N\setminus\Omega} J_\varepsilon(x-y)\,dy + C\varepsilon^\alpha
\end{aligned}
$$

if $\varepsilon < \bar{\varepsilon}$, by our choice of g_1.

Therefore, for every ε small enough, we obtain

$$|F_\varepsilon(x,t)| \le H(x,t,\varepsilon),$$

and, by a comparison argument, we conclude that

$$-M \le -v(x,t) \le u^\varepsilon(x,t) \le v(x,t) \le M,$$

for every $(x,t)\in\overline{\Omega}\times[0,T]$. This ends the proof. □

Finally, we prove our last result, Theorem 54.

Proof of Theorem 54. By Lemma 55 we have that

$$F_\varepsilon(x,t) \rightharpoonup 0 \quad \text{as measures in} \quad \Omega\times[0,T]$$

as $\varepsilon\to 0$.

Assume first that $\psi\in C_0^{2+\alpha}(\Omega)$ and let $\tilde{\varphi}_\varepsilon$ be the solution to $w_t - L_\varepsilon w = 0$ with $w(x,0)=\psi(x)$.

Let $\tilde{\varphi}$ be a solution to

$$
\begin{cases}
\varphi_t - \Delta\varphi = 0\\
\dfrac{\partial\varphi}{\partial\eta} = 0\\
\varphi(x,0) = \psi(x).
\end{cases}
$$

Then, by Theorem 48 we know that $\tilde{\varphi}_\varepsilon \to \tilde{\varphi}$ uniformly in $\Omega\times[0,T]$.

For a fixed $t>0$ set $\varphi_\varepsilon(x,s) = \tilde{\varphi}_\varepsilon(x,t-s)$. Then φ_ε satisfies

$$
\begin{aligned}
&\varphi_s + L_\varepsilon\varphi = 0, \qquad \text{for } s<t,\\
&\varphi(x,t) = \psi(x).
\end{aligned}
$$

Analogously, set $\varphi(x,s) = \tilde{\varphi}(x, t-s)$. Then φ satisfies

$$\begin{cases} \varphi_t + \Delta\varphi = 0 \\ \dfrac{\partial \varphi}{\partial \eta} = 0 \\ \varphi(x,t) = \psi(x). \end{cases}$$

Then, for $w^\varepsilon = u^\varepsilon - u$ we have

$$\int_\Omega w^\varepsilon(x,t)\,\psi(x)\,dx = \int_0^t \int_\Omega \frac{\partial w^\varepsilon}{\partial s}(x,s)\,\varphi_\varepsilon(x,s)\,dx\,ds + \int_0^t \int_\Omega \frac{\partial \varphi_\varepsilon}{\partial s}(x,s)\,w^\varepsilon(x,s)\,dx\,ds$$

$$= \int_0^t \int_\Omega L_\varepsilon(w^\varepsilon)(x,s)\varphi_\varepsilon(x,s)\,dx\,ds + \int_0^t \int_\Omega F_\varepsilon(x,s)\,\varphi_\varepsilon(x,s)\,dx\,ds$$
$$+ \int_0^t \int_\Omega \frac{\partial \varphi_\varepsilon}{\partial s}(x,s)\,w_\varepsilon(x,s)\,dx\,ds$$
$$= \int_0^t \int_\Omega L_\varepsilon(\varphi_\varepsilon)(x,s)w^\varepsilon(x,s)\,dx\,ds + \int_0^t \int_\Omega F_\varepsilon(x,s)\,\varphi_\varepsilon(x,s)\,dx\,ds$$
$$+ \int_0^t \int_\Omega \frac{\partial \varphi_\varepsilon}{\partial s}(x,s)\,w^\varepsilon(x,s)\,dx\,ds$$
$$= \int_0^t \int_\Omega F_\varepsilon(x,s)\varphi_\varepsilon(x,s)\,dx\,ds.$$

Now we observe that, by the Lemma 55,

$$\left| \int_0^t \int_\Omega F_\varepsilon(x,s)\varphi_\varepsilon(x,s)\,dx\,ds \right| \le \left| \int_0^t \int_\Omega F_\varepsilon(x,s)\varphi(x,s)\,dx\,ds \right|$$
$$+ \sup_{0<s<t} \|\varphi_\varepsilon(x,s) - \varphi(x,s)\|_{L^\infty(\Omega)} \int_0^t \int_\Omega |F_\varepsilon(x,s)|\,dx\,ds \to 0$$

as $\varepsilon \to 0$. This proves the result when $\psi \in C_0^{2+\alpha}(\Omega)$.

Now we deal with the general case. Let $\psi \in L^1(\Omega)$. Choose $\psi_n \in C_0^{2+\alpha}(\Omega)$ such that $\psi_n \to \psi$ in $L^1(\Omega)$. We have

$$\left| \int_\Omega w^\varepsilon(x,t)\,\psi(x)\,dx \right| \le \left| \int_\Omega w^\varepsilon(x,t)\,\psi_n(x)\,dx \right| + \|\psi_n - \psi\|_{L^1(\Omega)} \|w^\varepsilon\|_{L^\infty(\Omega)}.$$

By Lemma 56, $\{w^\varepsilon\}$ is uniformly bounded, and hence the result follows. $\square$

CHAPTER 6

A linear Dirichlet problem with a rescale of the kernel

In this chapter we propose the following nonlocal nonhomogeneous "Dirichlet" boundary value problem: Given $g(x,t)$ defined for $x \in \mathbb{R}^N \setminus \Omega$ and $u_0(x)$ defined for $x \in \Omega$, find $u(x,t)$ such that

$$\begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t)-u(x,t))dy, & x \in \Omega,\ t>0, \\ u(x,t) = g(x,t), & x \notin \Omega,\ t>0, \\ u(x,0) = u_0(x), & x \in \Omega. \end{cases} \tag{6.1}$$

In this model we prescribe the values of u outside Ω which is the analogous of prescribing the so called Dirichlet boundary conditions for the classical heat equation. However, the boundary data is not understood in the usual sense as we will see in Remark 17 below. As explained before in this model the right hand side models the diffusion, the integral $\int J(x-y)(u(y,t)-u(x,t))\,dy$ takes into account the individuals arriving or leaving position $x \in \Omega$ from or to other places while we are prescribing the values of u outside the domain Ω by imposing $u = g$ for $x \notin \Omega$. When $g = 0$ we get that any individuals that leave Ω die, this is the case when Ω is surrounded by a hostile environment.

Existence and uniqueness of solutions of (6.1) is proved by a fixed point argument. Also a comparison principle is obtained.

0.16. Existence, uniqueness and a comparison principle.

Existence and uniqueness of solutions is a consequence of Banach's fixed point theorem. We look for $u \in C([0,\infty); L^1(\Omega))$ satisfying (6.1). Fix $t_0 > 0$ and consider the Banach space $X_{t_0} = \{w \in C([0,t_0]; L^1(\Omega))\}$ with the norm

$$|||w||| = \max_{0 \le t \le t_0} \|w(\cdot,t)\|_{L^1(\Omega)}.$$

We will obtain the solution as a fixed point of the operator $\mathcal{T} : X_{t_0} \to X_{t_0}$ defined by

$$\mathcal{T}_{w_0}(w)(x,t) = w_0(x) + \int_0^t \int_{\mathbb{R}^N} J\,(x-y)\,(w(y,s)-w(x,s))\,dy\,ds,$$

where we impose

$$w(x,t) = g(x,t), \qquad \text{for } x \notin \Omega.$$

LEMMA 57. *Let $w_0, z_0 \in L^1(\Omega)$, then there exists a constant C depending on J and Ω such that*

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le Ct_0|||w - z||| + ||w_0 - z_0||_{L^1(\Omega)}$$

for all $w, z \in X_{t_0}$.

PROOF. We have

$$\int_\Omega |\mathcal{T}_{w_0}(w)(x,t) - \mathcal{T}_{z_0}(z)(x,t)|\,dx \le \int_\Omega |w_0 - z_0|(x)\,dx$$
$$+ \int_\Omega \left| \int_0^t \int_{\mathbb{R}^N} J\,(x-y) \Big[(w(y,s) - z(y,s)) \right.$$
$$\left. -(w(x,s) - z(x,s)) \Big] \,dy\,ds \right| \,dx.$$

Hence, taking into account that w and z vanish outside Ω,

$$|||\mathcal{T}_{w_0}(w) - \mathcal{T}_{z_0}(z)||| \le ||w_0 - z_0||_{L^1(\Omega)} + Ct_0|||w - z|||,$$

as we wanted to prove. □

THEOREM 58. *For every $u_0 \in L^1(\Omega)$ there exists a unique solution u, such that $u \in C([0,\infty); L^1(\Omega))$.*

PROOF. We check first that $\mathcal{T}_{u_0}$ maps X_{t_0} into X_{t_0}. Taking $z_0 \equiv 0$ and $z \equiv 0$ in Lemma 57 we get that $\mathcal{T}_{u_0}(w) \in C([0,t_0]; L^1(\Omega))$ for any $w \in X_{t_0}$.

Choose t_0 such that $Ct_0 < 1$. Now taking $z_0 \equiv w_0 \equiv u_0$ in Lemma 57 we get that $\mathcal{T}_{u_0}$ is a strict contraction in X_{t_0} and the existence and uniqueness part of the theorem follows from Banach's fixed point theorem in the interval $[0,t_0]$. To extend the solution to $[0,\infty)$ we may take as initial data $u(x,t_0) \in L^1(\Omega)$ and obtain a solution up to $[0,2t_0]$. Iterating this procedure we get a solution defined in $[0,\infty)$. □

REMARK 59. Note that in general a solution u with $u_0 > 0$ and $g = 0$ is strictly positive in $\overline{\Omega}$ (with a positive continuous extension to $\overline{\Omega}$) and vanishes outside $\overline{\Omega}$. Therefore a discontinuity occurs on $\partial\Omega$ and the boundary value is not taken in the usual "classical" sense, see [**34**].

We now define what we understand by sub and supersolutions.

DEFINITION 60. *A function $u \in C([0,T); L^1((\Omega))$ is a supersolution of (6.1) if*

$$\begin{cases} u_t(x,t) \ge \displaystyle\int_{\mathbb{R}^N} J(x-y)(u(y,t) - u(x,t))dy, & x \in \Omega,\ t > 0, \\ u(x,t) \ge g(x,t), & x \notin \Omega,\ t > 0, \\ u(x,0) \ge u_0(x), & x \in \Omega. \end{cases} \tag{6.2}$$

As usual, subsolutions are defined analogously by reversing the inequalities.

LEMMA 61. *Let $u_0 \in C(\overline{\Omega})$, $u_0 \geq 0$, and $u \in C(\overline{\Omega} \times [0,T])$ a supersolution to* (6.1) *with $g \geq 0$. Then, $u \geq 0$.*

PROOF. Assume for contradiction that $u(x,t)$ is negative somewhere. Let $v(x,t) = u(x,t) + \varepsilon t$ with ε so small such that v is still negative somewhere. Then, if (x_0, t_0) is a point where v attains its negative minimum, there holds that $t_0 > 0$ and

$$
\begin{aligned}
v_t(x_0,t_0) = u_t(x_0,t_0) + \varepsilon &> \int_{\mathbb{R}^N} J(x-y)(u(y,t_0) - u(x_0,t_0))\, dy \\
&= \int_{\mathbb{R}^N} J(x-y)(v(y,t_0) - v(x_0,t_0))\, dy \geq 0
\end{aligned}
$$

which is a contradiction. Thus, $u \geq 0$. □

COROLLARY 62. *Let $J \in L^\infty(\mathbb{R}^N)$. Let u_0 and v_0 in $L^1(\Omega)$ with $u_0 \geq v_0$ and $g, h \in L^\infty((0,T); L^1(\mathbb{R}^N \setminus \Omega))$ with $g \geq h$. Let u be a solution of* (6.1) *with $u(x,0) = u_0$ and Dirichlet datum g and v be a solution of* (6.1) *with $v(x,0) = v_0$ and datum h. Then, $u \geq v$ a.e.*

PROOF. Let $w = u - v$. Then, w is a supersolution with initial datum $u_0 - v_0 \geq 0$ and datum $g - h \geq 0$. Using the continuity of solutions with respect to the data and the fact that $J \in L^\infty(\mathbb{R}^N)$, we may assume that $u, v \in C(\overline{\Omega} \times [0,T])$. By Lemma 61 we obtain that $w = u - v \geq 0$. So the corollary is proved. □

COROLLARY 63. *Let $u \in C(\overline{\Omega} \times [0,T])$ (resp. v) be a supersolution (resp. subsolution) of* (6.1)*. Then, $u \geq v$.*

PROOF. It follows the lines of the proof of the previous corollary. □

0.17. Convergence to the heat equation when rescaling the kernel.

Let us consider the classical Dirichlet problem for the heat equation,

$$
\begin{cases}
v_t(x,t) - \Delta v(x,t) = 0, & x \in \Omega,\ t > 0, \\
v(x,t) = g(x,t), & x \in \partial\Omega,\ t > 0, \\
v(x,0) = u_0(x), & x \in \Omega.
\end{cases}
\tag{6.3}
$$

The nonlocal Dirichlet model (6.1) and the classical Dirichlet problem (6.3) share many properties, among them the asymptotic behavior of their solutions as $t \to \infty$ is similar as was proved in [**34**].

The main goal now is to show that the Dirichlet problem for the heat equation (6.3) can be approximated by suitable nonlocal problems of the form of (6.1).

More precisely, for a given J and a given $\varepsilon > 0$ we consider the rescaled kernel

$$
J_\varepsilon(\xi) = C_1 \frac{1}{\varepsilon^N} J\left(\frac{\xi}{\varepsilon}\right), \quad \text{with} \quad C_1^{-1} = \frac{1}{2}\int_{B(0,d)} J(z) z_N^2\, dz. \tag{6.4}
$$

Here C_1 is a normalizing constant in order to obtain the Laplacian in the limit instead of a multiple of it. Let $u^\varepsilon(x,t)$ be the solution of

$$\begin{cases} u^\varepsilon_t(x,t) = \displaystyle\int_\Omega \frac{J_\varepsilon(x-y)}{\varepsilon^2}(u^\varepsilon(y,t)-u^\varepsilon(x,t))dy, & x\in\Omega, t>0,\\ u(x,t)=g(x,t), & x\notin\Omega, t>0,\\ u(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{6.5}$$

Our main result now reads as follows.

THEOREM 64. *Let Ω be a bounded $C^{2+\alpha}$ domain for some $0<\alpha<1$.*

Let $v\in C^{2+\alpha,1+\alpha/2}(\overline{\Omega}\times[0,T])$ be the solution to (6.3) and let u^ε be the solution to (6.5) with J_ε as above. Then, there exists $C=C(T)$ such that

$$\sup_{t\in[0,T]} \|v-u^\varepsilon\|_{L^\infty(\Omega)} \le C\varepsilon^\alpha \to 0, \qquad \text{as } \varepsilon\to 0. \tag{6.6}$$

Related results for the Neumann problem where presented in the previous chapter (see also [**43**]).

Note that the assumed regularity of v is a consequence of regularity assumptions on the boundary data g, the domain Ω and the initial condition u_0, see [**62**].

In order to prove Theorem 64 let $\tilde v$ be a $C^{2+\alpha,1+\alpha/2}$ extension of v to $\mathbb{R}^N\times[0,T]$.

Let us define the operator

$$\tilde L_\varepsilon(z) = \frac{1}{\varepsilon^2}\int_{\mathbb{R}^N} J_\varepsilon(x-y)\big(z(y,t)-z(x,t)\big)dy.$$

Then $\tilde v$ verifies

$$\begin{cases} \tilde v_t(x,t) = \tilde L_\varepsilon(\tilde v)(x,t) + F_\varepsilon(x,t) & x\in\Omega,\ (0,T],\\ \tilde v(x,t) = g(x,t)+G(x,t), & x\notin\Omega,\ (0,T],\\ \tilde v(x,0)=u_0(x), & x\in\Omega. \end{cases} \tag{6.7}$$

where, since $\Delta v = \Delta\tilde v$ in Ω,

$$F_\varepsilon(x,t) = -\tilde L_\varepsilon(\tilde v)(x,t) + \Delta\tilde v(x,t).$$

Moreover as G is smooth and $G(x,t)=0$ if $x\in\partial\Omega$ we have

$$G(x,t)=O(\varepsilon), \qquad \text{for } x \text{ such that } \operatorname{dist}(x,\partial\Omega)\le\varepsilon d.$$

We set $w^\varepsilon = \tilde v - u^\varepsilon$ and we note that

$$\begin{cases} w^\varepsilon_t(x,t) = \tilde L_\varepsilon(w^\varepsilon)(x,t) + F_\varepsilon(x,t) & x\in\Omega,\ (0,T],\\ w^\varepsilon(x,t) = G(x,t), & x\notin\Omega,\ (0,T],\\ w^\varepsilon(x,0)=0, & x\in\Omega. \end{cases} \tag{6.8}$$

First, we claim that, by the choice of C_1, the fact that J is radially symmetric and $\tilde{u} \in C^{2+\alpha,1+\alpha/2}(\mathbb{R}^N \times [0,T])$, we have that

$$\sup_{t\in[0,T]} \|F_\varepsilon\|_{L^\infty(\Omega)} = \sup_{t\in[0,T]} \|\Delta\tilde{v} - \tilde{L}_\varepsilon(\tilde{v})\|_{L^\infty(\Omega)} = O(\varepsilon^\alpha)\,. \tag{6.9}$$

In fact,

$$\Delta\tilde{v}(x,t) - \frac{C_1}{\varepsilon^{N+2}}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right)(\tilde{v}(y,t) - \tilde{v}(x,t))\,dy$$

becomes, under the change variables $z = (x-y)/\varepsilon$,

$$\Delta\tilde{v}(x,t) - \frac{C_1}{\varepsilon^2}\int_{\mathbb{R}^N} J(z)\,(\tilde{v}(x-\varepsilon z,t) - \tilde{v}(x,t))\,dz$$

and hence (6.9) follows by a simple Taylor expansion. This proves the claim.

We proceed now to prove Theorem 64.

Proof of Theorem 64. In order to prove the theorem by a comparisonwe first look for a supersolution. Let $\overline{w}$ be given by

$$\overline{w}(x,t) = K_1\varepsilon^\alpha t + K_2\varepsilon. \tag{6.10}$$

For $x \in \Omega$ we have, if K_1 is large,

$$\overline{w}_t(x,t) - \tilde{L}(\overline{w})(x,t) = K_1\varepsilon^\alpha \geq F_\varepsilon(x,t) = w_t^\varepsilon(x,t) - \tilde{L}_\varepsilon(w^\varepsilon)(x,t). \tag{6.11}$$

Since

$$G_\varepsilon(x,t) = O(\varepsilon) \qquad \text{for } x \text{ such that } \operatorname{dist}(x,\partial\Omega) \leq \varepsilon$$

choosing K_2 large, we obtain

$$\overline{w}(x,t) \geq w^\varepsilon(x,t) \tag{6.12}$$

for $x \notin \Omega$ such that $\operatorname{dist}(x,\partial\Omega) \leq \varepsilon d$ and $t \in [0,T]$. Moreover it is clear that

$$\overline{w}(x,0) = K_2\varepsilon > w^\varepsilon(x,0) = 0. \tag{6.13}$$

Thanks to (6.11), (6.12) and (6.13) we can apply the comparison result and conclude that

$$w^\varepsilon(x,t) \leq \overline{w}(x,t) = K_1\varepsilon^\alpha t + K_2\varepsilon. \tag{6.14}$$

In a similar fashion we prove that $\underline{w}(x,t) = -K_1\varepsilon^\alpha t - K_2\varepsilon$ is a subsolution and hence

$$w^\varepsilon(x,t) \geq \underline{w}(x,t) = -K_1\varepsilon^\alpha t - K_2\varepsilon. \tag{6.15}$$

Therefore

$$\sup_{t\in[0,T]} \|u - u^\varepsilon\|_{L^\infty(\Omega)} \leq C(T)\varepsilon^\alpha, \tag{6.16}$$

as we wanted to prove. □

CHAPTER 7

Scaling the kernel in a higher order problem

In this short chapter we show that the problem $v_t(x,t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x,t)$ can be approximated by nonlocal problems like the one presented in Chapter 4 when rescaled in an appropriate way.

Let us recall that we have considered

$$\left\{ \begin{array}{rl} u_t(x,t) = & (-1)^{n-1} \left(J * Id - 1\right)^n (u(x,t)) \\ = & (-1)^{n-1} \left(\displaystyle\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (J*)^k(u) \right) (x,t), \\ u(x,0) = & u_0(x), \end{array} \right. \tag{7.1}$$

for $x \in \mathbb{R}^N$ and $t > 0$, as a nonlocal higher order equation.

THEOREM 65. *Let u_ε be the unique solution to*

$$\left\{ \begin{array}{rl} (u_\varepsilon)_t(x,t) = & (-1)^{n-1} \dfrac{(J_\varepsilon * Id - 1))^n}{\varepsilon^{\alpha n}} (u_\varepsilon(x,t)), \\ u(x,0) = & u_0(x), \end{array} \right. \tag{7.2}$$

where $J_\varepsilon(s) = \varepsilon^{-N} J(\frac{s}{\varepsilon})$. Then, for every $T > 0$, we have

$$\lim_{\varepsilon \to 0} \|u_\varepsilon - v\|_{L^\infty(\mathbb{R}^N \times (0,T))} = 0,$$

where v is the solution to the local problem $v_t(x,t) = -A^n(-\Delta)^{\frac{\alpha n}{2}} v(x,t)$ with the same initial condition $v(x,0) = u_0(x)$.

PROOF OF THEOREM 65. The proof uses once more the explicit formula for the solutions in Fourier variables. We have, arguing exactly as before,

$$\hat{u}_\varepsilon(\xi,t) = e^{(-1)^{n-1} \frac{(\widehat{J_\varepsilon}(\xi) - 1)^n}{\varepsilon^{\alpha n}} t} \hat{u}_0(\xi).$$

and

$$\hat{v}(\xi,t) = e^{-A^n |\xi|^{\alpha n} t} \hat{u}_0(\xi).$$

Now, we just observe that $\widehat{J}_\varepsilon(\xi) = \hat{J}(\varepsilon\xi)$ and therefore we obtain

$$\int_{\mathbb{R}^N} |\hat{u_\varepsilon} - \hat{v}|\,(\xi,t)\,d\xi = \int_{\mathbb{R}^N} \left|(e^{(-1)^{n-1}\frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{\alpha n}}t} - e^{-A^n|\xi|^{\alpha n}t})\hat{u_0}(\xi)\right|\,d\xi$$
$$\leq \|\hat{u_0}\|_{L^\infty(\mathbb{R}^N)}\left(\int_{|\xi|\geq r(\varepsilon)} \left|e^{(-1)^{n-1}\frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{\alpha n}}t} - e^{-A^n|\xi|^{\alpha n}t}\right|\,d\xi\right.$$
$$\left. + \int_{|\xi|<r(\varepsilon)} \left|e^{(-1)^{n-1}\frac{(\hat{J}(\varepsilon\xi)-1)^n}{\varepsilon^{\alpha n}}t} - e^{-A^n|\xi|^{\alpha n}t}\right|\,d\xi\right).$$

For $t \in [0,T]$ we can proceed as in the proof of Theorem 37 (Section 0.8) to obtain that

$$\max_x |u_\varepsilon(x,t) - v(x,t)| \leq \int_{\mathbb{R}^N} |\hat{u_\varepsilon} - \hat{v}|\,(\xi,t)\,d\xi \to 0, \ \ \varepsilon \to 0.$$

We leave the details to the reader. □

CHAPTER 8

A Non-local convection diffusion equation

In this chapter we analyze a nonlocal equation that takes into account convective and diffusive effects. We deal with the nonlocal evolution equation

$$(8.1)\qquad \begin{cases} u_t(x,t) = (J*u-u)\,(x,t) + (G*(f(u)) - f(u))\,(x,t), & t>0,\ x\in\mathbb{R}^N,\\ u(0,x) = u_0(x), & x\in\mathbb{R}^N. \end{cases}$$

In this paper we analyze a nonlocal equation that takes into account convective and diffusive effects. We deal with the nonlocal evolution equation

$$(8.2)\qquad \begin{cases} u_t(x,t) = (J*u-u)\,(x,t) + (G*(f(u)) - f(u))\,(x,t), & t>0,\ x\in\mathbb{R}^d,\\ u(0,x) = u_0(x), & x\in\mathbb{R}^d. \end{cases}$$

Let us state first our basic assumptions. The functions J and G are nonnegatives and verify $\int_{\mathbb{R}^d} J(x)dx = \int_{\mathbb{R}^d} G(x)dx = 1$. Moreover, we consider J smooth, $J\in\mathcal{S}(\mathbb{R}^d)$, the space of rapidly decreasing functions, and radially symmetric and G smooth, $G\in\mathcal{S}(\mathbb{R}^d)$, but not necessarily symmetric. To obtain a diffusion operator similar to the Laplacian we impose in addition that J verifies

$$\frac{1}{2}\partial^2_{\xi_i\xi_i}\widehat{J}(0) = \frac{1}{2}\int_{supp(J)} J(z) z_i^2\,dz = 1.$$

This implies that

$$\widehat{J}(\xi) - 1 + \xi^2 \sim |\xi|^3, \qquad \text{for } \xi \text{ close to } 0.$$

Here $\widehat{J}$ is the Fourier transform of J and the notation $A\sim B$ means that there exist constants C_1 and C_2 such that $C_1A\le B\le C_2A$. We can consider more general kernels J with expansions in Fourier variables of the form $\widehat{J}(\xi) - 1 + A\,\xi^2 \sim |\xi|^3$. Since the results (and the proofs) are almost the same, we do not include the details for this more general case, but we comment on how the results are modified by the appearance of A.

The nonlinearity f will de assumed nondecreasing with $f(0)=0$ and locally Lipschitz continuous (a typical example that we will consider below is $f(u) = |u|^{q-1}u$ with $q>1$).

In our case, see the equation in (8.2), we have a diffusion operator $J*u-u$ and a nonlinear convective part given by $G*(f(u)) - f(u)$. Concerning this last term, if G is not symmetric then individuals have greater probability of jumping in one direction than in others, provoking a convective effect.

We will call equation (8.2), a *nonlocal convection-diffusion equation.* It is nonlocal since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$ and its derivatives at that point (x,t), but on all the values of u in a fixed spatial neighborhood of x through the convolution terms $J * u$ and $G * (f(u))$ (this neighborhood depends on the supports of J and G).

First, we prove existence, uniqueness and well-possedness of a solution with an initial condition $u(0,x) = u_0(x) \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

THEOREM 66. *For any $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ there exists a unique global solution*

$$u \in C([0,\infty); L^1(\mathbb{R}^d)) \cap L^\infty([0,\infty); \mathbb{R}^d).$$

If u and v are solutions of (8.2) corresponding to initial data $u_0, v_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ respectively, then the following contraction property

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} \leq \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}$$

holds for any $t \geq 0$. In addition,

$$\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq \|u_0\|_{L^\infty(\mathbb{R}^d)}.$$

We have to emphasize that a lack of regularizing effect occurs. This has been already observed in [**34**] for the linear problem $w_t = J * w - w$. In [**54**], the authors prove that the solutions to the local convection-diffusion problem, $u_t = \Delta u + b \cdot \nabla f(u)$, satisfy an estimate of the form $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \leq C(\|u_0\|_{L^1(\mathbb{R}^d)})\, t^{-d/2}$ for any initial data $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. In our nonlocal model, we cannot prove such type of inequality independently of the $L^\infty(\mathbb{R}^d)$-norm of the initial data. Moreover, in the one-dimensional case with a suitable bound on the nonlinearity that appears in the convective part, f, we can prove that such an inequality does not hold in general, see Chapter 3. In addition, the $L^1(\mathbb{R}^d) - L^\infty(\mathbb{R}^d)$ regularizing effect is not available for the linear equation, $w_t = J * w - w$.

When J is nonnegative and compactly supported, the equation $w_t = J * w - w$ shares many properties with the classical heat equation, $w_t = \Delta w$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed, see [**59**]. However, there is no regularizing effect in general. Moreover, in [**42**] and [**43**] nonlocal Neumann boundary conditions where taken into account. It is proved there that solutions of the nonlocal problems converge to solutions of the heat equation with Neumann boundary conditions when a rescaling parameter goes to zero.

Concerning (8.2) we can obtain a solution to a standard convection-diffusion equation

$$v_t(x,t) = \Delta v(x,t) + b \cdot \nabla f(v)(x,t), \qquad t > 0,\ x \in \mathbb{R}^d, \tag{8.3}$$

as the limit of solutions to (8.2) when a scaling parameter goes to zero. In fact, let us consider

$$J_\varepsilon(s) = \frac{1}{\varepsilon^d} J\left(\frac{s}{\varepsilon}\right), \qquad G_\varepsilon(s) = \frac{1}{\varepsilon^d} G\left(\frac{s}{\varepsilon}\right),$$

and the solution $u_\varepsilon(x,t)$ to our convection-diffusion problem rescaled adequately,

$$(8.4)\qquad \begin{cases} (u_\varepsilon)_t(x,t) &= \dfrac{1}{\varepsilon^2}\displaystyle\int_{\mathbb{R}^d} J_\varepsilon(x-y)(u_\varepsilon(y,t)-u_\varepsilon(x,t))\,dy \\ & \qquad +\dfrac{1}{\varepsilon}\displaystyle\int_{\mathbb{R}^d} G_\varepsilon(x-y)(f(u_\varepsilon(y,t))-f(u_\varepsilon(x,t)))\,dy, \\ u_\varepsilon(x,0) &= u_0(x). \end{cases}$$

Remark that the scaling is different for the diffusive part of the equation $J*u-u$ and for the convective part $G*f(u)-f(u)$. The same different scaling properties can be observed for the local terms Δu and $b\cdot\nabla f(u)$.

THEOREM 67. *With the above notations, for any $T>0$, we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|u_\varepsilon - v\|_{L^2(\mathbb{R}^d)} = 0,$$

where $v(x,t)$ is the unique solution to the local convection-diffusion problem (8.3) *with initial condition $v(x,0)=u_0(x)\in L^1(\mathbb{R}^d)\cap L^\infty(\mathbb{R}^d)$ and $b=(b_1,...,b_d)$ given by*

$$b_j = \int_{\mathbb{R}^d} x_j\, G(x)\,dx, \qquad j=1,...,d.$$

This result justifies the use of the name "nonlocal convection-diffusion problem" when we refer to (8.2).

From our hypotheses on J and G it follows that they verify $|\widehat{G}(\xi)-1-ib\cdot\xi|\le C|\xi|^2$ and $|\widehat{J}(\xi)-1+\xi^2|\le C|\xi|^3$ for every $\xi\in\mathbb{R}^d$. These bounds are exactly what we are using in the proof of this convergence result.

Remark that when G is symmetric then $b=0$ and we obtain the heat equation in the limit. Of course the most interesting case is when $b\neq 0$ (this happens when G is not symmetric). Also we note that the conclusion of the theorem holds for other $L^p(\mathbb{R}^d)$-norms besides $L^2(\mathbb{R}^d)$, however the proof is more involved.

We can consider kernels J such that

$$A = \frac{1}{2}\int_{supp(J)} J(z)z_i^2\,dz \neq 1.$$

This gives the expansion $\widehat{J}(\xi)-1+A\xi^2\sim|\xi|^3$, for ξ close to 0. In this case we will arrive to a convection-diffusion equation with a multiple of the Laplacian as the diffusion operator, $v_t = A\Delta v + b\cdot\nabla f(v)$.

Next, we want to study the asymptotic behaviour as $t\to\infty$ of solutions to (8.2). To this end we first analyze the decay of solutions taking into account only the diffusive part (the linear part) of the equation. These solutions have a similar decay rate as the one that holds for the heat equation, see [**34**] and [**68**] where the Fourier transform play a key role. Using similar techniques we can prove the following result that deals with this asymptotic decay rate.

THEOREM 68. *Let $p \in [1,\infty]$. For any $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$ the solution $w(x,t)$ of the linear problem*

$$\begin{cases} w_t(x,t) = (J * w - w)(x,t), & t > 0,\ x \in \mathbb{R}^d, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \tag{8.5}$$

satisfies the decay estimate

$$\|w(t)\|_{L^p(\mathbb{R}^d)} \le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\, \langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})}.$$

Throughout this paper we will use the notation $A \le \langle t\rangle^{-\alpha}$ to denote $A \le (1+t)^{-\alpha}$.

Now we are ready to face the study of the asymptotic behaviour of the complete problem (8.2). To this end we have to impose some grow condition on f. Therefore, in the sequel we restrict ourselves to nonlinearities f that are pure powers

$$f(u) = |u|^{q-1}u \tag{8.6}$$

with $q > 1$.

The analysis is more involved than the one performed for the linear part and we cannot use here the Fourier transform directly (of course, by the presence of the nonlinear term). Our strategy is to write a variation of constants formula for the solution and then prove estimates that say that the nonlinear part decay faster than the linear one. For the local convection diffusion equation this analysis was performed by Escobedo and Zuazua in [**54**]. However, in the previously mentioned reference energy estimates were used together with Sobolev inequalities to obtain decay bounds. These Sobolev inequalities are not available for the nonlocal model, since the linear part does not have any regularizing effect, see Remark 87. Therefore, we have to avoid the use of energy estimates and tackle the problem using a variant of the Fourier splitting method proposed by Schonbek to deal with local problems, see [**78**], [**79**] and [**80**].

We state our result concerning the asymptotic behaviour (decay rate) of the complete nonlocal model as follows:

THEOREM 69. *Let f satisfies (8.6) with $q > 1$ and $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. Then, for every $p \in [1,\infty)$ the solution u of equation (8.2) verifies*

$$\|u(t)\|_{L^p(\mathbb{R}^d)} \le C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)})\, \langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})}. \tag{8.7}$$

Finally, we look at the first order term in the asymptotic expansion of the solution. For $q > (d+1)/d$, we find that this leading order term is the same as the one that appears in the linear local heat equation. This is due to the fact that the nonlinear convection is of higher order and that the radially symmetric diffusion leads to gaussian kernels in the asymptotic regime, see [**34**] and [**68**].

THEOREM 70. *Let f satisfies* (8.6) *with $q > (d+1)/d$ and let the initial condition u_0 belongs to $L^1(\mathbb{R}^d, 1+|x|) \cap L^\infty(\mathbb{R}^d)$. For any $p \in [2,\infty)$ the following holds*

$$t^{-\frac{d}{2}(1-\frac{1}{p})} \|u(t) - MH(t)\|_{L^p(\mathbb{R}^d)} \leq C(J,G,p,d)\, \alpha_q(t),$$

where

$$M = \int_{\mathbb{R}^d} u_0(x)\, dx,$$

$H(t)$ is the Gaussian,

$$H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}},$$

and

$$\alpha_q(t) = \begin{cases} \langle t \rangle^{-\frac{1}{2}} & \text{if } q \geq (d+2)/d, \\ \langle t \rangle^{\frac{1-d(q-1)}{2}} & \text{if } (d+1)/d < q < (d+2)/d. \end{cases}$$

Remark that we prove a weak nonlinear behaviour, in fact the decay rate and the first order term in the expansion are the same that appear in the linear model $w_t = J * w - w$, see [**68**].

As before, recall that our hypotheses on J imply that $\widehat{J}(\xi) - (1 - |\xi|^2) \sim B|\xi|^3$, for ξ close to 0. This is the key property of J used in the proof of Theorem 70. We note that when we have an expansion of the form $\widehat{J}(\xi) - (1 - A|\xi|^2) \sim B|\xi|^3$, for $\xi \sim 0$, we get as first order term a Gaussian profile of the form $H_A(t) = H(At)$.

Also note that $q = (d+1)/d$ is a critical exponent for the local convection-diffusion problem, $v_t = \Delta v + b \cdot \nabla(v^q)$, see [**54**]. When q is supercritical, $q > (d+1)/d$, for the local equation it also holds an asymptotic simplification to the heat semigroup as $t \to \infty$.

The first order term in the asymptotic behaviour for critical or subcritical exponents $1 < q \leq (d+1)/d$ is left open. One of the main difficulties that one has to face here is the absence of a self-similar profile due to the inhomogeneous behaviour of the convolution kernels.

0.18. The linear semigroup. In this chapter we analyze the asymptotic behavior of the solutions of the equation

$$\begin{cases} w_t(x,t) = (J * w - w)(x,t), & t > 0,\ x \in \mathbb{R}^d, \\ w(0,x) = u_0(x), & x \in \mathbb{R}^d. \end{cases} \tag{8.8}$$

As we have mentioned in the introduction, when J is nonnegative and compactly supported, this equation shares many properties with the classical heat equation, $w_t = \Delta w$, such as: bounded stationary solutions are constant, a maximum principle holds for both of them and perturbations propagate with infinite speed, see [**59**]. However, there is no regularizing effect in general. In fact, the singularity of the source solution, that is a solution to (8.8) with initial condition a delta measure, $u_0 = \delta_0$, remains with an exponential decay. In fact, this fundamental solution can be decomposed as $S(x,t) = e^{-t}\delta_0 + K_t(x)$ where $K_t(x)$ is smooth, see Lemma 5. In this way we see that there is no regularizing effect since the solution w of (8.8) can be written as $w(t) = S(t) * u_0 = e^{-t}u_0 + K_t * u_0$ with K_t smooth, which means that $w(\cdot,t)$ is as regular as u_0 is. This fact makes the analysis of (8.8) more involved.

LEMMA 71. *The fundamental solution of* (8.8), *that is the solution of* (8.8) *with initial condition* $u_0 = \delta_0$, *can be decomposed as*

$$S(x,t) = e^{-t}\delta_0(x) + K_t(x), \tag{8.9}$$

with $K_t(x) = K(x,t)$ *smooth. Moreover, if* u *is the solution of* (8.8) *it can be written as*

$$w(x,t) = (S * u_0)(x,t) = \int_{\mathbb{R}} S(x, t-y) u_0(y)\, dy.$$

PROOF. Applying the Fourier transform to (8.8) we obtain that

$$\widehat{w}_t(\xi,t) = \widehat{w}(\xi,t)(\widehat{J}(\xi) - 1).$$

Hence, as the initial datum verifies $\widehat{u_0} = \widehat{\delta_0} = 1$,

$$\widehat{w}(\xi,t) = e^{(\widehat{J}(\xi)-1)t} = e^{-t} + e^{-t}(e^{\widehat{J}(\xi)t} - 1).$$

The first part of the lemma follows applying the inverse Fourier transform.

To finish the proof we just observe that $S * u_0$ is a solution of (8.8) (just use Fubini's theorem) with $(S * u_0)(0,x) = u_0(x)$. ☐

In the following we will give estimates on the regular part of the fundamental solution K_t defined by:

$$K_t(x) = \int_{\mathbb{R}^d} (e^{t(\widehat{J}(\xi)-1)} - e^{-t})\, e^{ix\cdot\xi}\, d\xi, \tag{8.10}$$

that is, in the Fourier space,

$$\widehat{K}_t(\xi) = e^{t(\widehat{J}(\xi)-1)} - e^{-t}.$$

The behavior of $L^p(\mathbb{R}^d)$-norms of K_t will be obtained by analyzing the cases $p = \infty$ and $p = 1$. The case $p = \infty$ follows by Hausdorff-Young's inequality. The case $p = 1$ follows by using the fact that the $L^1(\mathbb{R}^d)$-norm of the solutions to (8.8) does not increase.

The analysis of the behaviour of the gradient ∇K_t is more involved. The behavior of $L^p(\mathbb{R}^d)$-norms with $2 \le p \le \infty$ follows by Hausdorff-Young's inequality in the case $p = \infty$ and Plancherel's identity for $p = 2$. However, the case $1 \le p < 2$ is more tricky. In order to evaluate the $L^1(\mathbb{R}^d)$-norm of ∇K_t we will use the Carlson inequality (see for instance [**25**])

$$\|\varphi\|_{L^1(\mathbb{R}^d)} \le C\, \|\varphi\|^{1-\frac{d}{2m}}_{L^2(\mathbb{R}^d)} \||x|^m \varphi\|^{\frac{d}{2m}}_{L^2(\mathbb{R})}, \tag{8.11}$$

which holds for $m > d/2$. The use of the above inequality with $\varphi = \nabla K_t$ imposes that $|x|^m \nabla K_t$ belongs to $L^2(\mathbb{R}^d)$. To guarantee this property and to obtain the decay rate for the $L^2(\mathbb{R}^d)$-norm of $|x|^m \nabla K_t$ we will use in Lemma 74 that $J \in \mathcal{S}(\mathbb{R}^d)$.

The following lemma gives us the decay rate of the $L^p(\mathbb{R}^d)$-norms of the kernel K_t for $1 \leq p \leq \infty$.

LEMMA 72. *Let J be such that $\widehat{J}(\xi) \in L^1(\mathbb{R}^d)$, $\partial_\xi \widehat{J}(\xi) \in L^2(\mathbb{R}^d)$ and*

$$\widehat{J}(\xi) - 1 + \xi^2 \sim |\xi|^3, \qquad \partial_\xi \widehat{J}(\xi) \sim -\xi \qquad \text{as } \xi \sim 0.$$

For any $p \geq 1$ there exists a positive constant $c(p, J)$ such that K_t, defined in (8.10), satisfies:

$$\|K_t\|_{L^p(\mathbb{R}^d)} \leq c(p, J) \, \langle t \rangle^{-\frac{d}{2}(1-\frac{1}{p})} \tag{8.12}$$

for any $t > 0$.

REMARK 73. *In fact, when $p = \infty$, a stronger inequality can be proven,*

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \leq Cte^{-\delta t} \|\widehat{J}\|_{L^1(\mathbb{R}^d)} + C \, \langle t \rangle^{-d/2},$$

for some positive $\delta = \delta(J)$.

Moreover, for $p = 1$ we have,

$$\|K_t\|_{L^1(\mathbb{R}^d)} \leq 2$$

and for any $p \in [1, \infty]$

$$\|S(t)\|_{L^p(\mathbb{R}^d) - L^p(\mathbb{R}^d)} \leq 3.$$

PROOF OF LEMMA 72. We analyze the cases $p = \infty$ and $p = 1$, the others can be easily obtained applying Hölder's inequality.

Case $p = \infty$. Using Hausdorff-Young's inequality we obtain that

$$\|K_t\|_{L^\infty(\mathbb{R}^d)} \leq \int_{\mathbb{R}^d} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}| d\xi.$$

Observe that the symmetry of J guarantees that $\widehat{J}$ is a real number. Let us choose $R > 0$ such that

$$|\widehat{J}(\xi)| \leq 1 - \frac{|\xi|^2}{2} \text{ for all } |\xi| \leq R. \tag{8.13}$$

Once R is fixed, there exists $\delta = \delta(J)$, $0 < \delta < 1$, with

$$|\widehat{J}(\xi)| \leq 1 - \delta \text{ for all } |\xi| \geq R. \tag{8.14}$$

Using that for any real numbers a and b the following inequality holds:

$$|e^a - e^b| \leq |a - b| \max\{e^a, e^b\}$$

we obtain for any $|\xi| \geq R$,

$$|e^{t(\widehat{J}(\xi)-1)} - e^{-t}| \leq t|\widehat{J}(\xi)| \max\{e^{-t}, e^{t(\widehat{J}(\xi)-1)}\} \leq te^{-\delta t}|\widehat{J}(\xi)|. \tag{8.15}$$

Then the following integral decays exponentially,

$$\int_{|\xi|\geq R} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}| d\xi \leq e^{-\delta t} t \int_{|\xi|\geq R} |\widehat{J}(\xi)| d\xi.$$

Using that this term is exponentially small, it remains to prove that

$$I(t) = \int_{|\xi|\leq R} |e^{t(\widehat{J}(\xi)-1)} - e^{-t}| d\xi \leq C\langle t\rangle^{-d/2}. \tag{8.16}$$

To handle this case we use the following estimates:

$$|I(t)| \leq \int_{|\xi|\leq R} e^{t(\widehat{J}(\xi)-1)} d\xi + e^{-t}C(R) \leq \int_{|\xi|\leq R} d\xi + e^{-t}C(R) \leq C(R)$$

and

$$\begin{aligned} |I(t)| &\leq \int_{|\xi|\leq R} e^{t(\widehat{J}(\xi)-1)} d\xi + e^{-t}C(R) \leq \int_{|\xi|\leq R} e^{-\frac{t|\xi|^2}{2}} + e^{-t}C(R) \\ &= t^{-d/2} \int_{|\eta|\leq Rt^{1/2}} e^{-\frac{|\eta|^2}{2}} + e^{-t}C(R) \leq Ct^{-d/2}. \end{aligned}$$

The last two estimates prove (8.16) and this finishes the analysis of this case.

Case $p = 1$**.** First we prove that the $L^1(\mathbb{R}^d)$-norm of the solutions to equation (8.5) does not increase. Multiplying equation (8.5) by $sgn(w(x,t))$ and integrating in space variable we obtain,

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^d} |w(x,t)|\, dx = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)w(y,t)sgn(w(x,t))\, dy\, ds - \int_{\mathbb{R}^d} |w(t,x)|\, dx \\ &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)|w(y,t)|\, dx\, dy - \int_{\mathbb{R}^d} |w(x,t)|\, dx \leq 0, \end{aligned}$$

which shows that the $L^1(\mathbb{R}^d)$-norm does not increase. Hence, for any $u_0 \in L^1(\mathbb{R}^d)$, the following holds:

$$\int_{\mathbb{R}^d} |e^{-t}u_0(x) + (K_t * u_0)(x)|\, dx \leq \int_{\mathbb{R}^d} |u_0(x)|\, dx,$$

and as a consequence,

$$\int_{\mathbb{R}^d} |(K_t * u_0)(x)|\, dx \leq 2\int_{\mathbb{R}^d} |u_0(x)|\, dx.$$

Choosing $(u_0)_n \in L^1(\mathbb{R}^d)$ such that $(u_0)_n \to \delta_0$ in $\mathcal{S}'(\mathbb{R}^d)$ we obtain in the limit that

$$\int_{\mathbb{R}^d} |K_t(x)| dx \leq 2.$$

This ends the proof of the L^1-case and finishes the proof. ☐

The following lemma will play a key role when analyzing the decay of the complete problem (8.2). In the sequel we will denote by $L^1(\mathbb{R}^d, a(x))$ the following space:

$$L^1(\mathbb{R}^d, a(x)) = \left\{ \varphi : \int_{\mathbb{R}^d} a(x)|\varphi(x)|dx < \infty \right\}.$$

LEMMA 74. *Let $p \geq 1$ and $J \in \mathcal{S}(\mathbb{R}^d)$. There exists a positive constant $c(p, J)$ such that*

$$\|K_t * \varphi - K_t\|_{L^p(\mathbb{R}^d)} \leq c(p)\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}} \|\varphi\|_{L^1(\mathbb{R}^d,|x|)}$$

holds for all $\varphi \in L^1(\mathbb{R}^d, 1 + |x|)$.

PROOF. Explicit computations shows that

$$\begin{aligned}
(K_t * \varphi - K_t)(x) &= \int_{\mathbb{R}^d} K_t(x-y)\varphi(y)dy - \int_{\mathbb{R}^d} K_t(x)\, dx \\
&= \int_{\mathbb{R}^d} \varphi(y)(K_t(x-y) - K_t(x))\, dy \\
&= \int_{\mathbb{R}^d} \varphi(y) \int_0^1 \nabla K_t(x - sy) \cdot (-y)\, ds\, dy. \qquad (8.17)
\end{aligned}$$

We will analyze the cases $p = 1$ and $p = \infty$, the others cases follow by interpolation.

For $p = \infty$ we have,

$$\|K_t * \varphi - K_t\|_{L^\infty(\mathbb{R}^d)} \leq \|\nabla K_t\|_{L^\infty(\mathbb{R}^d)} \int_{\mathbb{R}^d} |y||\varphi(y)|\, dy. \qquad (8.18)$$

In the case $p = 1$, by using (8.17) the following holds:

$$\begin{aligned}
\int_{\mathbb{R}^d} |(K_t * \varphi - K_t)(x)|\, dx &\leq \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |y||\varphi(y)| \int_0^1 |\nabla K_t(x - sy)|\, ds\, dy\, dx \\
&= \int_{\mathbb{R}^d} |y||\varphi(y)| \int_0^1 \int_{\mathbb{R}^d} |\nabla K_t(x - sy)|\, dx\, ds\, dy \\
&= \int_{\mathbb{R}^d} |y||\varphi(y)|\, dy \int_{\mathbb{R}^d} |\nabla K_t(x)|\, dx. \qquad (8.19)
\end{aligned}$$

In view of (8.18) and (8.19) it is sufficient to prove that

$$\|\nabla K_t\|_{L^\infty(\mathbb{R}^d)} \leq C\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}}$$

and

$$\|\nabla K_t\|_{L^1(\mathbb{R}^d)} \leq C\langle t\rangle^{-\frac{1}{2}}.$$

In the first case, with R and δ as in (8.13) and (8.14), by Hausdorff-Young's inequality and (8.15) we obtain:

$$\begin{aligned}
\|\nabla K_t\|_{L^\infty(\mathbb{R}^d)} &\leq \int_{\mathbb{R}^d} |\xi||e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi \\
&= \int_{|\xi|\leq R} |\xi||e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi + \int_{|\xi|\geq R} |\xi||e^{t(\widehat{J}(\xi)-1)} - e^{-t}|d\xi \\
&\leq \int_{|\xi|\leq R} |\xi|e^{-t|\xi|^2/2}d\xi + e^{-t}\int_{|\xi|\leq R} |\xi|d\xi + t\int_{|\xi|\geq R} |\xi||\widehat{J}(\xi)|e^{-\delta t}d\xi \\
&\leq C(R)\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}} + C(R)e^{-t} + C(J)\, t\, e^{-\delta t} \\
&\leq C(J)\,\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}},
\end{aligned}$$

provided that $|\xi|\widehat{J}(\xi)$ belongs to $L^1(\mathbb{R}^d)$.

In the second case it is enough to prove that the $L^1(\mathbb{R}^d)$-norm of $\partial_{x_1} K_t$ is controlled by $\langle t\rangle^{-1/2}$. In this case Carlson's inequality gives us

$$\|\partial_{x_1} K_t\|_{L^1(\mathbb{R}^d)} \leq C\,\|\partial_{x_1} K_t\|_{L^2(\mathbb{R}^d)}^{1-\frac{d}{2m}} \||x|^m \partial_{x_1} K_t\|_{L^2(\mathbb{R}^d)}^{\frac{d}{2m}},$$

for any $m > d/2$.

Now our aim is to prove that, for any $t > 0$, we have

$$\|\partial_{x_1} K_t\|_{L^2(\mathbb{R}^d)} \leq C(J)\langle t\rangle^{-\frac{d}{4}-\frac{1}{2}} \tag{8.20}$$

and

$$\||x|^m \partial_{x_1} K_t\|_{L^2(\mathbb{R}^d)} \leq C(J)\langle t\rangle^{\frac{m-1}{2}-\frac{d}{4}}. \tag{8.21}$$

By Plancherel's identity, estimate (8.15) and using that $|\xi|\widehat{J}(\xi)$ belongs to $L^2(\mathbb{R}^d)$ we obtain

$$\begin{aligned}
\|\partial_{x_1} K_t\|_{L^2(\mathbb{R}^d)}^2 &= \int_{\mathbb{R}^d} |\xi_1|^2 |e^{t(\widehat{J}(\xi)-1)} - e^{-t}|^2\, d\xi \\
&\leq 2\int_{|\xi|\leq R} |\xi_1|^2 e^{-t|\xi|^2} d\xi + e^{-2t}\int_{|\xi|\leq R} |\xi_1|^2 d\xi + \int_{|\xi|\geq R} |\xi_1|^2 e^{-2\delta t} t^2 |\widehat{J}(\xi)|^2 d\xi \\
&\leq C(R)\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}} + C(R)e^{-2t} + C(J)e^{-2\delta t}t^2 \\
&\leq C(J)\,\langle t\rangle^{-\frac{d}{2}-\frac{1}{2}}.
\end{aligned}$$

This shows (8.20).

To prove (8.21), observe that

$$\||x|^m \partial_{x_1} K_t\|_{L^2(\mathbb{R}^d)}^2 \leq c(d)\int_{\mathbb{R}^d} (x_1^{2m} + \cdots + x_d^{2m})|\partial_{x_1} K_t(x)|^2 dx.$$

Thus, by symmetry it is sufficient to prove that

$$\int_{\mathbb{R}^d} |\partial^m_{\xi_1}(\xi_1 \widehat{K}_t(\xi))|^2 d\xi \leq C(J)\, \langle t\rangle^{m-1-\frac{d}{2}}$$

and

$$\int_{\mathbb{R}^d} |\partial^m_{\xi_2}(\xi_1 \widehat{K}_t(\xi))|^2 d\xi \leq C(J)\, \langle t\rangle^{m-1-\frac{d}{2}}.$$

Observe that

$$|\partial^m_{\xi_1}(\xi_1 \widehat{K}_t(\xi))| = |\xi_1 \partial^m_{\xi_1}\widehat{K}_t(\xi) + m\partial^{m-1}_{\xi_1}\widehat{K}_t(\xi)| \leq |\xi||\partial^m_{\xi_1}\widehat{K}_t(\xi)| + m|\partial^{m-1}_{\xi_1}\widehat{K}_t(\xi)|$$

and

$$|\partial^m_{\xi_2}(\xi_1 \widehat{K}_t)| \leq |\xi||\partial^m_{\xi_2}\widehat{K}_t(\xi)|.$$

Hence we just have to prove that

$$\int_{\mathbb{R}^d} |\xi|^{2r}|\partial^n_{\xi_1}\widehat{K}_t(\xi)|^2 d\xi \leq C(J)\, \langle t\rangle^{n-r-\frac{d}{2}}, \qquad (r,n) \in \{(0,m-1),\,(1,m)\}\,.$$

Choosing $m = [d/2]+1$ (the notation $[\cdot]$ stands for the floor function) the above inequality has to hold for $n = [d/2], [d/2]+1$.

First we recall the following elementary identity

$$\partial^n_{\xi_1}(e^g) = e^g \sum_{i_1+2i_2+\ldots+ni_n=n} a_{i_1,\ldots,i_n}(\partial^1_{\xi_1}g)^{i_1}(\partial^2_{\xi_1}g)^{i_2}\ldots(\partial^n_{\xi_1}g)^{i_n},$$

where $a_{i_1,\ldots,i_n}$ are universal constants independent of g. Tacking into account that

$$\widehat{K_t}(\xi) = e^{t(\widehat{J}(\xi)-1)} - e^{-t}$$

we obtain

$$\partial^n_{\xi_1}\widehat{K_t}(\xi) \;=\; e^{t(\widehat{J}(\xi)-1)} \sum_{i_1+2i_2+\ldots+ni_n=n} a_{i_1,\ldots,i_n} t^{i_1+\cdots+i_n}\prod_{j=1}^{n}[\partial^j_{\xi_1}\widehat{J}(\xi)]^{i_j}$$

and hence

$$|\partial^n_{\xi_1}\widehat{K_t}(\xi)|^2 \;\leq\; C\, e^{2t(\widehat{J}(\xi)-1)} \sum_{i_1+2i_2+\ldots+ni_n=n} t^{2(i_1+\cdots+i_n)}\prod_{j=1}^{n}[\partial^j_{\xi_1}\widehat{J}(\xi)]^{2i_j}.$$

Using that all the partial derivatives of $\widehat{J}$ decay faster than any polinomial in $|\xi|$, as $|\xi|\to\infty$, we obtain that

$$\int_{|\xi|>R} |\xi|^{2r}|\partial^n_{\xi_1}\widehat{K_t}(\xi)|^2 d\xi \leq C(J)\, e^{-2\delta t}\langle t\rangle^{2n}$$

where R and δ are chosen as in (8.13) and (8.14). Tacking into account that $\widehat{J}(\xi)$ is smooth (since $J \in \mathcal{S}(\mathbb{R}^d)$) we obtain that for all $|\xi| \leq R$ the following hold:

$$|\partial_{\xi_1}\widehat{J}(\xi)| \leq C\,|\xi|$$

and

$$|\partial^j_{\xi_1} \widehat{J}(\xi)| \le C, \qquad j = 2, \dots, n.$$

Then for all $|\xi| \le R$ we have

$$|\partial^n_{\xi_1} \widehat{K_t}(\xi)|^2 \le C\, e^{-t|\xi|^2} \sum_{i_1+2i_2+\ldots+ni_n=n} t^{2(i_1+\dots+i_n)} |\xi|^{2i_1}.$$

Finally, using that for any $l \ge 0$

$$\int_{|\xi|\le R} e^{-t|\xi|^2} |\xi|^l d\xi \le C(R) \langle t\rangle^{-\frac{d}{2}-\frac{l}{2}},$$

we obtain

$$\int_{|\xi|\le R} |\xi|^{2r} |\partial^n_{\xi_1} K_t(\xi)|^2 d\xi \le C(R)\langle t\rangle^{-\frac{d}{2}} \sum_{i_1+2i_2+\dots+ni_n=n} \langle t\rangle^{2p(i_1,\dots,i_d)-r}$$

where

$$\begin{aligned} p(i_1,\dots,i_n) &= (i_1+\dots+i_n) - \frac{i_1}{2} \\ &= \frac{i_1}{2} + i_2 + \dots + i_n \le \frac{i_1+2i_2+\dots ni_n}{2} = \frac{n}{2}. \end{aligned}$$

This ends the proof. □

We now prove a decay estimate that takes into account the linear semigroup applied to the convolution with a kernel G.

LEMMA 75. *Let $1 \le p \le r \le \infty$, $J \in \mathcal{S}(\mathbb{R}^d)$ and $G \in L^1(\mathbb{R}^d, |x|)$. There exists a positive constant $C = C(p, J, G)$ such that the following estimate*

$$\|S(t) * G * \varphi - S(t) * \varphi\|_{L^r(\mathbb{R}^d)} \le C\langle t\rangle^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} (\|\varphi\|_{L^p(\mathbb{R}^d)} + \|\varphi\|_{L^r(\mathbb{R}^d)}). \tag{8.22}$$

holds for all $\varphi \in L^p(\mathbb{R}^d) \cap L^r(\mathbb{R}^d)$.

REMARK 76. *In fact the following stronger inequality holds:*

$$\|S(t) * G * \varphi - S(t) * \varphi\|_{L^r(\mathbb{R}^d)} \le C\,\langle t\rangle^{-\frac{d}{2}(\frac{1}{p}-\frac{1}{r})-\frac{1}{2}} \|\varphi\|_{L^p(\mathbb{R}^d)} + C\, e^{-t}\|\varphi\|_{L^r(\mathbb{R}^d)}.$$

PROOF. We write $S(t)$ as $S(t) = e^{-t}\delta_0 + K_t$ and we get

$$S(t) * G * \varphi - S(t) * \varphi = e^{-t}(G * \varphi - \varphi) + K_t * G * \varphi - K_t * \varphi.$$

The first term in the above right hand side verifies:

$$e^{-t}\|G * \varphi - \varphi\|_{L^r(\mathbb{R}^d)} \le e^{-t}(\|G\|_{L^1(\mathbb{R}^d)}\|\varphi\|_{L^r(\mathbb{R}^d)} + \|\varphi\|_{L^r(\mathbb{R}^d)}) \le 2e^{-t}\|\varphi\|_{L^r(\mathbb{R}^d)}.$$

For the second one, by Lemma 74 we get that K_t satisfies

$$\|K_t * G - K_t\|_{L^a(\mathbb{R}^d)} \le C(r, J)\|G\|_{L^1(\mathbb{R}^d,|x|)} \langle t\rangle^{-\frac{d}{2}(1-\frac{1}{a})-\frac{1}{2}}$$

for all $t \ge 0$ where a is such that $1/r = 1/a + 1/p - 1$. Then, using Young's inequality we end the proof. □

0.19. Existence and uniqueness. In this chapter we use the previous results and estimates on the linear semigroup to prove the existence and uniqueness of the solution to our nonlinear problem (8.2). The proof is based on the variation of constants formula and uses the previous properties of the linear diffusion semigroup.

PROOF OF THEOREM 66. Recall that we want prove the global existence of solutions for initial conditions $u_0 \in L^1(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$.

Let us consider the following integral equation associated with (8.2):

$$u(t) = S(t) * u_0 + \int_0^t S(t-s) * (G * (f(u)) - f(u))(s)\, ds, \tag{8.23}$$

the functional

$$\Phi[u](t) = S(t) * u_0 + \int_0^t S(t-s) * (G * (f(u)) - f(u))(s)\, ds$$

and the space

$$X(T) = C([0,T]; L^1(\mathbb{R}^d)) \cap L^\infty([0,T]; \mathbb{R}^d)$$

endowed with the norm

$$\|u\|_{X(T)} = \sup_{t\in[0,T]} \left(\|u(t)\|_{L^1(\mathbb{R}^d)} + \|u(t)\|_{L^\infty(\mathbb{R}^d)}\right).$$

We will prove that Φ is a contraction in the ball of radius R, B_R, of X_T, if T is small enough.

Step I. Local Existence. Let $M = \max\{\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}\}$ and $p = 1, \infty$. Then, using the results of Lemma 72 we obtain,

$$\begin{aligned}
\|\Phi[u](t)\|_{L^p(\mathbb{R}^d)} &\le \|S(t) * u_0\|_{L^p(\mathbb{R}^d)} \\
&\quad + \int_0^t \|S(t-s) * G * (f(u)) - S(t-s) * f(u)\|_{L^p(\mathbb{R}^d)}\, ds \\
&\le (e^{-t} + \|K_t\|_{L^1(\mathbb{R}^d)})\|u_0\|_{L^p(\mathbb{R}^d)} \\
&\quad + \int_0^t 2(e^{-(t-s)} + \|K_{t-s}\|_{L^1(\mathbb{R}^d)})\|f(u)(s)\|_{L^p(\mathbb{R}^d)}\, ds \\
&\le 3\,\|u_0\|_{L^p(\mathbb{R}^d)} + 6\,Tf(R) \le 3M + 6\,Tf(R).
\end{aligned}$$

This implies that

$$\|\Phi[u]\|_{X(T)} \le 6M + 12\,Tf(R).$$

Choosing $R = 12M$ and T such that $12\,Tf(R) < 6M$ we obtain that $\Phi(B_R) \subset B_R$.

Let us choose u and v in B_R. Then for $p = 1, \infty$ the following hold:

$$\begin{aligned}\|\Phi[u](t) - \Phi[v](t)\|_{L^p(\mathbb{R}^d)} &\leq \int_0^t \|(S(t-s) * G - S(t-s)) * (f(u) - f(v))\|_{L^p(\mathbb{R}^d)}\, ds \\ &\leq 6 \int_0^t \|f(u)(s) - f(v)(s)\|_{L^p(\mathbb{R}^d)}\, ds \\ &\leq C(R) \int_0^t \|u(s) - v(s)\|_{L^p(\mathbb{R}^d)}\, ds \\ &\leq C(R)\, T\, \|u - v\|_{X(T)}.\end{aligned}$$

Choosing T small we obtain that $\Phi[u]$ is a contraction in B_R and then there exists a unique local solution u of (8.23).

Step II. Global existence. To prove the global well posedness of the solutions we have to guarantee that both $L^1(\mathbb{R}^d)$ and $L^\infty(\mathbb{R}^d)$-norms of the solutions do not blow up in finite time. We will apply the following lemma to control the $L^\infty(\mathbb{R}^d)$-norm of the solutions.

LEMMA 77. *Let $\theta \in L^1(\mathbb{R}^d)$ and K be a nonnegative function with mass one. Then for any $\mu \geq 0$ the following hold:*

$$\int_{\theta(x)>\mu} \int_{\mathbb{R}^d} K(x-y)\theta(y)\, dy\, dx \leq \int_{\theta(x)>\mu} \theta(x)\, dx \tag{8.24}$$

and

$$\int_{\theta(x)<-\mu} \int_{\mathbb{R}^d} K(x-y)\theta(y)\, dy\, dx \geq \int_{\theta(x)<-\mu} \theta(x)\, dx. \tag{8.25}$$

PROOF OF LEMMA 77. First of all we point out that we only have to prove (8.24). Indeed, once it is proved, then (8.25) follows immediately applying (8.24) to the function $-\theta$.

First, we prove estimate (8.24) for $\mu = 0$ and then we apply this case to prove the general case, $\mu \neq 0$.

For $\mu = 0$ the following inequalities hold:

$$\begin{aligned}\int_{\theta(x)>0} \int_{\mathbb{R}^d} K(x-y)\theta(y)\, dy\, dx &\leq \int_{\theta(x)>0} \int_{\theta(y)>0} K(x-y)\theta(y)\, dy\, dx \\ &= \int_{\theta(y)>0} \theta(y) \int_{\theta(x)>0} K(x-y)\, dx\, dy \\ &\leq \int_{\theta(y)>0} \theta(y) \int_{\mathbb{R}^d} K(x-y)\, dx\, dy \\ &= \int_{\theta(y)>0} \theta(y)\, dy.\end{aligned}$$

Now let us analyze the general case $\mu > 0$. In this case the following inequality

$$\int_{\theta(x)>\mu} \theta(x)\,dx \le \int_{\mathbb{R}^d} |\theta(x)|\,dx$$

shows that the set $\{x \in \mathbb{R}^d : \theta(x) > \mu\}$ has finite measure. Then we obtain

$$\begin{aligned}\int_{\theta(x)>\mu}\int_{\mathbb{R}^d} K(x-y)\theta(y)\,dy\,dx &= \int_{\theta(x)>\mu}\int_{\mathbb{R}^d} K(x-y)(\theta(y)-\mu)\,dy\,dx + \int_{\theta(x)>\mu} \mu\,dx \\ &\le \int_{\theta(x)>\mu} (\theta(x)-\mu)\,dx + \int_{\theta(x)>\mu} \mu\,dx = \int_{\theta(x)>\mu} \theta(x)\,dx.\end{aligned}$$

This completes the proof of (8.24). □

Control of the L^1-norm. As in the previous chapter, we multiply equation (8.2) by $sgn(u(x,t))$ and integrate in $\mathbb{R}^d$ to obtain the following estimate

$$\begin{aligned}\frac{d}{dt}\int_{\mathbb{R}^d} |u(x,t)|\,dx &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(y,t)sgn(u(x,t))\,dy\,dx - \int_{\mathbb{R}^d} |u(t,x)|\,dx \\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(u(y,t))sgn(u(x,t))\,dy\,dx - \int_{\mathbb{R}^d} f(u(x,t))sgn(u(x,t))\,dx \\ &\le \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)|u(y,t)|\,dy\,dx - \int_{\mathbb{R}^d} |u(x,t)|\,dx \\ &\quad + \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)|f(u)(y,t)|\,dy\,dx - \int_{\mathbb{R}^d} |f(u)(x,t)|\,dx \\ &= \int_{\mathbb{R}^d} |u(y,t)| \int_{\mathbb{R}^d} J(x-y)\,dx\,dy - \int_{\mathbb{R}^d} |u(x,t)|\,dx \\ &\quad + \int_{\mathbb{R}^d} |f(u)(y,t)| \int_{\mathbb{R}^d} G(x-y)\,dx\,dy - \int_{\mathbb{R}^d} |f(u)(x,t)|\,dx \\ &\le 0,\end{aligned}$$

which shows that the L^1-norm does not increase.

Control of the L^∞-norm. Let us denote $m = \|u_0\|_{L^\infty(\mathbb{R}^d)}$. Multiplying the equation in (8.2) by $sgn(u-m)^+$ and integrating in the x variable we get,

$$\frac{d}{dt}\int_{\mathbb{R}^d} (u(x,t)-m)^+ dx = I_1(t) + I_2(t)$$

where

$$I_1(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(y,t)sgn(u(x,t)-m)^+\,dy\,dx - \int_{\mathbb{R}^d} u(x,t)sgn(u(x,t)-m)^+\,dx$$

and

$$I_2(t) = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(u)(y,t)sgn(u(x,t)-m)^+\,dy\,dx - \int_{\mathbb{R}^d} f(u)(x,t)sgn(u(x,t)-m)^+\,dx.$$

We claim that both I_1 and I_2 are negative. Thus $(u(x,t)-m)^+ = 0$ a.e. $x \in \mathbb{R}^d$ and then $u(x,t) \le m$ for all $t > 0$ and a.e. $x \in \mathbb{R}^d$.

In the case of I_1, applying Lemma 77 with $K = J$, $\theta = u(t)$ and $\mu = m$ we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)u(y,t)sgn(u(x,t)-m)^+\,dy\,dx &= \int_{u(x)>m}\int_{\mathbb{R}^d} J(x-y)u(y,t)\,dy\,dx \\ &\le \int_{u(x)>m} u(x,t)\,dx.\end{aligned}$$

To handle the second one, I_2, we proceed in a similar manner. Applying Lemma 77 with

$$\theta(x) = f(u)(x,t) \quad \text{and} \quad \mu = f(m)$$

we obtain

$$\int_{f(u(x,t))>f(m)}\int_{\mathbb{R}^d} G(x-y)f(u)(y,t)\,dy\,dx \le \int_{f(u(x,t))>f(m)} f(u)(x,t)\,dx.$$

Using that f is a nondecreasing function, we rewrite this inequality in an equivalent form te obtain the desired inequality:

$$\begin{aligned}&\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} G(x-y)f(u)(y,t)sgn(u(x,t)-m)^+\,dy\,dx \\ &\qquad = \int_{u(x,t)\ge m}\int_{\mathbb{R}^d} G(x-y)f(u)(y,t)\,dy\,dx \\ &\qquad = \int_{f(u)(x,t)\ge f(m)}\int_{\mathbb{R}^d} G(x-y)f(u)(y,t)\,dy\,dx \\ &\qquad \le \int_{u(x,t)\ge m} f(u)(x,t)\,dx.\end{aligned}$$

In a similar way, by using inequality (8.25) we get

$$\frac{d}{dt}\int_{\mathbb{R}^d}(u(x,t)+m)^-\,dx \le 0,$$

which implies that $u(x,t) \ge -m$ for all $t > 0$ and a.e. $x \in \mathbb{R}^d$.

We conclude that $\|u(t)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^\infty(\mathbb{R}^d)}$.

Step III. Uniqueness and contraction property. Let us consider u and v two solutions corresponding to initial data u_0 and v_0 respectively. We will prove that for any $t > 0$ the following holds:

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(x,t) - v(x,t)|\, dx \le 0.$$

To this end, we multiply by $sgn(u(x,t) - v(x,t))$ the equation satisfied by $u - v$ and using the symmetry of J, the positivity of J and G and that their mass equals one we obtain,

$$\begin{aligned}
\frac{d}{dt} \int_{\mathbb{R}^d} |u(x,t) - v(x,t)|\, dx &= \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x-y)(u(y,t) - v(y,t)) sgn(u(x,t) - v(x,t))\, dx\, dy \\
&\quad - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |u(x,t) - v(x,t)|\, dx \\
&\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)(f(u)(y,t) - f(v)(y,t)) sgn(u(x,t) - v(x,t))\, dx\, dy \\
&\quad - \int_{\mathbb{R}^d} |f(u)(x,t) - f(v)(x,t)|\, dx \\
&\le \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x-y)|u(y,t) - v(y,t)|\, dx\, dy - \int_{\mathbb{R}^d} |u(x,t) - v(x,t)|\, dx \\
&\quad + \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} G(x-y)|f(u)(y,t) - f(v)(y,t)|\, dx\, dy - \int_{\mathbb{R}^d} |f(u)(x,t) - f(v)(x,t)|\, dx \\
&= 0.
\end{aligned}$$

Thus we get the uniqueness of the solutions and the contraction property

$$\|u(t) - v(t)\|_{L^1(\mathbb{R}^d)} \le \|u_0 - v_0\|_{L^1(\mathbb{R}^d)}.$$

This ends the proof of Theorem 66. □

Now we prove that, due to the lack of regularizing effect, the $L^\infty(\mathbb{R})$-norm does not get bounded for positive times when we consider initial conditions in $L^1(\mathbb{R})$. This is in contrast to what happens for the local convection-diffusion problem, see [**54**].

PROPOSITION 78. *Let $d = 1$ and $|f(u)| \le C|u|^q$ with $1 \le q < 2$. Then*

$$\sup_{u_0 \in L^1(\mathbb{R})} \sup_{t \in [0,1]} \frac{t^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} = \infty.$$

PROOF. Assume by contradiction that

$$\sup_{u_0 \in L^1(\mathbb{R})} \sup_{t \in [0,1]} \frac{t^{\frac{1}{2}} \|u(t)\|_{L^\infty(\mathbb{R})}}{\|u_0\|_{L^1(\mathbb{R})}} = M < \infty. \tag{8.26}$$

Using the representation formula (8.23) we get:

$$\|u(1)\|_{L^\infty(\mathbb{R})} \geq \|S(1) * u_0\|_{L^\infty(\mathbb{R})} - \left\| \int_0^1 S(1-s) * (G * (f(u)) - f(u))(s)\, ds \right\|_{L^\infty(\mathbb{R})}$$

Using Lemma 75 the last term can be bounded as follows:

$$\begin{aligned}\left\| \int_0^1 S(1-s) * (G*(f(u)) - f(u))(s)\, ds \right\|_{L^\infty(\mathbb{R})} &\leq \int_0^1 \langle 1-s\rangle^{-\frac{1}{2}} \|f(u(s))\|_{L^\infty(\mathbb{R})}\, ds \\ &\leq C \int_0^1 \|u(s)\|^q_{L^\infty(\mathbb{R})} ds \leq CM^q \|u_0\|^q_{L^1(\mathbb{R})} \int_0^1 s^{-\frac{q}{2}} ds \\ &\leq CM^q \|u_0\|^q_{L^1(\mathbb{R})},\end{aligned}$$

provided that $q < 2$.

This implies that the $L^\infty(\mathbb{R})$-norm of the solution at time $t = 1$ satisfies

$$\begin{aligned}\|u(1)\|_{L^\infty(\mathbb{R})} &\geq \|S(1) * u_0\|_{L^\infty(\mathbb{R})} - CM^q \|u_0\|^q_{L^1(\mathbb{R})} \\ &\geq e^{-1}\|u_0\|_{L^\infty(\mathbb{R})} - \|K_1\|_{L^\infty(\mathbb{R})}\|u_0\|_{L^1(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})} \\ &\geq e^{-1}\|u_0\|_{L^\infty(\mathbb{R})} - C\|u_0\|_{L^1(\mathbb{R})} - CM^q\|u_0\|^q_{L^1(\mathbb{R})}.\end{aligned}$$

Choosing now a sequence $u_{0,\varepsilon}$ with $\|u_{0,\varepsilon}\|_{L^1(\mathbb{R})} = 1$ and $\|u_{0,\varepsilon}\|_{L^\infty(\mathbb{R})} \to \infty$ we obtain that

$$\|u_{0,\varepsilon}(1)\|_{L^\infty(\mathbb{R})} \to \infty,$$

a contradiction with our assumption (8.26). The proof of the result is now completed. □

0.20. Convergence to the local problem. In this chapter we prove the convergence of solutions of the nonlocal problem to solutions of the local convection-diffusion equation when we rescale the kernels and let the scaling parameter go to zero.

As we did in the previous sections we begin with the analysis of the linear part.

LEMMA 79. *Assume that $u_0 \in L^2(\mathbb{R}^d)$. Let w_ε be the solution to*

$$\begin{cases} (w_\varepsilon)_t(x,t) = \dfrac{1}{\varepsilon^2} \displaystyle\int_{\mathbb{R}^d} J_\varepsilon(x-y)(w_\varepsilon(y,t) - w_\varepsilon(x,t))\, dy, \\ w_\varepsilon(0,x) = u_0(x), \end{cases} \tag{8.27}$$

and w the solution to

$$\begin{cases} w_t(x,t) = \Delta w(x,t), \\ w(0,x) = u_0(x). \end{cases} \tag{8.28}$$

Then, for any positive T,

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|w_\varepsilon - w\|_{L^2(\mathbb{R}^d)} = 0.$$

PROOF. Taking the Fourier transform in (8.27) we get

$$\widehat{w_\varepsilon}(t,\xi) = \frac{1}{\varepsilon^2}\Big(\widehat{J}_\varepsilon(\xi)\widehat{w_\varepsilon}(t,\xi) - \widehat{w_\varepsilon}(t,\xi)\Big).$$

Therefore,

$$\widehat{w_\varepsilon}(t,\xi) = \exp\left(t\frac{\widehat{J}_\varepsilon(\xi)-1}{\varepsilon^2}\right)\widehat{u_0}(\xi).$$

But we have,

$$\widehat{J}_\varepsilon(\xi) = \widehat{J}(\varepsilon\xi).$$

Hence we get

$$\widehat{w_\varepsilon}(t,\xi) = \exp\left(t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}\right)\widehat{u_0}(\xi).$$

By Plancherel's identity, using the well known formula for solutions to (8.28),

$$\widehat{w}(t,\xi) = e^{-t\xi^2}\widehat{u_0}(\xi).$$

we obtain that

$$\|w_\varepsilon(t) - w(t)\|^2_{L^2(\mathbb{R}^d)} \;=\; \int_{\mathbb{R}^d}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}} - e^{-t\xi^2}\right|^2 |\widehat{u_0}(\xi)|^2\,d\xi$$

With R and δ as in (8.13) and (8.14) we get

$$\begin{aligned}\int_{|\xi|\geq R/\varepsilon}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}} - e^{-t\xi^2}\right|^2 |\widehat{u_0}(\xi)|^2 d\xi \;&\leq\; \int_{|\xi|\geq R/\varepsilon}(e^{-\frac{t\delta}{\varepsilon^2}} + e^{\frac{-tR^2}{\varepsilon^2}})^2|\widehat{u_0}(\xi)|^2 d\xi \\ &\leq\; (e^{-\frac{t\delta}{\varepsilon^2}} + e^{\frac{-tR^2}{\varepsilon^2}})^2\|u_0\|^2_{L^2(\mathbb{R}^d)} \underset{\varepsilon\to 0}{\longrightarrow} 0.\end{aligned} \tag{8.29}$$

To treat the integral on the set $\{\xi\in\mathbb{R}^d : |\xi|\leq R/\varepsilon\}$ we use the fact that on this set the following holds:

$$\begin{aligned}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}} - e^{-t\xi^2}\right| \;&\leq\; t\left|\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}+\xi^2\right|\max\{e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}, e^{-t\xi^2}\} \\ &\leq\; t\left|\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}+\xi^2\right|\max\{e^{-\frac{t\xi^2}{2}}, e^{-t\xi^2}\} \\ &\leq\; t\left|\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}+\xi^2\right|e^{-\frac{t\xi^2}{2}}.\end{aligned} \tag{8.30}$$

Thus:

$$\begin{aligned}\int_{|\xi|\le R/\varepsilon}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right|^2|\widehat{u_0}(\xi)|^2d\xi &\le \int_{|\xi|\le R/\varepsilon} e^{-t|\xi|^2}t^2\left|\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}+\xi^2\right|^2|\widehat{u_0}(\xi)|^2\,d\xi\\ &\le \int_{|\xi|\le R/\varepsilon} e^{-t\xi^2}t^2|\xi|^4\left|\frac{\widehat{J}(\varepsilon\xi)-1+\varepsilon^2\xi^2}{\varepsilon^2\xi^2}\right|^2|\widehat{u_0}(\xi)|^2\,d\xi.\end{aligned}$$

From $|\widehat{J}(\xi)-1|\le K|\xi|^2$ for all $\xi\in\mathbb{R}^d$ we get

$$\left|\frac{\widehat{J}(\varepsilon\xi)-1+\varepsilon^2\xi^2}{\varepsilon^2\xi^2}\right|\le\frac{(K+1)}{\varepsilon^2|\xi|^2}\varepsilon^2|\xi|^2\le K+1. \tag{8.31}$$

Using this bound and that $e^{-|s|}s^2\le C$, we get that

$$\sup_{t\in[0,T]}\int_{|\xi|\le R/\varepsilon}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right|^2|\widehat{u_0}(\xi)|^2d\xi\le C\int_{\mathbb{R}^d}\left|\frac{\widehat{J}(\varepsilon\xi)-1+\varepsilon^2|\xi|^2}{\varepsilon^2|\xi|^2}\right|^2|\widehat{u_0}(\xi)|^2\mathbf{1}_{\{|\xi|\le R/\varepsilon\}}d\xi.$$

By inequality (8.31) together with the fact that

$$\lim_{\varepsilon\to0}\frac{\widehat{J}(\varepsilon\xi)-1+\varepsilon^2|\xi|^2}{\varepsilon^2|\xi|^2}=0$$

and that $\widehat{u_0}\in L^2(\mathbb{R}^d)$, by Lebesgue dominated convergence theorem, we have that

$$\lim_{\varepsilon\to0}\sup_{t\in[0,T]}\int_{|\xi|\le R/\varepsilon}\left|e^{t\frac{\widehat{J}(\varepsilon\xi)-1}{\varepsilon^2}}-e^{-t\xi^2}\right|^2|\widehat{u_0}(\xi)|^2d\xi=0. \tag{8.32}$$

From (8.29) and (8.32) we obtain

$$\lim_{\varepsilon\to0}\sup_{t\in[0,T]}\|w_\varepsilon(t)-w(t)\|^2_{L^2(\mathbb{R}^d)}=0,$$

as we wanted to prove. □

Next we prove a lemma that provides us with a uniform (independent of ε) decay for the nonlocal convective part.

LEMMA 80. *There exists a positive constant $C=C(J,G)$ such that*

$$\left\|\Big(\frac{S_\varepsilon(t)*G_\varepsilon-S_\varepsilon(t)}{\varepsilon}\Big)*\varphi\right\|_{L^2(\mathbb{R}^d)}\le C\,t^{-\frac12}\|\varphi\|_{L^2(\mathbb{R}^d)}$$

holds for all $t>0$ and $\varphi\in L^2(\mathbb{R}^d)$, uniformly on $\varepsilon>0$. Here $S_\varepsilon(t)$ is the linear semigroup associated to (8.27).

PROOF. Let us denote by $\Phi_\varepsilon(x,t)$ the following quantity:

$$\Phi_\varepsilon(x,t) = \frac{(S_\varepsilon(t) * G_\varepsilon)(x) - S_\varepsilon(t)(x)}{\varepsilon}.$$

Then by the definition of S_ε and G_ε we obtain

$$\begin{aligned}\Phi_\varepsilon(x,t) &= \int_{\mathbb{R}^d} e^{ix\cdot\xi} \exp\Big(\frac{t(\widehat{J}(\varepsilon\xi)-1)}{\varepsilon^2}\Big)\frac{\widehat{G}(\xi\varepsilon)-1}{\varepsilon} d\xi \\ &= \varepsilon^{-d-1}\int_{\mathbb{R}^d} e^{i\varepsilon^{-1}x\cdot\xi}\exp\Big(\frac{t(\widehat{J}(\xi)-1)}{\varepsilon^2}\Big)(\widehat{G}(\xi)-1)\,d\xi \\ &= \varepsilon^{-d-1}\Phi_1(t\varepsilon^{-2}, x\varepsilon^{-1})\end{aligned}.$$

At this point, we observe that for $\varepsilon = 1$, Lemma 75 gives us

$$\|\Phi_1(t) * \varphi\|_{L^2(\mathbb{R}^d)} \le C(J,G)\langle t\rangle^{-\frac{1}{2}}\|\varphi\|_{L^2(\mathbb{R}^d)}.$$

Hence

$$\begin{aligned}\|\Phi_\varepsilon(t)*\varphi\|_{L^2(\mathbb{R}^d)} &= \varepsilon^{-d-1}\|\Phi_1(t\varepsilon^{-2},\varepsilon^{-1}\cdot)*\varphi\|_{L^2(\mathbb{R}^d)} = \varepsilon^{-1}\|[\Phi_1(t\varepsilon^{-2})*\varphi(\varepsilon\cdot)](\varepsilon^{-1}\cdot)\|_{L^2(\mathbb{R}^d)} \\ &= \varepsilon^{-1+\frac{d}{2}}\|\Phi_1(t\varepsilon^{-2}*\varphi(\varepsilon\cdot))\|_{L^2(\mathbb{R}^d)} \le \varepsilon^{-1+\frac{d}{2}}(t\varepsilon^{-2})^{-\frac{1}{2}}\|\varphi(\varepsilon\cdot)\|_{L^2(\mathbb{R}^d)} \\ &\le t^{-\frac{1}{2}}\|\varphi\|_{L^2(\mathbb{R}^d)}.\end{aligned}$$

This ends the proof. □

LEMMA 81. *Let be $T > 0$ and $M > 0$. Then the following*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\int_0^t \Big\|\Big(\frac{S_\varepsilon(s)*G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b\cdot\nabla H(s)\Big)*\varphi(s)\Big\|_{L^2(\mathbb{R}^d)} ds = 0,$$

holds uniformly for all $\|\varphi\|_{L^\infty([0,T];L^2(\mathbb{R}^d))} \le M$. Here H is the linear heat semigroup given by the Gaussian

$$H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}}$$

and $b = (b_1, ..., b_d)$ is given by

$$b_j = \int_{\mathbb{R}^d} x_j\, G(x)\,dx, \qquad j = 1, ..., d.$$

PROOF. Let us denote by $I_\varepsilon(t)$ the following quantity:

$$I_\varepsilon(t) = \int_0^t \Big\|\Big(\frac{S_\varepsilon(s)*G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b\cdot\nabla H(s)\Big)*\varphi(s)\Big\|_{L^2(\mathbb{R}^d)} ds.$$

Choose $\alpha \in (0,1)$. Then

$$I_\varepsilon(t) \le \begin{cases} I_{1,\varepsilon} & \text{if } t \le \varepsilon^\alpha, \\ I_{1,\varepsilon} + I_{2,\varepsilon}(t) & \text{if } t \ge \varepsilon^\alpha, \end{cases}$$

where

$$I_{1,\varepsilon} = \int_0^{\varepsilon^\alpha} \left\| \left(\frac{S_\varepsilon(s) * G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b \cdot \nabla H(s) \right) * \varphi(s) \right\|_{L^2(\mathbb{R}^d)} ds$$

and

$$I_{2,\varepsilon}(t) = \int_{\varepsilon^\alpha}^t \left\| \left(\frac{S_\varepsilon(s) * G_\varepsilon - S_\varepsilon(s)}{\varepsilon} - b \cdot \nabla H(s) \right) * \varphi(s) \right\|_{L^2(\mathbb{R}^d)} ds.$$

The first term $I_{1,\varepsilon}$ satisfies,

$$\begin{aligned} I_{1,\varepsilon} &\le \int_0^{\varepsilon^\alpha} \left\| \Big(\frac{S_\varepsilon(s) * G_\varepsilon - S_\varepsilon(s)}{\varepsilon} \Big) * \varphi \right\|_{L^2(\mathbb{R}^d)} ds + \int_0^{\varepsilon^\alpha} \| b \cdot \nabla H(s) * \varphi \|_{L^2(\mathbb{R}^d)} \, ds \\ &\le C \int_0^{\varepsilon^\alpha} s^{-\frac12} \|\varphi(s)\|_{L^2(\mathbb{R}^d)} ds + C \int_0^{\varepsilon^\alpha} \|\nabla H(s)\|_{L^1(\mathbb{R}^d)} \|\varphi(s)\|_{L^2(\mathbb{R}^d)} ds \\ &\le CM \int_0^{\varepsilon^\alpha} s^{-\frac12} \, ds = 2CM\varepsilon^{\frac{\alpha}{2}}. \end{aligned} \tag{8.33}$$

To bound $I_{2,\varepsilon}(t)$ we observe that, by Plancherel's identity, we get,

$$\begin{aligned} I_{2,\varepsilon}(t) &= \int_{\varepsilon^\alpha}^t \left\| \left(e^{s(\widehat{J}(\varepsilon\xi)-1)/\varepsilon^2} \left(\frac{\widehat{G}(\varepsilon\xi) - 1}{\varepsilon} \right) - i\, b \cdot \xi e^{-s|\xi|^2} \right) \widehat{\varphi}(s) \right\|_{L^2_\xi(\mathbb{R}^d)} ds \\ &\le \int_{\varepsilon^\alpha}^t \left\| \left(e^{s(\widehat{J}(\varepsilon\xi)-1)/\varepsilon^2} - e^{-s|\xi|^2} \right) \left(\frac{\widehat{G}(\varepsilon\xi) - 1}{\varepsilon} \right) \widehat{\varphi}(s) \right\|_{L^2_\xi(\mathbb{R}^d)} ds \\ &\quad + \int_{\varepsilon^\alpha}^t \left\| e^{-s|\xi|^2} \left(\frac{\widehat{G}(\varepsilon\xi) - 1}{\varepsilon} - i\, b \cdot \xi \right) \widehat{\varphi}(s) \right\|_{L^2_\xi(\mathbb{R}^d)} ds \\ &= \int_{\varepsilon^\alpha}^t R_{1,\varepsilon}(s)\, ds + \int_{\varepsilon^\alpha}^t R_{2,\varepsilon}(s)\, ds. \end{aligned}$$

In the following we obtain upper bounds for $R_{1,\varepsilon}$ and $R_{2,\varepsilon}$. Observe that $R_{1,\varepsilon}$ satisfies:

$$(R_{1,\varepsilon})^2(s) \le 2((R_{3,\varepsilon})^2(s) + (R_{4,\varepsilon})^2(s))$$

where

$$(R_{3,\varepsilon})^2(s) = \int_{|\xi| \le R/\varepsilon} \left(e^{s(\widehat{J}(\varepsilon\xi)-1)/\varepsilon^2} - e^{-s|\xi|^2} \right)^2 \left| \frac{\widehat{G}(\varepsilon\xi) - 1}{\varepsilon} \right|^2 |\widehat{\varphi}(s,\xi)|^2 d\xi$$

and

$$(R_{4,\varepsilon})^2(s) = \int_{|\xi|\geq R/\varepsilon} \left(e^{s(\widehat{J}(\varepsilon\xi)-1)/\varepsilon^2} - e^{-s|\xi|^2}\right)^2 \left|\frac{\widehat{G}(\varepsilon\xi)-1}{\varepsilon}\right|^2 |\widehat{\varphi}(s,\xi)|^2 d\xi.$$

With respect to $R_{3,\varepsilon}$ we proceed as in the proof of Lemma 80 by choosing δ and R as in (8.13) and (8.14). Using estimate (8.30) and the fact that $|\widehat{G}(\xi)-1|\leq C|\xi|$ and $|\widehat{J}(\xi)-1+\xi^2|\leq C|\xi|^3$ for every $\xi\in\mathbb{R}^d$ we obtain:

$$\begin{aligned}
(R_{3,\varepsilon})^2(s) &\leq C\int_{|\xi|\leq R/\varepsilon} e^{-s|\xi|^2}s^2\left|\frac{\widehat{J}(\varepsilon\xi)-1+\xi^2\varepsilon^2}{\varepsilon^2}\right|^2 |\xi|^2|\widehat{\varphi}(s,\xi)|^2 d\xi \\
&\leq C\int_{|\xi|\leq R/\varepsilon} e^{-s|\xi|^2}s^2\left[\frac{(\varepsilon\xi)^3}{\varepsilon^2}\right]^2 |\xi|^2|\widehat{\varphi}(s,\xi)|^2 d\xi \\
&= C\int_{|\xi|\leq R/\varepsilon} e^{-s|\xi|^2}s^2\varepsilon^2|\xi|^8|\widehat{\varphi}(s,\xi)|^2 d\xi \leq \varepsilon^2 s^{-2}\int_{\mathbb{R}^d} e^{-s|\xi|^2}s^4|\xi|^8|\widehat{\varphi}(s,\xi)|^2 d\xi \\
&\leq C\varepsilon^{2-2\alpha}\int_{\mathbb{R}^d}|\widehat{\varphi}(s,\xi)|^2 d\xi \leq C\varepsilon^{2-2\alpha}M^2.
\end{aligned}$$

In the case of $R_{4,\varepsilon}$, we use that $|\hat{G}(\xi)|\leq 1$ and we proceed as in the proof of (8.29):

$$\begin{aligned}
(R_{4,\varepsilon})^2(s) &\leq \int_{|\xi|\geq R/\varepsilon}(e^{-\frac{s\delta}{\varepsilon^2}}+e^{-\frac{sR^2}{\varepsilon^2}})^2\varepsilon^{-2}|\widehat{\varphi}(s,\xi)|^2 d\xi \\
&\leq (e^{-\frac{\delta}{\varepsilon^{2-\alpha}}}+e^{-\frac{R^2}{\varepsilon^{2-\alpha}}})^2\varepsilon^{-2}\int_{|\xi|\geq R/\varepsilon}|\widehat{\varphi}(s,\xi)|^2 d\xi \\
&\leq M^2(e^{-\frac{\delta}{\varepsilon^{2-\alpha}}}+e^{-\frac{R^2}{\varepsilon^{2-\alpha}}})^2\varepsilon^{-2} \\
&\leq CM^2\varepsilon^{2-2\alpha}
\end{aligned}$$

for sufficiently small ε.

Then

$$\int_{\varepsilon^\alpha}^t R_{1,\varepsilon}(s)ds \leq CTM\varepsilon^{1-\alpha}. \tag{8.34}$$

The second term can be estimated in a similar way, using that $|\widehat{G}(\xi) - 1 - ib\cdot\xi| \le C|\xi|^2$ for every $\xi \in \mathbb{R}^d$, we get

$$\begin{aligned}
(R_{2,\varepsilon})^2(s) &\le \int_{\mathbb{R}^d} e^{-2s|\xi|^2} \left| \frac{\widehat{G}(\varepsilon\xi) - 1 - i\, b\cdot \xi\varepsilon}{\varepsilon} \right|^2 |\widehat{\varphi}(s,\xi)|^2 d\xi \\
&\le C \int_{\mathbb{R}^d} e^{-2s|\xi|^2} \left[\frac{(\xi\varepsilon)^2}{\varepsilon} \right]^2 |\widehat{\varphi}(s,\xi)|^2 d\xi = C \int_{\mathbb{R}^d} e^{-2s|\xi|^2} \varepsilon^2 |\xi|^4 |\widehat{\varphi}(s,\xi)|^2 d\xi \\
&= C\varepsilon^2 s^{-2} \int_{\mathbb{R}^d} e^{-2s|\xi|^2} s^2 |\xi|^4 |\widehat{\varphi}(s,\xi)|^2 d\xi \le C\varepsilon^{2(1-\alpha)} \int_{\mathbb{R}^d} |\widehat{\varphi}(s,\xi)|^2 d\xi \\
&\le CM^2 \varepsilon^{2(1-\alpha)},
\end{aligned}$$

and we conclude that

$$\int_{\varepsilon^\alpha}^t R_{2,\varepsilon}(s)ds \le CTM\varepsilon^{1-\alpha}. \tag{8.35}$$

Now, by (8.33), (8.34) and (8.35) we obtain that

$$\sup_{t\in[0,T]} I_\varepsilon(t) \le CM(\varepsilon^{\frac{\alpha}{2}} + \varepsilon^{1-\alpha}) \to 0, \qquad \text{as } \varepsilon \to 0, \tag{8.36}$$

which finishes the proof. □

Now we are ready to prove Theorem 67.

Proof of Theorem 67. First we write the two problems in the semigroup formulation,

$$u_\varepsilon(t) = S_\varepsilon(t) * u_0 + \int_0^t \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * f(u_\varepsilon(s))\, ds$$

and

$$v(t) = H(t) * u_0 + \int_0^t b\cdot \nabla H(t-s) * f(v(s))\, ds.$$

Then

$$\sup_{t\in[0,T]} \|u_\varepsilon(t) - v(t)\|_{L^2(\mathbb{R}^d)} \le \sup_{t\in[0,T]} I_{1,\varepsilon}(t) + \sup_{t\in[0,T]} I_{2,\varepsilon}(t) \tag{8.37}$$

where

$$I_{1,\varepsilon}(t) = \|S_\varepsilon(t) * u_0 - H(t) * u_0\|_{L^2(\mathbb{R}^d)}$$

and

$$I_{2,\varepsilon}(t) = \left\| \int_0^t \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * f(u_\varepsilon(s)) - \int_0^t b\cdot\nabla H(t-s) * f(v(s)) \right\|_{L^2(\mathbb{R}^d)}.$$

In view of Lemma 79 we have

$$\sup_{t\in[0,T]} I_{1,\varepsilon}(t) \to 0 \qquad \text{as } \varepsilon \to 0.$$

So it remains to analyze the second term $I_{2,\varepsilon}$. To this end, we split it again

$$I_{2,\varepsilon}(t) \le I_{3,\varepsilon}(t) + I_{4,\varepsilon}(t)$$

where

$$I_{3,\varepsilon}(t) = \int_0^t \left\| \frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} * \big(f(u_\varepsilon(s)) - f(v(s))\big) \right\|_{L^2(\mathbb{R}^d)} ds$$

and

$$I_{4,\varepsilon}(t) = \int_0^t \left\| \left(\frac{S_\varepsilon(t-s) * G_\varepsilon - S_\varepsilon(t-s)}{\varepsilon} - b \cdot \nabla H(t-s) \right) * f(v(s)) \right\|_{L^2(\mathbb{R}^d)} ds.$$

Using Young's inequality and that from our hypotheses we have an uniform bound for u_ε and u in terms of $\|u_0\|_{L^1(\mathbb{R}^d)}$, $\|u_0\|_{L^\infty(\mathbb{R}^d)}$ we obtain

$$\begin{aligned} I_{3,\varepsilon}(t) &\le \int_0^t \frac{\|f(u_\varepsilon(s)) - f(v(s))\|_{L^2(\mathbb{R}^d)}}{|t-s|^{\frac{1}{2}}} ds \\ &\le \|f(u_\varepsilon) - f(v)\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \int_0^t \frac{ds}{|t-s|^{\frac{1}{2}}} \qquad (8.38) \\ &\le 2T^{1/2} \|u_\varepsilon - v\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}). \end{aligned}$$

By Lemma 81 we obtain, choosing $\alpha = 2/3$ in (8.36), that

$$\sup_{t\in[0,T]} I_{4,\varepsilon} \le C\varepsilon^{\frac{1}{3}} \|f(v)\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \le C\varepsilon^{\frac{1}{3}} C(\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R}^d)}). \qquad (8.39)$$

Using (8.37), (8.38) and (8.39) we get:

$$\begin{aligned} \|u_\varepsilon - v\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \le\ & \|I_{1,\varepsilon}\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \\ & + T^{\frac{1}{2}} C(\|u_0\|_{L^1(\mathbb{R})}, \|u_0\|_{L^\infty(\mathbb{R})}) \|u_\varepsilon - v\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))}. \end{aligned}$$

Choosing $T = T_0$ sufficiently small, depending on $\|u_0\|_{L^1(\mathbb{R})}$ and $\|u_0\|_{L^\infty(\mathbb{R})}$ we get

$$\|u_\varepsilon - v\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \le \|I_{1,\varepsilon}\|_{L^\infty((0,T);\,L^2(\mathbb{R}^d))} \to 0,$$

as $\varepsilon \to 0$.

Using the same argument in any interval $[\tau, \tau + T_0]$, the stability of the solutions of the equation (8.4) in $L^2(\mathbb{R}^d)$-norm and that for any time $\tau > 0$ it holds that

$$\|u_\varepsilon(\tau)\|_{L^1(\mathbb{R}^d)} + \|u_\varepsilon(\tau)\|_{L^\infty(\mathbb{R}^d)} \le \|u_0\|_{L^1(\mathbb{R}^d)} + \|u_0\|_{L^\infty(\mathbb{R}^d)},$$

we obtain

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_\varepsilon - v\|_{L^2(\mathbb{R}^d)} = 0,$$

as we wanted to prove. □

0.21. Long time behaviour of the solutions. The aim of this chapter is to obtain the first term in the asymptotic expansion of the solution u to (8.2). The main ingredient for our proofs is the following lemma inspired in the Fourier splitting method introduced by Schonbek, see [**78**], [**79**] and [**80**].

LEMMA 82. *Let R and δ be such that the function $\widehat{J}$ satisfies:*

$$\widehat{J}(\xi) \le 1 - \frac{|\xi|^2}{2}, \qquad |\xi| \le R \tag{8.40}$$

and

$$\widehat{J}(\xi) \le 1 - \delta, \qquad |\xi| \ge R. \tag{8.41}$$

Let us assume that the function $u : [0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ satisfies the following differential inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(x,t)|^2\,dx \le c\int_{\mathbb{R}^d} (J*u-u)(x,t)u(x,t)\,dx, \tag{8.42}$$

for any $t > 0$. Then for any $1 \le r < \infty$ there exists a constant $a = rd/c\delta$ such that

$$\int_{\mathbb{R}^d} |u(at,x)|^2 dx \le \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{rd\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}}\int_0^t (s+1)^{rd-\frac{d}{2}-1}\|u(as)\|^2_{L^1(\mathbb{R}^d)} ds \tag{8.43}$$

holds for all positive time t where ω_0 is the volume of the unit ball in $\mathbb{R}^d$. In particular

$$\|u(at)\|_{L^2(\mathbb{R}^d)} \le \frac{\|u(0)\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{rd}{2}}} + \frac{(2\omega_0)^{\frac{1}{2}}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}}\|u\|_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}. \tag{8.44}$$

REMARK 83. *Condition* (8.40) *can be replaced by $\widehat{J}(\xi) \le 1 - A|\xi|^2$ for $|\xi| \le R$ but omitting the constant A in the proof we simplify some formulas.*

REMARK 84. *The differential inequality* (8.42) *can be written in the following form:*

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(x,t)|^2\,dx \le -\frac{c}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x,t)-u(y,t))^2\,dx\,dy.$$

This is the nonlocal version of the energy method used in [**54**]. *However, in our case, exactly the same inequalities used in* [**54**] *could not be applied.*

PROOF. Let R and δ be as in (8.40) and (8.41). We set $a = rd/c\delta$ and consider the following set:

$$A(t) = \left\{\xi \in \mathbb{R}^d : |\xi| \le M(t) = \left(\frac{2rd}{c(t+a)}\right)^{1/2}\right\}.$$

Inequality (8.42) gives us:

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(x,t)|^2 dx \le c\int_{\mathbb{R}^d} (\widehat{J}(\xi)-1)|\widehat{u}(\xi)|^2 d\xi \le c\int_{A(t)^c} (\widehat{J}(\xi)-1)|\widehat{u}(\xi)|^2 d\xi. \tag{8.45}$$

Using the hypotheses (8.40) and (8.41) on the function $\widehat{J}$ the following inequality holds for all $\xi \in A(t)^c$:

$$c(\widehat{J}(\xi)-1) \leq -\frac{rd}{t+a}, \qquad \text{for every } \xi \in A(t)^c, \tag{8.46}$$

since for any $|\xi| \geq R$

$$c(\widehat{J}(\xi)-1) \leq -c\delta = -\frac{rd}{a} \leq -\frac{rd}{t+a}$$

and

$$c(\widehat{J}(\xi)-1) \leq -\frac{c|\xi|^2}{2} \leq -\frac{c}{2}\frac{2rd}{c(t+a)} = -\frac{rd}{t+a}$$

for all $\xi \in A(t)^c$ with $|\xi| \leq R$.

Introducing (8.46) in (8.45) we obtain

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{R}^d}|u(x,t)|^2dx &\leq -\frac{rd}{t+a}\int_{A(t)^c}|\widehat{u}(t,\xi)|^2d\xi \\
&\leq -\frac{rd}{t+a}\int_{\mathbb{R}^d}|\widehat{u}(t,\xi)|^2d\xi + \frac{rd}{t+a}\int_{|\xi|\leq M(t)}|\widehat{u}(t,\xi)|^2d\xi \\
&\leq -\frac{rd}{t+a}\int_{\mathbb{R}^d}|u(x,t)|^2dx + \frac{rd}{t+a}M(t)^d\omega_0\|\widehat{u}(t)\|^2_{L^\infty(\mathbb{R}^d)} \\
&\leq -\frac{rd}{t+a}\int_{\mathbb{R}^d}|u(x,t)|^2dx + \frac{rd}{t+a}\left[\frac{2rd}{c(t+a)}\right]^{\frac{d}{2}}\omega_0\|u(t)\|^2_{L^1(\mathbb{R}^d))}.
\end{aligned}$$

This implies that

$$\begin{aligned}
&\frac{d}{dt}\left[(t+a)^{rd}\int_{\mathbb{R}^d}|u(x,t)|^2dx\right] \\
&\qquad = (t+a)^{rd}\left[\frac{d}{dt}\int_{\mathbb{R}^d}|u(x,t)|^2dx\right] + rd(t+a)^{rd-1}\int_{\mathbb{R}^d}|u(x,t)|^2dx \\
&\qquad \leq (t+a)^{rd-\frac{d}{2}-1}rd\left(\frac{2rd}{c}\right)^{\frac{d}{2}}\omega_0\|u(t)\|^2_{L^1(\mathbb{R}^d)}.
\end{aligned}$$

Integrating on the time variable the last inequality we obtain:

$$(t+a)^{rd}\int_{\mathbb{R}^d}|u(x,t)|^2dx - a^{rd}\int_{\mathbb{R}^d}|u(0,x)|^2dx \leq rd\omega_0\left(\frac{2rd}{c}\right)^{\frac{d}{2}}\int_0^t(s+a)^{rd-\frac{d}{2}-1}\|u(s)\|^2_{L^1(\mathbb{R}^d)}ds$$

and hence

$$\begin{aligned}
\int_{\mathbb{R}^d}|u(x,t)|^2\,dx &\leq \frac{a^{rd}}{(t+a)^{rd}}\int_{\mathbb{R}^d}|u(0,x)|^2\,dx \\
&\quad + \frac{rd\omega_0}{(t+a)^{rd}}\left(\frac{2rd}{c}\right)^{\frac{d}{2}}\int_0^t(s+a)^{rd-\frac{d}{2}-1}\|u(s)\|^2_{L^1(\mathbb{R}^d)}\,ds.
\end{aligned}$$

Replacing t by ta we get:

$$\begin{aligned}\int_{\mathbb{R}^d} |u(at,x)|^2\,dx &\leq \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{rd\omega_0}{(t+1)^{rd}a^{rd}}\left(\frac{2rd}{c}\right)^{\frac{d}{2}} \int_0^{at} (s+a)^{rd-\frac{d}{2}-1}\|u(s)\|^2_{L^1(\mathbb{R}^d)}\,ds \\ &= \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{rd\omega_0}{(t+1)^{rd}}\left(\frac{2rd}{ca}\right)^{\frac{d}{2}} \int_0^{t} (s+1)^{rd-\frac{d}{2}-1}\|u(as)\|^2_{L^1(\mathbb{R}^d)}\,ds \\ &= \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{rd\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}} \int_0^{t} (s+1)^{rd-\frac{d}{2}-1}\|u(as)\|^2_{L^1(\mathbb{R}^d)}\,ds\end{aligned}$$

which proves (8.43).

Estimate (8.44) is obtained as follows:

$$\begin{aligned}\int_{\mathbb{R}^d} |u(at,x)|^2 dx &\leq \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{rd\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}}\|u\|^2_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}\int_0^t (s+1)^{rd-\frac{d}{2}-1}ds \\ &\leq \frac{\|u(0)\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}} + \frac{2\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{\frac{d}{2}}}\|u\|^2_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}.\end{aligned}$$

This ends the proof. □

LEMMA 85. *Let $2 \leq p < \infty$. For any function $u : \mathbb{R}^d \mapsto \mathbb{R}^d$, $I(u)$ defined by*

$$I(u) = \int_{\mathbb{R}^d} (J * u - u)(x)|u(x)|^{p-1} sgn(u(x))\,dx$$

satisfies

$$\begin{aligned}I(u) &\leq \frac{4(p-1)}{p^2}\int_{\mathbb{R}^d}(J*|u|^{p/2} - |u|^{p/2})(x)|u(x)|^{p/2}\,dx \\ &= -\frac{2(p-1)}{p^2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(|u(y)|^{p/2} - |u(x)|^{p/2})^2\,dx\,dy.\end{aligned}$$

REMARK 86. *This result is a nonlocal counterpart of the well known identity*

$$\int_{\mathbb{R}^d} \Delta u\,|u|^{p-1} sgn(u)\,dx = -\frac{4(p-1)}{p^2}\int_{\mathbb{R}^d} |\nabla(|u|^{p/2})|^2\,dx.$$

PROOF. Using the symmetry of J, $I(u)$ can be written in the following manner,

$$\begin{aligned}I(u) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(y)-u(x))|u(x)|^{p-1}sgn(u(x))\,dx\,dy \\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x)-u(y))|u(y)|^{p-1}sgn(u(y))\,dx\,dy.\end{aligned}$$

Thus

$$I(u) = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x)-u(y))\left(|u(x)|^{p-1}sgn(u(x)) - |u(y)|^{p-1}sgn(u(y))\right)\,dx\,dy.$$

Using the following inequality,

$$||\alpha|^{p/2} - |\beta|^{p/2}|^2 \leq \frac{p^2}{4(p-1)}(\alpha - \beta)(|\alpha|^{p-1} sgn(\alpha) - |\beta|^{p-1} sgn(\beta))$$

which holds for all real numbers α and β and for every $2 \leq p < \infty$, we obtain that $I(u)$ can be bounded from above as follows:

$$\begin{aligned} I(u) &\leq -\frac{4(p-1)}{2p^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x-y)(|u(y)|^{p/2} - |u(x)|^{p/2})^2 \, dx \, dy \\ &= -\frac{4(p-1)}{2p^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} J(x-y)(|u(y)|^p - 2|u(y)|^{p/2}|u(x)|^{p/2} + |u(x)|^p) \, dx \, dy \\ &= \frac{4(p-1)}{p^2} \int_{\mathbb{R}^d} (J * |u|^{p/2} - |u|^{p/2})(x)|u(x)|^{p/2} \, dx. \end{aligned}$$

The proof is finished. □

Now we are ready to proceed with the proof of Theorem 69.

Proof of Theorem 69. Let u be the solution to the nonlocal convection-diffusion problem. Then, by the same arguments that we used to control the $L^1(\mathbb{R}^d)$-norm, we obtain the following:

$$\begin{aligned} \frac{d}{dt} \int_{\mathbb{R}^d} |u(x,t)|^p dx &= p \int_{\mathbb{R}^d} (J * u - u)(x,t)|u(x,t)|^{p-1} sgn(u(x,t)) \, dx \\ &\quad + \int_{\mathbb{R}^d} (G * f(u) - f(u))(x,t)|u(x,t)|^{p-1} sgn(u(x,t)) \, dx \\ &\leq p \int_{\mathbb{R}^d} (J * u - u)(x,t)|u(x,t)|^{p-1} sgn(u(x,t)) \, dx. \end{aligned}$$

Using Lemma 85 we get that the $L^p(\mathbb{R}^d)$-norm of the solution u satisfies the following differential inequality:

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(x,t)|^p \, dx \leq \frac{4(p-1)}{p} \int_{\mathbb{R}^d} (J * |u|^{p/2} - |u|^{p/2})(x)|u(x)|^{p/2} \, dx. \tag{8.47}$$

First, let us consider $p = 2$. Then

$$\frac{d}{dt} \int_{\mathbb{R}^d} |u(x,t)|^2 \, dx \leq 2 \int_{\mathbb{R}^d} (J * |u| - |u|)(x,t)|u(x,t)| \, dx.$$

Applying Lemma 82 with $|u|$, $c=2$, $r=1$ and using that $\|u\|_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}\leq \|u_0\|_{L^1(\mathbb{R}^d)}$ we obtain

$$\begin{aligned}\|u(td/2\delta)\|_{L^2(\mathbb{R})} &\leq \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{d}{2}}}+\frac{(2\omega_0)^{\frac12}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}}\|u\|_{L^\infty([0,\infty);\,L^1(\mathbb{R}^d))}\\ &\leq \frac{\|u_0\|_{L^2(\mathbb{R}^d)}}{(t+1)^{\frac{d}{2}}}+\frac{(2\omega_0)^{\frac12}(2\delta)^{\frac{d}{4}}}{(t+1)^{\frac{d}{4}}}\|u_0\|_{L^1(\mathbb{R}^d))}\\ &\leq \frac{C(J,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})}{(t+1)^{\frac{d}{4}}},\end{aligned}$$

which proves (8.7) in the case $p=2$. Using that the $L^1(\mathbb{R}^d)$-norm of the solutions to (8.2), does not increase, $\|u(t)\|_{L^1(\mathbb{R}^d)}\leq\|u_0\|_{L^1(\mathbb{R}^d)}$, by Hölder's inequality we obtain the desired decay rate (8.7) in any $L^p(\mathbb{R}^d)$-norm with $p\in[1,2]$.

In the following, using an inductive argument, we will prove the result for any $r=2^m$, with $m\geq 1$ an integer. By Hölder's inequality this will give us the $L^p(\mathbb{R}^d)$-norm decay for any $2<p<\infty$.

Let us choose $r=2^m$ with $m\geq 1$ and assume that the following

$$\|u(t)\|_{L^r(\mathbb{R}^d)}\leq C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{r})}$$

holds for some positive constant $C=C(J,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})$ and for every positive time t. We want to show an analogous estimate for $p=2r=2^{m+1}$.

We use (8.47) with $p=2r$ to obtain the following differential inequality:

$$\frac{d}{dt}\int_{\mathbb{R}^d}|u(x,t)|^{2r}dx\leq\frac{4(2r-1)}{2r}\int_{\mathbb{R}^d}(J*|u|^r-|u|^r)(x,t)|u(x,t)|^r dx.$$

Applying Lemma (82) with $|u|^r$, $c(r)=2(2r-1)/r$ and $a=rd/c(r)\delta$ we get:

$$\begin{aligned}\int_{\mathbb{R}^d}|u(at)|^{2r} &\leq \frac{\|u_0^r\|^2_{L^2(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{d\omega_0(2\delta)^{\frac{d}{2}}}{(t+1)^{rd}}\int_0^t(s+1)^{rd-\frac{d}{2}-1}\|u^r(as)\|^2_{L^1(\mathbb{R}^d)}ds\\ &\leq \frac{\|u_0\|^{2r}_{L^{2r}(\mathbb{R}^d)}}{(t+1)^{rd}}+\frac{C(J)}{(t+1)^{rd}}\int_0^t(s+1)^{rd-\frac{d}{2}-1}\|u(as)\|^{2r}_{L^r(\mathbb{R}^d)}ds\\ &\leq \frac{C(J,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})}{(t+1)^{d}}\times\\ &\quad\left(1+\int_0^t(s+1)^{rd-\frac{d}{2}-1}(s+1)^{-dr(1-\frac{1}{r})}ds\right)\\ &\leq \frac{C}{(t+1)^{dr}}(1+(t+1)^{\frac{d}{2}})\leq C(t+1)^{\frac{d}{2}-dr}\end{aligned}$$

and then

$$\|u(at)\|_{L^{2r}(\mathbb{R}^d)}\leq C(J,\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})(t+1)^{-\frac{d}{2}(1-\frac{1}{2r})},$$

which finishes the proof. □

Let us close this chapter with a remark concerning the applicability of energy methods to study nonlocal problems.

REMARK 87. *If we want to use energy estimates to get decay rates (for example in $L^2(\mathbb{R}^d)$), we arrive easily to*

$$\frac{d}{dt}\int_{\mathbb{R}^d} |w(x,t)|^2\,dx = -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(w(x,t)-w(y,t))^2\,dx\,dy$$

when we deal with a solution of the linear equation $w_t = J*w - w$ and to

$$\frac{d}{dt}\int_{\mathbb{R}^d} |u(x,t)|^2\,dx \le -\frac{1}{2}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x,t)-u(y,t))^2\,dx\,dy$$

when we consider the complete convection-diffusion problem. However, we can not go further since an inequality of the form

$$\left(\int_{\mathbb{R}^d} |u(x)|^p\,dx\right)^{\frac{2}{p}} \le C\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} J(x-y)(u(x)-u(y))^2\,dx\,dy$$

is not available for $p>2$.

0.22. Weakly nonlinear behaviour. In this section we find the leading order term in the asymptotic expansion of the solution to (8.2). We use ideas from [**54**] showing that the nonlinear term decays faster than the linear part.

We recall a previous result of [**68**] that extends to nonlocal diffusion problems the result of [**52**] in the case of the heat equation.

LEMMA 88. *Let $J\in\mathcal{S}(\mathbb{R}^d)$ such that*

$$\widehat{J}(\xi) - (1-|\xi|^2) \sim B|\xi|^3, \qquad \xi\sim 0,$$

for some constant B. For every $p\in[2,\infty)$, there exists some positive constant $C=C(p,J)$ such that

$$\|S(t)*\varphi - MH(t)\|_{L^p(\mathbb{R}^d)} \le Ce^{-t}\|\varphi\|_{L^p(\mathbb{R}^d)} + C\|\varphi\|_{L^1(\mathbb{R}^d,|x|)}\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}},\ t>0, \tag{8.48}$$

for every $\varphi\in L^1(\mathbb{R}^d, 1+|x|)$ with $M=\int_{\mathbb{R}}\varphi(x)\,dx$, where

$$H(t) = \frac{e^{-\frac{x^2}{4t}}}{(2\pi t)^{\frac{d}{2}}},$$

is the gaussian.

REMARK 89. *We can consider a condition like $\widehat{J}(\xi)-(1-A|\xi|^2)\sim B|\xi|^3$ for $\xi\sim 0$ and obtain as profile a modified Gaussian $H_A(t)=H(At)$, but we omit the tedious details.*

REMARK 90. *The case $p \in [1,2)$ is more subtle. The analysis performed in the previous sections to handle the case $p=1$ can be also extended to cover this case when the dimension d verifies $1 \le d \le 3$. Indeed in this case, if J satisfies $\widehat{J}(\xi) \sim 1 - A|\xi|^s$, $\xi \sim 0$, then s has to be grater than $[d/2]+1$ and $s=2$ to obtain the Gaussian profile.*

PROOF. We write $S(t) = e^{-t}\delta_0 + K_t$. Then it is sufficient to prove that

$$\|K_t * \varphi - MK_t\|_{L^p(\mathbb{R}^d)} \le C\|\varphi\|_{L^1(\mathbb{R}^d,|x|)} \langle t \rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}$$

and

$$t^{\frac{d}{2}(1-\frac{1}{p})}\|K_t - H(t)\|_{L^p(\mathbb{R}^d)} \le C\langle t \rangle^{-\frac{1}{2}}.$$

The first estimate follows by Lemma 74. The second one uses the hypotheses on $\widehat{J}$. A detailed proof can be found in [**68**]. □

Now we are ready to prove that the same expansion holds for solutions to the complete problem (8.2) when $q > (d+1)/d$.

PROOF OF THEOREM 70. In view of (8.48) it is sufficient to prove that

$$t^{-\frac{d}{2}(1-\frac{1}{p})}\|u(t) - S(t) * u_0\|_{L^p(\mathbb{R}^d)} \le C\langle t \rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.$$

Using the representation (8.23) we get that

$$\|u(t) - S(t) * u_0\|_{L^p(\mathbb{R}^d)} \le \int_0^t \|[S(t-s) * G - S(t-s)] * |u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)}\, ds.$$

We now estimate the right hand side term as follows: we will split it in two parts, one in which we integrate on $(0, t/2)$ and another one where we integrate on $(t/2, t)$. Concerning the second term, by Lemma 75, Theorem 69 we have,

$$\begin{aligned}
&\int_{t/2}^t \|[S(t-s) * G - S(t-s)] * |u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)} ds \\
&\qquad \le C(J,G) \int_{t/2}^t \langle t-s \rangle^{-\frac{1}{2}} \|u(s)\|^q_{L^{pq}(\mathbb{R}^d)} ds \\
&\qquad \le C(J,G,\|u_0\|_{L^1(\mathbb{R}^d)}, \|u_0\|_{L^\infty(\mathbb{R})}) \int_{t/2}^t \langle t-s \rangle^{-\frac{1}{2}} \langle s \rangle^{-\frac{d}{2}(q-\frac{1}{p})} ds \\
&\qquad \le C\langle t \rangle^{-\frac{d}{2}(q-\frac{1}{p})+\frac{1}{2}} \le Ct^{-\frac{d}{2}(1-\frac{1}{p})}\langle t \rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.
\end{aligned}$$

To bound the first term we proceed as follows,

$$\begin{aligned}
&\int_0^{t/2} \|[S(t-s)*G - S(t-s)]*|u(s)|^{q-1}u(s)\|_{L^p(\mathbb{R}^d)}\,ds\\
&\quad\le C(p,J,G)\int_0^{t/2}\langle t-s\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}(\||u(s)|^q\|_{L^1(\mathbb{R}^d)}+\||u(s)|^q\|_{L^p(\mathbb{R}^d)})\,ds\\
&\quad\le C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}\Big(\int_0^{t/2}\|u(s)\|^q_{L^q(\mathbb{R}^d)}ds+\int_0^{t/2}\|u(s)\|^q_{L^{pq}(\mathbb{R}^d)}ds\Big)\\
&\quad= C\langle t\rangle^{-\frac{d}{2}(1-\frac{1}{p})-\frac{1}{2}}(I_1(t)+I_2(t)).
\end{aligned}$$

By Theorem 69, for the first integral, $I_1(t)$, we have the following estimate:

$$I_1(t)\le\int_0^{t/2}\|u(s)\|^q_{L^q(\mathbb{R}^d)}ds\le C(\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})\int_0^{t/2}\langle s\rangle^{-\frac{d}{2}(q-1)}ds,$$

and an explicit computation of the last integral shows that

$$\langle t\rangle^{-\frac{1}{2}}\int_0^{t/2}\langle s\rangle^{-\frac{d}{2}(q-1)}ds\le C\langle t\rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.$$

Arguing in the same manner for I_2 we get

$$\begin{aligned}
\langle t\rangle^{-\frac{1}{2}}I_2(t) &\le C(\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac{1}{2}}\int_0^{t/2}\langle s\rangle^{-\frac{dq}{2}(1-\frac{1}{pq})}ds\\
&\le C(\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac{d}{2}(q-\frac{1}{p})+\frac{1}{2}}\\
&\le C(\|u_0\|_{L^1(\mathbb{R}^d)},\|u_0\|_{L^\infty(\mathbb{R}^d)})\langle t\rangle^{-\frac{d}{2}(q-1)+\frac{1}{2}}.
\end{aligned}$$

This ends the proof. □

CHAPTER 9

A nonlinear Neumann problem

In this chapter we turn our attention to nonlinear equations with Neumann boundary conditions. We study

$$P_\gamma^J(z_0) \quad \begin{cases} z_t(t,x) = \displaystyle\int_\Omega J(x-y)(u(t,y)-u(t,x))\,dy, & x \in \Omega,\ t>0, \\ z(t,x) \in \gamma(u(t,x)), & x \in \Omega,\ t>0, \\ z(0,x) = z_0(x), & x \in \Omega. \end{cases}$$

Here Ω is a bounded domain, $z_0 \in L^1(\Omega)$ and γ is a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$.

Solutions to $P_\gamma^J(z_0)$ will be understood in the following sense.

Definition 91. *A solution of $P_\gamma^J(z_0)$ in $[0,T]$ is a function $z \in W^{1,1}(]0,T[;L^1(\Omega))$ which satisfies $z(0,x) = z_0(x)$, a.e. $x \in \Omega$, and for which there exists $u \in L^2(0,T;L^2(\Omega))$, $z \in \gamma(u)$ a.e. in $Q_T = \Omega \times]0,T[$, such that*

$$z_t(t,x) = \int_\Omega J(x-y)(u(t,y)-u(t,x))\,dy \quad a.e \ in \ \ ]0,T[\times\Omega.$$

The main results of this chapter can be summarized as follows.

"Under some natural assumptions about the initial condition z_0, there exists a unique global solution to $P_\gamma^J(z_0)$. Moreover, a contraction principle holds, given two solutions z_i of $P_\gamma^J(z_{i0})$, $i=1,2$, then

$$\int_\Omega (z_1(t)-z_2(t))^+ \le \int_\Omega (z_{10}-z_{20})^+.$$

Respect to the asymptotic behaviour of the solution we prove that if γ is a continuous function, then

$$\lim_{t\to\infty} z(t) = \frac{1}{|\Omega|}\int_\Omega z_0,$$

strongly in $L^1(\Omega)$".

We can consider different maximal monotone graphs γ. For example, if $\gamma(r) = r^m$, problem $P_\gamma^J(z_0)$ corresponds to the nonlocal version of the porous medium (or fast diffusion) problems. Note also that γ may be multivalued, so we are considering the nonlocal version of various phenomena with phase changes like the multiphase Stefan problem, for which

$$\gamma(r) = \begin{cases} r-1 & \text{if} \quad r<0, \\ [-1,0] & \text{if} \quad r=0, \\ r & \text{if} \quad r>0. \end{cases}$$

Even γ can have a domain different from $\mathbb{R}$, which corresponds to obstacle problems.

0.23. Notations and preliminaries. In this section we collect some preliminaries and notations that will be used in the sequel. For a maximal monotone graph η in $\mathbb{R}\times\mathbb{R}$ and $r\in\mathbb{N}$ we denote by η_r the *Yosida approximation* of η, given by $\eta_r = r(I-(I+\frac{1}{r}\eta)^{-1})$. The function η_r is maximal monotone and Lipschitz. We recall the definition of the *main section* η^0 of η

$$\eta^0(s) := \begin{cases} \text{the element of minimal absolute value of } \eta(s) & \text{if } \eta(s)\neq\emptyset, \\ +\infty \quad \text{if } [s,+\infty)\cap D(\eta)=\emptyset, \\ -\infty \quad \text{if } (-\infty,s]\cap D(\eta)=\emptyset, \end{cases}$$

where $D(\eta)$ denotes the domain of η. The following properties hold: if $s\in D(\eta)$, $|\eta_r(s)|\leq|\eta^0(s)|$ and $\eta_r(s)\to\eta^0(s)$ as $r\to+\infty$, and if $s\notin D(\eta)$, $|\eta_r(s)|\to+\infty$ as $r\to+\infty$.

We will use the following notations, $\eta_- := \inf\operatorname{Ran}(\eta)$ and $\eta_+ := \sup\operatorname{Ran}(\eta)$, where $\operatorname{Ran}(\eta)$ denotes the range of η. If $0\in D(\eta)$, $j_\eta(r)=\int_0^r\eta^0(s)ds$ defines a convex l.s.c. function such that $\eta=\partial j_\eta$. If j_η^* is the Legendre transform of j_η then $\eta^{-1}=\partial j_\eta^*$.

Also we will denote by J_0 and P_0 the following sets of functions,

$$J_0 = \{j:\mathbb{R}\to[0,+\infty], \text{ convex and lower semi-continuos with } j(0)=0\},$$

$$P_0 = \left\{p\in C^\infty(\mathbb{R}): 0\leq p'\leq 1, \operatorname{supp}(p') \text{ is compact, and } 0\notin\operatorname{supp}(p)\right\}.$$

In [**19**] the following relation for $u,v\in L^1(\Omega)$ is defined,

$$u\ll v \text{ if and only if } \int_\Omega j(u)\,dx \leq \int_\Omega j(v)\,dx,$$

and the following facts are proved.

Proposition 92. *Let Ω be a bounded domain in $\mathbb{R}^N$.*
(i) For any $u,v\in L^1(\Omega)$, if $\int_\Omega up(u)\leq\int_\Omega vp(u)$ for all $p\in P_0$, then $u\ll v$.
(ii) If $u,v\in L^1(\Omega)$ and $u\ll v$, then $\|u\|_q\leq\|v\|_q$ for any $q\in[1,+\infty]$.
(iii) If $v\in L^1(\Omega)$, then $\{u\in L^1(\Omega): u\ll v\}$ is a weakly compact subset of $L^1(\Omega)$.

The following Poincaré's type inequality is given in [**34**], see also Chapter 2.

PROPOSITION 93. *Given J and Ω the quantity*

$$\beta_1 := \beta_1(J,\Omega) = \inf_{u\in L^2(\Omega),\int_\Omega u=0} \frac{\dfrac{1}{2}\displaystyle\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2\,dy\,dx}{\displaystyle\int_\Omega (u(x))^2\,dx} \tag{9.1}$$

is strictly positive. Consequently

$$\beta_1 \int_\Omega \left| u - \frac{1}{|\Omega|}\int_\Omega u \right|^2 \le \frac{1}{2}\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))^2\,dy\,dx, \qquad \forall u\in L^2(\Omega). \tag{9.2}$$

In order to obtain a generalized Poincaré's type inequality we need the following result.

PROPOSITION 94. *Let $\Omega\subset\mathbb{R}^N$ be a bounded open set and $k>0$. There exists a constant $C>0$ such that for any $K\subset\Omega$ with $|K|>k$, it holds*

$$\|u\|_{L^2(\Omega)} \le C\left(\left\| u - \frac{1}{|\Omega|}\int_\Omega u\right\|_{L^2(\Omega)} + \left|\int_K u\right| \right), \qquad \forall\, u\in L^2(\Omega). \tag{9.3}$$

PROOF. Assume the conclusion does not hold. Then, for every $n\in\mathbb{N}$ there exists $K_n\subset\Omega$ with $|K_n|>k$, and $u_n\in L^2(\Omega)$ satisfying

$$\|u_n\|_{L^2(\Omega)} \ge n\left(\left\| u_n - \frac{1}{|\Omega|}\int_\Omega u_n\right\|_{L^2(\Omega)} + \left|\int_{K_n} u_n\right| \right), \quad \forall\, n\in\mathbb{N}. \tag{9.4}$$

We normalize u_n by $\|u_n\|_{L^2(\Omega)}=1$ for all $n\in\mathbb{N}$, and consequently we can assume that

$$u_n \rightharpoonup u \qquad \text{weakly in } \ L^2(\Omega). \tag{9.5}$$

Moreover, by (9.4), we have

$$\left\| u_n - \frac{1}{|\Omega|}\int_\Omega u_n\right\|_{L^2(\Omega)} \le \frac{1}{n}, \quad \text{and} \quad \left|\int_{K_n} u_n\right| \le \frac{1}{n}, \quad \forall\, n\in\mathbb{N}. \tag{9.6}$$

Hence

$$u_n - \frac{1}{|\Omega|}\int_\Omega u_n \to 0 \qquad \text{in } \ L^2(\Omega),$$

and by (9.5) we get $u(x) = \frac{1}{|\Omega|}\int_\Omega u = \alpha$ for almost all $x\in\Omega$, and $u_n\to\alpha$ strongly in $L^2(\Omega)$. Since $\|u_n\|_{L^2(\Omega)}=1$ for each $n\in\mathbb{N}$, $\alpha\neq 0$. On the other hand, (9.6) implies

$$\lim_{n\to\infty}\int_{K_n} u_n = 0.$$

Since χ_{K_n} is bounded in $L^2(\Omega)$, we can extract a subsequence (still denoted by χ_{K_n}) such that

$$\chi_{K_n} \rightharpoonup \phi \qquad \text{weakly in } \ L^2(\Omega).$$

Moreover, ϕ is nonnegative and verifies

$$k \leq \lim_{n\to\infty} |K_n| = \lim_{n\to\infty} \int_\Omega \chi_{K_n} = \int_\Omega \phi.$$

Now, since $u_n \to \alpha$ strongly in $L^2(\Omega)$ and $\chi_{K_n} \to \phi$ weakly in $L^2(\Omega)$ we have

$$0 = \lim_{n\to\infty} \int_{K_n} u_n = \lim_{n\to\infty} \int_\Omega \chi_{K_n} u_n = \alpha \int_\Omega \phi \neq 0,$$

a contradiction. □

To simplify the notation we define the linear self-adjoint operator $A : L^2(\Omega) \to L^2(\Omega)$ by

$$Au(x) = \int_\Omega J(x-y)(u(y) - u(x))\, dy, \qquad x \in \Omega.$$

As a consequence of the above results we have the next proposition, which plays the role of the classical generalized Poincaré's inequality for Sobolev spaces.

Proposition 95. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set and $k > 0$. There exists a constant $C = C(J, \Omega, k)$ such that, for any $K \subset \Omega$ with $|K| > k$,*

$$\|u\|_{L^2(\Omega)} \leq C \left(\left(-\int_\Omega Au\, u \right)^{1/2} + \|u\|_{L^2(K)} \right) \qquad \forall u \in L^2(\Omega). \tag{9.7}$$

Using the above result and working as in the proof of Lemma 4.2 in [**4**], we obtain the following lemma, of which we give the proof for the sake of completeness.

Lemma 96. *Let γ be a maximal monotone graph in $\mathbb{R}^2$ such that $0 \in \gamma(0)$. Let $\{u_n\}_{n\in\mathbb{N}} \subset L^2(\Omega)$ and $\{z_n\}_{n\in\mathbb{N}} \subset L^1(\Omega)$ such that, for every $n \in \mathbb{N}$, $z_n \in \gamma(u_n)$ a.e. in Ω.*

Let us suppose that

(i) if $\gamma_+ = +\infty$, there exists $M > 0$ such that

$$\int_\Omega z_n^+ < M, \qquad \forall n \in \mathbb{N},$$

(ii) if $\gamma_+ < +\infty$, there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega z_n < M < \gamma_+ |\Omega|, \qquad \forall n \in \mathbb{N}$$

and

$$\int_{\{x\in\Omega : z_n(x) < -h\}} |z_n| < \frac{\gamma_+|\Omega| - M}{4}, \qquad \forall n \in \mathbb{N}.$$

Then, there exists a constant $C = C(M,\Omega)$ in case (i), $C = C(M,\Omega,\gamma,h)$ in case (ii), such that

$$\|u_n^+\|_{L^2(\Omega)} \le C\left(\left(-\int_\Omega Au_n^+\, u_n^+\right)^{1/2} + 1\right), \qquad \forall n \in \mathbb{N}. \tag{9.8}$$

Let us suppose that

(iii) if $\gamma_- = -\infty$, there exists $M > 0$ such that

$$\int_\Omega z_n^- < M, \qquad \forall n \in \mathbb{N},$$

(iv) if $\gamma_- > -\infty$, there exists $M \in \mathbb{R}$ and $h > 0$ such that

$$\int_\Omega z_n > M > \gamma_-|\Omega|, \qquad \forall n \in \mathbb{N}$$

and

$$\int_{\{x\in\Omega : z_n(x) > h\}} z_n < \frac{M - \gamma_-|\Omega|}{4}, \qquad \forall n \in \mathbb{N}.$$

Then, there exists a constant $\tilde{C} = \tilde{C}(M,\Omega)$ in case (iii), $\tilde{C} = \tilde{C}(M,\Omega,\gamma,h)$ in case (iv), such that

$$\|u_n^-\|_{L^2(\Omega)} \le \tilde{C}\left(\left(-\int_\Omega Au_n^-\, u_n^-\right)^{1/2} + 1\right), \qquad \forall n \in \mathbb{N}. \tag{9.9}$$

PROOF. Let us only prove (9.8), since the proof of (9.9) is similar. First, consider the case $\gamma_+ = +\infty$. Then, by assumption, there exists $M > 0$ such that

$$\int_\Omega z_n^+ < M, \qquad \forall n \in \mathbb{N}.$$

For $n \in \mathbb{N}$ let $K_n = \left\{x \in \Omega : z_n^+(x) < \frac{2M}{|\Omega|}\right\}$. Then

$$0 \le \int_{K_n} z_n^+ = \int_\Omega z_n^+ - \int_{\Omega\setminus K_n} z_n^+ \le M - (|\Omega| - |K_n|)\frac{2M}{|\Omega|} = |K_n|\frac{2M}{|\Omega|} - M.$$

Therefore,

$$|K_n| \ge \frac{|\Omega|}{2},$$

and

$$\|u_n^+\|_{L^2(K_n)} \le |K_n|^{1/2} \sup \gamma^{-1}\left(\frac{2M}{|\Omega|}\right).$$

Then, by Proposition 95, for all $n \in \mathbb{N}$,

$$\|u_n^+\|_{L^2(\Omega)} \le \tilde{C}(J,\Omega)\left(\left(-\int_\Omega Au_n^+\, u_n^+\right)^{1/2} + |\Omega|^{1/2} \sup \gamma^{-1}\left(\frac{2M}{|\Omega|}\right)\right).$$

Now, let us consider the case $\gamma_+ < +\infty$. Let

$$\delta = \gamma_+|\Omega| - M.$$

By assumption, for every $n \in \mathbb{N}$, we have,

$$\int_\Omega z_n < \gamma_+|\Omega| - \delta. \tag{9.10}$$

For $n \in \mathbb{N}$, let $K_n = \left\{x \in \Omega : z_n(x) < \gamma_+ - \frac{\delta}{2|\Omega|}\right\}$. Then, by (9.10),

$$\int_{K_n} z_n = \int_\Omega z_n - \int_{\Omega\setminus K_n} z_n < -\frac{\delta}{2} + |K_n|\left(\gamma_+ - \frac{\delta}{2|\Omega|}\right).$$

Moreover,

$$\int_{K_n} z_n = -\int_{K_n\cap\{x\in\Omega: z_n<-h\}} |z_n| + \int_{K_n\cap\{x\in\Omega: z_n\geq -h\}} z_n \geq -\frac{\delta}{4} - h|K_n|.$$

Therefore,

$$|K_n|\left(h - \frac{\delta}{2|\Omega|} + \gamma_+\right) \geq \frac{\delta}{4}.$$

Hence $|K_n| > 0$, $h - \frac{\delta}{2|\Omega|} + \gamma_+ > 0$ and

$$|K_n| \geq \frac{\delta}{4\left(h - \frac{\delta}{2|\Omega|} + \gamma_+\right)}.$$

Consequently,

$$\|u_n^+\|_{L^2(K_n)} \leq |K_n|^{1/2} \sup\gamma^{-1}\left(\gamma_+ - \frac{\delta}{2|\Omega|}\right).$$

Then, by Proposition 95,

$$\|u_n^+\|_{L^2(\Omega)} \leq \tilde{C}(J,\Omega,\gamma,h)\left(\left(-\int_\Omega Au_n^+\,u_n^+\right)^{1/2} + |\Omega|^{1/2}\sup\gamma^{-1}\left(\gamma_+ - \frac{\delta}{2|\Omega|}\right)\right).$$

This ends the proof of (9.8). $\square$

Finally, we have the following monotonicity result. Its proof is straightforward.

LEMMA 97. *Let $T : \mathbb{R} \to \mathbb{R}$ a nondecreasing function. For every $u \in L^2(\Omega)$ such that $T(u) \in L^2(\Omega)$, it holds*

$$\begin{aligned} -\int_\Omega Au(x)T(u(x))\,dx = & -\int_\Omega\int_\Omega J(x-y)(u(y)-u(x))\,dy\,T(u(x))\,dx \\ = & \ \frac{1}{2}\int_\Omega\int_\Omega B(x,y)(u(y)-u(x))^2\,dy\,dx, \end{aligned}$$

where $B(x,y)$ is the non negative symmetric function given by

$$B(x,y) = \begin{cases} J(x-y)\dfrac{T(u(y))-T(u(x))}{u(y)-u(x)} & \text{if } u(y) \neq u(x), \\ 0 & \text{if } u(y) = u(x). \end{cases}$$

In particular we have

$$-\int_\Omega Au(x)\, u(x)\, dx = \frac{1}{2}\int_\Omega \int_\Omega J(x-y)(u(y)-u(x))^2\, dy\, dx.$$

0.24. Mild solutions and contraction principle. In this section we obtain a mild solution to our problem studying the associated integral operator.

Given a maximal monotone graph γ in $\mathbb{R}^2$ such that $0 \in \gamma(0)$, $\gamma_- < \gamma_+$, we consider the problem,

$$(S_\phi^\gamma) \qquad \gamma(u) - Au \ni \phi \quad \text{in } \Omega.$$

DEFINITION 98. *Let $\phi \in L^1(\Omega)$. A pair of functions $(u,z) \in L^2(\Omega)\times L^1(\Omega)$ is a solution of problem (S_ϕ^γ) if $z(x) \in \gamma(u(x))$ a.e. $x \in \Omega$ and $z(x) - Au(x) = \phi(x)$ a.e. $x \in \Omega$, that is,*

$$z(x) - \int_\Omega J(x-y)(u(y)-u(x))\, dy = \phi(x) \qquad \text{a.e. } x \in \Omega.$$

With respect to uniqueness of problem (S_ϕ^γ), we have the following maximum principle.

THEOREM 99.
(i) Let $\phi_1 \in L^1(\Omega)$ and (u_1,z_1) a subsolution of $(S_{\phi_1}^\gamma)$, that is, $z_1(x) \in \gamma(u_1(x))$ a.e. $x \in \Omega$ and $z_1(x) - Au_1(x) \le \phi_1(x)$ a.e. $x \in \Omega$, and let $\phi_2 \in L^1(\Omega)$ and (u_2,z_2) a supersolution of $(S_{\phi_2}^\gamma)$, that is, $z_2(x) \in \gamma(u_2(x))$ a.e. $x \in \Omega$ and $z_2(x) - Au_2(x) \ge \phi_2(x)$ a.e. $x \in \Omega$. Then

$$\int_\Omega (z_1 - z_2)^+ \le \int_\Omega (\phi_1 - \phi_2)^+.$$

Moreover, if $\phi_1 \le \phi_2$, $\phi_1 \neq \phi_2$, then $u_1(x) \le u_2(x)$ a.e. $x \in \Omega$.
(ii) Let $\phi \in L^1(\Omega)$, and (u_1,z_1), (u_2,z_2) two solutions of (S_ϕ^γ). Then, $z_1 = z_2$ a.e. and there exists a constant c such that $u_1 = u_2 + c$, a.e.

PROOF. To prove *(i)*, let (u_1,z_1) a subsolution of $(S_{\phi_1}^\gamma)$ and (u_2,z_2) a supersolution of $(S_{\phi_2}^\gamma)$. Then

$$-(Au_1(x) - Au_2(x)) + z_1(x) - z_2(x) \le \phi_1(x) - \phi_2(x).$$

Multiplying the above inequality by $\frac{1}{k}T_k^+(u_1 - u_2 + k\,\mathrm{sign}_0^+(z_1 - z_2))$ and integrating we get,

$$\begin{aligned}
&\int_\Omega (z_1 - z_2)\frac{1}{k}T_k^+(u_1 - u_2 + k\,\mathrm{sign}_0^+(z_1 - z_2)) \\
&\qquad - \int_\Omega (Au_1(x) - Au_2(x))\frac{1}{k}T_k^+(u_1(x) - u_2(x) + k\,\mathrm{sign}_0^+(z_1(x) - z_2(x)))\,dx \\
&\le \int_\Omega (\phi_1 - \phi_2)\frac{1}{k}T_k^+(u_1 - u_2 + k\,\mathrm{sign}_0^+(z_1 - z_2)) \le \int_\Omega (\phi_1(x) - \phi_2(x))^+\,dx.
\end{aligned} \tag{9.11}$$

Let us write $u = u_1 - u_2$ and $z = \mathrm{sign}_0^+(z_1 - z_2)$, then, by the monotonicity proved in Lemma 97,

$$\begin{aligned}
&\lim_{k\to 0}\int_\Omega (Au_1(x) - Au_2(x))\frac{1}{k}T_k^+(u_1(x) - u_2(x) + k\,\mathrm{sign}_0^+(z_1(x) - z_2(x)))\,dx \\
&\quad = \lim_{k\to 0}\int_\Omega Au(x)\frac{1}{k}T_k^+(u(x) + kz(x))\,dx \\
&\quad = \lim_{k\to 0}\int_\Omega A(u + kz)(x)\frac{1}{k}T_k^+(u(x) + kz(x))\,dx \le 0.
\end{aligned}$$

Therefore, taking limit as k goes to 0 in (9.11), we obtain

$$\int_\Omega (z_1 - z_2)^+ \le \int_\Omega (\phi_1 - \phi_2)^+.$$

Let us now suppose that $\phi_1 \le \phi_2$, $\phi_1 \ne \phi_2$. By the previous calculations we know that $z_1 \le z_2$. Since

$$\int_\Omega z_1 \le \int_\Omega \phi_1 < \int_\Omega \phi_2 \le \int_\Omega z_2,$$

$z_1 \ne z_2$. Going back to (9.11), if $u = u_1 - u_2$, we get

$$-\int_\Omega Au(x)\,T_k^+(u(x))\,dx = 0,$$

and therefore,

$$-\int_\Omega Au(x)\,u^+(x)\,dx = 0.$$

Consequently, by Lemma 97, there exists a null set $C \subset \Omega \times \Omega$ such that

$$J(x-y)(u^+(y) - u^+(x))(u(y) - u(x)) = 0 \quad \text{for all } (x,y) \in \Omega \times \Omega \setminus C. \tag{9.12}$$

Let B a null subset of Ω such that if $x \notin B$, the section $C_x = \{y \in \Omega : (x,y) \in C\}$ is null. Let $x \notin B$, if $u(x) > 0$ then, since there exists $r_0 > 0$ such that $J(z) > 0$ for every z such that $|z| \le r_0$, by a compactness argument and having in mind (9.12), it is easy to see that $u(y) = u(x) > 0$ for all $y \notin C_x$. Therefore $u_1(y) > u_2(y)$ for all $y \notin C_x$ in Ω and consequently $z_1(y) \ge z_2(y)$ *a.e.* in Ω which contradicts that $z_1 \le z_2$, $z_1 \ne z_2$.

Let us now prove *(ii)*. As (u_i, z_i) are solutions of (S_ϕ^γ) we have that

$$-(Au_1(x) - Au_2(x)) + z_1(x) - z_2(x) = 0.$$

Now, by *(i)*, $z_1 = z_2$, *a.e.* Consequently,

$$0 = \ -(Au_1(x) - Au_2(x)) = -A(u_1 - u_2)(x).$$

Therefore, multiplying the above equation by $u = u_1 - u_2$ and integrating we obtain

$$\frac{1}{2}\int_\Omega \int_\Omega J(x-y)(u(y)-u(x))^2\, dy\, dx = 0.$$

From here, by (9.2), u is constant *a.e.* in Ω. □

In particular we have the following result.

COROLLARY 100. *Let $k > 0$ and $u \in L^2(\Omega)$ such that*

$$ku - Au \geq 0 \quad a.e.\ in\ \Omega,$$

then $u \geq 0$ a.e. in Ω.

PROOF. Since (u, ku) is a supersolution of (S_0^γ), where $\gamma(r) = kr$, and $(0,0)$ is a subsolution of (S_0^γ), by Theorem 99, the result follows. □

To study the existence of solutions of problem (S_ϕ^γ) we start with the following two lemmas.

LEMMA 101. *Assume $\gamma : \mathbb{R} \to \mathbb{R}$ is a nondecreasing Lipschitz continuous function with $\gamma(0) = 0$ and $\gamma_- < \gamma_+$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution $(u, \gamma(u))$ of problem (S_ϕ^γ). Moreover, $\gamma(u) \ll \phi$.*

PROOF. Since $\gamma_- < \phi < \gamma_+$ and $\phi \in C(\overline{\Omega})$, we can find $c_1 \leq c_2$ such that

$$\gamma_- < \gamma(c_1) \leq \phi(x) \leq \gamma(c_2) < \gamma_+ \quad \forall\, x \in \Omega. \tag{9.13}$$

Since γ is a nondecreasing Lipschitz continuous function there exists $k > 0$ for which the function $s \mapsto ks - \gamma(s)$ is nondecreasing. Let us see by induction that we can find a sequence $\{u_i\} \subset L^2(\Omega)$ such that

$$u_0 = c_1, \qquad u_i \leq u_{i+1} \leq c_2,$$

$$ku_{i+1} - Au_{i+1} = \phi - \gamma(u_i) + ku_i, \qquad \forall\, i \in \mathbb{N}. \tag{9.14}$$

Since $k > 0$, as a consequence of being A self-adjoint, it is easy to see that k does not belong to the spectrum of A, then there exists $u_1 \in L^2(\Omega)$ such that

$$ku_1 - Au_1 = \phi - \gamma(c_1) + kc_1.$$

Then, by (9.13), we have

$$ku_1 - Au_1 = \phi - \gamma(c_1) + kc_1 \geq kc_1 = kc_1 - Ac_1.$$

Hence, from Corollary 100 we get that $u_0 = c_1 \leq u_1$. Analogously, there exists u_2 such that

$$ku_2 - Au_2 = \phi - \gamma(u_1) + ku_1.$$

Now, since $c_1 \leq u_1$, we get

$$ku_2 - Au_2 \geq \phi - \gamma(c_1) + kc_1 = ku_1 - Au_1.$$

Again by Corollary 100, we get $u_1 \leq u_2$, and by induction we obtain that $u_i \leq u_{i+1}$. On the other hand, since the function $s \mapsto ks - \gamma(s)$ is nondecreasing, $c_1 \leq c_2$ and (9.13), we have

$$kc_2 - Ac_2 \geq \phi - \gamma(c_2) + kc_2 \geq \phi - \gamma(c_1) + kc_1 = ku_1 - Au_1.$$

Applying again Corollary 100, we get $c_2 \geq u_1$, and by an inductive argument we deduce that $u_i \leq c_2$ for all $i \in \mathbb{N}$. Hence (9.14) holds. Consequently, there exists $u \in L^\infty(\Omega)$, such that $u(x) = \lim_{i\to+\infty} u_i(x)$ *a.e.* in Ω. Taking limits in (9.14), we obtain that

$$ku - Au = \phi - \gamma(u) + ku,$$

and $(u, \gamma(u))$ is a solution of problem (S_ϕ^γ), that is,

$$\gamma(u) - Au = \phi. \tag{9.15}$$

Finally, given $p \in P_0$, multiplying (9.15) by $p(\gamma(u))$, and integrating in Ω, we get

$$\int_\Omega \gamma(u(x))p(\gamma(u(x)))\,dx - \int_\Omega Au(x)p(\gamma(u(x)))\,dx = \int_\Omega \phi(x)p(\gamma(u(x)))\,dx.$$

Now, by Lemma 97, the second term in the above equality is nonnegative, therefore

$$\int_\Omega \gamma(u(x))p(\gamma(u(x)))\,dx \leq \int_\Omega \phi(x)p(\gamma(u(x)))\,dx.$$

By Proposition 92, we conclude that $\gamma(u) \ll \phi$. □

Lemma 102. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $]-\infty, 0] \subset D(\gamma)$, $0 \in \gamma(0)$, $\gamma_- < \gamma_+$. Let $\tilde{\gamma}(s) = \gamma(s)$ if $s < 0$, $\tilde{\gamma}(s) = 0$ if $s \geq 0$. Assume $\tilde{\gamma}$ is Lipschitz continuous in $]-\infty, 0]$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution (u, z) of (S_ϕ^γ). Moreover, $z \ll \phi$.*

Proof. If $\gamma_- < 0$, let c_1 such that $\gamma(c_1) = \{m_1\}$, $\gamma_- < m_1 < 0$ and $m_1 \leq \phi$. And if $\gamma_- = 0$ let $c_1 = m_1 = 0$. Let γ_r, $r \in \mathbb{N}$, be the Yosida approximation of γ and let the maximal monotone graph

$$\gamma^r(s) = \begin{cases} \gamma(s) & \text{if } s < 0, \\ \gamma_r(s) & \text{if } s \geq 0. \end{cases}$$

Observe that γ^r is a nondecreasing Lipschitz continuous function with $\gamma^r(0) = 0$ and, for r large enough, $\gamma_- = \gamma_-^r < \phi < \gamma_+^r$, $\gamma^r \leq \gamma^{r+1}$, and converges in the sense of maximal monotone graphs to γ. From the previous lemma, for each γ^r we obtain a solution (u_r, z_r) of $(S_\phi^{\gamma^r})$, that is, $z_r = \gamma^r(u_r)$ *a.e.* and

$$z_r - Au_r = \phi. \tag{9.16}$$

Moreover, $z_r \ll \phi$, and consequently, $z_r \geq m_1$. Moreover, $u_r \geq c_1$. Let

$$\hat{z}_r(x) = \begin{cases} z_r(x) & \text{if } \ u_r(x) \leq 0, \\ \gamma_{r+1}(u_r(x)) & \text{if } \ u_r(x) > 0. \end{cases}$$

Then, since γ_r is nondecreasing,

$$\hat{z}_r \geq z_r,$$

and also,

$$\hat{z}_r \in \gamma^{r+1}(u_r).$$

Therefore, $(u_r, \hat{z}_r)$ is a supersolution to $(S_\phi^{\gamma^{r+1}})$. Using Theorem 99, we obtain that

$$\hat{z}_r \geq z_{r+1}.$$

Now, if $\hat{z}_r = z_r$ then

$$z_r \geq z_{r+1},$$

and if $\hat{z}_r \neq z_r$, by Theorem 99,

$$u_r \geq u_{r+1}.$$

So, there exists a monotone non increasing subsequence of $\{u_r\}$, denoted equal, with $u_r \geq \hat{c}_1$, or there exists a monotone non increasing subsequence of $\{z_r\}$, denoted equal, with $z_r \geq m_1$. In the first case, we have that

$$u_r \to u \quad \text{in } L^2(\Omega),$$

and also, since $z_r \ll \phi$,

$$z_r \to z \qquad \text{weakly in } L^1(\Omega).$$

And in the second case, we obtain

$$z_r \to z \quad \text{in } L^1(\Omega). \tag{9.17}$$

In fact, since $z_r \ll \phi$, we get that

$$z_r \to z \quad \text{in } L^2(\Omega). \tag{9.18}$$

Now, in this second case, multiplying (9.16) by $u_r - u_s$ and integrating we get

$$-\int_\Omega Au_r(u_r - u_s) = \int_\Omega \phi\,(u_r - u_s) - \int_\Omega z_r(u_r - u_s).$$

Moreover,

$$-\int_\Omega Au_s(u_r - u_s) = \int_\Omega \phi\,(u_r - u_s) - \int_\Omega z_s(u_r - u_s).$$

Hence, since $\int_\Omega z_r = \int_\Omega z_s$,

$$-\int_\Omega A(u_r - u_s)(u_r - u_s) = -\int_\Omega (z_r - z_s)(u_r - u_s)$$
$$= -\int_\Omega (z_r - z_s)\left(u_r - \frac{1}{|\Omega|}\int_\Omega u_r - \left(u_s - \frac{1}{|\Omega|}\int_\Omega u_s\right)\right)$$

and, by Proposition 93,

$$\beta_1 \left\| \left(u_r - \frac{1}{|\Omega|}\int_\Omega u_r\right) - \left(u_s - \frac{1}{|\Omega|}\int_\Omega u_s\right) \right\|_{L^2(\Omega)} \leq \|z_r - z_s\|_{L^2(\Omega)}.$$

From (9.18) we get,

$$u_r - \frac{1}{|\Omega|}\int_\Omega u_r \to w \quad \text{in } L^2(\Omega).$$

Let us see that $\left\{\frac{1}{|\Omega|}\int_\Omega u_r\right\}$ is bounded. If not, we can assume, passing to a subsequence if necessary, that it converges to $-\infty$. Then, $u_r \to -\infty$ *a.e.* in Ω. Since $z_r \in \gamma^r(u_r)$, $\gamma^r \to \gamma$ and (9.17), $z = \gamma_-$ *a.e.* in Ω. Consequently, $\int_\Omega \phi = \int_\Omega z = |\Omega|\gamma_-$ which contradicts that $\phi > \gamma_-$. Thus, $\left\{\frac{1}{|\Omega|}\int_\Omega u_r\right\}$ is bounded and we have that there exists a subsequence of $\{u_r\}$, denoted equal, such that

$$u_r \to u \quad \text{in } L^2(\Omega).$$

Therefore, in both cases, $z \in \gamma(u)$ *a.e.* in Ω, $z \ll \phi$, and, taking limit in

$$z_r - Au_r \ni \phi,$$

we obtain

$$z - Au \ni \phi,$$

which concludes the proof. □

With this lemma in mind we proceed to extend the result for general monotone graphs.

THEOREM 103. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $0 \in \gamma(0)$ and $\gamma_- < \gamma_+$. Let $\phi \in C(\overline{\Omega})$ such that $\gamma_- < \phi < \gamma_+$. Then, there exists a solution (u, z) of (S_ϕ^γ). Moreover, $z \ll \phi$.*

PROOF. Let γ_r, $r \in \mathbb{N}$, be the Yosida approximation of γ and let the maximal monotone graph

$$\gamma^r(s) = \begin{cases} \gamma(s) & \text{if } s > 0, \\ \gamma_r(s) & \text{if } s \leq 0. \end{cases}$$

Observe that γ^r satisfies the hypothesis of Lemma 102, $\gamma^r_- < \phi < \gamma^r_+$ for r large enough, $\gamma^r \geq \gamma^{r+1}$ and converges in the sense of maximal monotone graphs to γ. From the previous lemma, for each γ^r we obtain a solution (u_r, z_r) of $(S_\phi^{\gamma^r})$, $z_r \ll \phi$. Now, we can proceed similarly to the previous lemma passing to the limit to conclude. □

The natural space to study the problem $P_\gamma^J(z_0)$ from the point of view of Nonlinear Semigroup Theory is $L^1(\Omega)$. In this space we define the following operator,

$$B^\gamma := \left\{(z,\hat{z}) \in L^1(\Omega)\times L^1(\Omega) \ : \ \exists u \in L^2(\Omega) \text{ such that } (u,z) \text{ is a solution of } (S^\gamma_{z+\hat{z}})\right\},$$

in other words, $\hat{z} \in B^\gamma(z)$ if and only if there exists $u \in L^2(\Omega)$ such that $z(x) \in \gamma(u(x))$ *a.e.* in Ω, and

$$-\int_\Omega J(x-y)(u(y)-u(x))\,dy = \hat{z}(x), \qquad a.e. \ \ x \in \Omega. \tag{9.19}$$

The operator B^γ allows us to rewrite $P_\gamma^J(z_0)$ as the following abstract Cauchy problem in $L^1(\Omega)$,

$$\begin{cases} z'(t) + B^\gamma(z(t)) \ni 0 & t \in (0,T) \\ z(0) = z_0. \end{cases} \tag{9.20}$$

A direct consequence of Theorems 99 and 103 is the following result.

COROLLARY 104. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$, $0 \in \gamma(0)$. Then, the operator B^γ is T-accretive in $L^1(\Omega)$ and satisfies*

$$\left\{\phi \in C(\overline{\Omega}) : \gamma_- < \phi < \gamma_+\right\} \subset Ran(I + B^\gamma).$$

The following theorem is a consequence of the above result.

THEOREM 105. *Let $T > 0$ and $z_{i0} \in L^1(\Omega)$, $i = 1, 2$. Let z_i be a solution in $[0,T]$ of $P_\gamma^J(z_{i0})$, $i = 1, 2$. Then*

$$\int_\Omega (z_1(t) - z_2(t))^+ \le \int_\Omega (z_{10} - z_{20})^+ \tag{9.21}$$

for almost every $t \in]0,T[$.

PROOF. Let $(u_i(t), z_i(t))$ be solutions of $P_\gamma^J(z_{0i})$, $i = 1, 2$. Then, since they are strong solutions of (9.20) and A is T-accretive, (9.21) follows from the Nonlinear Semigroup Theory ([**20**]). ☐

In the next result we characterize $\overline{D(B^\gamma)}^{L^1(\Omega)}$.

THEOREM 106. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$. Then, we have*

$$\overline{D(B^\gamma)}^{L^1(\Omega)} = \left\{z \in L^1(\Omega) : \ \gamma_- \le z \le \gamma_+\right\}.$$

PROOF. It is obvious that

$$\overline{D(B^\gamma)}^{L^1(\Omega)} \subset \left\{z \in L^1(\Omega) : \ \gamma_- \le z \le \gamma_+\right\}.$$

To obtain the another inclusion, it is enough to take $\phi \in C(\overline{\Omega})$, satisfying $\gamma_- < \phi < \gamma_+$, and to prove that $\phi \in \overline{D(B^\gamma)}^{L^1(\Omega)}$. Let $a, b \in \mathbb{R}$ such that $\gamma_- < a < \phi < b < \gamma_+$.

Now, by Theorem 103, for any $n \in \mathbb{N}$, there exists $v_n := \left(I + \frac{1}{n}B^\gamma\right)^{-1}\phi \in D(B^\gamma)$. Then, $(v_n, n(\phi - v_n)) \in B^\gamma$, thus there exists $u_n \in L^2(\Omega)$ such that $v_n \in \gamma(u_n)$ *a.e.* in Ω and

$$v_n(x) - \frac{1}{n}\int_\Omega J(x-y)(u_n(y) - u_n(x))\, dy = \phi(x) \qquad \forall\, x \in \Omega. \tag{9.22}$$

Moreover, $v_n \ll \phi$. Then,

$$-\infty < \inf \gamma^{-1}(a) \leq u_n \leq \sup \gamma^{-1}(b) < +\infty. \tag{9.23}$$

Hence, from (9.22) and (9.23) it follows that $v_n \to \phi$ in $L^1(\Omega)$. □

As a consequence of the above results we have the following theorem concerning mild solutions (see [**20**]).

THEOREM 107. *Assume γ is a maximal monotone graph in $\mathbb{R}^2$. Let $T > 0$ and let $z_0 \in L^1(\Omega)$ satisfying $\gamma_- \leq z_0 \leq \gamma_+$. Then, there exists a unique mild solution of* (9.20). *Moreover $z \ll z_0$.*

PROOF. For $n \in \mathbb{N}$, let $\varepsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \varepsilon$. Let $z_0^\varepsilon \in C(\overline{\Omega})$ with

$$\gamma_- < z_0^\varepsilon < \gamma_+$$

and

$$\|z_0^\varepsilon - z_0\|_{L^1(\Omega)} \leq \varepsilon.$$

By Theorem 103, for n large enough, there exists a solution $(u_i^\varepsilon, z_i^\varepsilon)$ of

$$\gamma(u_i^\varepsilon) - \varepsilon A u_i^\varepsilon \ni z_{i-1}^\varepsilon \tag{9.24}$$

for $i = 1, \ldots, n$, with

$$z_i^\varepsilon \ll z_{i-1}^\varepsilon. \tag{9.25}$$

That is, there exists a unique solution $z_i^\varepsilon \in L^1(\Omega)$ of the time discretized scheme associated with (9.20),

$$z_i^\varepsilon + \varepsilon B^\gamma z_i^\varepsilon \ni \varepsilon z_{i-1}^\varepsilon, \qquad \text{for} \quad i = 1, \ldots, n.$$

Therefore, if we define $z_\varepsilon(t)$ by

$$\begin{cases} z_\varepsilon(0) = z_0^\varepsilon, \\ z_\varepsilon(t) = z_i^\varepsilon, \quad \text{for} \quad t \in]t_{i-1}, t_i], \quad i = 1, \ldots, n, \end{cases}$$

it is an ε-approximate solution of problem (10.3).

By using now the Nonlinear Semigroup Theory (see [**18**], [**20**], [**49**]), on account of Corollary 104 and Theorem 106, problem (9.20) has a unique mild-solution $z(t) \in C([0,T] : L^1(\Omega))$, obtained as $z(t) = L^1(\Omega)\text{-}\lim_{\varepsilon \to 0} z_\varepsilon(t)$ uniformly for $t \in [0,T]$. Finally, from (9.25) we get $z \ll z_0$. □

By Crandall-Liggett's Theorem, [**49**], the mild solution obtained above is given by the well-known exponential formula,

$$e^{-tB^\gamma} z_0 = \lim_{n\to\infty} \left(I + \frac{t}{n} B^\gamma\right)^{-n} z_0. \tag{9.26}$$

The nonlinear contraction semigroup e^{-tB^γ} generated by the operator $-B^\gamma$ will be denoted in the sequel by $(S(t))_{t\geq 0}$.

In principle, it is not clear how these mild solutions have to be interpreted respect to $P^J_\gamma(z_0)$. In the next section we will see that they coincide with the solutions defined in the Introduction.

0.25. Existence of solutions. In this section we prove that the mild solution of (9.20) is in fact a solution of the problem $P^J_\gamma(z_0)$.

THEOREM 108. *Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|}\int_\Omega z_0 < \gamma_+$ and $\int_\Omega j^*_\gamma(z_0) < +\infty$. Then, there exists a unique solution to $P^J_\gamma(z_0)$ in $[0,T]$ for every $T > 0$. Moreover, $z \ll z_0$.*

PROOF. We divide the proof in three steps.

Step 1. First, let us suppose that

$$\begin{array}{l}\text{there exist } c_1, c_2 \text{ such that } c_1 \leq c_2,\ m_1 \in \gamma(c_1), m_2 \in \gamma(c_2) \\ \text{and } \gamma_- < m_1 \leq z_0 \leq m_2 < \gamma_+.\end{array} \tag{9.27}$$

Let $z(t)$ be the mild solution of (9.20) given by Theorem 107. We shall show that z is a solution of problem $P^J_\gamma(z_0)$.

For $n \in \mathbb{N}$, let $\varepsilon = T/n$, and consider a subdivision $t_0 = 0 < t_1 < \cdots < t_{n-1} < T = t_n$ with $t_i - t_{i-1} = \varepsilon$. Then, it follows that

$$z(t) = L^1(\Omega)\text{-}\lim\nolimits_\varepsilon z_\varepsilon(t) \quad \text{uniformly for } t \in [0,T], \tag{9.28}$$

where $z_\varepsilon(t)$ is given, for ε small enough, by

$$\begin{cases} z_\varepsilon(t) = z_0 & \text{for } \ t \in]-\infty, 0], \\ z_\varepsilon(t) = z_i^n, & \text{for } \ t \in]t_{i-1}, t_i], \quad i = 1, \ldots, n, \end{cases} \tag{9.29}$$

where $(u_i^n, z_i^n) \in L^2(\Omega) \times L^1(\Omega)$ is the solution of

$$-Au_i^n + \frac{z_i^n - z_{i-1}^n}{\varepsilon} = 0, \quad i = 1, 2, \ldots, n. \tag{9.30}$$

Moreover, $z_i^n \ll z_0$. Hence $\gamma_- < m_1 \leq z_i^n \leq m_2 < \gamma_+$ and consequently,

$$\inf \gamma^{-1}(m_1) \leq u_i^n \leq \sup \gamma^{-1}(m_2).$$

Therefore, if we write $u_\varepsilon(t) = u_i^n$, $t \in]t_{i-1}, t_i]$, $i = 1, \ldots, n$, we can suppose that

$$u_\varepsilon \rightharpoonup u \quad \text{weakly in } \ L^2(0,T;L^2(\Omega)) \ \text{ as } \ \varepsilon \to 0^+. \tag{9.31}$$

Since $z_\varepsilon \in \gamma(u_\varepsilon)$ *a.e.* in Q_T, $z_\varepsilon \to z$ in $L^1(Q_T)$, having in mind (9.31), we obtain that $z \in \gamma(u)$ *a.e.* in Q_T. On the other hand, from (9.30),

$$\frac{z_\varepsilon(t) - z_\varepsilon(t-\varepsilon)}{\varepsilon} \rightharpoonup z_t \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad \varepsilon \to 0^+.$$

Step 2. Let now $z_0 \in L^1(\Omega)$ such that $\gamma_- \le z_0 \le \gamma_+$, $\gamma_-|\Omega| < \int_\Omega z_0 < \gamma_-|\Omega|$, $\int_\Omega j^*_\gamma(z_0) < +\infty$, and

$$\text{there exists } c_1 \text{ and } m_1 \in \gamma(c_1) \text{ with } \gamma_- < m_1 \le z_0 \text{ and (9.27) is not satisfied.} \tag{9.32}$$

Let $z_{0n} \in L^\infty(\Omega)$,

$$z_{0n} \nearrow z_0 \quad \text{as } n \text{ goes to} + \infty,$$

such that $\int_\Omega z_{0n} < \int_\Omega z_{0n+1}$ and $z_{0n} \le m_2(n) < \gamma_+$, $m_2(n) \in \gamma(c_2(n))$ for some $c_2(n)$. By *Step 1*, there exists a solution z_n of problem $P^J_\gamma(z_{0n})$, which is the mild solution of (9.20) with initial datum z_{0n} and satisfies $z_n \ll z_{0n}$. It is obvious that

$$\lim_{n\to\infty} z_n = z \quad \text{in } C([0,T] : L^1(\Omega)),$$

being z the mild solution of (9.20) with initial datum z_0, moreover $z \ll z_0$. Next we prove that z is the solution of $P^J_\gamma(z_0)$.

Since z_n is a solution of problem $P^J_\gamma(z_{0n})$, there exists $u_n \in L^2(0,T,L^2(\Omega))$, $z_n \in \gamma(u_n)$ *a.e.* in $\Omega\times]0,T[$, such that

$$(z_n)_t - Au_n = 0. \tag{9.33}$$

Moreover, we can suppose that (see Theorem 99)

$$u_n \text{ is non decreasing in } n. \tag{9.34}$$

Multiplying (9.33) by u_n, we obtain

$$\frac{d}{dt}\int_\Omega \left(\int_0^{z_n(t)} (\gamma^{-1})^0(s)ds \right) = \int_\Omega Au_n(t)u_n(t)dt \tag{9.35}$$

in $\mathcal{D}'(]0,T[)$. Indeed, since $u_n(t) \in \gamma^{-1}(z_n(t)) = \partial j^*_\gamma(z_n(t))$,

$$(z_n(t+\tau) - z_n(t))u_n(t) \le \int_{z_n(t)}^{z_n(t+\tau)} (\gamma^{-1})^0(s)ds \quad \text{for all } \tau.$$

Consequently,

$$\int_\Omega (z_n)_t(t)u_n(t) = \frac{d}{dt}\int_\Omega \left(\int_0^{z_n(t)} (\gamma^{-1})^0(s)ds \right)$$

and (9.35) holds.

Integrating now (9.35) between 0 and T we get

$$-\int_0^T \int_\Omega Au_n(t)\, u_n(t)\, dt \leq \int_\Omega j_\gamma^*(z_0). \tag{9.36}$$

Let us see that $\{u_n\}$ is bounded in $L^2(Q_T)$. In the case $\gamma_+ = +\infty$, let

$$M = \sup_{t\in[0,T]} \int_\Omega z^+(t) + 1.$$

Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \int_\Omega (z_n)^+(t) < M, \qquad \forall n \geq n_0.$$

In the case $\gamma_+ < +\infty$, since we have conservation of mass, there exists $M \in \mathbb{R}$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\sup_{t\in[0,T]} \int_\Omega z_n(t) < M < \gamma_+|\Omega|,$$

moreover, since $z_n \ll z_0$, we have that $z_n \geq m_1$ and, since it is not difficult to see that $|m_1| < \frac{\gamma_+|\Omega|-M}{4|\Omega|}$, we have

$$\sup_{t\in[0,T]} \int_{\left\{x\in\Omega: z_n(t)(x) < -4(m_1^2+1)|\Omega|/(\gamma_+|\Omega|-M)\right\}} |z_n(t)| < \frac{\gamma_+|\Omega|-M}{4}, \qquad \forall n \in \mathbb{N}.$$

Therefore, in both cases, by Lemma 96, there exists $C > 0$ such that

$$\|(u_n(t))^+\|_{L^2(\Omega)} \leq C\left(\left(-\int_\Omega A(u_n(t))^+\,(u_n(t))^+\right)^{1/2} + 1\right), \qquad \forall\, t \in [0,T]. \tag{9.37}$$

Hence, by (9.36), since u_n is non decreasing in n, $\{u_n\}$ is bounded in $L^2(Q_T)$.

Passing to a subsequence if necessary, we can assume

$$u_n \rightharpoonup u \quad \text{weakly in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad n \to +\infty,$$

and, by (9.34),

$$u_n \to u \quad \text{in} \quad L^2(0,T;L^2(\Omega)) \quad \text{as} \quad n \to +\infty.$$

Consequently,

$$z \in \gamma(u) \quad a.e. \text{ in } Q_T.$$

Since also $\{Au_n\}$ is bounded in $L^2(Q_T)$, passing to the limit in (9.33) we get

$$z_t - Au = 0.$$

Step 3. Let now $z_0 \in L^1(\Omega)$, $\gamma_- \leq z_0 \leq \gamma_+$ and $\gamma_-|\Omega| < \int_\Omega z_0 < \gamma_-|\Omega|$, $\int_\Omega j_\gamma^*(z_0) < +\infty$ such that (9.32) is not satisfied. Let $z_{0n} \in L^\infty(\Omega)$,

$$z_{0n} \searrow z_0 \quad \text{as } n \text{ goes to } +\infty,$$

such that $\int_\Omega z_{0n} > \int_\Omega z_{0n+1}$ and $z_{0n} \geq m_1(n) > \gamma_-$, $m_1(n) \in \gamma(c_1(n))$ for some $c_1(n)$. By *Step 2*, there exist a solution z_n of problem $P_\gamma^J(z_{0n})$, which is the mild solution of (9.20) with initial datum z_{0n} and satisfies $z_n \ll z_0$. It is obvious that

$$\lim_{n\to\infty} z_n = z \quad \text{in } C([0,T] : L^1(\Omega)), \tag{9.38}$$

being z the mild solution of (9.20) with initial datum z_0. Moreover $z \ll z_0$. We shall see that z is the solution of $P_\gamma^J(z_0)$. The proof is similar to the above step and we only need to take care in the proof of the boundedness of $\{u_n\}$ in $L^2(Q_T)$. To this end we need a formula like (9.37) for u_n^-, that is, we need to prove that there exists $C > 0$ such that

$$\|(u_n(t))^-\|_{L^2(\Omega)} \leq C\left(\left(-\int_\Omega A(u_n(t))^-\,(u_n(t))^-\right)^{1/2} + 1\right), \qquad \forall\, t \in [0,T]. \tag{9.39}$$

Let us consider first that $\gamma_- = -\infty$, and let

$$M = \sup_{t\in[0,T]} \int_\Omega z^-(t) + 1.$$

Then, there exists $n_0 \in \mathbb{N}$ such that

$$\sup_{t\in[0,T]} \int_\Omega (z_n)^-(t) < M, \qquad \forall n \geq n_0.$$

In the case $\gamma_- > -\infty$, there exists $M \in \mathbb{R}$, $h > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\inf_{t\in[0,T]} \int_\Omega z_n(t) > M > \gamma_-|\Omega| \tag{9.40}$$

and

$$\sup_{t\in[0,T]} \int_{\{x\in\Omega : z_n(t)(x) > h\}} z_n(t) < \frac{M - \gamma_-|\Omega|}{4}. \tag{9.41}$$

Formula (9.40) is straightforward and (9.41) follows from (9.38). Indeed, by (9.38), there exists $n_0 \in \mathbb{N}$, $\delta > 0$ and $h > 0$ such that, for all $n \geq n_0$ and for all $t \in [0,T]$,

$$\int_E |z_n(t)| < \frac{M - \gamma_-|\Omega|}{4}, \qquad \forall E \subset \Omega,\ |E| < \delta,$$

and we can take h satisfying

$$|\{x \in \Omega : z_n(t)(x) > h\}| < \delta.$$

Therefore, in both cases, by Lemma 96, (9.39) is proved.

Uniqueness of solutions follows from Theorem 105. □

0.26. Asymptotic behaviour. In this section we study the asymptotic behaviour of the solutions to $P_\gamma^J(z_0)$. Note that since the solution preserves the total mass it is natural to expect that solutions to our diffusion problem converge to the mean value of the initial condition as $t \to \infty$. We shall see that this is the case, for instance, when γ is a continuous function, nevertheless this fails when γ has jumps.

Let us introduce the $\omega-$limit set for a given initial condition z_0,

$$\omega(z_0) = \left\{ w \in L^1(\Omega) \, : \, \exists \, t_n \to \infty \text{ with } S(t_n)z_0 \to w, \text{ strongly in } L^1(\Omega) \right\}$$

and the weak $\omega-$limit set

$$\omega_\sigma(z_0) = \left\{ w \in L^1(\Omega) \, : \, \exists \, t_n \to \infty \text{ with } S(t_n)z_0 \rightharpoonup w, \text{ weakly in } L^1(\Omega) \right\}.$$

Since $S(t)z_0 \ll z_0$, $\omega_\sigma(z_0) \neq \emptyset$ always. Moreover since $S(t)$ preserves the total mass, for all $w \in \omega_\sigma(z_0)$,

$$\int_\Omega w = \int_\Omega z_0.$$

We denote by F the set of fixed points of the semigroup $(S(t))$, that is,

$$F = \left\{ w \in \overline{D(B^\gamma)}^{L^1(\Omega)} \, : \, S(t)w = w \quad \forall \, t \geq 0 \right\}.$$

It is easy to see that

$$F = \left\{ w \in L^1(\Omega) \, : \, \exists k \in D(\gamma) \text{ such that } w \in \gamma(k) \right\}. \tag{9.42}$$

THEOREM 109. *Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|} \int_\Omega z_0 < \gamma_+$ and $\int_\Omega j_\gamma^*(z_0) < +\infty$. Then, $\omega_\sigma(z_0) \subset F$. Moreover, if $\omega(z_0) \neq \emptyset$, then $\omega(z_0)$ consists of a unique $w \in F$, and consequently,*

$$\lim_{t \to \infty} S(t)z_0 = w \qquad \textit{strongly in } L^1(\Omega).$$

PROOF. Along this proof we denote by $z(t) = S(t)z_0$ the solution to problem $P_\gamma^J(z_0)$ and $u(t)$ the corresponding function that appears in Definition 1.1.

Multiplying the equation in $P_\gamma^J(z_0)$ by $u(t)$ and integrating, we deduce

$$-\int_0^{+\infty} \int_\Omega Au(t)\, u(t) \, dt \leq \int_\Omega j_\gamma^*(z_0). \tag{9.43}$$

Therefore, thanks to (93), we obtain that there exists a constant C such that

$$\int_0^{+\infty} \int_\Omega \left| u(t) - \frac{1}{|\Omega|} \int_\Omega u(t) \right|^2 dt \leq C. \tag{9.44}$$

Let $w \in \omega_\sigma(z_0)$, then there exists a sequence $t_n \to +\infty$ such that $S(t_n)z_0 \rightharpoonup w$. By (9.44), we have

$$\alpha_n := \int_{t_n}^{+\infty} \int_\Omega \left| u(t) - \frac{1}{|\Omega|} \int_\Omega u(t) \right|^2 dt \to 0.$$

Take $s_n \to 0$ such that

$$\lim_{n\to\infty} \frac{\alpha_n}{s_n} = 0. \tag{9.45}$$

By contradiction it is easy to see that there exists $\bar{t}_n \in [t_n, t_n + \frac{C}{s_n}]$ such that

$$\int_\Omega \left| u(\bar{t}_n) - \frac{1}{|\Omega|}\int_\Omega u(\bar{t}_n)\right|^2 \le s_n,$$

and consequently,

$$\int_\Omega \left| u(\bar{t}_n) - \frac{1}{|\Omega|}\int_\Omega u(\bar{t}_n)\right|^2 \to 0. \tag{9.46}$$

Let us prove that

$$\frac{1}{|\Omega|}\int_\Omega u(\bar{t}_n)$$

is bounded. In fact, assume there exists a subsequence (still denoted by $\bar{t}_n$) such that

$$\frac{1}{|\Omega|}\int_\Omega u(\bar{t}_n) \to +\infty.$$

By (9.46) we get that $u(\bar{t}_n) \to +\infty$ *a.e.* Since $z(\bar{t}_n) \in \gamma(u(\bar{t}_n))$, then $z(\bar{t}_n) \to \gamma_+$ *a.e.* Moreover, as $z(\bar{t}_n) \ll z_0$ and $\gamma^+ \ge 0$, we can deduce that $\lim_{n\to\infty} z(\bar{t}_n) = \lim_{n\to\infty} z(\bar{t}_n)^+$ weakly in $L^1(\Omega)$. Hence, applying Fatou's Lemma, we get

$$\int_\Omega z_0 = \lim_{n\to\infty}\int_\Omega z(\bar{t}_n)^+ \ge \gamma_+|\Omega|,$$

a contradiction. A similar argument shows that $\frac{1}{|\Omega|}\int_\Omega u(\bar{t}_n)$ is bounded from below. Therefore, passing to a subsequence if necessary, we may assume that

$$\frac{1}{|\Omega|}\int_\Omega u(\bar{t}_n) \to k$$

for some constant k. Using again (9.46),

$$u(\bar{t}_n) \to k, \qquad \text{strongly in } L^2(\Omega) \text{ and } a.e. \tag{9.47}$$

Since $z(\bar{t}_n) \ll z_0$, we can assume, taking a subsequence if necessary, that $z(\bar{t}_n) \rightharpoonup \hat{w}$ weakly in $L^1(\Omega)$. Then, from (9.47) it follows that $\hat{w} \in \gamma(k)$, and consequently $\hat{w} \in F$. Let us show now that $w = \hat{w}$. By (9.45), we have

$$\|z(\bar{t}_n) - z(t_n)\| = \left\| \int_{t_n}^{\bar{t}_n} z_t(s)\,ds \right\|_{L^1(\Omega)} = \left\| \int_{t_n}^{\bar{t}_n} Au(s)\,ds \right\|_{L^1(\Omega)}$$

$$\le M(\bar{t}_n - t_n)^{1/2}\left(\int_{t_n}^{+\infty}\int_\Omega \left| u(s) - \frac{1}{|\Omega|}\int_\Omega u(s)\right|^2 ds\right)^{1/2} \le M\left(C\,\frac{\alpha_n}{s_n}\right)^{1/2} \to 0,$$

where M is a constant depending of $|\Omega|$. Therefore, taking limit, we get $z(\bar{t}_n) - z(t_n) \to 0$, strongly in $L^1(\Omega)$, since it converges weakly to $\overline{w} - w$, it follows that $w = \overline{w}$, which is a fixed point. Finally, if $\omega(z_0) \neq \emptyset$, since $\omega(z_0) \subset \omega_\sigma(z_0) \subset F$ and $(S(t))$ is a contraction semigroup, we have that $\omega(z_0) = \{w\} \subset F$ and

$$\lim_{t\to\infty} S(t)z_0 = w \quad \text{strongly in } L^1(\Omega).$$

□

Remark 110. Note that in order to proof that $\omega(z_0) \neq \emptyset$, a usual tool is to show that the resolvent of B^γ is compact. In our case this fails in general as the following example shows. Let γ any maximal monotone graph with $\gamma(0) = [0,1]$, $z_n \in L^\infty(\Omega)$, $0 \leq z_n \leq 1$ such that $\{z_n\}$ is not relatively compact in $L^1(\Omega)$. It is easy to check that $z_n = (I + B^\gamma)^{-1}(z_n)$. Hence $(I+B^\gamma)^{-1}$ is not a compact operator in $L^1(\Omega)$. On the other hand, since the nonlocal operator does not have regularizing effects, here we cannot prove regularity properties of the solutions that would help to find compactness of the orbits. Nevertheless, we shall see in the next result that when γ is a continuous function we are able to prove that $\omega(z_0) \neq \emptyset$.

Let us see now some cases in which $\omega(z_0) \neq \emptyset$ and

$$\lim_{t\to\infty} S(t)z_0 = \frac{1}{|\Omega|}\int_\Omega z_0 \quad \text{strongly in } L^1(\Omega).$$

Given a maximal monotone graph γ in $\mathbb{R}\times\mathbb{R}$, we set

$$\gamma(r+) := \inf \gamma(]r, +\infty[), \quad \gamma(r-) := \sup \gamma(]-\infty, r[)$$

for $r \in \mathbb{R}$, where we use the conventions $\inf \emptyset = +\infty$ and $\sup \emptyset = -\infty$. It is easy to see that

$$\gamma(r) = [\gamma(r-), \gamma(r+)] \cap \mathbb{R} \quad \text{for} \quad r \in \mathbb{R}.$$

Moreover, $\gamma(r-) = \gamma(r+)$ except at a countable set of points, which we denote by $J(\gamma)$.

Corollary 111. *Let $z_0 \in L^1(\Omega)$ such that $\gamma_- \leq z_0 \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|}\int_\Omega z_0 < \gamma_+$ and $\int_\Omega j_\gamma^*(z_0) < +\infty$. The following statements hold.*

(1) *If $\frac{1}{|\Omega|}\int_\Omega z_0 \notin \gamma(J(\gamma))$ or $\frac{1}{|\Omega|}\int_\Omega z_0 \in \{\gamma(k+), \gamma(k-)\}$ for some $k \in J(\gamma)$, then*
$$\lim_{t\to\infty} S(t)z_0 = \frac{1}{|\Omega|}\int_\Omega z_0 \quad \textit{strongly in } L^1(\Omega).$$
(2) *If γ is a continuous function then*
$$\lim_{t\to\infty} S(t)z_0 = \frac{1}{|\Omega|}\int_\Omega z_0 \quad \textit{strongly in } L^1(\Omega).$$
(3) *If $\frac{1}{|\Omega|}\int_\Omega z_0 \in]\gamma(k-), \gamma(k+)[$ for some $k \in J(\gamma)$, then*
$$\omega_\sigma(z_0) \subset \left\{ w \in L^1(\Omega) \ : w \in [\gamma(k-), \gamma(k+)] \ \textit{a.e.}, \int_\Omega w = \int_\Omega z_0 \right\}.$$

and consequently, for any $w \in \omega_\sigma(z_0)$*, there exists a non null set in which* $w \in]\gamma(k-), \gamma(k+)[$.

PROOF. (1). Along this proof we denote by $z(t) = S(t)z_0$ the solution to problem $P_\gamma^J(z_0)$ and $u(t)$ the corresponding function that appears in Definition 1.1. First, let us assume that $\frac{1}{|\Omega|}\int_\Omega z_0 \notin \gamma(J(\gamma))$ and $z_0 \in L^\infty(\Omega)$. Working as in the above theorem, we have that there exists a constat k such that

$$u(t_n) \to k, \quad \text{strongly in } L^2(\Omega) \text{ and } a.e. \tag{9.48}$$

Since $z(t_n) \ll z_0$, there exists a subsequence such that $z(t_n) \rightharpoonup w$ weakly in $L^1(\Omega)$. Now, from $z(t_n) \in \gamma(u(t_n))$ we deduce that $w \in \gamma(k)$ and consequently, since $\frac{1}{|\Omega|}\int_\Omega z_0 = \frac{1}{|\Omega|}\int_\Omega w$, $k \notin J(\gamma)$. Then, there exists $\delta > 0$ such that γ is univalued and continuous on $]k-\delta, k+\delta[$. Hence, $w = \gamma(k)$ and $z(t_n) \to \gamma(k)$ *a.e.* Therefore, Since $z(t_n)$ is bounded in $L^\infty(\Omega)$, $z(t_n) \to \gamma(k) = \frac{1}{|\Omega|}\int_\Omega z_0$ strongly in $L^1(\Omega)$. Then, by the above theorem we get that

$$z(t) \to \frac{1}{|\Omega|}\int_\Omega z_0, \qquad \text{as } t \to \infty.$$

The general case $z_0 \in L^1(\Omega)$ follows easily from the previous arguments using again that we deal with a contraction semigroup.

Assume now that $\frac{1}{|\Omega|}\int_\Omega z_0 \in \{\gamma(k+), \gamma(k-)\}$ for some $k \in J(\gamma)$. It is easy to see that we can find $z_{0,n} \in L^1(\Omega)$, with $\gamma_- \leq z_{0,n} \leq \gamma_+$, $\gamma_- < \frac{1}{|\Omega|}\int_\Omega z_{0,n} < \gamma_+$ and $\int_\Omega j_\gamma^*(z_{0,n}) < +\infty$, such that $z_{0,n} \to z_0$ strongly in $L^1(\Omega)$ and verifying $\frac{1}{|\Omega|}\int_\Omega z_{0,n} \notin \gamma(J(\gamma))$ for all n. Then, by the above step, we have

$$S(t)z_{0,n} \to \frac{1}{|\Omega|}\int_\Omega z_{0,n}, \quad \text{strongly in } L^1(\Omega),$$

from where it follows, using again that $(S(t))$ is a contraction semigroup, that

$$S(t)z_0 \to \frac{1}{|\Omega|}\int_\Omega z_0, \quad \text{strongly in } L^1(\Omega).$$

Statement (2) is an obvious consequence of (1) since in this case $J(\gamma) = \emptyset$.

Finally, we prove (3). Given $w \in \omega_\sigma(z_0)$, by Theorem 109, there exists $k_0 \in D(\gamma)$, such that $w \in \gamma(k_0)$. Then, $k_0 = k$. In fact, if we assume, for instance, that $k_0 < k$, then

$$\gamma(k_0+) \geq \frac{1}{|\Omega|}\int_\Omega w = \frac{1}{|\Omega|}\int_\Omega z_0 > \gamma(k-) > \gamma(k_0+),$$

a contradiction. Hence, we have $w \in \gamma(k)$, and

$$\frac{1}{|\Omega|}\int_\Omega w = \frac{1}{|\Omega|}\int_\Omega z_0 \in]\gamma(k-), \gamma(k+)[.$$

Thus, $w \in [\gamma(k-), \gamma(k+)]$ *a.e.* and, moreover, there exists a non null set in which $w \in]\gamma(k-), \gamma(k+)[$. □

REMARK 112. An alternative proof of the fact that $\omega(z_0) \subset F$ is the following. Let

$$\Psi : L^1(\Omega) \to]-\infty, +\infty]$$

the functional defined by

$$\Psi(z) := \begin{cases} \int_\Omega j_\gamma^*(z) & \text{if } \; j_\gamma^*(z) \in L^1(\Omega), \\ +\infty & \text{if } \; j_\gamma^*(z) \notin L^1(\Omega). \end{cases}$$

Since j_γ^* is continuous and convex, Ψ is lower semi-continuous ([**24**], pag.160). Moreover, since $S(t)z_0 \ll z_0$ for all $t \geq 0$, we have $\Psi(S(t)z_0) \leq \Psi(z_0)$ for all $t \geq 0$. Therefore, Ψ is a lower semi-continuous Liapunov functional for $(S(t))$. Then, by the Invariance Principle of Dafermos ([**51**]), Ψ is constant on $\omega(z_0)$. Consequently, given $w_0 \in \omega(z_0)$, if $w(t) = S(t)w_0$, we have $\Psi(w(t))$ is constant for all $t \geq 0$. Let $u(t)$ such that $w(t) \in \gamma(u(t))$ and $w_t = A(u(t))$. Working as in the proof of (9.35), we get

$$0 = \frac{d}{dt}\Psi(w(t)) = \frac{d}{dt}\int_\Omega j_\gamma^*(w(t)) = \frac{d}{dt}\int_\Omega j_{\gamma^{-1}}(w(t)) = \int_\Omega Au(t)\, u(t).$$

Then, by Proposition 93, we obtain that

$$u(t) = \frac{1}{|\Omega|}\int_\Omega u(t).$$

Hence, $w(t) \in F$ for all $t > 0$, and consequently, $w_0 \in F$.

CHAPTER 10

A non-local $p-$Laplacian with Neumann boundary conditions

Our main goal in this chapter is to study the following nonlocal nonlinear diffusion problem, which we call the *nonlocal p-Laplacian problem* (with homogeneous Neumann boundary conditions),

$$P_p^J(u_0) \quad \begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy, \\ u(x,0) = u_0(x). \end{cases}$$

Here $J:\mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $J(0)>0$ and $\int_{\mathbb{R}^N} J(x)dx = 1$ (this last condition is not necessary to prove the results of this chapter, it is imposed to simplify the exposition), $1 \le p < +\infty$ and $\Omega \subset \mathbb{R}^N$ is a bounded domain.

When dealing with local evolution equations, two models of nonlinear diffusion has been extensively studied in the literature, the porous medium equation, $u_t = \Delta u^m$, and the $p-$Laplacian evolution, $u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$. In the first case (for the porous medium equation) a nonlocal analogous equation was studied in [**7**] (see also [**40**]). Our main objective in this paper is to study the nonlocal equation P_p^J, that is, the nonlocal analogous to the $p-$Laplacian evolution.

First, let us state the precise definition of a solution. Solutions to $P_p^J(u_0)$ will be understood in the following sense.

DEFINITION 113. *Let $1 < p < +\infty$. A solution of $P_p^J(u_0)$ in $[0,T]$ is a function $u \in C([0,T];L^1(\Omega)) \cap W^{1,1}(]0,T[;L^1(\Omega))$ which satisfies $u(0,x) = u_0(x)$ a.e. $x \in \Omega$ and*

$$u_t(x,t) = \int_\Omega J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy \quad \text{a.e in }]0,T[\times\Omega.$$

Let us note that, with this definition of solution, the evolution problem $P_p^J(u_0)$ is the gradient flow associated to the functional

$$J_p(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^p\,dy\,dx,$$

which is the nonlocal analogous to the energy functional associated to the $p-$Laplacian

$$F_p(u) = \frac{1}{p}\int_\Omega |\nabla u(y)|^p\,dy.$$

Our first result shows existence and uniqueness of a global solution for this problem. Moreover, a contraction principle holds.

THEOREM 114. *Assume* $p > 1$ *and let* $u_0 \in L^p(\Omega)$. *Then, there exists a unique solution to* $P_p^J(u_0)$ *in the sense of Definition 113.*

Moreover, if $u_{i0} \in L^1(\Omega)$, $i = 1, 2$, *and* u_i *is a solution in* $[0, T]$ *of* $P_p^J(u_{i0})$. *Then*

$$\int_\Omega (u_1(t) - u_2(t))^+ \le \int_\Omega (u_{10} - u_{20})^+ \quad \text{for every } t \in]0, T[.$$

If $u_{i0} \in L^p(\Omega)$, $i = 1, 2$, *then*

$$\|u_1(t) - u_2(t)\|_{L^p(\Omega)} \le \|u_{10} - u_{20}\|_{L^p(\Omega)} \quad \text{for every } t \in]0, T[.$$

Let us finish the introduction by collecting some preliminaries and notations that will be used in the sequel.

We denote by J_0 and P_0 the following sets of functions,

$$J_0 = \{j : \mathbb{R} \to [0, +\infty], \text{ convex and lower semi-continuos with } j(0) = 0\},$$

$$P_0 = \left\{q \in C^\infty(\mathbb{R}) : 0 \le q' \le 1, \text{supp}(q') \text{ is compact, and } 0 \notin \text{supp}(q)\right\}.$$

In [**19**] the following relation for $u, v \in L^1(\Omega)$ is defined,

$$u \ll v \text{ if and only if } \int_\Omega j(u)\,dx \le \int_\Omega j(v)\,dx,$$

and the following facts are proved.

PROPOSITION 115. *Let* Ω *be a bounded domain in* $\mathbb{R}^N$.
(i) For any $u, v \in L^1(\Omega)$, *if* $\int_\Omega uq(u) \le \int_\Omega vq(u)$ *for all* $q \in P_0$, *then* $u \ll v$.
(ii) If $u, v \in L^1(\Omega)$ *and* $u \ll v$, *then* $\|u\|_r \le \|v\|_r$ *for any* $r \in [1, +\infty]$.
(iii) If $v \in L^1(\Omega)$, *then* $\{u \in L^1(\Omega) : u \ll v\}$ *is a weakly compact subset of* $L^1(\Omega)$.

0.27. Existence of solutions for the nonlocal problems.

The case $p > 1$**.**

We first study the problem $P_p^J(u_0)$ from the point of view of Nonlinear Semigroup Theory. For this we introduce in $L^1(\Omega)$ the following operator associated with our problem.

DEFINITION 116. For $1 < p < +\infty$ we define in $L^1(\Omega)$ the operator B_p^J by

$$B_p^J u(x) = -\int_\Omega J(x-y)|u(y) - u(x)|^{p-2}(u(y) - u(x))\,dy, \qquad x \in \Omega.$$

REMARK 117. *It is easy to see that,*

1. B_p^J *is positively homogeneous of degree* $p-1$,

2. $L^{p-1}(\Omega) \subset Dom(B_p^J)$, *if* $p > 2$.

3. for $1 < p \leq 2$, $Dom(B_p^J) = L^1(\Omega)$ *and* B_p^J *is closed in* $L^1(\Omega) \times L^1(\Omega)$.

We have the following monotonicity lemma, whose proof is straightforward.

LEMMA 118. *Let* $1 < p < +\infty$, *and let* $T : \mathbb{R} \to \mathbb{R}$ *a nondecreasing function. Then,*

(i) *for every* u, $v \in L^p(\Omega)$ *such that* $T(u-v) \in L^p(\Omega)$, *it holds*

$$\begin{aligned}&\int_\Omega (B_p^J u(x) - B_p^J v(x)) T(u(x) - v(x)) dx = \\ &\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\,(T(u(y)-v(y)) - T(u(x)-v(x))) \times \\ &\times \big(|u(y)-u(x)|^{p-2}(u(y)-u(x)) - |v(y)-v(x)|^{p-2}(v(y)-v(x))\big)\, dy\, dx.\end{aligned} \tag{10.1}$$

(ii) *Moreover, if* T *is bounded,* (10.1) *holds for* u, $v \in Dom(B_p^J)$.

In the next result we prove that B_p^J is completely accretive and verifies a range condition. In short, this means that for any $\phi \in L^p(\Omega)$ there is a unique solution of the problem $u + B_p^J u = \phi$ and the resolvent $(I + B_p^J)^{-1}$ is a contraction in $L^q(\Omega)$ for all $1 \leq q \leq +\infty$.

THEOREM 119. *For* $1 < p < +\infty$, *the operator* B_p^J *is completely accretive and verifies the range condition*

$$L^p(\Omega) \subset Ran(I + B_p^J). \tag{10.2}$$

PROOF. Given $u_i \in \mathrm{Dom}(B_p^J)$, $i = 1, 2$ and $q \in P_0$, by the monotonicity Lemma 118, we have

$$\int_\Omega (B_p^J u_1(x) - B_p^J u_2(x)) q(u_1(x) - u_2(x))\, dx \geq 0,$$

from where it follows that B_p^J is a completely accretive operator (see [**19**]).

To show that B_p^J satisfies the range condition we have to prove that for any $\phi \in L^p(\Omega)$ there exists $u \in \mathrm{Dom}(B_p^J)$ such that $u = (I + B_p^J)^{-1}\phi$. Let us first take $\phi \in L^\infty(\Omega)$. Let $A_{n,m} : L^p(\Omega) \to L^{p'}(\Omega)$ the continuous monotone operator defined by

$$A_{n,m}(u) := T_c(u) + B_p^J u + \frac{1}{n}|u|^{p-2}u^+ - \frac{1}{m}|u|^{p-2}u^-.$$

We have that $A_{n,m}$ is coercive in $L^p(\Omega)$. In fact,

$$\lim_{\|u\|_{L^p(\Omega)} \to +\infty} \frac{\displaystyle\int_\Omega A_{n,m}(u) u}{\|u\|_{L^p(\Omega)}} = +\infty.$$

Then, by Corollary 30 in [**26**], there exists $u_{n,m} \in L^p(\Omega)$, such that

$$T_c(u_{n,m}) + B_p^J u_{n,m} + \frac{1}{n}|u_{n,m}|^{p-2}u_{n,m}^+ - \frac{1}{m}|u_{n,m}|^{p-2}u_{n,m}^- = \phi.$$

Using the monotonicity of $B_p^J u_{n,m} + \frac{1}{n}|u_{n,m}|^{p-2}u_{n,m}^+ - \frac{1}{m}|u_{n,m}|^{p-2}u_{n,m}^-$, we obtain that $T_c(u_{n,m}) \ll \phi$. Consequently, taking $c > \|\phi\|_{L^\infty(\Omega)}$, $u_{n,m} \ll \phi$ and

$$u_{n,m} + B_p^J u_{n,m} + \frac{1}{n}|u_{n,m}|^{p-2}u_{n,m}^+ - \frac{1}{m}|u_{n,m}|^{p-2}u_{n,m}^- = \phi.$$

Moreover, since $u_{n,m}$ is increasing in n and decreasing in m. As $u_{n,m} \ll \phi$, we can pass to the limit as $n \to \infty$ (using the monotone convergence to handle the term $B_p^J u_{n,m}$) obtaining u_m is a solution to

$$u_m + B_p^J u_m - \frac{1}{m}|u_m|^{p-2}u_m^- = \phi.$$

Using u_m is decreasing in m we can pass again to the limit and to obtain

$$u + B_p^J u = \phi.$$

Let now $\phi \in L^p(\Omega)$. Take $\phi_n \in L^\infty(\Omega)$, $\phi_n \to \phi$ in $L^p(\Omega)$. Then, by our previous step, there exists $u_n = (I + B_p^J)^{-1}\phi_n$, $u_n \ll \phi_n$. Since B_p^J is completely accretive, $u_n \to u$ in $L^p(\Omega)$, also $B_p^J u_n \to B_p^J u$ in $L^{p'}(\Omega)$ and we conclude that $u + B_p^J u = \phi$. □

If $\mathcal{B}_p^J$ denotes the closure of B_p^J in $L^1(\Omega)$, then by Theorem 119 we obtain $\mathcal{B}_p^J$ is m-completely accretive in $L^1(\Omega)$.

As a consequence of the above results we get the following theorems (see [**20**] and [**19**]), from which Theorem 114 can be derived.

THEOREM 120. *Assume $p > 1$. Let $T > 0$ and let $u_0 \in L^1(\Omega)$. Then, there exists a unique mild solution u of*

$$\begin{cases} u'(t) + B_p^J u(t) = 0, & t \in (0,T), \\ u(0) = u_0. \end{cases} \tag{10.3}$$

THEOREM 121. *Assume $p > 1$. Let $T > 0$.*

(1) *Let $u_0 \in L^p(\Omega)$. Then, the unique mild solution u of (10.3) is a solution of $P_p^J(u_0)$ in the sense of Definition 113. If $1 < p \le 2$, this is true for any $u_0 \in L^1(\Omega)$.*

(2) *Let $u_{i0} \in L^1(\Omega)$, $i = 1,2$, and u_i a solution in $[0,T]$ of $P_p^J(u_{i0})$, $i = 1,2$. Then*

$$\int_\Omega (u_1(t) - u_2(t))^+ \le \int_\Omega (u_{10} - u_{20})^+ \quad \textit{for every } t \in]0,T[.$$

Moreover, for $q \in [1,+\infty]$, if $u_{i0} \in L^q(\Omega)$, $i = 1,2$, then

$$\|u_1(t) - u_2(t)\|_{L^q(\Omega)} \le \|u_{10} - u_{20}\|_{L^q(\Omega)} \quad \textit{for every } t \in]0,T[.$$

PROOF. The result follows from the fact that $u(t)$ is a solution of $P_p^J(u_0)$ if and only if $u(t)$ is a strong solution of the abstract Cauchy problem (10.3). Now, $u(t)$ is a strong solution under the hypothesis of the theorem thanks to the completely accretivity of B_p^J and the range condition (10.2). Moreover, the result follows for $1 < p \leq 2$, since in this case $\mathrm{Dom}(B_p^J) = L^1(\Omega)$ and B_p^J is closed in $L^1(\Omega) \times L^1(\Omega)$. □

REMARK 122. *Observe that our results can be extended (with minor modifications) to obtain existence and uniqueness for*

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x,y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy, \\ u(x,0) = u_0(x), \end{cases}$$

with J symmetric, that is, $J(x,y) = J(y,x)$, bounded and nonnegative.

The case $p = 1$

This chapter deals with the existence and uniqueness of solutions for the nonlocal 1-Lapla-cian problem with homogeneous Neumann boundary conditions,

$$P_1^J(u_0) \qquad \begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)\frac{u(y,t)-u(x,t)}{|u(y,t)-u(x,t)|}\,dy. \\ u(x,0) = u_0(x). \end{cases}$$

We have that the formal evolution problem

$$u_t(x,t) = \int_\Omega J(x-y)\frac{u(y,t)-u(x,t)}{|u(y,t)-u(x,t)|}\,dy,$$

is the gradient flow associated to the functional

$$J_1(u) = \frac{1}{2}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|\,dy\,dx,$$

which is the nonlocal analogous to the energy functional associated to the total variation

$$F_1(u) = \int_\Omega |\nabla u(y)|\,dy.$$

For $p = 1$ we give the following definition of what we understand as a solution.

DEFINITION 123. *A solution of $P_1^J(z_0)$ in $[0,T]$ is a function*

$$u \in C([0,T];L^1(\Omega)) \cap W^{1,1}(]0,T[;L^1(\Omega))$$

which satisfies $u(0,x) = u_0(x)$ a.e. $x \in \Omega$ and

$$u_t(x,t) = \int_\Omega J(x-y)g(x,y,t)\,dy \qquad a.e\ in\ \ ]0,T[\times\Omega,$$

for some $g \in L^\infty(0,T;L^\infty(\Omega\times\Omega))$ *with* $\|g\|_\infty \le 1$ *such that* $g(x,y,t) = -g(y,t,x)$ *and*

$$J(x-y)g(x,y,t) \in J(x-y)sign(u(y,t)-u(x,t)).$$

To get existence and uniqueness of these kind of solutions, the idea is to take the limit as $p \searrow 1$ of solutions to P_p^J with $p > 1$.

THEOREM 124. *Assume* $p = 1$ *and let* $u_0 \in L^1(\Omega)$. *Then, there exists a unique solution to* $P_1^J(u_0)$ *in the sense of Definition 123.*

Moreover, for $i = 1,2$, *let* $u_{i0} \in L^1(\Omega)$ *and* u_i *be a solution in* $[0,T]$ *of* $P_1^J(u_{i0})$. *Then*

$$\int_\Omega (u_1(t)-u_2(t))^+ \le \int_\Omega (u_{10}-u_{20})^+ \quad \text{for almost every } t \in]0,T[.$$

As in the case $p > 1$, to prove the existence and uniqueness of solutions of $P_1^J(u_0)$ we use the Nonlinear Semigroup Theory, so we start introducing the following operator in $L^1(\Omega)$.

DEFINITION 125. *We define the operator* B_1^J *in* $L^1(\Omega)\times L^1(\Omega)$ *by* $\hat{u} \in B_1^J u$ *if and only if* $u, \hat{u} \in L^1(\Omega)$, *there exists* $g \in L^\infty(\Omega\times\Omega)$, $g(x,y) = -g(y,x)$ *for almost all* $(x,y) \in \Omega\times\Omega$, $\|g\|_\infty \le 1$,

$$\hat{u}(x) = -\int_\Omega J(x-y)g(x,y)\,dy, \quad a.e.\ x \in \Omega$$

and

$$J(x-y)g(x,y) \in J(x-y)\,\mathrm{sign}(u(y)-u(x)) \quad a.e.\ (x,y) \in \Omega\times\Omega. \tag{10.4}$$

REMARK 126.

1. It is not difficult to see that (10.4) *is equivalent to*

$$-\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx = \frac{1}{2}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|\,dy\,dx,$$

2. $L^1(\Omega) = Dom(B_1^J)$ *and* B_1^J *is closed in* $L^1(\Omega)\times L^1(\Omega)$.

3. B_1^J *is positively homogeneous of degree zero, that is, if* $\hat{u} \in B_1^J u$ *and* $\lambda > 0$ *then* $\hat{u} \in B_1^J(\lambda u)$.

THEOREM 127. *The operator* B_1^J *is completely accretive and satisfies the range condition*

$$L^\infty(\Omega) \subset Ran(I + B_1^J).$$

PROOF. Let $\hat{u}_i \in B_1^J u_i$, $i = 1,2$. Then there exists $g_i \in L^\infty(\Omega\times\Omega)$, $\|g_i\|_\infty \le 1$, $g_i(x,y) = -g_i(y,x)$, $J(x-y)g_i(x,y) \in J(x-y)\mathrm{sign}(u_i(y)-u_i(x))$ for almost all $(x,y) \in \Omega\times\Omega$, such that

$$\hat{u}_i(x) = -\int_\Omega J(x-y)g_i(x,y)\,dy, \quad \text{a.e. } x \in \Omega,$$

for $i = 1, 2$. Then, given $q \in P_0$, we have

$$\begin{aligned}
&\int_\Omega (\hat{u}_1(x) - \hat{u}_2(x)) q(u_1(x) - u_2(x))\, dx \\
&= \frac{1}{2} \int_\Omega \int_\Omega J(x-y)(g_1(x,y) - g_2(x,y)) \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x)) \right) dxdy \\
&= \frac{1}{2} \int \int_{\{(x,y): u_1(y) \neq u_1(x), u_2(y) = u_2(x)\}} J(x-y)(g_1(x,y) - g_2(x,y)) \times \\
&\qquad\qquad \times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x)) \right) dxdy \\
&\quad + \frac{1}{2} \int \int_{\{(x,y): u_1(y) = u_1(x), u_2(y) \neq u_2(x)\}} J(x-y)(g_1(x,y) - g_2(x,y)) \times \\
&\qquad\qquad \times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x)) \right) dxdy \\
&\quad + \frac{1}{2} \int \int_{\{(x,y): u_1(y) \neq u_1(x), u_2(y) \neq u_2(x)\}} J(x-y)(g_1(x,y) - g_2(x,y)) \times \\
&\qquad\qquad \times \left(q(u_1(y) - u_2(y)) - q(u_1(x) - u_2(x)) \right) dxdy,
\end{aligned}$$

and the last three integrals are nonnegative. Hence

$$\int_\Omega (\hat{u}_1(x) - \hat{u}_2(x)) q(u_1(x) - u_2(x))\, dx \geq 0,$$

from where it follows that B_1^J is a completely accretive operator.

To show that B_1^J satisfies the range condition, let us see that for any $\phi \in L^\infty(\Omega)$,

$$\lim_{p \to 1+} (I + B_p^J)^{-1} \phi = (I + B_1^J)^{-1} \phi \quad \text{weakly in } L^1(\Omega).$$

Let $\phi \in L^\infty(\Omega)$, and write, for $1 < p < +\infty$, $u_p = \left(I + B_p^J \right)^{-1} \phi$. Then,

$$u_p(x) - \int_\Omega J(x-y) |u_p(y) - u_p(x)|^{p-2} (u_p(y) - u_p(x))\, dy = \phi(x) \quad \text{a.e. } x \in \Omega. \tag{10.5}$$

Thus, for every $v \in L^\infty(\Omega)$, we can write

$$\int_\Omega u_p v - \int_\Omega \int_\Omega J(x-y) |u_p(y) - u_p(x)|^{p-2} (u_p(y) - u_p(x))\, dy\, v(x)\, dx = \int_\Omega \phi v.$$

Since $u_p \ll \phi$, by Proposition 115, we have that there exists a sequence $p_n \to 1$ such that

$$u_{p_n} \rightharpoonup u \quad \text{weakly in } L^1(\Omega), \quad u \ll \phi.$$

Observe that $\|u_{p_n}\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)} \leq \|\phi\|_{L^\infty(\Omega)}$.

Now, since

$$-\int_\Omega\int_\Omega J\,(x-y)\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n-2}(u_{p_n}(y)-u_{p_n}(x))\,dy\,v(x)\,dx$$
$$=\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n-2}\,(u_{p_n}(y)-u_{p_n}(x))\,(v(y)-v(x))\;dy\,dx,$$

taking $v=u_{p_n}$, by (10.5), we get that

$$\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n}\;dy\,dx\le M_1,\qquad\forall\, n\in\mathbb{N}.$$

Therefore, for any measurable subset $E\subset\Omega\times\Omega$, we have

$$\left|\int\int_E J(x-y)|u_{p_n}(y)-u_{p_n}(x)|^{p_n-2}\,(u_{p_n}(y)-u_{p_n}(x))\right|$$
$$\le\int\int_E J(x-y)|u_{p_n}(y)-u_{p_n}(x)|^{p_n-1}\le M_2|E|^{\frac{1}{p_n}}.$$

Hence, by the Dunford-Pettis Theorem we may assume that there exists $g(x,y)$ such that

$$J(x-y)|u_{p_n}(y)-u_{p_n}(x)|^{p_n-2}\,(u_{p_n}(y)-u_{p_n}(x))\rightharpoonup J(x-y)g(x,y),$$

weakly in $L^1(\Omega\times\Omega)$, $g(x,y)=-g(y,x)$ for almost all $(x,y)\in\Omega\times\Omega$, and $\|g\|_\infty\le 1$.

Therefore, passing to the limit in (10.5) for $p=p_n$, we get

$$\int_\Omega uv-\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,v(x)\,dx=\int_\Omega\phi v\tag{10.6}$$

for every $v\in L^\infty(\Omega)$, and consequently we get

$$u(x)-\int_\Omega J(x-y)g(x,y)\,dy=\phi(x)\qquad\text{a.e. } x\in\Omega.$$

Then, to finish the proof we have to show that

$$-\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx=\frac{1}{2}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|\,dy\,dx.\tag{10.7}$$

In fact, by (10.6) with $v=u$,

$$\frac{1}{2}\int_\Omega\int_\Omega J(x-y)\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n}\;dy\,dx$$
$$=\int_\Omega\phi u_{p_n}-\int_\Omega u_{p_n}u_{p_n}=\int_\Omega\phi u-\int_\Omega uu-\int_\Omega\phi(u-u_{p_n})$$
$$+\int_\Omega 2u(u-u_{p_n})-\int_\Omega(u-u_{p_n})(u-u_{p_n})$$
$$\le-\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx-\int_\Omega\phi(u-u_{p_n})+\int_\Omega 2u(u-u_{p_n}),$$

so,

$$\limsup_{n\to+\infty} \frac{1}{2}\int_\Omega\int_\Omega J(x-y)\,|u_{p_n}(y)-u_{p_n}(x)|^{p_n}\,dy\,dx \le -\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,u(x)\,dx.$$

Now, by the monotonicity Lemma 118,

$$\begin{aligned}&-\int_\Omega\int_\Omega J(x-y)|\rho(y)-\rho(x)|^{p_n-2}(\rho(y)-\rho(x))\,dy\,(u_{p_n}(x)-\rho(x))\,dx\\ &\le -\int_\Omega\int_\Omega J(x-y)|u_{p_n}(y)-u_{p_n}(x)|^{p_n-2}(u_{p_n}(y)-u_{p_n}(x))\,dy\,(u_{p_n}(x)-\rho(x))\,dx.\end{aligned}$$

Therefore, taking limits,

$$\begin{aligned}&-\int_\Omega\int_\Omega J(x-y)\,\mathrm{sign}_0(\rho(y)-\rho(x))\,dy\,(u(x)-\rho(x))\,dx\\ &\qquad\le -\int_\Omega\int_\Omega J(x-y)g(x,y)\,dy\,(u(x)-\rho(x))\,dx.\end{aligned}$$

Taking now, $\rho = u \pm \lambda u$, $\lambda > 0$, and letting $\lambda \to 0$, we get (10.7), and the proof is finished. □

As a consequence of the above results we have the following theorems (see [**20**]), from which Theorem 124 follows.

THEOREM 128. *Let $T > 0$ and $u_0 \in L^1(\Omega)$. Then, there exists a unique mild solution u of*

$$\begin{cases} u'(t) + B_1^J u(t) \ni 0, & t\in(0,T),\\ u(0)=u_0. \end{cases} \tag{10.8}$$

THEOREM 129. *Let $T > 0$.*

(1) *Let $u_0 \in L^1(\Omega)$. Then, the unique mild solution u of (10.8) is a solution of $P_1^J(u_0)$ in the sense of Definition 123.*

(2) *Let $u_{i0} \in L^1(\Omega)$, $i = 1,2$, and u_i a solution in $[0,T]$ of $P_1^J(u_{i0})$, $i = 1,2$. Then*

$$\int_\Omega (u_1(t)-u_2(t))^+ \le \int_\Omega (u_{10}-u_{20})^+ \quad \text{for almost every } t\in]0,T[.$$

0.28. Convergence to the p-laplacian.

Our next step is to rescale the kernel J appropriately and take the limit as the scaling parameter goes to zero. To be more precise, for every $p \ge 1$, we consider the local

$p-$Laplace evolution equation with homogeneous Neumann boundary conditions

$$N_p(u_0) \quad \begin{cases} u_t = \Delta_p u & \text{in }]0,T[\times\Omega, \\ |\nabla u|^{p-2}\nabla u \cdot \eta = 0 & \text{on }]0,T[\times\partial\Omega, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where η is the unit outward normal on $\partial\Omega$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-laplacian of u. We obtain that the solutions of this local problem, $N_p(u_0)$, can be approximated by solutions of a sequence of nonlocal *p-Laplacian* problems of the form P_p^J.

Problem $N_1(u_0)$, that is, the Neumann problem for the Total Variation Flow, was studied in [**2**] (see also [**3**]), motivated by problems in image processing. This PDE appears when one uses the steepest descent method to minimize the Total Variation, a method introduced by L. Rudin, S. Osher and E. Fatemi [**77**] in the context of image denoising and reconstruction. Then, solving $N_1(u_0)$ amounts to regularize or, in other words, to filter the initial datum u_0. This filtering process has less destructive effect on the edges than filtering with a Gaussian, i.e., than solving the heat equation with initial condition u_0. In this context the given *image* u_0 is a function defined on a bounded, smooth or piecewise smooth open subset Ω of $\mathbb{R}^N$, typically, Ω will be a rectangle in $\mathbb{R}^2$.

S. Kindermann, S. Osher and P. W. Jones in [**70**] have studied deblurring and denoising of images by nonlocal functionals, motivated by the use of neighborhood filters [**29**]. Such filters have originally been proposed by Yaroslavsky, [**85**], [**86**], and further generalized by C. Tomasi and R. Manduchi, [**83**], as bilateral filter. The main aim of [**70**] is to relate the neighborhood filter to an energy minimization. Now in this case the Euler-Lagrange equations are not partial differential equations but include integrals. The functional considered in [**70**] takes the general form

$$J_g(u) = \int_{\Omega\times\Omega} g\left(\frac{|u(x)-u(y)|^2}{h^2}\right) w(|x-y|)\, dxdy, \tag{10.9}$$

with $w \in L^\infty(\Omega)$, $g \in C^1(\mathbb{R}^+)$ and $h > 0$ is a parameter. The Fréchet derivative of J_g as a functional from $L^2(\Omega)$ into $\mathbb{R}$ is given by

$$J_g'(u)(x) = \frac{4}{h^2}\int_\Omega g'\left(\frac{|u(x)-u(y)|^2}{h^2}\right)(u(x)-u(y))w(|x-y|)\, dy.$$

Note that the nonlocal functional J_p is of the form (10.9) with $g(t) = \frac{1}{2p}|t|^{\frac{p}{2}}$, $w = J$ and $h = 1$. Then, problem $P_p^J(u_0)$ appears when one uses the steepest descent method to minimize this particular nonlocal functional.

For given $p \geq 1$ and J we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right),$$

where

$$C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p\,dz$$

is a normalizing constant in order to obtain the p-Laplacian in the limit instead a multiple of it.

Associated with these rescaled kernels we have solutions u_ε to the equation in P_p^J with J replaced by $J_{p,\varepsilon}$ and the same initial condition u_0 (we shall call this problem $P_p^{J_{p,\varepsilon}}$). Our next result states that these functions u_ε converge strongly in $L^p(\Omega)$ to the solution to the local $p-$Laplacian $N_p(u_0)$.

THEOREM 130. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $p \geq 1$. Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$, $u_0 \in L^p(\Omega)$ and u_ε the unique solution of $P_p^{J_{p,\varepsilon}}(u_0)$. Then, if u is the unique solution of $N_p(u_0)$,*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]}\|u_\varepsilon(t,.) - u(t,.)\|_{L^p(\Omega)} = 0.$$

Note that the above result states that P_p^J is a nonlocal analogous to the $p-$Laplacian.

Recall that for the linear case, $p = 2$, under additional regularity hypothesis on the involved data, the convergence of the solutions of rescaled nonlocal problems of the form P_2^J to the solution of the heat equation is proved in Chapter 5, see also [**43**].

Convergence to the p-laplacian for $p > 1$

Our main goal in this chapter is to show that the Neumann problem for the p-Laplacian equation $N_p(u_0)$ can be approximated by suitable nonlocal Neumann problems $P_p^J(u_0)$.

Let us start recalling some results about the p-Laplacian equation

$$N_p(u_0)\qquad \begin{cases} u_t = \Delta_p u & \text{in }]0,T[\times\Omega,\\ |\nabla u|^{p-2}\nabla u\cdot\eta = 0 & \text{on }]0,T[\times\partial\Omega,\\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases}$$

obtained in [**5**], [**6**] and [**4**]. We have the two following concepts of solutions.

A *weak solution* of $N_p(u_0)$ in $[0,T]$ is a function

$$u \in C([0,T] : L^1(\Omega)) \cap L^p(0,T; W^{1,p}(\Omega)) \cap W^{1,1}(0,T; L^1(\Omega))$$

with $u(0) = u_0$, satisfying

$$\int_\Omega u'(t)\xi + \int_\Omega |\nabla u(t)|^{p-2}\nabla u(t)\cdot\nabla\xi = 0 \qquad \text{for almost all } \; t \in]0,T[$$

for any $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

An *entropy solution* of $N_p(u_0)$ in the time interval $[0,T]$ is a function

$$u \in C([0,T] : L^1(\Omega)) \cap W^{1,1}(0,T; L^1(\Omega)),$$

with $T_k(u) \in L^p(0,T; W^{1,p}(\Omega))$ for all $k > 0$ such that $u(0) = u_0$ and

$$\int_\Omega u'(t) T_k(u(t) - \xi) + \int_\Omega |\nabla u(t)|^{p-2} \nabla u(t) \cdot \nabla T_k(u(t) - \xi) = 0 \qquad \text{for almost all } \ t \in]0,T[$$

for any $\xi \in W^{1,p}(\Omega) \cap L^\infty(\Omega)$.

Here the truncature functions T_k are defined by $T_k(r) = k \wedge (r \vee (-k))$, $k \geq 0$, $r \in \mathbb{R}$.

THEOREM 131 ([**6**], [**4**]). *Let $T > 0$. For any $u_0 \in L^1(\Omega)$ there exists a unique entropy solution $u(t)$ of $N_p(u_0)$. Moreover, if $u_0 \in L^{p'}(\Omega) \cap L^2(\Omega)$ the entropy solution $u(t)$ is a weak solution.*

Let us perform a formal calculation just to convince the reader that the convergence result, Theorem 130, is correct. Let $N = 1$. Let $u(x)$ be a smooth function and consider

$$A_\varepsilon(u) = \frac{1}{\varepsilon^{p+1}} \int_{\mathbb{R}} J\left(\frac{x-y}{\varepsilon}\right) |u(y) - u(x)|^{p-2} (u(y) - u(x))\, dy.$$

Changing variables, $y = x - \varepsilon z$, we get

$$A_\varepsilon(u) = \frac{1}{\varepsilon^p} \int_{\mathbb{R}} J(z) |u(x - \varepsilon z) - u(x)|^{p-2} (u(x - \varepsilon z) - u(x))\, dz. \tag{10.10}$$

Now, we expand in powers of ε to obtain

$$|u(x - \varepsilon z) - u(x)|^{p-2} = \varepsilon^{p-2} \left| u'(x) z + \tfrac{u''(x)}{2} \varepsilon z^2 + O(\varepsilon^2) \right|$$

$$= \varepsilon^{p-2} |u'(x)|^{p-2} |z|^{p-2} + \varepsilon^{p-1} (p-2) |u'(x) z|^{p-4} u'(x) z \frac{u''(x)}{2} z^2 + O(\varepsilon^p),$$

and

$$u(x - \varepsilon z) - u(x) = \varepsilon u'(x) z + \frac{u''(x)}{2} \varepsilon^2 z^2 + O(\varepsilon^3).$$

Hence, (10.10) becomes

$$\begin{aligned} A_\varepsilon(u) = \ & \frac{1}{\varepsilon} \int_{\mathbb{R}} J(z) |z|^{p-2} z\, dz |u'(x)|^{p-2} u'(x) \\ & + \frac{1}{2} \int_{\mathbb{R}} J(z) |z|^p\, dz \left((p-2) |u'(x)|^{p-2} u''(x) + |u'(x)|^{p-2} u''(x) \right) + O(\varepsilon). \end{aligned}$$

Using that J is radially symmetric, the first integral vanishes and therefore,

$$\lim_{\varepsilon \to 0} A_\varepsilon(u) = C (|u'(x)|^{p-2} u'(x))',$$

where

$$C = \frac{1}{2} \int_{\mathbb{R}} J(z) |z|^p\, dz.$$

To do this formal calculation rigorous we need to obtain the following result which is a variant of [**23**, Theorem 4].

PROPOSITION 132. *Let $1 \leq q < +\infty$. Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, non-identically zero, and $\rho_n(x) := n^N \rho(nx)$. Let $\{f_n\}$ be a sequence of functions in $L^q(\Omega)$ such that*

$$\int_\Omega \int_\Omega |f_n(y) - f_n(x)|^q \rho_n(y-x)\, dx\, dy \leq M \frac{1}{n^q}. \tag{10.11}$$

1. If $\{f_n\}$ is weakly convergent in $L^q(\Omega)$ to f then

(i) if $q > 1$, $f \in W^{1,q}(\Omega)$, and moreover

$$(\rho(z))^{1/q} \chi_\Omega \left(x + \frac{1}{n} z \right) \frac{f_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup (\rho(z))^{1/q}\, z \cdot \nabla f$$

weakly in $L^q(\Omega) \times L^q(\mathbb{R}^N)$.

(ii) If $q = 1$, $f \in BV(\Omega)$, and moreover

$$\rho(z) \chi_\Omega \left(x + \frac{1}{n} z \right) \frac{f_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup \rho(z) z \cdot Df$$

weakly as measures.

2. Assume Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\rho(x) \geq \rho(y)$ if $|x| \leq |y|$. Then $\{f_n\}$ is relatively compact in $L^q(\Omega)$, and consequently, there exists a subsequence $\{f_{n_k}\}$ such that

(i) if $q > 1$, $f_{n_k} \to f$ in $L^q(\Omega)$ with $f \in W^{1,q}(\Omega)$,

(ii) if $q = 1$, $f_{n_k} \to f$ in $L^1(\Omega)$ with $f \in BV(\Omega)$.

PROOF. We suppose $f_n \to f$ weakly in $L^q(\Omega)$ and write (10.11) as

$$\begin{aligned} &\int_\Omega \int_\Omega n^N \rho(n(x-y)) \left| \frac{f_n(y) - f_n(x)}{1/n} \right|^q dx\, dy \\ &= \int_{\mathbb{R}^N} \int_\Omega \rho(z) \chi_\Omega \left(x + \frac{1}{n} z \right) \left| \frac{f_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \right|^q dx\, dz \leq M. \end{aligned} \tag{10.12}$$

On the other hand, if $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in \mathcal{D}(\mathbb{R}^N)$, taking n large enough,

$$\begin{aligned} &\int_{\mathbb{R}^N} (\rho(z))^{1/q} \int_\Omega \chi_\Omega \left(x + \frac{1}{n} z \right) \frac{f_n(x + 1/nz) - f_n(x)}{1/n} \varphi(x) dx \psi(z) dz \\ &= \int_{\mathbb{R}^N} (\rho(z))^{1/q} \int_\Omega \frac{f_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \varphi(x) dx \psi(z) dz \\ &= -\int_{\mathbb{R}^N} (\rho(z))^{1/q} \int_\Omega f_n(x) \frac{\varphi(x) - \varphi\left(x - \frac{1}{n}z\right)}{1/n} dx \psi(z) dz. \end{aligned} \tag{10.13}$$

Let us start with the case *1.(i)*. By (10.12), up to a subsequence,

$$(\rho(z))^{1/q}\,\chi_\Omega\left(x+\frac{1}{n}z\right)\frac{f_n\left(x+\frac{1}{n}z\right)-f_n(x)}{1/n} \rightharpoonup (\rho(z))^{1/q}\,g(x,z)$$

weakly in $L^q(\Omega)\times L^q(\mathbb{R}^N)$. Therefore, passing to the limit in (10.13), we get

$$\int_{\mathbb{R}^N}(\rho(z))^{1/q}\int_\Omega g(x,z)\varphi(x)\,dx\psi(z)\,dz=-\int_{\mathbb{R}^N}(\rho(z))^{1/q}\int_\Omega f(x)\,z\cdot\nabla\varphi(x)dx\psi(z)dz.$$

Consequently,

$$\int_\Omega g(x,z)\varphi(x)dx=-\int_\Omega f(x)\,z\cdot\nabla\varphi(x)dx\quad\forall z\in\operatorname{int}(\operatorname{supp}(J)).$$

And from here, for s small,

$$\int_\Omega g(x,se_i)\varphi(x)dx=-\int_\Omega f(x)\,s\frac{\partial}{\partial x_i}\varphi(x)dx,$$

which implies $f\in W^{1,q}(\Omega)$ and $(\rho(z))^{1/q}\,g(x,z)=(\rho(z))^{1/q}\,z\cdot\nabla f(x)$.

Let us now prove *1.(ii)*. By (10.12), there exists a bounded Radon measure $\mu\in\mathcal{M}(\Omega\times\mathbb{R}^N)$ such that, up to a subsequence,

$$\rho(z)\chi_\Omega\left(x+\frac{1}{n}z\right)\frac{f_n\left(x+\frac{1}{n}z\right)-f_n(x)}{1/n}\rightharpoonup\mu(x,z)$$

weakly in $\mathcal{M}(\Omega\times\mathbb{R}^N)$. Hence, passing to the limit in (10.13), we get

$$\int_{\Omega\times\mathbb{R}^N}\varphi(x)\psi(z)d\mu(x,z)=-\int_{\Omega\times\mathbb{R}^N}\rho(z)\psi(z)\,z\cdot\nabla\varphi(x)f(x)\,dxdz. \tag{10.14}$$

Now, applying the disintegration theorem (Theorem 2.28 in [**1**]) to the measure μ, we get that if $\pi:\Omega\times\mathbb{R}^N\to\mathbb{R}^N$ is the projection on the first factor and $\nu=\pi_\#|\mu|$, then there exists a Radon measures μ_x in $\mathbb{R}^N$ such that $x\mapsto\mu_x$ is ν-measurable,

$$|\mu_x|(\mathbb{R}^N)\le 1\qquad \nu-\text{a.e.}\quad\text{in}\;\;\Omega$$

and, for any $h\in L^1(\Omega\times\mathbb{R}^N,|\mu|)$,

$$h(x,\cdot)\in L^1(\mathbb{R}^N,|\mu_x|)\qquad\nu-\text{a.e.}\quad\text{in}\;\;x\in\Omega,$$

$$x\mapsto\int_\Omega h(x,z)d\mu_x(z)\in L^1(\Omega,\nu)$$

and

$$\int_{\Omega\times\mathbb{R}^N}h(x,z)d\mu(x,z)=\int_\Omega\left(\int_{\mathbb{R}^N}h(x,z)d\mu_x(z)\right)d\nu(x). \tag{10.15}$$

From (10.14) and (10.15), we get, for $\varphi \in \mathcal{D}(\Omega)$ and $\psi \in \mathcal{D}(\mathbb{R}^N)$,

$$\int_\Omega \left(\int_{\mathbb{R}^N} \psi(z) d\mu_x(z) \right) \varphi(x) d\nu(x) = \left(\sum_{i=1}^N \int_{\mathbb{R}^N} \rho(z) z_i \psi(z) dz \, \frac{\partial f}{\partial x_i} \, , \, \varphi(x) \right).$$

Hence, as measures,

$$\sum_{i=1}^N \int_{\mathbb{R}^N} \rho(z) z_i \psi(z) dz \, \frac{\partial f}{\partial x_i} = \int_{\mathbb{R}^N} \psi(z) d\mu_x(z) \, \nu.$$

Let now $\tilde{\psi} \in \mathcal{D}(\mathbb{R}^N)$ a radial function such that $\tilde{\psi} = 1$ in $\operatorname{supp}(\rho)$. We set $\psi(z) = \tilde{\psi}(z) z_i$. Then

$$\int_{\mathbb{R}^N} \rho(z) {z_i}^2 \tilde{\psi}(z) dz \, \frac{\partial f}{\partial x_i} = \int_{\mathbb{R}^N} \tilde{\psi}(z) z_i d\mu_x(z) \, \nu.$$

Since $\nu \in M_b(\Omega)$ and $x \mapsto \int_{\mathbb{R}^N} \tilde{\psi}(z) z_i d\mu_x(z) \in L^1(\Omega, \nu)$, $f \in BV(\Omega)$. Going back to (10.15) we obtain that

$$\mu(x,z) = \sum_{i=1}^N \frac{\partial f}{\partial x_i}(x) \cdot \rho(z) z_i \mathcal{L}^N(z).$$

As in the proof of [**23**, Theorem 4], to prove *2* it is enough to show that for any $\delta > 0$ there exists $n_\delta \in \mathbb{N}$ such that

$$\delta^{-N} \int_0^\delta t^{N-1} F_n(t) \, dt \leq C \delta^q \qquad \text{for } n \geq n_\delta \tag{10.16}$$

for some constant C independent of n and δ, being F_n the function defined for $t > 0$ as

$$\begin{aligned} F_n(t) = & \int_{w \in S^{N-1}} \int_{\mathbb{R}^N} |f_n((x+tw) - f_n(x)|^q \, dx \, d\sigma \\ = & \frac{1}{t^{N-1}} \int_{|h|=t} \int_{\mathbb{R}^N} |f_n((x+h) - f_n(x)|^q \, dx \, d\sigma. \end{aligned}$$

In terms of F_n assumption (10.11) can be expressed as

$$\int_0^1 t^{N+q-1} \frac{F_n(t)}{t^q} \rho_n(t) \, dt \leq M \frac{1}{n^q}. \tag{10.17}$$

On the other hand, applying [**23**, Lemma 2] with $g(t) = F_n(t)/t^q$ and $h(t) = \rho_n(t)$, there exists a constant $K = K(N+q) > 0$ such that

$$\delta^{-N-q} \int_0^\delta t^{N+q-1} \frac{F_n(t)}{t^q} dt \leq K \left(\int_0^\delta t^{N+q-1} \frac{F_n(t)}{t^q} \rho_n(t) \right) \Big/ \left(\int_{[|x|<\delta]} |x|^q \rho_n(x) dx \right). \tag{10.18}$$

Now, since ρ is a function with compact support, given $\delta > 0$, we can find $n_\delta \in \mathbb{N}$ such that

$$\begin{aligned}\int_{[|x|<\delta]} |x|^q \rho_n(x)\,dx &= \int_{[|x|<\delta]} |x|^q n^N \rho(nx)\,dx \\ &= \int_{[|y|<n\delta]} n^{-q}|y|^q \rho(y)\,dy = \frac{1}{n^q}\int_{\mathbb{R}^N} |y|^q\rho(y)\,dy, \quad \text{for } n \geq n_\delta.\end{aligned}$$

Hence, by (10.17) and (10.18), (10.16) follows. □

For given $p > 1$ and J, we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right),$$

where

$$C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p\,dz$$

is a normalizing constant in order to obtain the p-Laplacian in the limit instead a multiple of it. Observe, that, using spherical coordinates,

$$C_{J,p}^{-1} = \omega_{N-1}\int_0^{+\infty}\int_0^{\pi} \frac{1}{2} J(\rho)|\rho\cos\theta|^p \rho^{N-1}\sin^{N-2}\theta\,d\theta\,d\rho.$$

In [**5**], associated to the p-Laplacian with homogeneous boundary condition, we defined the operator $B_p \subset L^1(\Omega)\times L^1(\Omega)$ as $(u,\hat{u}) \in B_p$ if and only if $\hat{u} \in L^1(\Omega)$, $u \in W^{1,p}(\Omega)$ and

$$\int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla v = \int_\Omega \hat{u} v \quad \text{for every} \quad v \in W^{1,p}(\Omega)\cap L^\infty(\Omega).$$

Moreover, since B_p is a completely accretive operator in $L^1(\Omega)$ with dense domain satisfying the range condition (see [**5**]), its closure $\mathcal{B}_p$ in $L^1(\Omega)$ is an m-completely accretive operator in $L^1(\Omega)$ with dense domain. In [**6**], it was proved that for any $u_0 \in L^1(\Omega)$, the unique entropy solution $u(t)$ of problem $N_p(u_0)$ (see Theorem 131) coincides with the unique mild-solution $e^{t\mathcal{B}_p}u_0$ given by the Crandall-Liggett's exponential formula.

PROPOSITION 133. *For any $\phi \in L^\infty(\Omega)$, we have that*

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1}\phi \rightharpoonup (I+B_p)^{-1}\phi \quad \textit{weakly in } L^p(\Omega) \textit{ as } \varepsilon \to 0.$$

PROOF. For $\varepsilon > 0$, let $u_\varepsilon = \left(I + B_p^{J_{p,\varepsilon}}\right)^{-1}\phi$. Then,

$$\begin{aligned}\int_\Omega u_\varepsilon v - \frac{C_{J,p}}{\varepsilon^{p+N}} \int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right)|u_\varepsilon(y)-u_\varepsilon(x)|^{p-2}\times \\ \times(u_\varepsilon(y)-u_\varepsilon(x))\,dy\,v(x)\,dx = \int_\Omega \phi v\end{aligned} \tag{10.19}$$

for every $v \in L^\infty(\Omega)$.

Changing variables, we get

$$
\begin{aligned}
&-\frac{C_{J,p}}{\varepsilon^{p+N}} \int_\Omega \int_\Omega J\left(\frac{x-y}{\varepsilon}\right) |u_\varepsilon(y)-u_\varepsilon(x)|^{p-2}(u_\varepsilon(y)-u_\varepsilon(x))\,dy\,v(x)\,dx \\
&= \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi_\Omega(x+\varepsilon z)\left|\frac{u_\varepsilon(x+\varepsilon z)-u_\varepsilon(x)}{\varepsilon}\right|^{p-2}\frac{u_\varepsilon(x+\varepsilon z)-u_\varepsilon(x)}{\varepsilon}\times \\
&\qquad\qquad \times\frac{v(x+\varepsilon z)-v(x)}{\varepsilon}\,dx\,dz.
\end{aligned} \tag{10.20}
$$

So we can write (10.19) as

$$
\begin{aligned}
&\int_\Omega \phi(x)v(x)\,dx - \int_\Omega u_\varepsilon(x)v(x)\,dx \\
&= \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi_\Omega(x+\varepsilon z)\left|\frac{u_\varepsilon(x+\varepsilon z)-u_\varepsilon(x)}{\varepsilon}\right|^{p-2}\frac{u_\varepsilon(x+\varepsilon z)-u_\varepsilon(x)}{\varepsilon}\times \\
&\qquad\qquad \times\frac{v(x+\varepsilon z)-v(x)}{\varepsilon}\,dx\,dz.
\end{aligned} \tag{10.21}
$$

We shall see that there exists a sequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n} \to u$ weakly in $L^p(\Omega)$, $u \in W^{1,p}(\Omega)$ and $u = (I+B_p)^{-1}\phi$, that is,

$$
\int_\Omega uv + \int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla v = \int_\Omega \phi v \qquad \text{for every } v\in W^{1,p}(\Omega)\cap L^\infty(\Omega).
$$

Since $u_\varepsilon \ll \phi$, there exists a sequence $\varepsilon_n \to 0$ such that

$$
u_{\varepsilon_n} \rightharpoonup u, \quad \text{weakly in } L^p(\Omega), \quad u \ll \phi.
$$

Observe that $\|u_{\varepsilon_n}\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)} \le \|\phi\|_{L^\infty(\Omega)}$. Taking $\varepsilon = \varepsilon_n$ and $v = u_{\varepsilon_n}$ in (10.21), we get

$$
\begin{aligned}
&\int_\Omega\int_\Omega \frac{1}{2}\frac{C_{J,p}}{{\varepsilon_n}^N} J\left(\frac{x-y}{\varepsilon_n}\right)\left|\frac{u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dy \\
&= \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dz \le M.
\end{aligned} \tag{10.22}
$$

Therefore, by Proposition 132, $u \in W^{1,p}(\Omega)$ and

$$
\left(\frac{C_{J,p}}{2}J(z)\right)^{1/p}\chi_\Omega(x+\varepsilon_n z)\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n} \rightharpoonup \left(\frac{C_{J,p}}{2}J(z)\right)^{1/p} z\cdot\nabla u(x) \tag{10.23}
$$

weakly in $L^p(\Omega)\times L^p(\mathbb{R}^N)$. Moreover, we can also assume that

$$
\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^{p-2}\chi_\Omega(x+\varepsilon_n z)\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n} \rightharpoonup \chi(x,z)
$$

weakly in $L^{p'}(\Omega)\times L^{p'}(\mathbb{R}^N)$. Therefore, passing to the limit in (10.21) for $\varepsilon=\varepsilon_n$, we get

$$\int_\Omega uv+\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla v(x)\,dx\,dz=\int_\Omega \phi v \tag{10.24}$$

for every v smooth and by approximation for every $v\in W^{1,p}(\Omega)$.

Let us see now that

$$\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi(x,z)z\cdot\nabla v(x)\,dx\,dz=\int_\Omega |\nabla u|^{p-2}\,\nabla u\cdot\nabla v. \tag{10.25}$$

In fact, taking $v=u$ in (10.24), we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p\,dx\,dz\\
&=\int_\Omega \phi u_{\varepsilon_n}-\int_\Omega u_{\varepsilon_n}u_{\varepsilon_n}\\
&=\int_\Omega \phi u-\int_\Omega uu-\int_\Omega \phi(u-u_{\varepsilon_n})+\int_\Omega 2u(u-u_{\varepsilon_n})-\int_\Omega (u-u_{\varepsilon_n})(u-u_{\varepsilon_n})\\
&\le \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla u(x)\,dx\,dz-\int_\Omega \phi(u-u_{\varepsilon_n})+\int_\Omega 2u(u-u_{\varepsilon_n}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\limsup_n \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p\,dx\,dz\\
&\qquad\le \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla u(x)\,dx\,dz.
\end{aligned} \tag{10.26}$$

Now, by the monotonicity Lemma 118, for every ρ smooth,

$$\begin{aligned}
&-\frac{C_{J,p}}{{\varepsilon_n}^{p+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right)|\rho(y)-\rho(x)|^{p-2}(\rho(y)-\rho(x))\,dy\,(u_{\varepsilon_n}(x)-\rho(x))\,dx\\
&\le -\frac{C_{J,p}}{{\varepsilon_n}^{p+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon_n}\right)|u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)|^{p-2}(u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x))\,dy\,(u_{\varepsilon_n}(x)-\rho(x))\,dx.
\end{aligned}$$

Using the change of variable (10.20) and taking limits, on account of (10.23) and (10.26), we obtain for every ρ smooth,

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)|z\cdot\nabla\rho|^{p-2}z\cdot\nabla\rho\,z\cdot(\nabla u-\nabla\rho)\\
&\le \int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot(\nabla u(x)-\nabla\rho(x))\,dx\,dz,
\end{aligned}$$

and then for every $\rho \in W^{1,p}(\Omega)$. Taking now, $\rho = u \pm \lambda v$, $\lambda > 0$ and $v \in W^{1,p}(\Omega)$, and letting $\lambda \to 0$, we get

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi(x,z) z\cdot\nabla v(x)\,dx\,dz\\ &= \int_{\mathbb{R}^N}\frac{C_{J,p}}{2} J(z)\int_\Omega |z\cdot\nabla u(x)|^{p-2}\left(z\cdot\nabla u(x)\right)\left(z\cdot\nabla v(x)\right)\,dx\,dz.\end{aligned}$$

Consequently,

$$\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,p}}{2} J(z)\chi(x,z) z\cdot\nabla v(x)\,dx\,dz = C_{J,p}\int_\Omega \mathbf{a}(\nabla u)\cdot\nabla v \quad \text{for every } v\in W^{1,p}(\Omega),$$

where

$$\mathbf{a}_j(\xi) = C_{J,p}\int_{\mathbb{R}^N}\frac{1}{2}J(z)\,|z\cdot\xi|^{p-2}\; z\cdot\xi\, z_j\,dz.$$

Then, if we prove that

$$\mathbf{a}(\xi) = |\xi|^{p-2}\xi, \tag{10.27}$$

then (10.25) is true and $u = (I+B_p)^{-1}\phi$. So, to finish the proof we only need to show that (10.27) holds. Obviously, $\mathbf{a}$ is positively homogeneous of degree $p-1$, that is,

$$\mathbf{a}(t\xi) = t^{p-1}\mathbf{a}(\xi) \qquad \text{for all } \ \xi\in\mathbb{R}^N \quad \text{and all } t>0.$$

Therefore, in order to prove (10.27) it is enough to see that

$$\mathbf{a}_i(\xi) = \xi_i \quad \text{for all } \xi\in\mathbb{R}^N, |\xi| = 1 \qquad \text{for each } i.$$

Now, let $R_{\xi,i}$ be the rotation such that $R^t_{\xi,i}(\xi) = \mathbf{e}_i$, where $\mathbf{e}_i$ is the vector with components $(\mathbf{e}_i)_i = 1$, $(\mathbf{e}_i)_j = 0$ for $j\neq i$, and $R^t_{\xi,i}$ is the transpose of $R_{\xi,i}$. Observe that

$$\xi_i = \xi\cdot\mathbf{e}_i = R^t_{\xi,i}(\xi)\cdot R^t_{\xi,i}(\mathbf{e}_i) = \mathbf{e}_i\cdot R^t_{\xi,i}(\mathbf{e}_i).$$

On the other hand, since J is radial, $C^{-1}_{J,p} = \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_i|^p\,dz$ and

$$\mathbf{a}(\mathbf{e}_i) = \mathbf{e}_i \quad \text{for any } i.$$

Then, if we make the change of variables $z = R_{\xi,i}(y)$, since J is a radial function, we obtain

$$\begin{aligned}\mathbf{a}_i(\xi) &= C_{J,p}\int_{\mathbb{R}^N}\frac{1}{2}J(z)|z\cdot\xi|^{p-2}z\cdot\xi\; z\cdot\mathbf{e}_i\,dz\\ &= C_{J,p}\int_{\mathbb{R}^N}\frac{1}{2}J(y)|y\cdot\mathbf{e}_i|^{p-2}y\cdot\mathbf{e}_i\; y\cdot R^t_{\xi,i}(\mathbf{e}_i)\,dy\\ &= \mathbf{a}(\mathbf{e}_i)\cdot R^t_{\xi,i}(\mathbf{e}_i) = \mathbf{e}_i\cdot R^t_{\xi,i}(\mathbf{e}_i) = \xi_i.\end{aligned}$$

This ends the proof. □

THEOREM 134. *Let Ω a smooth bounded domain in $\mathbb{R}^N$. Assume $J(x)\geq J(y)$ if $|x|\leq|y|$. For any $\phi\in L^\infty(\Omega)$,*

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1}\phi \to (I+B_p)^{-1}\phi \quad \textit{in } L^p(\Omega) \textit{ as } \varepsilon\to 0. \tag{10.28}$$

PROOF. The proof is a consequence of Proposition 133, (10.22), and Proposition 132. □

From the above theorem by standard results of the Nonlinear Semigroup Theory (see [**50**], [**19**] and [**20**]) we obtain the following result, which gives Theorem 130 in the case $p > 1$.

THEOREM 135. *Let Ω be a smooth bounded domain in $\mathbb{R}^N$. Assume $J(x) \geq J(y)$ if $|x| \leq |y|$. Let $T > 0$ and $u_0 \in L^q(\Omega)$, $p \leq q < +\infty$. Let u_ε the unique solution of $P_p^{J_{p,\varepsilon}}(u_0)$ and u the unique solution of $N_p(u_0)$. Then*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_\varepsilon(.,t) - u(.,t)\|_{L^q(\Omega)} = 0. \tag{10.29}$$

Moreover, if $1 < p \leq 2$, (10.29) holds for any $u_0 \in L^q(\Omega)$, $1 \leq q < +\infty$.

PROOF. Since B_p^J is completely accretive and satisfies the range condition (10.2), to get (10.29) it is enough to see

$$\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \to (I + B_p)^{-1} \phi \qquad \text{in } L^q(\Omega) \text{ as } \varepsilon \to 0$$

for any $\phi \in L^\infty(\Omega)$. Taking into account that $\left(I + B_p^{J_{p,\varepsilon}}\right)^{-1} \phi \ll \phi$, the above convergence follows by (10.28). □

Convergence to the total variation flow for $p = 1$

As was mentioned in the introduction, motivated by problems in image processing, the problem $N_1(u_0)$, that is, the Neumann problem for the Total Variation Flow, was studied in [**2**] (see also [**3**]).

DEFINITION 136. A measurable function $u : (0,T)\times\Omega \to \mathbb{R}$ is a *weak solution* of $N_1(u_0)$ in $(0,T) \times \Omega$ if $u \in C([0,T], L^1(\Omega)) \cap W^{1,1}_{loc}(0,T; L^1(\Omega))$, $T_k(u) \in L^1_w(0,T; BV(\Omega))$ for all $k > 0$ and there exists $z \in L^\infty((0,T) \times \Omega)$ with $\|z\|_\infty \leq 1$, $u_t = \mathrm{div}(z)$ in $\mathcal{D}'((0,T) \times \Omega)$ such that

$$\int_\Omega (T_k(u(t)) - w)u_t(t)\,dx \leq \int_\Omega z(t) \cdot \nabla w\,dx - |DT_k(u(t))|(\Omega)$$

for every $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$ and *a.e.* on $[0,T]$.

The main result of [**2**] is the following.

THEOREM 137. *Let $u_0 \in L^1(\Omega)$. Then there exists a unique weak solution of $N_1(u_0)$ in $(0,T) \times \Omega$ for every $T > 0$ such that $u(0) = u_0$. Moreover, if $u(t), \hat{u}(t)$ are weak solutions corresponding to initial data $u_0, \hat{u}_0$, respectively, then*

$$\|(u(t) - \hat{u}(t))^+\|_1 \leq \|(u_0 - \hat{u}_0)^+\|_1 \quad \text{and} \quad \|u(t) - \hat{u}(t)\|_1 \leq \|u_0 - \hat{u}_0\|_1,$$

for all $t \geq 0$.

Theorem 137 is proved using the techniques of completely accretive operators ([**19**]) and the Crandall-Liggett's semigroup generation Theorem. To this end, the following operator B_1 in $L^1(\Omega)$ was defined in [**2**] by the following rule:

$$(u,v) \in B_1 \quad \text{if and only if } u,v \in L^1(\Omega),\ T_k(u) \in BV(\Omega) \text{ for all } k>0 \text{ and}$$

$$\text{there exists } z \in L^\infty(\Omega,\mathbb{R}^N) \text{ with } \|z\|_\infty \le 1,\ v = -\mathrm{div}(z) \text{ in } \mathcal{D}'(\Omega) \text{ such that}$$

$$\int_\Omega (w - T_k(u))v\,dx \le \int_\Omega z\cdot\nabla w dx - |DT_k(u)|(\Omega),$$

$\forall w \in W^{1,1}(\Omega)\cap L^\infty(\Omega)$, $\forall k>0$.

Theorem 137 follows from the following result given in [**2**].

THEOREM 138. *The operator B_1 is m-completely accretive in $L^1(\Omega)$ with dense domain. For any $u_0 \in L^1(\Omega)$ the semigroup solution $u(t) = e^{-tB_1}u_0$ is a strong solution of*

$$\begin{cases} \dfrac{du}{dt} + B_1 u \ni 0, \\ u(0) = u_0. \end{cases}$$

Set

$$J_{1,\varepsilon}(x) := \frac{C_{J,1}}{\varepsilon^{1+N}} J\left(\frac{x}{\varepsilon}\right), \qquad \text{with} \qquad \frac{1}{C_{J,1}} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|\,dz.$$

THEOREM 139. *Assume Ω is a smooth bounded domain in $\mathbb{R}^N$ and $J(x) \ge J(y)$ if $|x| \le |y|$. For any $\phi \in L^\infty(\Omega)$, we have*

$$\left(I + B_1^{J_{1,\varepsilon}}\right)^{-1}\phi \to (I+B_1)^{-1}\phi \quad \textit{in } L^1(\Omega) \textit{ as } \varepsilon \to 0.$$

PROOF. Given $\varepsilon>0$, we set $u_\varepsilon = \left(I + B_1^{J_{1,\varepsilon}}\right)^{-1}\phi$. Then, there exists $g_\epsilon \in L^\infty(\Omega\times\Omega)$, $g_\varepsilon(x,y) = -g_\varepsilon(y,x)$ for almost all $x,y\in\Omega$, $\|g_\varepsilon\|_\infty \le 1$,

$$J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y) \in J\left(\frac{x-y}{\varepsilon}\right)\mathrm{sign}(u_\varepsilon(y) - u_\varepsilon(x)) \quad \text{a.e. } x,y\in\Omega$$

and

$$-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_\Omega J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)dy = \phi(x) - u_\varepsilon(x) \qquad \text{a.e. } x\in\Omega. \tag{10.30}$$

Observe that

$$\begin{aligned} &-\frac{C_{J,1}}{\varepsilon^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)dy\, u_\varepsilon(x)\,dx \\ &\qquad = \frac{C_{J,1}}{\varepsilon^{1+N}}\frac{1}{2}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right)|u_\varepsilon(y) - u_\varepsilon(x)|\,dy\,dx. \end{aligned} \tag{10.31}$$

By (10.30), we can write

$$\begin{aligned}
&\frac{C_{J,1}}{2\varepsilon^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)(v(y)-v(x))\,dxdy \\
&= -\frac{C_{J,1}}{\varepsilon^{1+N}}\int_\Omega\int_\Omega J\left(\frac{x-y}{\varepsilon}\right) g_\varepsilon(x,y)dyv(x)\,dx \qquad (10.32)\\
&= \int_\Omega (\phi(x)-u_\varepsilon(x))v(x)\,dx, \qquad \forall v\in L^\infty(\Omega).
\end{aligned}$$

Since $u_\varepsilon \ll \phi$, there exists a sequence $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \rightharpoonup u \quad \text{weakly in } L^1(\Omega), \quad u \ll \phi.$$

Observe that $\|u_{\varepsilon_n}\|_{L^\infty(\Omega)}, \|u\|_{L^\infty(\Omega)} \le \|\phi\|_{L^\infty(\Omega)}$. Hence taking $\varepsilon = \varepsilon_n$ and $v = u_{\varepsilon_n}$ in (10.32), changing variables and having in mind (10.31), we get

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_\Omega \frac{C_{J,1}}{2} J(z)\chi_\Omega(x+\varepsilon_n z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right| dx\,dz \\
&= \int_\Omega\int_\Omega \frac{1}{2}\frac{C_{J,1}}{{\varepsilon_n}^N} J\left(\frac{x-y}{\varepsilon_n}\right)\left|\frac{u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right| dx\,dy \\
&= \int_\Omega (\phi(x)-u_{\varepsilon_n}(x))u_{\varepsilon_n}(x)\,dx \le M, \qquad \forall\, n\in\mathbb{N}.
\end{aligned}$$

Therefore, by Proposition 132, $u \in BV(\Omega)$,

$$\frac{C_{J,1}}{2} J(z)\chi_\Omega(x+\varepsilon_n z)\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n} \rightharpoonup \frac{C_{J,1}}{2} J(z) z\cdot Du \qquad (10.33)$$

weakly as measures and

$$u_{\varepsilon_n} \to u, \qquad \text{strongly in } L^1(\Omega).$$

Moreover, we also can assume that

$$J(z)\chi_\Omega(x+\varepsilon_n z) g_{\varepsilon_n}(x,x+\varepsilon_n z) \rightharpoonup \Lambda(x,z) \qquad (10.34)$$

weakly* in $L^\infty(\Omega)\times L^\infty(\mathbb{R}^N)$, and $\Lambda(x,z)\le J(z)$ almost every where in $\Omega\times\mathbb{R}^N$. Changing variables and having in mind (10.32), we can write

$$\begin{aligned}
&\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_\Omega J(z)\chi_\Omega(x+\varepsilon_n z) g_{\varepsilon_n}(x,x+\varepsilon_n z)\,dz \frac{v(x+\varepsilon_n z)-v(x)}{\varepsilon_n}\,dx \\
&= -\frac{C_{J,1}}{\varepsilon_n}\int_{\mathbb{R}^N}\int_\Omega J(z)\chi_\Omega(x+\varepsilon_n z) g_{\varepsilon_n}(x,x+\varepsilon_n z)\,dz\,v(x)\,dx \qquad (10.35)\\
&= \int_\Omega (\phi(x)-u_{\varepsilon_n}(x))v(x)\,dx \qquad \forall v\in L^\infty(\Omega).
\end{aligned}$$

By (10.34), passing to the limit in (10.35), we get

$$\begin{aligned}(10.36)\qquad &\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\int_{\Omega}\Lambda(x,z)z\cdot\nabla v(x)\,dx\,dz\\ &=\int_{\Omega}(\phi(x)-u(x))v(x)\,dx \qquad\qquad \forall v\in L^{\infty}(\Omega)\cap W^{1,1}(\Omega).\end{aligned}$$

We set $\zeta=(\zeta_1,\ldots,\zeta_N)$, the vector field defined by

$$\zeta_i(x):=\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,z)z_i\,dz,\qquad i=1,\ldots,N.$$

Then, $\zeta\in L^{\infty}(\Omega,\mathbb{R}^N)$, and from (10.36),

$$-\mathrm{div}(\zeta)=\phi-u \quad \text{ in } \mathcal{D}'(\Omega).$$

Let us see that $\|\zeta\|_{\infty}\leq 1$. Given $\xi\in\mathbb{R}^N\setminus\{0\}$, let R_{ξ} be the rotation such that $R_{\xi}^t(\xi)=\mathbf{e}_1|\xi|$. Then, if we make the change of variables $z=R_{\xi}(y)$, we obtain

$$\zeta(x)\cdot\xi=\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,z)z\cdot\xi\,dz=\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,R_{\xi}(y))R_{\xi}(y)\cdot\xi\,dy$$

$$=\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}\Lambda(x,R_{\xi}(y))y_1|\xi|\,dy.$$

On the other hand, since J is a radial function and $\Lambda(x,z)\leq J(z)$ almost every where,

$${C_{J,1}}^{-1}=\frac{1}{2}\int_{\mathbb{R}^N}J(z)|z_1|\,dz$$

and

$$|\zeta(x)\cdot\xi|\leq\frac{C_{J,1}}{2}\int_{\mathbb{R}^N}J(y)|y_1|\,dy|\xi|=|\xi|\quad \text{ a.e. } x\in\Omega.$$

Therefore, we obtain that $\|\zeta\|_{\infty}\leq 1$.

Since $u\in L^{\infty}(\Omega)$, to finish the proof we only need to show that

$$(10.37)\qquad \int_{\Omega}(w-u)(\phi-u)\,dx\leq\int_{\Omega}\zeta\cdot\nabla w dx-|Du|(\Omega),\qquad \forall w\in W^{1,1}(\Omega)\cap L^{\infty}(\Omega).$$

Given $w \in W^{1,1}(\Omega) \cap L^\infty(\Omega)$, taking $v = w - u_{\varepsilon_n}$ in (10.35), we get

$$
\begin{aligned}
&\int_\Omega (\phi(x) - u_{\varepsilon_n}(x))(w(x) - u_{\varepsilon_n}(x))\, dx \\
&= \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} \int_\Omega J(z) \chi_\Omega(x + \varepsilon_n z) g_{\varepsilon_n}(x, x + \varepsilon_n z)\, dz \times \\
&\qquad\qquad \times \left(\frac{w(x + \varepsilon_n z) - w(x)}{\varepsilon_n} - \frac{u_{\varepsilon_n}(x + \varepsilon_n z) - u_{\varepsilon_n}(x)}{\varepsilon_n} \right) dx \qquad (10.38) \\
&= \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} \int_\Omega J(z) \chi_\Omega(x + \varepsilon_n z) g_{\varepsilon_n}(x, x + \varepsilon_n z)\, dz \frac{w(x + \varepsilon_n z) - w(x)}{\varepsilon_n}\, dx \\
&- \frac{C_{J,1}}{2} \int_{\mathbb{R}^N} \int_\Omega J(z) \chi_\Omega(x + \varepsilon_n z) \left| \frac{u_{\varepsilon_n}(x + \varepsilon_n z) - u_{\varepsilon_n}(x)}{\varepsilon_n} \right| dx.
\end{aligned}
$$

Having in mind (10.33) and (10.34) and taking limit in (10.38) as $n \to \infty$, we obtain that

$$
\int_\Omega (w - u)(\phi - u)\, dx \le \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} \Lambda(x, z) z \cdot \nabla w(x)\, dx\, dz - \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} |J(z) z \cdot Du|
$$

$$
= \int_\Omega \zeta \cdot \nabla w\, dx - \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} |J(z) z \cdot Du|.
$$

Now, for every $x \in \Omega$ such that the Radon-Nikodym derivative $\frac{Du}{|Du|}(x) \neq 0$, let R_x be the rotation such that $R_x^t[\frac{Du}{|Du|}(x)] = \mathbf{e}_1 |\frac{Du}{|Du|}(x)|$. Then, since J is a radial function and $|\frac{Du}{|Du|}(x)| = 1$ $|Du|$-a.e. in Ω, if we make the change of variables $y = R_x(z)$, we obtain

$$
\frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} |J(z) z \cdot Du| = \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} J(z) \left| z \cdot \frac{Du}{|Du|}(x) \right| dz\, d|Du|(x)
$$

$$
= \frac{C_{J,1}}{2} \int_\Omega \int_{\mathbb{R}^N} J(y)\, |y_1|\ dy\, d|Du|(x) = \int_\Omega |Du|.
$$

Consequently, (10.37) holds and the proof concludes. □

From the above theorem, arguing as in Theorem 135, by standard results of the Nonlinear Semigroup Theory ([**50**], [**20**]), we obtain the following result, from which Theorem 130 holds in the case $p = 1$.

THEOREM 140. *Let Ω a smooth bounded domain in $\mathbb{R}^N$. Assume $J(x) \ge J(y)$ if $|x| \le |y|$. Let $T > 0$ and $u_0 \in L^1(\Omega)$. Let u_ε the unique solution in $[0, T]$ of $P_1^J(u_0)$ and u the unique weak solution of $N_1(u_0)$. Then*

$$
\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_\varepsilon(., t) - u(., t)\|_{L^1(\Omega)} = 0.
$$

0.29. Asymptotic behaviour. Now we study the asymptotic behaviour as $t \to \infty$ of the solutions of the nonlocal problems. We show that the solutions of the nonlocal problems converge to the mean value of the initial condition.

THEOREM 141. *Let $p \geq 1$ and $u_0 \in L^\infty(\Omega)$. Let u be the solution to $P_p^J(u_0)$, then*

$$\|u(t) - \overline{u_0}\|_{L^p(\Omega)} \leq \left(\frac{||u_0||^2_{L^2(\Omega)}}{t} \right)^{1/p} \to 0, \qquad \text{as } t \to \infty,$$

where $\overline{u_0}$ is the mean value of the initial condition,

$$\overline{u_0} = \frac{1}{|\Omega|} \int_\Omega u_0(x)\, dx.$$

To prove Theorem 141, we start by showing the following Poincaré's type inequality. In the linear case, that is, for $p = 2$, Poincaré's type inequality has been proved using spectral theory in [**34**].

PROPOSITION 142. *Given $p \geq 1$, J and Ω, the quantity*

$$\beta_{p-1} := \beta_{p-1}(J, \Omega, p) = \inf_{u \in L^p(\Omega), \int_\Omega u = 0} \frac{\displaystyle\frac{1}{2} \int_\Omega \int_\Omega J(x-y)|u(y) - u(x)|^p \, dy\, dx}{\displaystyle\int_\Omega |u(x)|^p \, dx}$$

is strictly positive. Consequently

$$\beta_{p-1} \int_\Omega \left| u - \frac{1}{|\Omega|} \int_\Omega u \right|^p \leq \frac{1}{2} \int_\Omega \int_\Omega J(x-y)|u(y) - u(x)|^p \, dy\, dx \quad \forall u \in L^p(\Omega). \tag{10.39}$$

PROOF. It is enough to prove that there exists a constant c such that

$$\|u\|_p \leq c \left(\left(\int_\Omega \int_\Omega J(x-y)|u(y) - u(x)|^p dy dx \right)^{1/p} + \left| \int_\Omega u \right| \right) \quad \forall u \in L^p(\Omega). \tag{10.40}$$

Let $r > 0$ such that $J(z) \geq \alpha > 0$ in $B(0, r)$. Since $\overline{\Omega} \subset \cup_{x \in \Omega} B(x, r/2)$, there exists $\{x_i\}_{i=1}^m \subset \Omega$ such that $\Omega \subset \cup_{i=1}^m B(x_i, r/2)$. Let $0 < \delta < r/2$ such that $B(x_i, \delta) \subset \Omega$ for all $i = 1, ...m$. Then, for any $\hat{x}_i \in B(x_i, \delta)$, $i = 1, ..., m$,

$$\Omega = \bigcup_{i=1}^m (B(\hat{x}_i, r) \cap \Omega). \tag{10.41}$$

Let us argue by contradiction. Suppose that (10.40) is false. Then, there exists $u_n \in L^p(\Omega)$, $\|u_n\|_p = 1$, and satisfying

$$1 \geq n \left(\left(\int_\Omega \int_\Omega J(x-y)|u_n(y) - u_n(x)|^p dy dx \right)^{1/p} + \left| \int_\Omega u_n \right| \right) \qquad \forall n \in \mathbb{N}.$$

Consequently,

$$\lim_n \int_\Omega \int_\Omega J(x-y)|u_n(y)-u_n(x)|^p dydx = 0 \tag{10.42}$$

and

$$\lim_n \int_\Omega u_n = 0. \tag{10.43}$$

Let

$$F_n(x,y) = J(x-y)^{1/p}|u_n(y)-u_n(x)|$$

and

$$f_n(x) = \int_\Omega J(x-y)|u_n(y)-u_n(x)|^p dy.$$

From (10.42), it follows that

$$f_n \to 0 \quad \text{in } L^1(\Omega),$$

so we can assume there exists a subsequence, denoted equal, such that

$$f_n(x) \to 0 \quad \forall x \in \Omega \setminus B_1, \quad B_1 \text{ null.} \tag{10.44}$$

On the other hand, by (10.42), we also have that

$$F_n \to 0 \quad \text{en } L^p(\Omega \times \Omega).$$

So we can suppose, passing to a subsequence if necessary,

$$F_n(x,y) \to 0 \quad \forall (x,y) \in \Omega \times \Omega \setminus C, \quad C \text{ null.} \tag{10.45}$$

Let $B_2 \subset \Omega$ a null set satisfying that,

$$\text{for all } x \in \Omega \setminus B_2, \text{ the chapter } C_x \text{ of } C \text{ is null.} \tag{10.46}$$

Let $\hat{x}_1 \in B(x_1,\delta) \setminus (B_1 \cup B_2)$, then there exists a subsequence, denoted equal, such that

$$u_n(\hat{x}_1) \to \lambda_1 \in [-\infty, +\infty].$$

Consider now $\hat{x}_2 \in B(x_2,\delta) \setminus (B_1 \cup B_2)$, then up to a subsequence, we can assume

$$u_n(\hat{x}_2) \to \lambda_2 \in [-\infty, +\infty].$$

So, successively (up to m), for $\hat{x}_m \in B(x_m,\delta) \setminus (B_1 \cup B_2)$, there exists a subsequence, againdenoted equal, such that

$$u_n(\hat{x}_m) \to \lambda_m \in [-\infty, +\infty].$$

By (10.45) and (10.46),

$$u_n(y) \to \lambda_i \quad \forall y \in (B(\hat{x}_i, r) \cap \Omega) \setminus C_{\hat{x}_i}.$$

Now, by (10.41),

$$\Omega = (B(\hat{x}_1, r) \cap \Omega) \cup (\cup_{i=2}^m (B(\hat{x}_i, r) \cap \Omega)).$$

Hence, since Ω is a domain, there exists $i_2 \in \{2, .., m\}$ such that

$$(B(\hat{x}_1, r) \cap \Omega) \cap (B(\hat{x}_{i_2}, r) \cap \Omega) \neq \emptyset.$$

Therefore, $\lambda_1 = \lambda_{i_2}$. Let us call $i_1 := 1$. Again, since

$$\Omega = ((B(\hat{x}_{i_1}, r) \cap \Omega) \cup ((B(\hat{x}_{i_1}, r) \cap \Omega)) \cup (\cup_{i \in \{1,...,m\} \setminus \{i_1, i_2\}} (B(\hat{x}_i, r) \cap \Omega)),$$

and there exists $i_3 \in \{1, ..., m\} \setminus \{i_1, i_2\}$ such that

$$((B(\hat{x}_{i_1}, r) \cap \Omega) \cup ((B(\hat{x}_{i_1}, r) \cap \Omega)) \cap (B(\hat{x}_{i_3}, r) \cap \Omega) \neq \emptyset.$$

Consequently

$$\lambda_{i_1} = \lambda_{i_2} = \lambda_{i_3}.$$

Using the same argument we arrive at

$$\lambda_1 = \lambda_2 = ... = \lambda_m = \lambda.$$

If $|\lambda| = +\infty$, we have shown that

$$|u_n(y)|^p \to +\infty \quad \text{for almost every } y \in \Omega,$$

which contradicts $\|u_n\|_p = 1$ for all $n \in \mathbb{N}$. Hence λ is finite.

On the other hand, by (10.44), $f_n(\hat{x}_i) \to 0$, $i = 1, ..., m$, hence,

$$F_n(\hat{x}_1, .) \to 0 \quad \text{in } L^p(\Omega).$$

Since $u_n(\hat{x}_1) \to \lambda$, from the above we conclude that

$$u_n \to \lambda \quad \text{in } L^p(B(\hat{x}_i, r) \cap \Omega).$$

Using again the compactness argument we get

$$u_n \to \lambda \quad \text{in } L^p(\Omega).$$

Now, by (10.43), $\lambda = 0$, so

$$u_n \to 0 \quad \text{in } L^p(\Omega),$$

which contradicts $\|u_n\|_p = 1$. ☐

REMARK 143. *The above Poincaré's type inequality fails to be true in general if* $0 \notin$ *supp*(J)*, as the following example shows. Let* $\Omega = (0, 3)$ *and* J *be such that*

$$supp(J) \subset (-3, -2) \cup (2, 3).$$

Then, if

$$u(x) = \begin{cases} 1 & \text{if } 0 < x < 1 \text{ or } 2 < x < 3, \\ 2 & 1 \leq x \leq 2, \end{cases}$$

we have that

$$\int_0^3 \int_0^3 J(x - y)|u(y) - u(x)|^p \, dx \, dy = 0,$$

but clearly

$$u(x) - \frac{1}{3}\int_0^3 u(y)\,dy \neq 0.$$

Therefore there is no Poincaré's type inequality available for this J.

This example can be easily extended for any domain in any dimension just by considering functions u that are constant on an annuli intersected with Ω.

Next we prove Theorem 141.

PROOF OF THEOREM 141. First we observe that a simple integration in space of the equation gives that the total mass is preserved, that is,

$$\frac{1}{|\Omega|}\int_\Omega u(x,t)\,dx = \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx.$$

Let

$$w(x,t) = u(x,t) - \frac{1}{|\Omega|}\int_\Omega u_0(x)\,dx.$$

Then,

$$\frac{d}{dt}\int_\Omega |w(x,t)|^p\,dx = p\int_\Omega |w|^{p-2}w(x,t)\int_\Omega J(x-y)|w(y,t)-w(x,t)|^{p-2}(w(y,t)-w(x,t))\,dydx$$

$$= -\frac{p}{2}\int_\Omega\int_\Omega J(x-y)|w(y,t)-w(x,t)|^{p-2}(w(y,t)-w(x,t))(|w|^{p-2}w(y,t)-|w|^{p-2}w(x,t))\,dydx.$$

Therefore the $L^p(\Omega)$-norm of w is decreasing with t.

Moreover, as the solution preserves the total mass, using Poincaré's type inequality (10.39), we have,

$$\int_\Omega |w(x,t)|^p\,dx \le C\int_\Omega\int_\Omega J(x-y)|u(y,t)-u(x,t)|^p\,dy\,dx.$$

Consequently,

$$t\int_\Omega |w(x,t)|^p\,dx \le \int_0^t\int_\Omega |w(x,s)|^p\,dx\,ds \le C\int_0^t\int_\Omega\int_\Omega J(x-y)|u(y,s)-u(x,s)|^p\,dy\,dx\,ds.$$

On the other hand, multiplying the equation by $u(x,t)$ and integrating in space and time, we get

$$\int_\Omega |u(x,t)|^2 - \int_\Omega |u_0(x)|^2\,dx = -\int_0^t\int_\Omega\int_\Omega J(x-y)|u(y,s)-u(x,s)|^p\,dy\,dx\,ds,$$

which implies

$$\int_0^t\int_\Omega\int_\Omega J(x-y)|u(y,s)-u(x,s)|^p\,dy\,dx\,ds \le ||u_0||^2_{L^2(\Omega)},$$

and therefore

$$\int_\Omega |w(x,t)|^p\,dx \le \frac{||u_0||^2_{L^2(\Omega)}}{t}.$$

□

REMARK 144. *Observe that using Poincaré's type inequality* (10.39)*, we can obtain*

$$u + B_p^J u = \phi, \tag{10.47}$$

for $p \ge 2$ *in the following manner: let*

$$\mathcal{K} := \left\{ u \in L^p(\Omega) \ : \ \int_\Omega u = 0 \right\}$$

and $A : \mathcal{K} \to L^{p'}(\Omega)$ *the continuous monotone operator defined by* $A(u) := u + B_p^J u$*. By* (10.39)*, we have*

$$\lim_{\substack{\|u\|_p \to +\infty \\ u \in \mathcal{K}}} \frac{\displaystyle\int_\Omega A(u)u}{\|u\|_p} = +\infty.$$

Then, by Corollary 30 in [**26**]*, for* $\phi \in L^\infty(\Omega)$*,* $\int_\Omega \phi = 0$*, there exists* $u \in \mathcal{K}$*, such that*

$$\int_\Omega uv + \int_\Omega B_p^J uv = \int_\Omega \phi v \quad \forall v \in \mathcal{K}.$$

Since $\int_\Omega u = 0$*,* $\int_\Omega \phi = 0$ *and* $\int_\Omega B_p^J u = 0$*, we have that*

$$\begin{aligned} \int_\Omega uv + \int_\Omega B_p^J uv &= \int_\Omega u \left(v - \frac{1}{|\Omega|} \int_\Omega v \right) + \int_\Omega B_p^J u \left(v - \frac{1}{|\Omega|} \int_\Omega v \right) \\ &= \int_\Omega \phi \left(v - \frac{1}{|\Omega|} \int_\Omega v \right) = \int_\Omega \phi v, \end{aligned}$$

for any $v \in L^p(\Omega)$*, and consequently* (10.47) *holds.*

CHAPTER 11

The limit as $p \to \infty$ in a nonlocal $p-$Laplacian evolution equation. A nonlocal approximation of a model for sandpiles

Our main purpose in this chapter is to study a nonlocal $\infty-$Laplacian type diffusion equation obtained as the limit as $p \to \infty$ to the nonlocal analogous to the $p-$Laplacian evolution.

First, let us recall some known results on local evolution problems. In [**56**] (see also [**10**] and [**55**]) was investigated the limiting behavior as $p \to \infty$ of solutions to the quasilinear parabolic problem

$$P_p(u_0) \qquad \begin{cases} v_{p,t} - \Delta_p v_p = f, & \text{in }]0,T[\times \mathbb{R}^N, \\ v_p(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ and f is nonnegative and represents a given source term, which is interpreted physically as adding material to an evolving system, within which mass particles are continually rearranged by diffusion.

We hereafter take the space $H = L^2(\mathbb{R}^N)$ and define for $1 < p < \infty$ the functional

$$F_p(v) = \begin{cases} \dfrac{1}{p}\displaystyle\int_{\mathbb{R}^N} |\nabla v(y)|^p \, dy, & \text{if } \ u \in L^2(\mathbb{R}^N) \cap W^{1,p}(\mathbb{R}^N), \\ +\infty & \text{if } \ u \in L^2(\mathbb{R}^N) \setminus W^{1,p}(\mathbb{R}^N). \end{cases}$$

Therefore, the PDE problem $P_p(u_0)$ has the standard reinterpretation

$$\begin{cases} f(t) - v_{p,t} = \partial F_p(v_p(t)), & \text{a.e. } \ t \in]0,T[, \\ v_p(0,x) = u_0(x), & \text{in } \mathbb{R}^N. \end{cases}$$

In [**56**], assuming that u_0 is a Lipschitz function with compact support, satisfying

$$\|\nabla u_0\|_\infty \le 1,$$

and for f a smooth nonnegative function with compact support in $[0,T] \times \mathbb{R}^N$, it is proved that we can extract a sequence $p_i \to +\infty$ and obtain a limit function v_∞, such that for each $T > 0$,

$$\begin{cases} v_{p_i} \to v_\infty, & \text{a.e. and in } L^2(\mathbb{R}^N \times (0,T)), \\ \nabla v_{p_i} \rightharpoonup \nabla v_\infty, \ v_{p_i,t} \rightharpoonup v_{\infty,t} & \text{weakly in } \ L^2(\mathbb{R}^N \times (0,T)). \end{cases}$$

Moreover, the limit function v_∞ satisfies

$$P_\infty(u_0) \quad \begin{cases} f(t) - v_{\infty,t} \in \partial F_\infty(v_\infty(t)), & \text{a.e. } t \in]0,T[, \\ v_\infty(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$F_\infty(v) = \begin{cases} 0 & \text{if } v \in L^2(\mathbb{R}^N),\ |\nabla v| \le 1, \\ +\infty & \text{in other case.} \end{cases}$$

This limit problem $P_\infty(u_0)$ explains the movement of a sandpile ($v_\infty(x,t)$ describes the amount of the sand at the point x at time t), the main assumption being that the sandpile is stable when the slope is less or equal than one and unstable if not.

On the other hand, we have the following nonlocal nonlinear diffusion problem, which we call the *nonlocal p-Laplacian problem*,

$$P_p^J(u_0) \quad \begin{cases} u_{p,t}(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)|u_p(y,t) - u_p(x,t)|^{p-2}(u_p(y,t) - u_p(x,t))dy + f(x,t), \\ u_p(0,x) = u_0(x). \end{cases}$$

Here $J : \mathbb{R}^N \to \mathbb{R}$ is a nonnegative continuous radial function with compact support, $J(0) > 0$ and $\int_{\mathbb{R}^N} J(x)\,dx = 1$ (this last condition is not necessary to prove our results, it is imposed to simplify the exposition).

In the previous chapter (see aso [**9**]) we have studied this problem when the integral is taken in a bounded domain Ω (hence dealing with homogeneous Neumann boundary conditions). We have obtained existence and uniqueness of solutions and, if the kernel J is rescaled in an appropriate way, that the solutions to the corresponding nonlocal problems converge to the solution of the $p-$laplacian with homogeneous Neumann boundary conditions. We have also studied the asymptotic behaviour of the solutions as t goes to infinity, showing the convergence to the mean value of the initial condition.

Let us note that the evolution problem $P_p^J(u_0)$ is the gradient flow associated to the functional

$$G_p^J(u) = \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u(y) - u(x)|^p\,dy\,dx,$$

which is the nonlocal analogous to the functional F_p associated to the $p-$Laplacian.

Following [**9**], we obtain existence and uniqueness of a global solution for this nonlocal problem.

Our next result in this article concerns the limit as $p \to \infty$ in $P_p^J(u_0)$. We obtain that the limit functional is given by

$$G_\infty^J(u) = \begin{cases} 0 & \text{if } u \in L^2(\mathbb{R}^N), \quad |u(x) - u(y)| \le 1, \text{ for } x - y \in \text{supp}(J), \\ +\infty & \text{in other case.} \end{cases}$$

Then, the nonlocal limit problem can be written as

$$P_\infty^J(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial G_\infty^J(u(t)), & \text{a.e. } t \in]0,T[, \\ u(0,x) = u_0(x). \end{cases}$$

With these notations, we obtain the following result.

THEOREM 145. *Let $T > 0$, $f \in BV(0,T; L^p(\mathbb{R}^N))$, $u_0 \in L^2(\mathbb{R}^N) \cap L^p(\mathbb{R}^N)$ such that $|u_0(x) - u_0(y)| \leq 1$, for $x - y \in \operatorname{supp}(J)$, and u_p the unique solution of $P_p^J(u_0)$. Then, if u_∞ is the unique solution to $P_\infty^J(u_0)$,*

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(t,\cdot) - u_\infty(t,\cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

0.30. Preliminaries. To identify the limit of the solutions u_p of problem $P_p^J(u_0)$ we will use the methods of convex analysis, and so we first recall some terminology (see [**53**], [**27**] and [**11**]).

If H is a real Hilbert space with inner product $(\ ,\ )$ and $\Psi : H \to (-\infty, +\infty]$ is convex, then the subdifferential of Ψ is defined as the multivalued operator $\partial\Psi$ given by

$$v \in \partial\Psi(u) \iff \Psi(w) - \Psi(u) \geq (v, w-u) \quad \forall\, w \in H.$$

The epigraph of Ψ is defined by

$$\operatorname{Epi}(\Psi) = \{(u,\lambda) \in H \times \mathbb{R} \ :\ \lambda \geq \Psi(u)\}.$$

Given K a closed convex subset of H, the indicator function of K is defined by

$$I_K(u) = \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Then it is easy to see that the subdifferential is characterized as follows,

$$v \in \partial I_K(u) \iff u \in K \ \text{ and } \ (v, w-u) \leq 0 \quad \forall\, w \in K.$$

In case the convex functional $\Psi : H \to (-\infty, +\infty]$ is proper , lower-semicontinuous and $\min \Psi = 0$, it is well known (see [**27**]) that the abstract Cauchy problem

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f(t), & \text{a.e } t \in]0,T[, \\ u(0) = u_0, \end{cases}$$

has a unique strong solution for any $f \in L^2(0,T;H)$ and $u_0 \in \overline{D(\partial\Psi)}$.

The following convergence was studied by Mosco in [**72**] (see [**11**]). Suppose X is a metric space and $A_n \subset X$. We define

$$\liminf_{n\to\infty} A_n = \{x \in X \ :\ \exists x_n \in A_n,\ x_n \to x\}$$

and

$$\limsup_{n\to\infty} A_n = \{x \in X \ : \ \exists x_{n_k} \in A_{n_k}, \ x_{n_k} \to x\}.$$

In the case X is a normed space, we note by $s-\lim$ and $w-\lim$ the above limits associated respectively to the strong and to the weak topology of X.

Given a sequence $\Psi_n, \Psi : H \to (-\infty, +\infty]$ of convex lower-semicontinuous functionals, we say that Ψ_n converges to Ψ in the sense of Mosco if

$$w - \limsup_{n\to\infty} \mathrm{Epi}(\Psi_n) \subset \mathrm{Epi}(\Psi) \subset s - \liminf_{n\to\infty} \mathrm{Epi}(\Psi_n). \tag{11.1}$$

It is easy to see that (11.1) is equivalent to the two following conditions:

$$\forall\, u \in D(\Psi) \ \ \exists u_n \in D(\Psi_n) \ : \ u_n \to u \ \text{ and } \ \Psi(u) \geq \limsup_{n\to\infty} \Psi_n(u_n); \tag{11.2}$$

$$\text{for every subsequence } n_k, \text{ when } u_k \rightharpoonup u, \text{ it holds } \Psi(u) \leq \liminf_k \Psi_{n_k}(u_k). \tag{11.3}$$

As consequence of results in [**28**] and [**11**] we can write the following result.

THEOREM 146. *Let $\Psi_n, \Psi : H \to (-\infty, +\infty]$ convex lower-semicontinuous functionals. Then the following statements are equivalent:*

(i) *Ψ_n converges to Ψ in the sense of Mosco.*

(ii) *$(I + \lambda\partial\Psi_n)^{-1}u \to (I + \lambda\partial\Psi)^{-1}u, \quad \forall\, \lambda > 0, \ u \in H.$*

Moreover, any of these two conditions (i) or (ii) imply that

(iii) *for every $u_0 \in \overline{D(\partial\Psi)}$ and $u_{0,n} \in \overline{D(\partial\Psi_n)}$ such that $u_{0,n} \to u_0$, and every $f_n, f \in L^1(0,T;H)$ with $f_n \to f$, if $u_n(t)$, $u(t)$ are the strong solutions of the abstract Cauchy problems*

$$\begin{cases} u_n'(t) + \partial\Psi_n(u_n(t)) \ni f_n, & a.e. \ \ t \in]0,T[, \\ u_n(0) = u_{0,n}, \end{cases}$$

and

$$\begin{cases} u'(t) + \partial\Psi(u(t)) \ni f, & a.e. \ \ t \in]0,T[, \\ u(0) = u_0, \end{cases}$$

respectively, then

$$u_n \to u \qquad in \ C([0,T] : H).$$

Let us also collect some preliminaries and notations concerning completely accretive operators that will be used afterwards (see [**19**]).

We denote by J_0 and P_0 the following sets of functions,

$J_0 = \{j : \mathbb{R} \to [0, +\infty], \text{ such that } j \text{ is convex, lower semi-continuos and } j(0) = 0\},$

$$P_0 = \{q \in C^\infty(\mathbb{R}) : 0 \le q' \le 1, \mathrm{supp}(q') \text{ is compact, and } 0 \notin \mathrm{supp}(q)\}.$$

Let $M(\mathbb{R}^N)$ denote the space of measurable functions from $\mathbb{R}^N$ into $\mathbb{R}$. We set $L(\mathbb{R}^N) := L^1(\mathbb{R}^N) + L^\infty(\mathbb{R}^N)$. Note that $L(\mathbb{R}^N)$ is a Banach space with the norm

$$\|u\|_{1+\infty} := \inf\{\|f\|_1 + \|g\|_\infty \ : f \in L^1(\mathbb{R}^N),\ g \in L^\infty(\mathbb{R}^N),\ f + g = u\}.$$

The closure of $L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ in $L(\mathbb{R}^N)$ is denoted by $L_0(\mathbb{R}^N)$.

Given $u, v \in M(\mathbb{R}^N)$ we say

$$u \ll v \ \text{ if and only if } \int_{\mathbb{R}^N} j(u)\,dx \le \int_{\mathbb{R}^N} j(v)\,dx \quad \forall j \in J_0.$$

An operator $A \subset M(\mathbb{R}^N) \times M(\mathbb{R}^N)$ is said to be completely accretive if

$$u - \hat{u} \ll u - \hat{u} + \lambda(v - \hat{v}) \qquad \forall \lambda > 0 \text{ and } \forall (u, v), (\hat{u}, \hat{v}) \in A.$$

The following facts are proved in [**19**].

Proposition 147.
(i) Let $u \in L_0(\mathbb{R}^N)$, $v \in L(\mathbb{R}^N)$, then

$$u \ll u + \lambda v \quad \forall \lambda > 0 \quad \text{if and only if} \quad \int_{\mathbb{R}^N} q(u)v \ge 0, \qquad \forall q \in P_0.$$

(ii) If $v \in L_0(\mathbb{R}^N)$, then $\{u \in M(\mathbb{R}^N) : u \ll v\}$ is a weak sequentially compact subset of $L_0(\mathbb{R}^N)$.

Concerning nonlocal models, in [**9**] we have studied the following nonlocal nonlinear diffusion problem, which we call the nonlocal p-Laplacian problem with homogeneous Neumann boundary conditions,

$$\begin{cases} u_t(x,t) = \displaystyle\int_\Omega J(x-y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\,dy, \\ u(0,x) = u_0(x). \end{cases}$$

Using similar ideas and techniques we can deal with the nonlocal problem in $\mathbb{R}^N$.

Solutions to $P_p^J(u_0)$ are to be understood in the following sense.

Definition 148. *Let $1 < p < +\infty$. Let $f \in L^1(0,T; L^p(\mathbb{R}^N))$ and $u_0 \in L^p(\mathbb{R}^N)$. A solution of $P_p^J(u_0)$ in $[0,T]$ is a function $u \in C([0,T]; L^p(\mathbb{R}^N)) \cap W^{1,1}(]0,T[; L^p(\mathbb{R}^N))$ which satisfies $u(0,x) = u_0(x)$ a.e. $x \in \mathbb{R}^N$ and*

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)|u(y,t)-u(x,t)|^{p-2}(u(y,t)-u(x,t))\,dy + f(x,t) \quad \text{a.e. in }]0,T[\times\mathbb{R}^N.$$

Working as in [**9**], we can obtain the following result about existence and uniqueness of a global solution for this problem. Let us first define $B_p^J : L^p(\mathbb{R}^N) \to L^{p'}(\mathbb{R}^N)$ by

$$B_p^J u(x) = -\int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, \qquad x \in \mathbb{R}^N.$$

Observe that, for every u, $v \in L^p(\mathbb{R}^N)$ and $T : \mathbb{R} \to \mathbb{R}$ such that $T(u-v) \in L^p(\mathbb{R}^N)$, it holds

$$\begin{aligned} &\int_{\mathbb{R}^N} (B_p^J u(x) - B_p^J v(x)) T(u(x)-v(x))dx = \\ &\qquad \frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)\,(T(u(y)-v(y)) - T(u(x)-v(x))) \times \\ &\qquad \times \left(|u(y)-u(x)|^{p-2}(u(y)-u(x)) - |v(y)-v(x)|^{p-2}(v(y)-v(x))\right)\,dy\,dx. \end{aligned} \tag{11.4}$$

Let us also define the operator

$$\mathcal{B}_p^J = \left\{(u,v) \in L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N) \ : \ v = B_p^J(u)\right\}.$$

It is easy to see that $\overline{\mathrm{Dom}(\mathcal{B}_p^J)} = L^p(\mathbb{R}^N)$ and $\mathcal{B}_p^J$ is positively homogeneous of degree $p-1$.

THEOREM 149. *Let $1 < p < +\infty$. If $f \in BV(0,T; L^p(\mathbb{R}^N))$ and $u_0 \in D(\mathcal{B}_p^J)$ then there exists a unique solution to $P_p^J(u_0)$. If $f = 0$ then there exists a unique solution to $P_p^J(u_0)$ for all $u_0 \in L^p(\mathbb{R}^N)$.*

Moreover, if $u_i(t)$ is a solution of $P_p^J(u_{i0})$ with $f = f_i$, $f_i \in L^1(0,T; L^p(\mathbb{R}^N))$ and $u_{i0} \in L^p(\mathbb{R}^N)$, $i = 1, 2$, then, for every $t \in [0,T]$,

$$\|(u_1(t) - u_2(t))^+\|_{L^p(\mathbb{R}^N)} \le \|(u_{10} - u_{20})^+\|_{L^p(\mathbb{R}^N)} + \int_0^t \|f_1(s) - f_2(s)\|_{L^p(\mathbb{R}^N)} ds.$$

PROOF. Let us first show that $\mathcal{B}_p^J$ is completely accretive and verifies the following range condition

$$L^p(\mathbb{R}^N) = \mathrm{Ran}(I + \mathcal{B}_p^J). \tag{11.5}$$

Indeed, given $u_i \in \mathrm{Dom}(\mathcal{B}_p^J)$, $i = 1, 2$ and $q \in P_0$, by (11.4) we have

$$\int_{\mathbb{R}^N} (B_p^J u_1(x) - B_p^J u_2(x))\, q(u_1(x) - u_2(x))\,dx \ge 0,$$

from where it follows that $\mathcal{B}_p^J$ is a completely accretive operator. To show that $\mathcal{B}_p^J$ satisfies the range condition we have to prove that for any $\phi \in L^p(\mathbb{R}^N)$ there exists $u \in \mathrm{Dom}(\mathcal{B}_p^J)$ such that $u = (I + B_p^J)^{-1}\phi$. Let us first take $\phi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. For every $n \in \mathbb{N}$, let $\phi_n := \phi \chi_{B_n(0)}$. By the results in [**9**], the operator $B_{p,n}^J$ defined by

$$B_{p,n}^J u(x) = -\int_{B_n(0)} J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy, \qquad x \in B_n(0),$$

is m-completely accretive in $L^p(B_n(0))$. Then, there exists $u_n \in L^p(B_n(0))$, such that

$$u_n(x) + B^J_{p,n} u_n(x) = \phi_n(x), \qquad \text{a.e. in } \ B_n(0). \tag{11.6}$$

Moreover, $u_n \ll \phi_n$.

We denote by $\tilde{u}_n$ and H_n the extensions

$$\tilde{u}_n(x) = \begin{cases} u_n(x) & \text{if } \ x \in B_n(0), \\ 0 & \text{if } \ x \in \mathbb{R}^N \setminus B_n(0), \end{cases}$$

and

$$H_n(x) = \begin{cases} B^J_{p,n} u_n(x), & \text{if } \ x \in B_n(0), \\ 0 & \text{if } \ x \in \mathbb{R}^N \setminus B_n(0). \end{cases}$$

Since, $u_n \ll \phi_n$, we have

$$\|\tilde{u}_n\|_q \leq \|\phi\|_q \qquad \text{for all } 1 \leq q \leq \infty, \forall\, n \in \mathbb{N}. \tag{11.7}$$

Hence, we can suppose

$$\tilde{u}_n \rightharpoonup u \quad \text{in } L^{p'}(\mathbb{R}^N). \tag{11.8}$$

On the other hand, multiplying (11.6) by u_n and integrating, we get

$$\int_{B_n(0)} \int_{B_n(0)} J(x-y)|u_n(y) - u_n(x)|^p \, dydx \leq \|\phi\|_2 \quad \forall\, n \in \mathbb{N}, \tag{11.9}$$

which implies, by Hölder's inequality, that $\{H_n \ : \ n \in \mathbb{N}\}$ is bounded in $L^{p'}(\mathbb{R}^N)$. Therefore, we can assume that

$$H_n \rightharpoonup H \quad \text{in } L^{p'}(\mathbb{R}^N). \tag{11.10}$$

By (11.8) and (11.10), taking limit in (11.6), we get

$$u + H = \phi \qquad \text{a.e. in } \mathbb{R}^N. \tag{11.11}$$

Let us see that

$$H(x) = -\int_{\mathbb{R}^N} J(x-y)|u(y) - u(x)|^{p-2}(u(y) - u(x))\, dy \quad \text{a.e. in } x \in \mathbb{R}^N. \tag{11.12}$$

In fact, multiplying (11.6) by u_n and integrating, we get

$$\begin{aligned} &\int_{B(0,n)} B^J_{p,n} u_n \, u_n = \int_{B(0,n)} (\phi - u_n) u_n \\ &= \int_{B(0,n)} (\phi - u) u - \int_{B(0,n)} \phi(u - u_n) + \int_{B(0,n)} 2u(u - u_n) - \int_{B(0,n)} (u - u_n)(u - u_n). \end{aligned}$$

Therefore, by (11.11),

$$\limsup \int_{B_n(0)} B^J_{p,n} u_n \, u_n \leq \int_{\mathbb{R}^N} (\phi - u) u = \int_{\mathbb{R}^N} H\, u. \tag{11.13}$$

On the other hand, for any $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, since

$$0 \leq \int_{B_n(0)} \left(B^J_{p,n} u_n - B^J_{p,n} v\right)(u_n - v),$$

we have that,

$$\int_{B_n(0)} B^J_{p,n} u_n \, u_n + \int_{B_n(0)} B^J_{p,n} v \, v \geq \int_{B_n(0)} B^J_{p,n} u_n \, v + \int_{B_n(0)} B^J_{p,n} v \, u_n.$$

Therefore, by (11.13),

$$\int_{\mathbb{R}^N} H\, u + \int_{\mathbb{R}^N} B^J_p v \, v \geq \int_{\mathbb{R}^N} H\, v + \int_{\mathbb{R}^N} B^J_p v \, u. \tag{11.14}$$

Taking now $v = u \pm \lambda w$, $\lambda > 0$ and $w \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, and letting $\lambda \to 0$, we get

$$\int_{\mathbb{R}^N} H\, w = \int_{\mathbb{R}^N} B^J_p u \, w,$$

and consequently (11.12) is proved. Therefore, by (11.11), the range condition is satisfied for $\phi \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Let now $\phi \in L^p(\mathbb{R}^N)$. Take $\phi_n \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$, $\phi_n \to \phi$ in $L^p(\mathbb{R}^N)$. Then, by our previous step, there exists $u_n = (I + \mathcal{B}^J_p)^{-1} \phi_n$. Since $\mathcal{B}^J_p$ is completely accretive, $u_n \to u$ in $L^p(\mathbb{R}^N)$, also $B^J_p u_n \to B^J_p u$ in $L^{p'}(\mathbb{R}^N)$ and we conclude that $u + \mathcal{B}^J_p u = \phi$.

Consequently (see [**19**] and [**20**]) we have that $\mathcal{B}^J_p$ is an m-accretive operator in $L^p(\mathbb{R}^N)$ and we get the existence of mild solution $u(t)$ of the abstract Cauchy problem

$$\begin{cases} u'(t) + \mathcal{B}^J_p u(t) = f(t), & t \in]0, T[, \\ u(0) = u_0, \end{cases} \tag{11.15}$$

Now, by the Nonlinear Semigroup Theory (see [**20**], [**49**] or [**50**]), if $f \in BV(0, T; L^p(\mathbb{R}^N))$ and $u_0 \in D(\mathcal{B}^J_p)$, $u(t)$ is a strong solution of (11.15), that is, a solution of $P^J_p(u_0)$ in the sense of Definition 148. The same is true for all $u_0 \in L^p(\mathbb{R}^N)$ in the case $f = 0$ by the complete accretivity of $\mathcal{B}^J_p$, since $\mathrm{Dom}(\mathcal{B}^J_2) = L^2(\mathbb{R}^N)$ and for $p \neq 2$ the operator $\mathcal{B}^J_p$ is homogeneous of degree $p - 1$ (see [**19**]). Finally, the contraction principle follows from the general Nonlinear Semigroup Theory since the solutions u_i, $i = 1, 2$, are mild-solutions of (11.15). $\square$

0.31. Limit as $p \to \infty$. Recall from the Introduction that the *nonlocal p-Laplacian evolution problem*

$$P^J_p(u_0) \quad \begin{cases} u_t(x,t) = \displaystyle\int_{\mathbb{R}^N} J(x-y)|u(y,t) - u(x,t)|^{p-2}(u(y,t) - u(x,t))\, dy + f(x,t), \\ u(0,x) = u_0(x). \end{cases}$$

is the gradient flow associated to the functional

$$G_p^J(u) = \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^p\,dy\,dx.$$

With a formal calculation, taking limit as $p \to \infty$, we arrive to the functional

$$G_\infty^J(u) = \begin{cases} 0 & \text{if } |u(x)-u(y)| \le 1, \text{ for } x-y \in \operatorname{supp}(J), \\ \\ +\infty & \text{in other case.} \end{cases}$$

Hence, if we define

$$K_\infty^J := \{u \in L^2(\mathbb{R}^N) \ : \ |u(x)-u(y)| \le 1, \text{ for } x-y \in \operatorname{supp}(J)\},$$

we have that the functional G_∞^J is given by the indicator function of K_∞^J, that is, $G_\infty^J = I_{K_\infty^J}$. Then, the *nonlocal limit problem* can be written as

$$P_\infty^J(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t) \in \partial I_{K_\infty^J}(u(t)), & \text{a.e. } t \in]0,T[, \\ \\ u(0,x) = u_0(x). \end{cases}$$

Proof of Theorem 145. Let $T > 0$. Recall that we want to prove that, given $f \in BV(0,T;L^p(\mathbb{R}^N))$, $u_0 \in K_\infty^J \cap L^p(\mathbb{R}^N)$ and u_p the unique solution of $P_p^J(u_0)$, if u_∞ is the unique solution of $P_\infty^J(u_0)$, then

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(t,\cdot) - u_\infty(t,\cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

By Theorem 146, to prove the result it is enough to show that the functionals

$$G_p^J(u) = \frac{1}{2p}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u(y)-u(x)|^p\,dy\,dx$$

converge to

$$G_\infty^J(u) = \begin{cases} 0 & \text{if } |u(x)-u(y)| \le 1, \text{ for } x-y \in \operatorname{supp}(J), \\ \\ +\infty & \text{in other case,} \end{cases}$$

as $p \to \infty$, in the sense of Mosco.

First, let us check that

$$\operatorname{Epi}(G_\infty^J) \subset s - \liminf_{p\to\infty} \operatorname{Epi}(G_p^J). \tag{11.16}$$

To this end let $(u,\lambda) \in \operatorname{Epi}(G_\infty^J)$. We can assume that $u \in K_\infty^J$ and $\lambda \ge 0$ (as $G_\infty^J(u) = 0$). Now take

$$v_p = u\chi_{B_{R(p)}(0)} \qquad \text{and} \qquad \lambda_p = G_p^J(u_p) + \lambda. \tag{11.17}$$

Then, as $\lambda \geq 0$ we have $(v_p, \lambda_p) \in \mathrm{Epi}(G_p^J)$. It is obvious that if $R(p) \to \infty$ as $p \to \infty$ we have

$$v_p \to u \quad \text{in} \quad L^2(\mathbb{R}^N),$$

and, if we choose $R(p) = p^{\frac{1}{4N}}$ we get

$$G_p^J(v_p) = \frac{1}{2p} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y) \left| v_p(y) - v_p(x) \right|^p \, dy \, dx \leq C \frac{R(p)^{2N}}{p} \to 0,$$

as $p \to \infty$, we get (11.16).

Finally, let us prove that

$$w - \limsup_{p \to \infty} \mathrm{Epi}(G_p^J) \subset \mathrm{Epi}(G_\infty^J). \tag{11.18}$$

To this end, let us consider a sequence $(u_{p_j}, \lambda_{p_j}) \in \mathrm{Epi}(G_{p_j}^J)$ $(p_j \to \infty)$, that is,

$$G_{p_j}^J(u_{p_j}) \leq \lambda_{p_j},$$

with

$$u_{p_j} \rightharpoonup u, \qquad \text{and} \qquad \lambda_{p_j} \to \lambda.$$

Therefore we obtain that $0 \leq \lambda$, since

$$0 \leq G_{p_j}^J(u_{p_j}) \leq \lambda_{p_j} \to \lambda.$$

On the other hand, we have that

$$\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y) \left| u_{p_j}(y) - u_{p_j}(x) \right|^{p_j} \, dy \, dx \right)^{1/p_j} \leq (C p_j)^{1/p_j}.$$

Now, fix a bounded domain $\Omega \subset \mathbb{R}^N$ and $q < p_j$. Then, by the above inequality,

$$\begin{aligned} &\left(\int_\Omega \int_\Omega J(x-y) \left| u_{p_j}(y) - u_{p_j}(x) \right|^q \, dy \, dx \right)^{1/q} \\ &\leq \left(\int_\Omega \int_\Omega J(x-y) \, dy \, dx \right)^{(p_j - q)/p_j q} \\ &\qquad \times \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} J(x-y) \left| u_{p_j}(y) - u_{p_j}(x) \right|^{p_j} \, dy \, dx \right)^{1/p_j} \\ &\leq \left(\int_\Omega \int_\Omega J(x-y) \, dy \, dx \right)^{(p_j - q)/p_j q} (C p_j)^{1/p_j}. \end{aligned}$$

Hence, we can extract a subsequence (if necessary) and let $p_j \to \infty$ to obtain

$$\left(\int_\Omega \int_\Omega J(x-y) \left| u(y) - u(x) \right|^q \, dy \, dx \right)^{1/q} \leq \left(\int_\Omega \int_\Omega J(x-y) \, dy \, dx \right)^{1/q}.$$

Now, just taking $q \to \infty$, we get

$$|u(x) - u(y)| \le 1 \qquad \text{a.e. } (x,y) \in \Omega \times \Omega,\ x - y \in \operatorname{supp}(J).$$

As Ω is arbitrary we conclude that

$$u \in K_\infty^J.$$

This ends the proof. □

0.32. Limit as $\varepsilon \to 0$. Our next step is to rescale the kernel J appropriately and take the limit as the scaling parameter goes to zero.

In the sequel we assume that $\operatorname{supp}(J) = \overline{B}_1(0)$. For given $p > 1$ and J we consider the rescaled kernels

$$J_{p,\varepsilon}(x) := \frac{C_{J,p}}{\varepsilon^{p+N}} J\left(\frac{x}{\varepsilon}\right), \quad \text{where } C_{J,p}^{-1} := \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_N|^p\, dz$$

is a normalizing constant in order to obtain the p-Laplacian in the limit instead a multiple of it. Associated to these kernels we have solutions $u_{p,\varepsilon}$ to the nonlocal problems $P_p^{J_{p,\varepsilon}}(u_0)$. Let us also consider the solution to the local problem $P_p(u_0)$. Working as in [**9**] again, we can prove the following result.

Theorem 150. *Let $p > N$ and assume $J(x) \ge J(y)$ if $|x| \le |y|$. Let $T > 0$, $f \in BV(0,T;L^p(\mathbb{R}^N))$, $u_0 \in L^p(\mathbb{R}^N)$ and $u_{p,\varepsilon}$ the unique solution of $P_p^{J_{p,\varepsilon}}(u_0)$. Then, if v_p is the unique solution of $P_p(u_0)$,*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{p,\varepsilon}(t,\cdot) - v_p(t,\cdot)\|_{L^p(\mathbb{R}^N)} = 0.$$

We will use the following result from [**9**] which is a variant of [**23**, Theorem 4] (the first statement is given in [**9**] for bounded domains Ω but it also holds for general open sets).

Proposition 151. *Let Ω an open subset of $\mathbb{R}^N$. Let $1 \le p < +\infty$. Let $\rho : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative continuous radial function with compact support, non-identically zero, and $\rho_n(x) := n^N \rho(nx)$. Let $\{f_n\}$ be a sequence of functions in $L^p(\Omega)$ such that*

$$\int_\Omega \int_\Omega |f_n(y) - f_n(x)|^p \rho_n(y-x)\, dx\, dy \le M \frac{1}{n^p}.$$

(1) If $\{f_n\}$ is weakly convergent in $L^p(\Omega)$ to f, then $f \in W^{1,q}(\Omega)$ and moreover

$$(\rho(z))^{1/p} \chi_\Omega\left(x + \frac{1}{n}z\right) \frac{f_n\left(x + \frac{1}{n}z\right) - f_n(x)}{1/n} \rightharpoonup (\rho(z))^{1/p}\, z \cdot \nabla f$$

weakly in $L^p(\Omega) \times L^p(\Omega)$.

(2) If we further assume Ω is a smooth bounded domain in $\mathbb{R}^N$ and $\rho(x) \ge \rho(y)$ if $|x| \le |y|$ then $\{f_n\}$ is relatively compact in $L^p(\Omega)$, and consequently, there exists a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ in $L^p(\Omega)$ with $f \in W^{1,p}(\Omega)$.

Using the above proposition we can take the limit as $\varepsilon \to 0$ for a fixed $p > N$.

Proof of Theorem 150. Recall that we have $p > N$ and $J(x) \geq J(y)$ if $|x| \leq |y|$, $T > 0$, $f \in L^1(0,T;L^p(\mathbb{R}^N))$, $u_0 \in L^p(\mathbb{R}^N)$ and $u_{p,\varepsilon}$ the unique solution of $P_p^{J_{p,\varepsilon}}(u_0)$. We want to show that, if u_p is the unique solution of $N_p(u_0)$, then

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{p,\varepsilon}(t,\cdot) - u_p(t,\cdot)\|_{L^p(\mathbb{R}^N)} = 0. \tag{11.19}$$

Since $\mathcal{B}_p^J$ is m-accretive, to get (11.19) it is enough to see (see [**20**] or [**49**])

$$\left(I + \mathcal{B}_p^{J_{p,\varepsilon}}\right)^{-1} \phi \to \left(I + \mathcal{B}_p\right)^{-1} \phi \quad \text{in } L^p(\mathbb{R}^N) \text{ as } \varepsilon \to 0$$

for any $\phi \in C_c(\mathbb{R}^N)$.

Let $\phi \in C_c(\mathbb{R}^N)$ and $u_\varepsilon := \left(I + \mathcal{B}_p^{J_{p,\varepsilon}}\right)^{-1} \phi$. Then,

$$\begin{aligned}\int_{\mathbb{R}^N} u_\varepsilon v - \frac{C_{J,p}}{\varepsilon^{p+N}} \quad \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right) |u_\varepsilon(y) - u_\varepsilon(x)|^{p-2} \times \\ \times (u_\varepsilon(y) - u_\varepsilon(x))\, dy\, v(x)\, dx = \int_{\mathbb{R}^N} \phi v\end{aligned} \tag{11.20}$$

for every $v \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$.

Changing variables, we get

$$\begin{aligned}&-\frac{C_{J,p}}{\varepsilon^{p+N}} \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right) |u_\varepsilon(y) - u_\varepsilon(x)|^{p-2} (u_\varepsilon(y) - u_\varepsilon(x))\, dy\, v(x)\, dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z) \left|\frac{u_\varepsilon(x+\varepsilon z) - u_\varepsilon(x)}{\varepsilon}\right|^{p-2} \frac{u_\varepsilon(x+\varepsilon z) - u_\varepsilon(x)}{\varepsilon} \times \\ &\qquad \times \frac{v(x+\varepsilon z) - v(x)}{\varepsilon}\, dx\, dz.\end{aligned} \tag{11.21}$$

So we can rewrite (11.20) as

$$\begin{aligned}&\int_{\mathbb{R}^N} \phi(x) v(x)\, dx - \int_{\mathbb{R}^N} u_\varepsilon(x) v(x)\, dx \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z) \left|\frac{u_\varepsilon(x+\varepsilon z) - u_\varepsilon(x)}{\varepsilon}\right|^{p-2} \frac{u_\varepsilon(x+\varepsilon z) - u_\varepsilon(x)}{\varepsilon} \times \\ &\qquad \times \frac{v(x+\varepsilon z) - v(x)}{\varepsilon}\, dx\, dz.\end{aligned} \tag{11.22}$$

We shall see that there exists a sequence $\varepsilon_n \to 0$ such that $u_{\varepsilon_n} \to u$ in $L^p(\mathbb{R}^N)$, $u \in W^{1,p}(\mathbb{R}^N)$ and $u = (I + B_p)^{-1} \phi$, that is,

$$\int_{\mathbb{R}^N} uv + \int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla v = \int_{\mathbb{R}^N} \phi v \quad \text{for every } v \in W^{1,p}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N).$$

Since $u_\varepsilon \ll \phi$, there exists a sequence $\varepsilon_n \to 0$ such that

$$u_{\varepsilon_n} \rightharpoonup u, \quad \text{weakly in } L^p(\mathbb{R}^N), \quad u \ll \phi.$$

Observe that $\|u_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^N)}, \|u\|_{L^\infty(\mathbb{R}^N)} \le \|\phi\|_{L^\infty(\mathbb{R}^N)}$. Taking $\varepsilon = \varepsilon_n$ and $v = u_{\varepsilon_n}$ in (11.22) and applying Young's inequality, we get

$$\begin{aligned} &\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{1}{2}\frac{C_{J,p}}{{\varepsilon_n}^N} J\left(\frac{x-y}{\varepsilon_n}\right)\left|\frac{u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dy \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{C_{J,p}}{2} J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dz \le M := \frac{1}{2}\int_{\mathbb{R}^N} |\phi(x)|^2\,dx. \end{aligned} \tag{11.23}$$

Therefore, by Proposition 151,

$$u \in W^{1,p}(\mathbb{R}^N),$$

$$u_{\varepsilon_n} \to u \text{ in } L^p_{\mathrm{loc}}(\mathbb{R}^N)$$

and

$$\left(\frac{C_{J,p}}{2}J(z)\right)^{1/p} \frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n} \rightharpoonup \left(\frac{C_{J,p}}{2}J(z)\right)^{1/p} z\cdot\nabla u(x) \tag{11.24}$$

weakly in $L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$. Let us prove now the tightness of $\{u_{\varepsilon_n}\}$, which is to say, that no mass moves to infinity as $p \to +\infty$. For this, assume $\mathrm{supp}(\phi) \subset B_R(0)$ and fix $S > 2R$. Select a smooth function $\varphi \in C^\infty(\mathbb{R}^N)$ such that $0 \le \varphi \le 1$, $\varphi \equiv 0$ on $B_R(0)$, $\varphi \equiv 1$ on $\mathbb{R}^N \setminus B_S(0)$ and $|\nabla\varphi| \le \frac{2}{S}$. Taking in (11.22) $\varphi|u_{\varepsilon_n}|^{p-2}u_{\varepsilon_n}$, and having in mind that $\|u_{\varepsilon_n}\|_{L^\infty(\mathbb{R}^N)} \le \|\phi\|_{L^\infty(\mathbb{R}^N)}$, we get

$$\begin{aligned} &\int_{\mathbb{R}^N} |u_{\varepsilon_n}|^p(x)\varphi(x)\,dx \\ &\le \frac{C_{J,p}}{2{\varepsilon_n}^p}\int_{\mathbb{R}^N}\int_{B_1(0)} J(z)|u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)|^{p-1}|u_{\varepsilon_n}(x+\varepsilon_n z)|^{p-1} \\ &\qquad\qquad\qquad \times|\varphi(x+\varepsilon_n z)-\varphi(x)|\,dzdx \\ &\le \frac{C_{J,p}\|\phi\|_{L^\infty}^{p-1}}{S}\int_{\{|x|\le S+1\}}\int_{B_1(0)} J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^{p-1} dy\,dx \\ &\le \frac{C_{J,p}\|\phi\|_{L^\infty}^{p-1}}{S}\left(\int_{\{|x|\le S+1\}}\int_{B_1(0)} J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^{p} dy\right)^{\frac{1}{p'}} \\ &\qquad \times\left(\int_{\{|x|\le S+1\}}\int_{B_1(0)} J(z)dz\right)^{\frac{1}{p}} dx \\ &= O(S^{-1+\frac{N}{p}}), \end{aligned}$$

the last equality being true by (11.23) and since

$$\left(\int_{\{|x|\le S+1\}}\int_{B_1(0)} J(z)dz\right)^{\frac{1}{p}} dx \le C(S+1)^{\frac{N}{p}}.$$

Consequently,

$$\int_{\{|x|\ge S\}} |u_{\varepsilon_n}|^p(x)\,dx = O(S^{-1+\frac{N}{p}})$$

uniformly in ε_n. Therefore,

$$u_{\varepsilon_n} \to u \text{ in } L^p(\mathbb{R}^N).$$

Moreover, from (11.23), we can also assume that

$$\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^{p-2}\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n} \rightharpoonup \chi(x,z)$$

weakly in $L^{p'}(\mathbb{R}^N)\times L^{p'}(\mathbb{R}^N)$. Therefore, passing to the limit in (11.22) for $\varepsilon=\varepsilon_n$, we get

$$\int_{\mathbb{R}^N} uv + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla v(x)\,dx\,dz = \int_{\mathbb{R}^N}\phi v \tag{11.25}$$

for every v smooth and by approximation for every $v\in W^{1,p}(\mathbb{R}^N)\cap L^\infty(\mathbb{R}^N)$.

From now on we follow closely the arguments in [**9**], but we include some details here for the sake of completeness.

Let us see now that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)z\cdot\nabla v(x)\,dx\,dz = \int_{\mathbb{R}^N}|\nabla u|^{p-2}\nabla u\cdot\nabla v. \tag{11.26}$$

In fact, taking $v=u_{\varepsilon_n}$ in (11.22), by (11.25) we have

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{\mathbb{R}^n}\frac{C_{J,p}}{2}J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dz = \int_{\mathbb{R}^N}\phi u_{\varepsilon_n} - \int_{\mathbb{R}^N}u_{\varepsilon_n}u_{\varepsilon_n}\\
&= \int_{\mathbb{R}^N}\phi u - \int_{\mathbb{R}^n}uu - \int_{\mathbb{R}^N}\phi(u-u_{\varepsilon_n}) + \int_{\mathbb{R}^N}2u(u-u_{\varepsilon_n}) - \int_{\mathbb{R}^N}(u-u_{\varepsilon_n})(u-u_{\varepsilon_n})\\
&\le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla u(x)\,dx\,dz - \int_{\mathbb{R}^N}\phi(u-u_{\varepsilon_n}) + \int_{\mathbb{R}^nN}2u(u-u_{\varepsilon_n}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\limsup_n \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\left|\frac{u_{\varepsilon_n}(x+\varepsilon_n z)-u_{\varepsilon_n}(x)}{\varepsilon_n}\right|^p dx\,dz\\
&\qquad \le \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot\nabla u(x)\,dx\,dz.
\end{aligned} \tag{11.27}$$

Now, by the monotonicity property (11.4), for every ρ smooth,

$$-\frac{C_{J,p}}{{\varepsilon_n}^{p+N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon_n}\right)|\rho(y)-\rho(x)|^{p-2}(\rho(y)-\rho(x))\,dy\,(u_{\varepsilon_n}(x)-\rho(x))\,dx$$
$$\leq -\frac{C_{J,p}}{{\varepsilon_n}^{p+N}}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon_n}\right)|u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x)|^{p-2}(u_{\varepsilon_n}(y)-u_{\varepsilon_n}(x))\,dy\,(u_{\varepsilon_n}(x)-\rho(x))\,dx.$$

Using the change of variable (11.21) and taking limits, on account of (11.24) and (11.27), we obtain for every ρ smooth,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)|z\cdot\nabla\rho|^{p-2}z\cdot\nabla\rho\,z\cdot(\nabla u-\nabla\rho)$$
$$\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)\,z\cdot(\nabla u(x)-\nabla\rho(x))\,dx\,dz,$$

and then, by approximation, for every $\rho\in W^{1,p}(\mathbb{R}^N)$. Taking now, $\rho=u\pm\lambda v$, $\lambda>0$ and $v\in W^{1,p}(\mathbb{R}^N)$, and letting $\lambda\to 0$, we get

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)z\cdot\nabla v(x)\,dx\,dz$$
$$=\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\int_{\Omega}|z\cdot\nabla u(x)|^{p-2}\left(z\cdot\nabla u(x)\right)\left(z\cdot\nabla v(x)\right)\,dx\,dz.$$

Consequently,

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\frac{C_{J,p}}{2}J(z)\chi(x,z)z\cdot\nabla v(x)\,dx\,dz=C_{J,p}\int_{\mathbb{R}^N}\mathbf{a}(\nabla u)\cdot\nabla v\quad\text{ for every } v\in W^{1,p}(\mathbb{R}^N),$$

where

$$\mathbf{a}_j(\xi)=C_{J,p}\int_{\mathbb{R}^N}\frac{1}{2}J(z)\,|z\cdot\xi|^{p-2}\;z\cdot\xi\,z_j\,dz.$$

Hence, if we prove that

$$\mathbf{a}(\xi)=|\xi|^{p-2}\xi, \tag{11.28}$$

then (11.26) is true and $u=(I+B_p)^{-1}\phi$. So, to finish the proof we only need to show that (11.28) holds. Obviously, $\mathbf{a}$ is positively homogeneous of degree $p-1$, that is,

$$\mathbf{a}(t\xi)=t^{p-1}\mathbf{a}(\xi)\qquad\text{for all }\ \xi\in\mathbb{R}^N\quad\text{and all } t>0.$$

Therefore, in order to prove (11.28) it is enough to see that

$$\mathbf{a}_i(\xi)=\xi_i\quad\text{for all } \xi\in\mathbb{R}^N, |\xi|=1,\qquad i=1,\ldots,N.$$

Now, let $R_{\xi,i}$ be the rotation such that $R^t_{\xi,i}(\xi)=\mathbf{e}_i$, where $\mathbf{e}_i$ is the vector with components $(\mathbf{e}_i)_i=1$, $(\mathbf{e}_i)_j=0$ for $j\neq i$, being $R^t_{\xi,i}$ the transpose of $R_{\xi,i}$. Observe that

$$\xi_i=\xi\cdot\mathbf{e}_i=R^t_{\xi,i}(\xi)\cdot R^t_{\xi,i}(\mathbf{e}_i)=\mathbf{e}_i\cdot R^t_{\xi,i}(\mathbf{e}_i).$$

On the other hand, since J is radial, $C_{J,p}^{-1} = \frac{1}{2}\int_{\mathbb{R}^N} J(z)|z_i|^p\,dz$ and

$$\mathbf{a}(\mathbf{e}_i) = \mathbf{e}_i \quad \text{for every } i.$$

Making the change of variables $z = R_{\xi,i}(y)$, since J is a radial function, we obtain

$$\begin{aligned}
\mathbf{a}_i(\xi) &= C_{J,p}\int_{\mathbb{R}^N} \frac{1}{2}J(z)|z\cdot\xi|^{p-2}z\cdot\xi\, z\cdot\mathbf{e}_i\,dz \\
&= C_{J,p}\int_{\mathbb{R}^N} \frac{1}{2}J(y)|y\cdot\mathbf{e}_i|^{p-2}y\cdot\mathbf{e}_i\, y\cdot R^t_{\xi,i}(\mathbf{e}_i)\,dy \\
&= \mathbf{a}(\mathbf{e}_i)\cdot R^t_{\xi,i}(\mathbf{e}_i) = \mathbf{e}_i\cdot R^t_{\xi,i}(\mathbf{e}_i) = \xi_i,
\end{aligned}$$

and the proof finishes. □

0.33. The limit as $\varepsilon \to 0$ in the sandpile model. Let us rescale the limit problem $P_\infty^J(u_0)$ considering the functionals

$$G_\infty^\varepsilon(u) = \begin{cases} 0 & \text{if } u\in L^2(\mathbb{R}^N), \quad |u(x)-u(y)|\le\varepsilon, \text{ for } |x-y|\le\varepsilon, \\ +\infty & \text{in other case,} \end{cases}$$

and the gradient flow associated to this functional,

$$P_\infty^\varepsilon(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial G_\infty^\varepsilon(u(t)), & \text{a.e } t\in]0,T[, \\ u(0,x) = u_0(x). \end{cases}$$

We have the following theorem.

THEOREM 152. *Let $T>0$, $f\in L^1(0,T;L^2(\mathbb{R}^N))$, $u_0\in L^2(\mathbb{R}^N)\cap W^{1,\infty}(\mathbb{R}^N)$ such that $\|\nabla u_0\|_\infty \le 1$ and consider $u_{\infty,\varepsilon}$ the unique solution of $P_\infty^\varepsilon(u_0)$. Then, if v_∞ is the unique solution of $P_\infty(u_0)$, we have*

$$\lim_{\varepsilon\to 0}\sup_{t\in[0,T]} \|u_{\infty,\varepsilon}(t,\cdot) - v_\infty(t,\cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

Hence, we have approximated the sandpile model described in [**10**] and [**56**] by a nonlocal equation. In this nonlocal approximation a configuration of sand is stable when its height u verifies $|u(x)-u(y)|\le\varepsilon$ when $|x-y|\le\varepsilon$. This is a sort of measure of how large is the size of irregularities of the sand; the sand can be completely irregular for sizes smaller than ε but it has to be arranged for sizes greater than ε.

For $\varepsilon > 0$, we rescale the functional G_∞^J as follows

$$G_\infty^\varepsilon(u) = \begin{cases} 0 & \text{if } \ |u(x)-u(y)|\le\varepsilon, \text{ for } |x-y|\le\varepsilon, \\ +\infty & \text{in other case.} \end{cases}$$

In other words, $G_\infty^\varepsilon = I_{K_\varepsilon}$, where

$$K_\varepsilon := \{u\in L^2(\mathbb{R}^N) \ : \ |u(x)-u(y)|\le\varepsilon, \text{ for } |x-y|\le\varepsilon\}.$$

Consider the gradient flow associated to the functional G^{ε}_{∞}

$$P^{\varepsilon}_{\infty}(u_0) \quad \begin{cases} f(t,\cdot) - u_t(t,\cdot) \in \partial I_{K_\varepsilon}(u(t)), & \text{a.e. } t \in]0,T[, \\ u(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

and the problem

$$P_{\infty}(u_0) \quad \begin{cases} f(t,\cdot) - u_{\infty,t} \in \partial I_{K_0}(u_\infty), & \text{a.e. } t \in]0,T[, \\ u_\infty(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$K_0 := \left\{ u \in L^2(\mathbb{R}^N) \cap W^{1,\infty}(\mathbb{R}^N) \ : \ |\nabla u| \leq 1 \right\}.$$

Observe that if $u \in K_0$, $|\nabla u| \leq 1$. Hence, $|u(x) - u(y)| \leq |x - y|$, from where it follows that $u \in K_\varepsilon$, that is, $K_0 \subset K_\varepsilon$.

With all these definitions and notations, we can proceed with the limit as $\varepsilon \to 0$ for the sandpile model ($p = \infty$).

Proof of Theorem 152. We have $T > 0$, $f \in L^1(0,T;L^2(\mathbb{R}^N))$, $u_0 \in K_0$ and $u_{\infty,\varepsilon}$ the unique solution of $P^{\varepsilon}_{\infty}(u_0)$. We have to show that if v_∞ is the unique solution of $P_\infty(u_0)$, then

$$\lim_{\varepsilon \to 0} \sup_{t \in [0,T]} \|u_{\infty,\varepsilon}(t,\cdot) - v_\infty(t,\cdot)\|_{L^2(\mathbb{R}^N)} = 0.$$

Since $u_0 \in K_0$, $u_0 \in K_\varepsilon$ for all $\varepsilon > 0$, and consequently there exists $u_{\infty,\varepsilon}$ the unique solution of $P^{\varepsilon}_{\infty}(u_0)$.

By Theorem 146 to prove the result it is enough to show that I_{K_ε} converges to I_{K_0} in the sense of Mosco. It is easy to see that

$$K_{\varepsilon_1} \subset K_{\varepsilon_2}, \quad \text{if } \ \varepsilon_1 \leq \varepsilon_2. \tag{11.29}$$

Since $K_0 \subset K_\varepsilon$ for all $\varepsilon > 0$, we have

$$K_0 \subset \bigcap_{\varepsilon > 0} K_\varepsilon.$$

On the other hand, if

$$u \in \bigcap_{\varepsilon > 0} K_\varepsilon,$$

we have

$$\frac{|u(y) - u(x)|}{|y - x|} \leq 1, \quad \text{a.e } \ x, y \in \mathbb{R}^N,$$

from where it follows that $u \in K_0$. Therefore, we have

$$K_0 = \bigcap_{\varepsilon > 0} K_\varepsilon. \tag{11.30}$$

Note that

$$\text{(11.31)} \qquad \mathrm{Epi}(I_{K_0}) = K_0 \times [0,\infty[, \quad \mathrm{Epi}(I_{K_\varepsilon}) = K_\varepsilon \times [0,\infty[\ \ \forall\, \varepsilon > 0.$$

By (11.30) and (11.31), we have

$$\text{(11.32)} \qquad \mathrm{Epi}(I_{K_0}) \subset s - \liminf_{\varepsilon\to 0} \mathrm{Epi}(I_{K_\varepsilon}).$$

On the other hand, given $(u,\lambda) \in w - \limsup_{\varepsilon\to 0} \mathrm{Epi}(I_{K_\varepsilon})$ there exists $(u_{\varepsilon_k}, \lambda_k) \in K_{\varepsilon_k} \times [0,\infty[$, such that $\varepsilon_k \to 0$ and

$$u_{\varepsilon_k} \rightharpoonup u \quad \text{in } L^2(\mathbb{R}^N), \quad \lambda_k \to \lambda \ \text{ in } \mathbb{R}.$$

By (11.29), given $\varepsilon > 0$, there exists k_0, such that $u_{\varepsilon_k} \in K_\varepsilon$ for all $k \ge k_0$. Then, since K_ε is a closed convex set, we get $u \in K_\varepsilon$, and, by (11.30), we obtain that $u \in K_0$. Consequently,

$$\text{(11.33)} \qquad w - \limsup_{n\to\infty} \mathrm{Epi}(I_{K_\varepsilon}) \subset \mathrm{Epi}(I_{K_0}).$$

Finally, by (11.32) and (11.33), and having in mind (11.1), we obtain that I_{K_ε} converges to I_{K_0} in the sense of Mosco. □

0.34. Collapse of the initial condition.

In [**56**] the authors studied the collapsing of the initial condition phenomena for the local problem $P_p(u_0)$ when the initial condition u_0 satisfies $\|\nabla u_0\|_\infty > 1$. They find that the limit of the solutions $v_p(x,t)$ to $P_p(u_0)$ is independent of time but does not coincide with u_0. They also describe the small layer in which the solution rapidly changes from being u_0 at $t = 0$ to something close to the final stationary limit for $t > 0$.

Now, our task is to perform a similar analysis for the nonlocal problem. To this end let us take $\varepsilon = 1$ and $f = 0$ and look for the limit as $p \to \infty$ of the solutions to the nonlocal problem u_p when the initial condition u_0 does not verify that $|u_0(x) - u_0(y)| \le 1$ for $x - y \in \mathrm{supp}(J)$. We get that the nonlinear nature of the problem creates an initial short-time layer in which the solution changes very rapidly. We describe this layer by means of a limit evolution problem. We have the following result.

THEOREM 153. *Let u_p be the solution to $P_p^J(u_0)$ with initial condition $u_0 \in L^2(\mathbb{R}^N)$ such that*

$$1 < L = \sup_{|x-y|\in supp(J)} |u_0(x) - u_0(y)|.$$

Then there exists the limit

$$\lim_{p\to\infty} u_p(x,t) = u_\infty(x) \qquad \text{in } L^2(\mathbb{R}^N),$$

which is a function independent of t such that $|u_\infty(x) - u_\infty(y)| \le 1$ for $x - y \in supp(J)$. Moreover, $u_\infty(x) = v(1,x)$, where v is the unique strong solution of the evolution equation

$$\begin{cases} \dfrac{v}{t} - v_t \in \partial G_\infty^J(v), & t \in]\tau,\infty[, \\[2mm] v(\tau,x) = \tau u_0(x), \end{cases}$$

with $\tau = L^{-1}$.

Remark that when u_0 verifies $|u_0(x) - u_0(y)| \leq 1$ for $x - y \in \mathrm{supp}(J)$ then it is an immediate consequence of Theorem 145 that the limit exists and is given by

$$\lim_{p\to\infty} u_p(x,t) = u_0(x).$$

Recall that we have mentioned that Evans, Feldman and Gariepy in [**56**] study the behavior of the solution v_p of the initial value problem

$$\begin{cases} v_{p,t} - \Delta_p v_p = 0, & t \in]0,T[, \\ v_p(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases}$$

in the "infinitely fast diffusion" limit $p \to \infty$, that is, when the initial condition u_0 is a Lipschitz function with compact support, satisfying

$$\mathrm{ess\ sup}_{\mathbb{R}^N} |\nabla u_0| = L > 1.$$

They prove that for each time $t > 0$

$$v_p(t,\cdot) \to v_\infty(\cdot), \qquad \text{uniformly as } \ p \to +\infty,$$

where v_∞ is independent of time and satisfies

$$\mathrm{ess\ sup}_{\mathbb{R}^N} |\nabla v_\infty| \leq 1.$$

Moreover, $v_\infty(x) = v(1,x)$, v solving the nonautonomous evolution equation

$$\begin{cases} \dfrac{v}{t} - v_t \in \partial I_{K_0}(v), & t \in]\tau, \infty[\\ v(\tau,x) = \tau u_0(x), \end{cases}$$

where $\tau = L^{-1}$. They interpreted this as a crude model for the collapse of a sandpile from an initially unstable configuration. The proof of this result is based in a scaling argument, which was extended by Bénilan, Evans and Gariepy in [**21**], to cover general nonlinear evolution equations governed by homogeneous accretive operators. Here, using this general result, we prove similar results for our nonlocal model.

We look for the limit as $p \to \infty$ of the solutions to the nonlocal problem $P_p^J(u_0)$ when the initial datum u_0 satisfies

$$1 < L = \sup_{x-y\in \mathrm{supp}(J)} |u_0(x) - u_0(y)|.$$

For $p > 2$, we consider in the Banach space $X = L^2(\mathbb{R}^N)$ the operators ∂G_p^J. Then, ∂G_p^J are m-accretive operators in $L^2(\mathbb{R}^n)$ and also positively homogeneous of degree $p-1$.

Moreover, the solution u_p to the nonlocal problem $P_p^J(u_0)$ coincides with the strong solution of the abstract Cauchy problem

$$\begin{cases} -u_t(x,t) \in \partial G_p^J(u(t)), & \text{a.e } t \in]0,T[, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N. \end{cases}$$

Let

$$C := \left\{ u \in L^2(\mathbb{R}^N) \ : \ \exists (u_p, v_p) \in \partial G_p^J \text{ with } u_p \to u, \ v_p \to 0 \text{ as } p \to \infty \right\}.$$

It is easy to see that

$$C = K_\infty^J = \left\{ u \in L^2(\mathbb{R}^N) \ : \ |u(x) - u(y)| \leq 1, \text{ for } x - y \in \operatorname{supp}(J) \right\}.$$

Then,

$$X_0 := \overline{\bigcup_{\lambda > 0} \lambda C}^{L^2(\mathbb{R}^N)} = L^2(\mathbb{R}^N).$$

LEMMA 154. *For $f \in L^2(\mathbb{R}^N)$ and $p > N$, let $u_p := (I + \partial G_p^J)^{-1} f$. Then, the set of functions $\{u_p \ : \ p > N\}$ is precompact in $L^2(\mathbb{R}^N)$.*

PROOF. First assume that f is bounded and the support of f lies in the ball $B_R(0)$. Since the operator ∂G_p^J is completely accretive, observe that

$$\partial G_p^J = \overline{\mathcal{B}_p^J \cap (L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N))}^{L^2(\mathbb{R}^N)},$$

we have the estimates

$$\|u_p\|_{L^\infty} \leq \|f\|_{L^\infty}, \quad \|u_p\|_{L^2} \leq \|f\|_{L^2}$$

and

$$\|u_p(\cdot) - u_p(\cdot + h)\|_{L^2} \leq \|f(\cdot) - f(\cdot + h)\|_{L^2}$$

for each $h \in \mathbb{R}^N$. Consequently, $\{u_p \ : \ p > N\}$ is precompact in $L^2(K)$ for each compact set $K \subset \mathbb{R}^N$. We must show that $\{u_p \ : \ p > N\}$ is tight. For this, fix $S > 2R$ and select a smooth function $\varphi \in C^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 0$ on $B_R(0)$, $\varphi \equiv 1$ on $\mathbb{R}^N \setminus B_S(0)$ and $|\nabla \varphi| \leq \frac{2}{S}$.

We have

$$u_p(x) = \int_{\mathbb{R}^N} J(x-y) |u_p(y) - u_p(x)|^{p-2} (u_p(y) - u_p(x)) \, dy + f(x).$$

Then, multiplying by φu_p and integrating, we get

$$\begin{aligned}&\int_{\mathbb{R}^N} u_p^2(x)\varphi(x)\,dx\\&=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))u_p(x)\varphi(x)\,dydx\\&=-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))(u_p(y)\varphi(y)-u_p(x)\varphi(x))\,dydx\\&\le-\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))u_p(y)(\varphi(y)-\varphi(x))\,dydx.\end{aligned}$$

Now, since $|\nabla\varphi|\le\frac{2}{S}$, by Hölder's inequality we obtain

$$\begin{aligned}&\left|\frac{1}{2}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} J(x-y)|u_p(y)-u_p(x)|^{p-2}(u_p(y)-u_p(x))u_p(y)(\varphi(y)-\varphi(x))\,dydx\right|\\&\le\frac{\|f\|_{L^\infty}}{S}\int_{\{|x|\le S+1\}}\left(\int_{B_1(x)} J(x-y)|u_p(y)-u_p(x)|^{p-1}\,dy\right)dx\\&\le\frac{\|f\|_{L^\infty}}{S}\left(\int_{\{|x|\le S+1\}}\int_{B_1(x)} J(x-y)|u_p(y)-u_p(x)|^{p}dy\right)^{\frac{1}{p'}}\\&\qquad\times\left(\int_{\{|x|\le S+1\}}\int_{B_1(x)} J(x-y)dy\right)^{\frac{1}{p}}dx\\&\le M(S+1)^{\frac{N}{p}-1}=O(S^{-1+\frac{N}{p}}),\end{aligned}$$

the last inequality being true since $\int\int J(x-y)|u_p(y)-u_p(x)|^p$ is bounded uniformly in p. Hence,

$$\int_{\{|x|\ge S\}} u_p^2(x)\,dx=O(S^{-1+\frac{N}{p}})$$

uniformly in $p>N$. This proves tightness and we have established compactness in $L^2(\mathbb{R}^N)$ provided f is bounded and has compact support. The general case follows, since such functions are dense in $L^2(\mathbb{R}^N)$. □

Proof of Theorem 153. By the above Lemma, given $f\in L^2(\mathbb{R}^N)$ if $u_p:=(I+\partial G_p^J)^{-1}f$, there exists a sequence $p_j\to+\infty$, such that $u_{p_j}\to v$ in $L^2(\mathbb{R}^N)$ as $j\to\infty$. In the proof of Theorem 145 we have established that the functionals G_p^J converge to $I_{K_\infty^J}$, as $p\to\infty$, in the sense of Mosco. Then, by Theorem 146, we have $v=(I+I_{K_\infty^J})^{-1}f$. Therefore, the limit

$$Pf:=\lim_{p\to\infty}(I+\partial G_p^J)^{-1}f$$

exists in $L^2(\mathbb{R}^N)$, for all $f\in X_0=L^2(\mathbb{R}^N)$, and $Pf=f$ if $f\in C=K_\infty^J$. Moreover,

$$P^{-1}-I=\partial I_{K_\infty^J}$$

and $u = Pf$ is the unique solution of

$$u + \partial I_{K^J_\infty} u \ni f.$$

Therefore, as consequence of the main result of [**21**], we have obtained Theorem 153. □

0.35. Explicit solutions. In this chapter we show some explicit examples of solutions to

$$P^\varepsilon_\infty(u_0) \quad \begin{cases} f(x,t) - u_t(x,t) \in \partial G^\varepsilon_\infty(u(t)), & \text{a.e. } t \in]0,T[, \\ u(0,x) = u_0(x), & \text{in } \mathbb{R}^N, \end{cases}$$

where

$$G^\varepsilon_\infty(u) = \begin{cases} 0 & \text{if } u \in L^2(\mathbb{R}^N),\ |u(x) - u(y)| \le \varepsilon,\ \text{for } |x-y| \le \varepsilon, \\ +\infty & \text{in other case.} \end{cases}$$

In order to verify that a function $u(x,t)$ is a solution to $P^\varepsilon_\infty(u_0)$ we need to check that

$$G^\varepsilon_\infty(v) \ge G^\varepsilon_\infty(u) + \langle f - u_t,\ v - u \rangle, \qquad \text{for all } v \in L^2(\mathbb{R}^N). \tag{11.34}$$

To this end we can assume that $v \in K_\varepsilon$ (otherwise $G^\varepsilon_\infty(v) = +\infty$ and then (11.34) becomes trivial). Therefore,

$$u(t,\cdot) \in K_\varepsilon \tag{11.35}$$

and (11.34) can be rewritten as

$$0 \ge \int_{\mathbb{R}} (f(x,t) - u_t(x,t))(v(x) - u(x,t))\, dx \tag{11.36}$$

for every $v \in K_\varepsilon$.

Example 1. Let us consider, in one space dimension, as source an approximation of a delta function

$$f(x,t) = f_\eta(x,t) = \frac{1}{\eta}\chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x), \quad 0 < \eta \le 2\varepsilon,$$

and as initial datum

$$u_0(x) = 0.$$

Now, let us find the solution by looking at its evolution between some critical times.

First, for small times, the solution to $P^\varepsilon_\infty(u_0)$ is given by

$$u(x,t) = \frac{t}{\eta}\chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x), \tag{11.37}$$

for

$$t \in [0, \eta\varepsilon).$$

Remark that $t_1 = \eta\varepsilon$ is the first time when $u(x,t) = \varepsilon$ and hence it is immediate that $u(t,\cdot) \in K_\varepsilon$. Moreover, as $u_t(x,t) = f(x,t)$ then (11.36) holds.

For times greater than t_1 the support of the solution is greater than the support of f. Indeed the solution can not be larger than ε in $[-\frac{\eta}{2}, \frac{\eta}{2}]$ without being larger than zero in the adjacent intervals of size ε, $[\frac{\eta}{2}, \frac{\eta}{2}+\varepsilon]$ and $[-\frac{\eta}{2}-\varepsilon, -\frac{\eta}{2}]$.

We have

$$u(x,t) = \begin{cases} \varepsilon + k_1(t-t_1) & \text{for } x \in [-\frac{\eta}{2}, \frac{\eta}{2}], \\ k_1(t-t_1) & \text{for } x \in [-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon] \setminus [-\frac{\eta}{2}, \frac{\eta}{2}], \\ 0 & \text{for } x \notin [-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon], \end{cases} \tag{11.38}$$

for times t such that

$$t \in [t_1, t_2)$$

where

$$k_1 = \frac{1}{2\varepsilon + \eta} \quad \text{and} \quad t_2 = t_1 + \frac{\varepsilon}{k_1} = 2\varepsilon^2 + 2\varepsilon\eta.$$

Note that t_2 is the first time when $u(x,t) = 2\varepsilon$ for $x \in [-\frac{\eta}{2}, \frac{\eta}{2}]$. Again it is immediate to see that $u(t,\cdot) \in K_\varepsilon$, since for $|x-y| < \varepsilon$ the maximum of the difference $u(x,t) - u(y,t)$ is exactly ε. Now let us check (11.36).

Using the explicit formula for $u(x,t)$ given in (11.38), we obtain

(11.39)

$$\begin{aligned}
&\int_{\mathbb{R}} (f(x,t) - u_t(x,t))(v(x) - u(x,t))\,dx = \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \left(\frac{1}{\eta} - u_t(x,t)\right)(v(x) - u(x,t))\,dx \\
&\quad + \int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} (-u_t(x,t))(v(x) - u(x,t))\,dx + \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} (-u_t(x,t))(v(x) - u(x,t))\,dx \\
&= \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \left(\frac{1}{\eta} - k_1\right)(v(x) - (\varepsilon + k_1(t-t_1)))\,dx + \int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} (-k_1)(v(x) - (k_1(t-t_1)))\,dx \\
&\quad + \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} (-k_1)(v(x) - (k_1(t-t_1)))\,dx \\
&= \left(-\eta\left(\frac{1}{\eta} - k_1\right) + 2\varepsilon k_1\right) k_1(t-t_1) - \varepsilon\eta\left(\frac{1}{\eta} - k_1\right) + \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} \left(\frac{1}{\eta} - k_1\right) v(x)\,dx \\
&\qquad - \int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} k_1 v(x)\,dx - \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} k_1 v(x)\,dx.
\end{aligned}$$

From our choice of k_1 we get

$$-\eta\left(\frac{1}{\eta} - k_1\right) + 2\varepsilon k_1 = 0$$

and, since $v \in K_\varepsilon$, we have

$$\int_{\mathbb{R}} (f(x,t) - u_t(x,t))(v(x) - u(x,t))\,dx$$

(11.40)
$$= -2\varepsilon^2 k_1 + \frac{2\varepsilon k_1}{\eta} \int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} v(x)\,dx - k_1 \int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} v(x)\,dx - k_1 \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} v(x)\,dx \le 0.$$

In fact, without loss of generality we can suppose that

$$\int_{-\frac{\eta}{2}}^{\frac{\eta}{2}} v(x)\,dx = 0.$$

Then

$$\int_0^{\eta/2} (-v) = a, \quad \int_{-\eta/2}^{0} (-v) = -a. \tag{11.41}$$

Consequently,

$$-v \le \frac{2}{\eta} a + \varepsilon \quad \text{in } [0, \varepsilon]. \tag{11.42}$$

Indeed, if (11.42) does not hold, then $-v > \frac{2}{\eta} a$ in $[0, \varepsilon]$ which contradicts (11.41).

Now, by (11.41), since $v \in K_\varepsilon$,

(11.43)
$$\begin{aligned} \int_{\varepsilon}^{\varepsilon+\eta/2} (-v(x))dx &= \int_0^{\eta/2} (-v(y+\varepsilon))dy \\ &= \int_0^{\eta/2} (-v(y+\varepsilon) + v(y))dy + \int_0^{\eta/2} (-v(y))dy \\ &\le \varepsilon \frac{\eta}{2} + a. \end{aligned}$$

Therefore, by (11.42) and (11.43),

(11.44)
$$\begin{aligned} \int_{\eta/2}^{\varepsilon+\eta/2} (-v) &= \int_{\eta/2}^{\varepsilon} (-v) + \int_{\varepsilon}^{\varepsilon+\eta/2} (-v) \\ &\le \left(\frac{2}{\eta} a + \varepsilon\right)\left(\varepsilon - \frac{\eta}{2}\right) + \varepsilon \frac{\eta}{2} + a = \frac{2}{\eta} a\varepsilon + \varepsilon^2. \end{aligned}$$

Similarly,

$$\int_{-\varepsilon-\eta/2}^{-\eta/2} (-v) \le -\frac{2}{\eta} a\varepsilon + \varepsilon^2. \tag{11.45}$$

Consequently, by (11.44) and (11.45),

$$\int_{\frac{\eta}{2}}^{\frac{\eta}{2}+\varepsilon} (-v) + \int_{-\frac{\eta}{2}-\varepsilon}^{-\frac{\eta}{2}} (-v) \le 2\varepsilon^2.$$

Now, it is easy to generalize and verify the following general formula that describes the solution for every $t \geq 0$. For any given integer $l \geq 0$ we have

$$u(x,t) = \begin{cases} l\varepsilon + k_l(t - t_l), & x \in [-\frac{\eta}{2}, \frac{\eta}{2}], \\ (l-1)\varepsilon + k_l(t - t_l), & x \in [-\frac{\eta}{2} - \varepsilon, \frac{\eta}{2} + \varepsilon] \setminus [-\frac{\eta}{2}, \frac{\eta}{2}], \\ \dots & \\ k_l(t - t_l), & x \in [-\frac{\eta}{2} - l\varepsilon, \frac{\eta}{2} + l\varepsilon] \setminus [-\frac{\eta}{2} - (l-1)\varepsilon, \frac{\eta}{2} + (l-1)\varepsilon], \\ 0, & x \notin [-\frac{\eta}{2} - l\varepsilon, \frac{\eta}{2} + l\varepsilon], \end{cases} \tag{11.46}$$

for

$$t \in [t_l, t_{l+1}),$$

where

$$k_l = \frac{1}{2l\varepsilon + \eta} \quad \text{and} \quad t_{l+1} = t_l + \frac{\varepsilon}{k_l}, \quad t_0 = 0.$$

From formula (11.46) we get, taking the limit as $\eta \to 0$, that the expected solution to (11.34) with $f = \delta_0$ is given by, for any given integer $l \geq 1$,

$$u(x,t) = \begin{cases} (l-1)\varepsilon + k_l(t - t_l), & x \in [-\varepsilon, \varepsilon], \\ (l-2)\varepsilon + k_l(t - t_l), & x \in [-2\varepsilon, 2\varepsilon] \setminus [-\varepsilon, \varepsilon], \\ \dots & \\ k_l(t - t_l), & x \in [-l\varepsilon, l\varepsilon] \cup [-(l-1)\varepsilon, (l-1)\varepsilon], \\ 0, & x \notin [-l\varepsilon, l\varepsilon], \end{cases} \tag{11.47}$$

for

$$t \in [t_l, t_{l+1})$$

where

$$k_l = \frac{1}{2l\varepsilon}, \quad t_{l+1} = t_l + \frac{\varepsilon}{k_l}, \quad t_1 = 0.$$

Remark that, since the space of functions K_ε is not contained into $C(\mathbb{R})$, the formulation (11.36) with $f = \delta_0$ does not make sense. Hence the function $u(x,t)$ described by (11.47) is to be understood as a generalized solution to (11.34) (it is obtained as a limit of solutions to approximating problems).

Note that the function $u(t_l, x)$ is a "regular and symmetric pyramid" composed by squares of side ε.

Recovering the sandpile model as $\varepsilon \to 0$. Now, to recover the sandpile model, let us fix

$$l\varepsilon = L,$$

and take the limit as $\varepsilon \to 0$ in the previous example. We get that $u(x,t) \to v(x,t)$, where

$$v(x,t) = (L - |x|)_+, \qquad \text{for } t = L^2,$$

that is exactly the evolution given by the sandpile model with initial datum $u_0 = 0$ and a point source δ_0, see [**10**].

Therefore, this concrete example illustrates the general convergence result Theorem 152.

Example 2. The explicit formula (11.46) can be easily generalized to the case in where the source depends on t in the form

$$f(x,t) = \varphi(t)\chi_{[-\frac{\eta}{2},\frac{\eta}{2}]}(x),$$

with φ a nonnegative integrable function and $0 < \eta \leq \varepsilon$. We arrive to the following formulas, setting

$$g(t) = \int_0^t \varphi(s)ds,$$

for any given integer $l \geq 0$,

$$u(x,t) = \begin{cases} l\varepsilon + \hat{k}_l\left(g(t) - g(t_l)\right), & x \in [-\frac{\eta}{2}, \frac{\eta}{2}], \\ (l-1)\varepsilon + \hat{k}_l\left(g(t) - g(t_l)\right), & x \in [-\frac{\eta}{2} - \varepsilon, \frac{\eta}{2} + \varepsilon] \setminus [-\frac{\eta}{2}, \frac{\eta}{2}], \\ \dots & \\ \hat{k}_l\left(g(t) - g(t_l)\right), & x \in [-\frac{\eta}{2} - l\varepsilon, \frac{\eta}{2} + l\varepsilon] \setminus [-\frac{\eta}{2} - (l-1)\varepsilon, \frac{\eta}{2} + (l-1)\varepsilon] \\ 0, & x \notin [-\frac{\eta}{2} - l\varepsilon, \frac{\eta}{2} + l\varepsilon], \end{cases}$$

for

$$t \in [t_l, t_{l+1}),$$

where

$$\hat{k}_l = \frac{\eta}{\eta + 2l\varepsilon} \quad \text{and} \quad g(t_{l+1}) - g(t_l) = \frac{\varepsilon}{\hat{k}_l}, \quad t_0 = 0.$$

Observe that t_l is the first time at which the solution reaches level $l\varepsilon$.

We can also consider φ changing sign. In this case the solution increases if $\varphi(t)$ is positive in every interval of size ε (around the support of the source $[-\frac{\eta}{2}, \frac{\eta}{2}]$) for which $u(x) - u(y) = i\varepsilon$ with $|x - y| = i\varepsilon$ for some $x \in [-\frac{\eta}{2}, \frac{\eta}{2}]$ (here i is any integer). While if $\varphi(t)$ is negative the solution decreases in every interval of size ε for which $u(x) - u(y) = -i\varepsilon$ with $|x - y| = i\varepsilon$ for some $x \in [-\frac{\eta}{2}, \frac{\eta}{2}]$.

Example 3. Observe that if $\eta > 2\varepsilon$, then $u(x,t)$ given in (11.38) does not satisfy (11.36) for a test function $v \in K_\varepsilon$ whose values in $[-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon]$ are

$$v(x) = \begin{cases} -\beta\dfrac{\varepsilon}{2} + 2\varepsilon & \text{for } x \in [-\frac{\eta}{2}+\varepsilon, \frac{\eta}{2}-\varepsilon], \\ -\beta\dfrac{\varepsilon}{2} + \varepsilon & \text{for } x \in [-\frac{\eta}{2}, \frac{\eta}{2}] \setminus [-\frac{\eta}{2}+\varepsilon, \frac{\eta}{2}-\varepsilon], \\ -\beta\dfrac{\varepsilon}{2} & \text{for } x \in [-\frac{\eta}{2}-\varepsilon, \frac{\eta}{2}+\varepsilon] \setminus [-\frac{\eta}{2}, \frac{\eta}{2}], \end{cases}$$

for $\beta = 4(1-\varepsilon/\eta)$ which is greater that 2.

At this point one can ask what happens in the previous situation when $\eta > 2\varepsilon$. In this case the solution begins to grow as before with constant speed in the support of f but after the first time when it reaches level ε the situation changes. Consider, for example, that the source is given by

$$f(x,t) = \frac{1}{\varepsilon}\chi_{[-2\varepsilon,2\varepsilon]}(x).$$

In this case the solution to our nonlocal problem with $u_0(x) = 0$, $u(x,t)$, can be described as follows. Firstly we have

$$u(x,t) = \frac{t}{\varepsilon}\chi_{[-2\varepsilon,2\varepsilon]}(x),$$

for

$$t \in [0, \varepsilon^2).$$

Remark that $t_1 = \varepsilon^2$ is the first time when $u(x,t) = \varepsilon$ and hence it is immediate that $u(t,\cdot) \in K_\varepsilon$. Moreover, as $u_t(x,t) = f(x,t)$ then (11.36) holds.

For times greater that t_1 we have

$$u(x,t) = \begin{cases} \varepsilon + \dfrac{1}{\varepsilon}(t-t_1) & \text{for } x \in [-\varepsilon, \varepsilon], \\ \varepsilon + k_1(t-t_1) & \text{for } x \in [-2\varepsilon, -\varepsilon] \cup [\varepsilon, 2\varepsilon], \\ k_1(t-t_1) & \text{for } x \in [-3\varepsilon, -2\varepsilon] \cup [2\varepsilon, 3\varepsilon], \\ 0 & \text{for } x \notin [-3\varepsilon, 3\varepsilon], \end{cases}$$

for

$$t \in [t_1, t_2),$$

where

$$k_1 = \frac{1}{2\varepsilon} \quad \text{and} \quad t_2 = \varepsilon^2 + 2\varepsilon^2 = 3\varepsilon^2.$$

With this expression of $u(x,t)$ it is easy to see that it verifies (11.36).

For times greater than t_2 an expression similar to (11.46) holds. We leave the details to the reader.

Example 4. For two or more dimensions we can get similar formulas. Given a bounded domain $\Omega_0 \subset \mathbb{R}^N$ let us define inductively

$$\Omega_1 = \left\{ x \in \mathbb{R}^N \ : \ \exists y \in \Omega_0, \ \text{with } |x-y| < \varepsilon \right\}$$

and

$$\Omega_j = \left\{ x \in \mathbb{R}^N \ : \ \exists y \in \Omega_{j-1}, \ \text{with } |x-y| < \varepsilon \right\}.$$

In the sequel, for simplicity, we consider the two dimensional case $N = 2$. Let us take as source

$$f(x,t) = \chi_{\Omega_0}(x), \quad \Omega_0 = B(0, \varepsilon/2),$$

and, as initial datum,

$$u_0(x) = 0.$$

In this case, for any integer $l \geq 0$, the solution to (11.34) is given by

$$u(x,t) = \begin{cases} l\varepsilon + \hat{k}_l(t - t_l), & x \in \Omega_0, \\ (l-1)\varepsilon + \hat{k}_l(t-t_l), & x \in \Omega_1 \setminus \Omega_0, \\ \dots & \\ \hat{k}_l(t-t_l), & x \in \Omega_l \setminus \bigcup_{j=1}^{l-1} \Omega_j, \\ 0, & x \notin \Omega_l, \end{cases} \tag{11.48}$$

for

$$t \in [t_l, t_{l+1}),$$

where

$$\hat{k}_l = \frac{|\Omega_0|}{|\Omega_l|}, \qquad t_{l+1} = t_l + \frac{\varepsilon}{\hat{k}_l}, \quad t_0 = 0.$$

Note that the solution grows in strips of width ε around the set Ω_0 where the source is localized.

As in the previous examples, the result is evident for $t \in [0, t_1)$. Let us see it for $t \in [t_1, t_2)$, a similar argument works for later times. It is clear that $u(t, \cdot) \in K_\varepsilon$, let us check (11.36). Working as in Example 1, we must show that

$$(1 - \hat{k}_1) \int_{\Omega_0} v - \hat{k}_1 \int_{\Omega_1 \setminus \Omega_0} v \leq (1 - \hat{k}_1) \varepsilon |\Omega_0| \quad \forall v \in K_\varepsilon,$$

where $\Omega_1 = B(0, 3\varepsilon/2)$. Since $\hat{k}_1 = |\Omega_0|/|\Omega_1|$, the last inequality is equivalent to

$$\left| \frac{1}{|\Omega_0|} \int_{\Omega_0} v - \frac{1}{|\Omega_1 \setminus \Omega_0|} \int_{\Omega_1 \setminus \Omega_0} v \right| \leq \varepsilon \quad \forall v \in K_\varepsilon. \tag{11.49}$$

By density, it is enough to prove (11.49) for any $v \in K_\varepsilon$ continuous.

Let us now divide $\Omega_0 = \{r(\cos\theta, \sin\theta) : 0 \leq \theta \leq 2\pi, 0 \leq r < \frac{\varepsilon}{2}\}$ and $\Omega_1 \setminus \Omega_0 = \{r(\cos\theta, \sin\theta) : 0 \leq \theta \leq 2\pi, \varepsilon \leq r < \frac{3}{2}\varepsilon\}$ as follows. Consider the partitions

$$0 = \theta_0 < \theta_1 < .... < \theta_N = 2\pi,$$

with $\theta_i - \theta_{i-1} = 2\pi/N$, $N \in \mathbb{N}$,

$$0 = r_0 < r_1 < .... < r_N = \varepsilon/2$$

and

$$\varepsilon/2 = \tilde{r}_0 < \tilde{r}_1 < .... < \tilde{r}_N = 3\varepsilon/2,$$

such that the measure of

$$B_{ij} = \{r(\cos\theta, \sin\theta) : \theta_{i-1} < \theta < \theta_i \ r_{j-1} < r < r_j\}$$

is constant, that is, $|B_{ij}| = |\Omega_0|/N^2$, and the measure of

$$A_{ij} = \{r(\cos\theta, \sin\theta) : \theta_{i-1} < \theta < \theta_i \ \tilde{r}_{j-1} < r < \tilde{r}_j\}$$

is also constant, that is, $|A_{ij}| = |\Omega_1 \setminus \Omega_0|/N^2$. In this way we have partitioned Ω_0 and $\Omega_1 \setminus \Omega_0$ as a disjoint family of N^2 sets such that

$$\left|\Omega_0 \setminus \bigcup_{i,j=1}^{N} B_{ij}\right| = 0, \qquad \left|(\Omega_1 \setminus \Omega_0) \setminus \bigcup_{i,j=1}^{N} A_{ij}\right| = 0.$$

By construction, if we take

$$x_{ij} = r_j(\cos\theta_{i-1}, \sin\theta_{i-1}) \in B_{ij}, \quad \tilde{x}_{ij} = \tilde{r}_{j-1}(\cos\theta_{i-1}, \sin\theta_{i-1}) \in A_{ij},$$

then $|x_{ij} - \tilde{x}_{ij}| \leq \varepsilon$ for all $i, j = 1, \dots N$.

Given a continuous function $v \in K_\varepsilon$, by uniform continuity of v, for $\delta > 0$, there exists $\rho > 0$ such that

$$|v(x) - v(y)| \leq \frac{\delta}{2} \qquad \text{if } |x - y| \leq \rho.$$

Hence, if we take N big enough such that diameter$(B_{ij}) \leq \rho$ and diameter$(A_{ij}) \leq \rho$, we have

$$\left|\int_{\Omega_0} v(x) - \sum_{i,j=1}^{N} v(x_{ij})|B_{ij}|\right| \leq \frac{\delta|\Omega_0|}{2}$$

and

$$\left|\int_{\Omega_1\setminus\Omega_0} v(x) - \sum_{i,j=1}^{N} v(\tilde{x}_{ij})|A_{ij}|\right| \leq \frac{\delta|\Omega_1 \setminus \Omega_0|}{2}.$$

Since $v \in K_\varepsilon$ and $|x_{ij} - \tilde{x}_{ij}| \leq \varepsilon$, $|v(x_{ij}) - v(\tilde{x}_{ij})| \leq \varepsilon$. Consequently,

$$
\begin{aligned}
&\left|\frac{1}{|\Omega_0|}\int_{\Omega_0} v - \frac{1}{|\Omega_1\setminus\Omega_0|}\int_{\Omega_1\setminus\Omega_0} v\right| \\
&\le \left|\frac{1}{|\Omega_0|}\sum_{i,j=1}^{N} v(x_{ij})|B_{ij}| - \frac{1}{|\Omega_1\setminus\Omega_0|}\sum_{i,j=1}^{N} v(\tilde{x}_{ij})|A_{ij}|\right| + \delta \\
&= \left|\frac{1}{N^2}\sum_{i,j=1}^{N} v(x_{ij}) - \frac{1}{N^2}\sum_{i,j=1}^{N} v(\tilde{x}_{ij})\right| + \delta \\
&\le \varepsilon + \delta.
\end{aligned}
$$

Therefore, since $\delta > 0$ is arbitrary, (11.49) is obtained.

Again the explicit formula (11.48) can be easily generalized to the case where the source depends on t in the form

$$f(x,t) = \varphi(t)\chi_{\Omega_0}(x).$$

An estimate of the support of u_t. Taking a source $f \ge 0$ supported in a set A, let us see where the material is added (places where u_t is positive). Let us compute a set that we will call $\Omega^*(t)$ as follows. Let

$$\Omega_0(t) = A,$$

and define inductively

$$\Omega_1(t) = \left\{x \in \mathbb{R}^N \setminus \Omega_0(t) \ : \ \exists y \in \Omega_0(t) \text{ with } |x-y| < \varepsilon \text{ and } u(y,t) - u(x,t) = \varepsilon\right\}$$

and

$$\Omega_j(t) = \left\{x \in \mathbb{R}^N \setminus \Omega_{j-1}(t) \ : \ \exists y \in \Omega_{j-1}(t) \text{ with } |x-y| < \varepsilon \text{ and } u(y,t) - u(x,t) = \varepsilon\right\}.$$

With these sets $\Omega_i(t)$ (observe that there exists a finite number of such sets, since $u(x,t)$ is bounded) let

$$\Omega^*(t) = \bigcup_i \Omega_i(t).$$

We have that

$$u_t(x,t) = 0, \qquad \text{for } x \notin \Omega^*(t).$$

Indeed, this can be easily deduced using an appropriate test function v in (11.36). Just take $v(x) = u(x,t)$ but for a small neighborhood near $x \notin \Omega^*(t)$.

Example 5. Finally, note that an analogous description like in the above examples can be made for an initial condition that is of the form

$$u_0(x) = \sum_{i=-K}^{K} a_i \chi_{[i\varepsilon,(i+1)\varepsilon]}(x),$$

with

$$|a_i - a_{i\pm 1}| \le \varepsilon, \qquad a_{-K} = a_K = 0,$$

(this last condition is needed just to imply that $u_0 \in K_\varepsilon$) together with the sum of a finite number of delta functions placed at points $x_l = l\varepsilon$ (or a finite sum of functions of time times the characteristic functions of some intervals of the form $[l\varepsilon, (l+1)\varepsilon]$) as the source term.

For example, let us consider a source placed in just one interval, $f(x,t) = \chi_{[0,\varepsilon]}(x)$. Initially, $u(0,x) = l\varepsilon$ for $x \in [0,\varepsilon]$. Let us take $w_1(x)$ the regular and symmetric pyramid centered at $[0,\varepsilon]$ of height $(l+1)\varepsilon$ (and base of length $(2l-1)\varepsilon$). With this pyramid and the initial condition let us consider the set

$$\Lambda_1 = \{j \,:\, w_1(x) > u(0,x) \text{ for } x \in (j\varepsilon,(j+1)\varepsilon)\}.$$

This set contains the indexes of the intervals in which the sand is being added in the first stage. During this first stage, $u(x,t)$ is given by

$$u(x,t) = u(0,x) + \frac{t}{\mathrm{Card}(\Lambda_1)} \sum_{j\in\Lambda_1} \chi_{[j\varepsilon,(j+1)\varepsilon]}(x),$$

for $t \in [0,t_1]$, where $t_1 = \mathrm{Card}(\Lambda_1)\varepsilon$ is the first time at which u is of size $(l+1)\varepsilon$ in the interval $[0,\varepsilon]$.

From now on the evolution follows the same scheme. In fact,

$$u(x,t) = u(t_i,x) + \frac{t-t_i}{\mathrm{Card}(\Lambda_i)} \sum_{j\in\Lambda_i} \chi_{[j\varepsilon,(j+1)\varepsilon]}(x),$$

for

$$t \in [t_i,t_{i+1}], \qquad t_{i+1} - t_i = \mathrm{Card}(\Lambda_i)\varepsilon.$$

Where, from the pyramid w_i of height $(l+i)\varepsilon$, we obtain

$$\Lambda_i = \{j \,:\, w_i(x) > u(t_i,x) \text{ for } x \in (j\varepsilon,(j+1)\varepsilon)\}.$$

Remark that eventually the pyramid w_k is bigger than the initial condition, from this time on the evolution is the same as described for $u_0 = 0$ in the first example.

In case we have two sources, the pyramids w_i, $\tilde{w}_i$ corresponding to the two sources eventually intersect. In the interval where the intersection takes place, u_t is given by the greater of the two possible speeds (that correspond to the different sources). If both possible speeds are the same this interval has to be computed as corresponding to both sources simultaneously.

Recovering the sandpile model. Note that any initial condition w_0 with $|\nabla w_0| \le 1$ can be approximated by an u_0 like the one described above. Hence we can obtain an explicit solution of the nonlocal model that approximates the solutions constructed in [**10**].

Compact support of the solutions. Note also that when the source f and the initial condition u_0 are compactly supported and bounded then also the solution is compactly supported and bounded for all positive times. This property has to be contrasted with the fact that solutions to the nonlocal $p-$laplacian $P_p^J(u_0)$ are not compactly supported even if u_0 is.

0.36. A mass transport interpretation. We can also give an interpretation of the limit problem $P_\infty(u_0)$ in terms of Monge-Kantorovich mass transport theory as in [**56**], [**58**] (see [**84**] for a general introduction to mass transportation problems). To this end let us consider the distance

$$d(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \\ [|x-y|] + 1 & \text{if } x \neq y. \end{cases}$$

Here $[\cdot]$ means the entire part of the number. Note that this function d measures distances with jumps of length one. Then, given two measures (that for simplicity we will take absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^N$) f_+, f_- in $\mathbb{R}^N$, and supposing the overall condition of mass balance

$$\int_{\mathbb{R}^N} f^+ \, dx = \int_{\mathbb{R}^N} f^- \, dy,$$

the Monge's problem associated to the distance d is given by: minimize

$$\int d(x, s(x))\, f_+(x) dx$$

among the set of maps s that transport f_+ into f_-, which means

$$\int_{\mathbb{R}^N} h(s(x)) f^+(x)\, dx = \int_{\mathbb{R}^N} h(y) f^-(y)\, dy$$

for each continuous function $h : \mathbb{R}^N \to \mathbb{R}$. The dual formulation of this minimization problem, introduced by Kantorovich (see [**55**]), is given by

$$\max_{u \in K_\infty} \int_{\mathbb{R}^N} u(x)(f_+(x) - f_-(x)) dx$$

where the set K_∞ is given by

$$K_\infty := \{u \in L^2(\mathbb{R}^N) \ : \ |u(x) - u(y)| \leq 1, \text{ for } |x-y| \leq 1\}.$$

We are assuming that $\operatorname{supp}(J) = \overline{B}_1(0)$ (in other case we have to redefine the distance d accordingly).

With these definitions and notations we have the following result.

THEOREM 155. *The solution $u_\infty(t,\cdot)$ of the limit problem $P_\infty^J(u_0)$ is a solution to the dual problem*

$$\max_{u \in K_\infty} \int_{\mathbb{R}^N} u(x)(f_+(x) - f_-(x)) dx$$

when the involved measures are the source term $f_+ = f(x,t)$ and the time derivative of the solution $f_- = u_t(x,t)$.

Proof of Theorem 155. Let

$$d(x,y)=\begin{cases} 0 & \text{if } x=y,\\ \\ [|x-y|]+1 & \text{if } x\neq y. \end{cases}$$

Here $[\cdot]$ means the entire part of the number. Note that this function d measures distances with jumps of length one.

Then, given two measures (that for simplicity we will take absolutely continuous with respect to the Lebesgue measure in $\mathbb{R}^N$) f_+, f_- in $\mathbb{R}^N$, and supposing the overall condition of mass balance

$$\int_{\mathbb{R}^N} f^+\,dx=\int_{\mathbb{R}^N} f^-\,dy,$$

the Monge's problem associated to the distance d is given by: minimize

$$\int d(x,s(x))\,f_+(x)dx \tag{11.50}$$

among maps s that transport f_+ into f_-.

The dual formulation of this problem is given by

$$\max_{u\in K_\infty}\int_{\mathbb{R}^N} u(x)(f_+(x)-f_-(x))dx \tag{11.51}$$

where, as before, K_∞ is given by

$$K_\infty:=\left\{u\in L^2(\mathbb{R}^N)\ :\ |u(x)-u(y)|\le 1,\ \text{for } |x-y|\le 1\right\}.$$

We are assuming that $\operatorname{supp}(J)=\overline{B}_1(0)$ (in other case we may redefine the distance d accordingly).

Then it is easy to obtain that the solution $u_\infty(t,\cdot)$ of the limit problem $G_\infty^J(u_0)$ is a solution to the dual problem (11.51) when the involved measures are the source $f(x,t)$ and the time derivative of the solution $u_{\infty,t}(x,t)$. In fact, we have

$$G_\infty^J(v)\ge G_\infty^J(u_\infty)+\langle f-u_{\infty,t},\ v-u_\infty\rangle,\qquad \text{for all } v\in L^2(\mathbb{R}^N).$$

That is equivalent to

$$u_\infty(t,\cdot)\in K_\infty$$

and

$$0\ge\int_{\mathbb{R}^N}(f(x,t)-u_{\infty,t}(x,t))(v(x)-u_\infty(x,t))\,dx \tag{11.52}$$

for every $v\in K_\infty$. Now, we just observe that (11.52) is

$$\int_{\mathbb{R}^N}(f(x,t)-u_{\infty,t}(x,t))u_\infty(x,t)\,dx\ge\int_{\mathbb{R}^N}(f(x,t)-u_{\infty,t}(x,t))v(x)\,dx.$$

Therefore, we have that $u_\infty(t,\cdot)$ is a solution to the dual mass transport problem.

Consequently, we conclude that the mass of sand added by the source $f(t,\cdot)$ is transported (via $u(t,\cdot)$ as the transport potential) to $u_{\infty,t}(t,\cdot)$ at each time t. $\square$

This mass transport interpretation of the problem can be clearly observed looking at the previous concrete examples.

0.37. Neumann boundary conditions. Finally, let us observe that analogous results are also valid when we consider the Neumann problem in a bounded convex domain Ω, that is, when all the involved integrals are taken in Ω.

Let Ω be a convex domain in $\mathbb{R}^N$. As we have mentioned, in [**9**] we have studied the evolution problem

$$P_p^{J,\Omega}(u_0)\ \begin{cases} u_{p,t}(x,t) = \displaystyle\int_\Omega J(x-y)|u_p(y,t)-u_p(x,t)|^{p-2}(u_p(y,t)-u_p(x,t))dy + f(x,t), \\ u_p(0,x) = u_0(x), \quad \text{in } \Omega. \end{cases}$$

The associated functional being

$$G_p^{J,\Omega}(u) = \frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^p\,dy\,dx.$$

This is the nonlocal analogous to the Neumann problem for the $p-$Laplacian since in this evolution problem, we have imposed a zero flux condition across the boundary of Ω, see [**9**].

Also, let us consider the rescaled problems with J_ε, that we call $P_p^{J_\varepsilon,\Omega}(u_0)$, and the corresponding limit problems

$$P_\infty^{\varepsilon,\Omega}(u_0)\quad \begin{cases} f(t,\cdot)-u_t(t,\cdot)\in\partial G_\infty^{J,\Omega}(u(t)), & \text{a.e. } t\in]0,T[, \\ u(0,x)=u_0(x), & \text{in } \Omega. \end{cases}$$

With associated functionals

$$G_\infty^{\varepsilon,\Omega}(u) = \begin{cases} 0 & \text{if } \ |u(x)-u(y)|\le\varepsilon, \text{ for } |x-y|\le\varepsilon; \quad x,y\in\Omega, \\ +\infty & \text{in other case.} \end{cases}$$

The limit problem of the local $p-$Laplacians being

$$P_\infty^{\Omega}(u_0)\quad \begin{cases} f(t)-v_{\infty,t}\in\partial F_\infty^{\Omega}(v_\infty(t)), & \text{a.e. } t\in]0,T[, \\ v_\infty(0,x)=g(x), & \text{in } \Omega, \end{cases}$$

where the functional F_∞^Ω is defined in $L^2(\Omega)$ by

$$F_\infty^\Omega(v) = \begin{cases} 0 & \text{if } \ |\nabla v|\le 1, \\ +\infty & \text{in other case.} \end{cases}$$

In these limit problems we assume that the material is confined in a domain Ω, thus we are looking at models for sandpiles inside a container, see also [**58**].

Working as in the previous sections we can prove that

THEOREM 156. *Let Ω be a convex domain in $\mathbb{R}^N$.*

(1) *Let $T > 0$, $u_0 \in L^2(\Omega)$ such that $|u_0(x) - u_0(y)| \le 1$, for $x - y \in \Omega \cap \operatorname{supp}(J)$ and u_p the unique solution of $P_p^{J,\Omega}(u_0)$. Then, if u_∞ is the unique solution to $P_\infty^{J,\Omega}(u_0)$,*

$$\lim_{p\to\infty} \sup_{t\in[0,T]} \|u_p(t,\cdot) - u_\infty(t,\cdot)\|_{L^2(\Omega)} = 0.$$

(2) *Let $p > 1$ be and assume $J(x) \ge J(y)$ if $|x| \le |y|$. Let $T > 0$, $u_0 \in L^p(\Omega)$ and $u_{p,\varepsilon}$ the unique solution of $P_p^{J_\varepsilon,\Omega}(u_0)$. Then, if v_p is the unique solution of $P_p(u_0)$,*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{p,\varepsilon}(t,\cdot) - v_p(t,\cdot)\|_{L^p(\Omega)} = 0.$$

(3) *Let $T > 0$, $u_0 \in L^2(\Omega) \cap W^{1,\infty}(\Omega)$ such that $|\nabla u_0| \le 1$ and consider $u_{\infty,\varepsilon}$ the unique solution of $P_\infty^{\varepsilon,\Omega}(u_0)$. Then, if v_∞ is the unique solution of $P_\infty^{\Omega}(u_0)$, we have*

$$\lim_{\varepsilon\to 0} \sup_{t\in[0,T]} \|u_{\infty,\varepsilon}(t,\cdot) - v_\infty(t,\cdot)\|_{L^2(\Omega)} = 0.$$

Part *(2)* was proved in [**9**], the other statements follows just by considering, as we did before, the Mosco convergence of the associated functionals. We leave the details to the reader.

Example 6. In this case, let us also compute an explicit solution to the limit problem $P_\infty^{1,\Omega}(u_0)$ (to simplify we have considered $\varepsilon = 1$ in this example). Let us consider a recipient $\Omega = (0,l)$ with l an integer greater than 1, $u_0 = 0$ and a source given by $f(x,t) = \chi_{[0,1]}(x)$. Then the solution is given by

$$u(x,t) = t\chi_{[0,1]}(x),$$

for times $t \in [0,1]$. For $t \in [1,3]$ we get

$$u(x,t) = \begin{cases} 1 + \dfrac{t-1}{2} & x \in [0,1), \\[2mm] \dfrac{t-1}{2} & x \in [1,2), \\[2mm] 0 & x \notin [0,2). \end{cases}$$

In general we have, until the recipient is full, for any $k = 1, ..., l$ and for $t \in [t_{k-1}, t_k)$

$$u(x,t) = \begin{cases} k - 1 + \dfrac{t - t_{k-1}}{k} & x \in [0,1), \\ k - 2 + \dfrac{t - t_{k-1}}{k} & x \in [1,2) \\ \dots & \\ \dfrac{t - t_{k-1}}{k} & x \in [k-1,k), \\ 0 & x \notin [0,k) \end{cases}$$

Here $t_k = t_{k-1} + k$ is the first time when the solution reaches level k, that is $u(t_k, 0) = k$.

For times even greater, $t \geq t_l$, the solution turns out to be

$$u(x,t) = \begin{cases} l + \dfrac{t - t_l}{l} & x \in [0,1), \\ l - 1 + \dfrac{t - t_l}{l} & x \in [1,2), \\ \dots & \\ 1 + \dfrac{t - t_l}{l} & x \in [l-1,l). \end{cases}$$

Hence, when the recipient is full the solution grows with speed $1/l$ uniformly in $(0, l)$.

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