RANDOM WALKS AND THE POROUS MEDIUM EQUATION.

C. CORTÁZAR, M. ELGUETA, S. MARTINEZ AND J. D, ROSSI.

Introduction.

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, smooth function such that $\int_{\mathbb{R}^N} J(y)dy = 1$. Equations of the form

$$u_t(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)\,dy - u(x,t) \quad \text{in } \mathbb{R}^N \times [0,\infty) \tag{1.1}$$

have been widely used to model diffusion processes, see [2], [4], [10], [11], [12]. As stated in [10] if $u(x,t)$ is thought of as a density at the point x at time t and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $\int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places. On the other hand $-u(x,t) = -\int_{\mathbb{R}^N} J(x-y)u(x,t)dy$ is the rate at which they are leaving location x to travel to all other sites. This consideration, in the absence of external sources, leads immediately to the fact that the density u satisfies equation (1.1).

Throughout this note we shall assume also that J is a decreasing radial function whose support is the unit ball. Under these hypotheses we have that individuals that are at a position x are not allowed, with probability 1, to jump to a locations y such that $||x-y|| > 1$. In this fashion we like to think of this process as a, continuos in time, random walk of step's size 1.

Another equation that has been widely used to model diffusion processes is the classical heat equation

$$v_t = D\Delta v. \tag{1.2}$$

Equations (1.1) and (1.2) share several properties. For example the initial value problem is well posed for suitable initial data and a maximum principle, and hence a comparison one, holds for both of them.

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An important relation between equations (1.1) and (1.2) is that solutions of a properly rescaled version of (1.2) converge to the solution of (1.1) when the scaling parameter tends to zero. More precisely, let $\varepsilon > 0$ and consider the rescaled equation

$$\varepsilon^2 u_t(x,t) = \int_{\mathbb{R}^N} J\left(\frac{x-y}{\varepsilon}\right) \frac{u(y,t)}{\varepsilon^N} dy - u(x,t). \tag{1.3}$$

We observe that solutions to (1.3) are random walks, in the sense explained above, where the probability density of jumping from position y to position x is given by

$$J\left(\frac{x-y}{\varepsilon}\right) \frac{1}{\varepsilon^N}$$

so in this case the size of the step of the random walk is ε. On the other hand the ε^2 that appears in the left hand side of (1.3) stands for an increment in the number of steps per unit of time.

The following result is classic:

Theorem 1.1. *Let u_ε be a solution of* (1.3) *and let v be a solution of* (1.2)*. Assume that*

$$u_\varepsilon(x,0) = v(x,0) \in L^1(\mathbb{R}^N),$$

then

$$\lim_{\varepsilon \to 0} u_\varepsilon(x,t) = v(x,t)$$

in $L^1(\mathbb{R}^N \times [0,T])$ for any $T > 0$.

The aim of this note is to exhibit a similar relation between solutions of the porous medium equation, $v_t = \Delta(v^m)$, and suitable random walks. Since solutions v of the porous medium equation have the property that if $v(\cdot,0)$ is compactly supported in $\mathbb{R}^N$ then so is $v(\cdot,t)$ for all $t > 0$, see [1], we would like to approximate them by random walks that also have this property. Such a random walk was constructed in [6].

Our result.

The well known porous medium equation

$$v_t = D\Delta(v^m) \text{ in } \mathbb{R}^N \times [0,\infty), \tag{1.4}$$

where $m > 1$ and $D > 0$ is any constant, has been also widely used to model diffusion.

Solutions to (1.4) and (1.2) share several properties such as the fact that the initial value problem is well posed for suitable initial data and a comparison principle holds for both of them. However there exists quite a difference in their behavior as we explain now. While solutions to (1.2) do have the so called infinite speed of propagation of disturbances, this means that if u is a non trivial solution of (1.2) with $u(x,0) \geq 0$ then $u(x,t) > 0$ for all (x,t) with $t > 0$, solutions of (1.4) do not. See [1]. It is actually proved that if $u \geq 0$ is a solution of (1.4) such that the support of $u(\cdot,0)$ is contained in a finite interval $[-a,a]$, then there exits a non decreasing function $s : [0,\infty) \to [0,\infty)$ such that $u(\cdot,t)$ is supported inside $[-s(t),s(t)]$.

At this point it is worth to note here that solutions of (1.3) do have infinite speed of propagation of disturbances. This is no difficult to prove and is left as an exercise.

In [6] a random walk was proposed, in the case $N = 1$, which has the property that if the initial condition has compact support, then the spacial support of the solution is compact for any $t > 0$. This was achieved by considering functions of the form

$$J\left(\frac{x-y}{u(y,t)}\right)\frac{1}{u(y,t)}$$

to represent the probability of jumping from position y to position x.

In this case the random walk, continuous in time, is regulated by the equation

$$u_t(x,y) = \int_{\mathbb{R}} J\left(\frac{x-y}{u(y,t)}\right) dy - u(x,t) \quad \text{in } \mathbb{R}\times[0,\infty). \tag{1.5}$$

Existence, uniqueness and a comparison principle for (1.5) have been proved in [6]. See [3] for the case of arbitrary dimension, that is, in $\mathbb{R}^N$.

We are now in a position to describe the main result of this note. To do this let an initial condition u_0 be given and let $\varepsilon > 0$. Consider the re-scaled problem

$$u_t(x,y) = \frac{1}{\varepsilon^2}\left(\int_{\mathbb{R}} J\left(\frac{x-y}{\varepsilon u(y,t)}\right)\frac{dy}{\varepsilon} - u(x,t)\right) \quad \text{in } \mathbb{R}\times[0,\infty). \tag{1.6}$$

and let u_ε be a solution of (1.6) with initial condition $u_\varepsilon(x,0) = u_0(x)$.

We want to determine the escaled limit

$$\lim_{\varepsilon\to 0} u_\varepsilon.$$

Unfortunately at this point we have to make an extra assumption that we are sure it can be removed. The extra assumption is

$$u_\varepsilon(x,0) = w_0(x) + \delta$$

where $w_0(x)$ is a non negative compactly supported C^∞ function and $\delta > 0$.

We need first the following definition. For $\varepsilon > 0$ and a positive bounded function h we define

$$T_\varepsilon h(x) = \int_{\mathbb{R}} J\left(\frac{y-x}{\varepsilon h(y)}\right)\frac{dy}{\varepsilon} - h(x).$$

We also the need the following lemma.

Lemma 1.1. *Let $\delta > 0$ and let h be a smooth function such that $h(x) \geq \delta$. Then*

$$T_\varepsilon h(x) = C(h^3)_{xx}\varepsilon^2 + R(x,\varepsilon)$$

where $R(\cdot,\varepsilon) \in L^1(\mathbb{R})$ and $C = \int J(z)z^2 dz$. Moreover

$$||R(\cdot,\varepsilon)||_1 \leq K\frac{\varepsilon^3}{\delta^6}$$

where K depends on J, $||h||_\infty$, $||h'||_\infty$, $||h''||_\infty$, $||h'''||_\infty$, $||h'||_1$, $||h''||_1$ and $||h'''||_1$ but is independent of ε, δ and $||h||_1$.

Proof: The proof consists of expanding the function $T_\varepsilon h(x)$ in a Taylor series in ε about the point $\varepsilon = 0$.

Performing the change of variables $z = \frac{y-x}{\varepsilon}$ we get

$$T_\varepsilon h(x) = \int_{\mathbb{R}} J\left(\frac{z}{h(x+\varepsilon z)}\right) dz - h(x).$$

Hence

$$T_0 h(x) = 0. \tag{1.7}$$

Now

$$\frac{d}{d\varepsilon}T_\varepsilon h(x) = -\int_{\mathbb{R}} J'\left(\frac{z}{h(x+\varepsilon z)}\right) z^2 \frac{h'(x+\varepsilon z)}{h^2(x+\varepsilon z)} dz.$$

and using the fact that J' is an odd function we get

$$\frac{d}{d\varepsilon}T_\varepsilon h(x)|_{\varepsilon=0} = 0. \tag{1.8}$$

Differentiating with respect to ε one more time we obtain

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}T_\varepsilon h(x) = &\int_{\mathbb{R}} J''\left(\frac{z}{h(x+\varepsilon z)}\right) z^4 \left(\frac{h'(x+\varepsilon z)}{h^2(x+\varepsilon z)}\right)^2 dz \\ &- \int_{\mathbb{R}} J'\left(\frac{z}{h(x+\varepsilon z)}\right) z^3 \left(\frac{h'(x+\varepsilon z)}{h^2(x+\varepsilon z)}\right)' dz.\end{aligned}$$

Thus

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}T_\varepsilon h(x)|_{\varepsilon=0} &= \int_{\mathbb{R}} J''\left(\frac{z}{h(x)}\right) z^4 dz \left(\frac{h'(x)}{h^2(x)}\right)^2 \\ &\quad - \int_{\mathbb{R}} J'\left(\frac{z}{h(x)}\right) z^3 dz \left(\frac{h'(x)}{h^2(x)}\right)' \\ &= \int_{\mathbb{R}} J''(w)w^4 dw h(x)(h'(x))^2 - \int_{\mathbb{R}} J'(w)w^3 dw h^4(x)\left(\frac{h'(x)}{h^2(x)}\right)'.\end{aligned}$$

But

$$\int_{\mathbb{R}} J''\left(\frac{z}{h(x)}\right) z^4 dz = -4\int_{\mathbb{R}} J'(\frac{z}{h(x)})z^3 dz = 12\int_{\mathbb{R}} J(\frac{z}{h(x)})z^2 dz$$

and

$$\int_{\mathbb{R}} J'(\frac{z}{h(x)})z^3 dz = -3\int_{\mathbb{R}} J(\frac{z}{h(x)})z^2 dz.$$

So setting

$$C = \int_{\mathbb{R}} J(z)z^2 dz$$

we get

$$\begin{aligned}\frac{d^2}{d\varepsilon^2}T_\varepsilon h(x)|_{\varepsilon=0} &= C\left(12h(x)(h'(x))^2 + 3h^4(x)\left(\frac{h'(x)}{h^2(x)}\right)'\right) \\ &= C(6h(x)(h'(x))^2 + 3h^2(x)h''(x)) = C(h^3(x))''.\end{aligned}$$

Or

$$\frac{d^2}{d\varepsilon^2}T_\varepsilon h(x)|_{\varepsilon=0} = C(h^3(x))''. \tag{1.9}$$

Using (1.7), (1.8) and (1.9) we obtain, by Taylor's theorem,

$$T_\varepsilon h(x) = C(h^3)_{xx}\varepsilon^2 + R(x,\varepsilon)$$

where the remainder can be written as

$$R(x,\varepsilon) = \frac{1}{2}\int_0^\varepsilon \frac{d^3}{ds^3}T_s h(x)(\varepsilon - s)^2 ds.$$

Differentiating the formula for $\frac{d^2}{d\varepsilon^2}T_\varepsilon h(x)$ one more time with respect to ε and plugging in $\frac{d^3}{ds^3}T_s h(x)$ into the above integral the desired estimate is obtained after a straightforward calculation that uses the fact that J is compactly supported. □

We will give now a proof of our result. We recall that we are assuming that the initial condition is of the form $u_0(x) = w_0(x) + \delta$ with $\delta > 0$ and w_0 a compactly supported non negative C^∞ function.

Theorem 1.2. *Let $w_0 \in C^\infty(\mathbb{R})$ be a nonnegative compactly supported function and let $\delta > 0$. For each $\varepsilon > 0$ let u_ε be the solution* (1.6) *with initial condition $u_\varepsilon(x,0) = u_0(x) = w_0(x) + \delta$. Let v be a solution of*

$$v_t = C(v^3)_{xx}$$

with initial condition $v(x,0) = u_0(x) = w_0(x) + \delta$.

Then

$$\lim_{\varepsilon \to 0} u_\varepsilon = v$$

in $L^1(\mathbb{R} \times [0,\infty])$ for each $T > 0$.

Proof: Existence and uniqueness of the solutions u_ε can be found in [6]. By Lemma 1.1 we can write

$$v_t(x,t) = \frac{1}{\varepsilon^2}\left(\int_{\mathbb{R}} J\left(\frac{y-x}{\varepsilon v(y,t)}\right)\frac{dy}{\varepsilon} - v(x,t)\right) + \frac{1}{\varepsilon^2} R(x,t,\varepsilon)$$

where, due to our hypotheses on w_0,

$$||R(\cdot,t,\varepsilon)||_1 \le K\frac{\varepsilon^3}{\delta^6} \text{ for all } t \in [0,T]$$

with K independent of ε and δ.

Let now $\underline{w}$ y $\overline{w}$ solutions of

$$\underline{w}_t(x,t) = \frac{1}{\varepsilon^2}\left(\int_{\mathbb{R}} J\left(\frac{y-x}{\varepsilon \underline{w}(y,t)}\right)\frac{dy}{\varepsilon} - \underline{w}(x,t)\right) - \frac{1}{\varepsilon^2}|R(x,t,\varepsilon)|$$

and

$$\overline{w}_t(x,t) = \frac{1}{\varepsilon^2}\left(\int_{\mathbb{R}} J\left(\frac{y-x}{\varepsilon \overline{w}(y,t)}\right)\frac{dy}{\varepsilon} - \overline{w}(x,t)\right) + \frac{1}{\varepsilon^2}|R(x,t,\varepsilon)|$$

with initial data

$$\underline{w}(x,0) = \overline{w}(x,0) = u_0(x) + \delta,$$

respectively.

Clearly, by comparison,

$$\underline{w}(x,t) \le v(x,t) \le \overline{w}(x,t),$$

and

$$\underline{w}(x,t) \le u_\varepsilon(x,t) \le \overline{w}(x,t).$$

Now since the quantity inside the double integral is non negative, in this step we are using strongly the hypothesis that J is decreasing in

$[0,1]$, we can apply Fubini's theorem and get

$$\begin{aligned}
&\frac{d}{dt}\int_{\mathbb{R}}(\overline{w}-\underline{w})(x,t)\,dx = \\
&= \frac{1}{\varepsilon^2}\left(\int_{\mathbb{R}} J\left(\frac{y-x}{\varepsilon\overline{w}(y,t)}\right)\frac{dy}{\varepsilon} - \overline{w}(x,t)\right)dx \\
&\qquad -\int_{\mathbb{R}}\frac{1}{\varepsilon^2}\left(\int_{\mathbb{R}} J\left(\frac{y-x}{\varepsilon\underline{w}(y,t)}\right)\frac{dy}{\varepsilon} - \underline{w}(x,t)\right) \\
&+2\frac{1}{\varepsilon^2}\int_{\mathbb{R}}|R(x,t,\varepsilon)|dx \\
&= 2\frac{1}{\varepsilon^2}\int_{\mathbb{R}}|R(x,t,\varepsilon)|dx \le 2K\frac{\varepsilon}{\delta^6}.
\end{aligned}$$

This implies

$$\|\overline{w}-\underline{w}\|_1 \to 0, \qquad \text{as } \varepsilon \to 0.$$

and hence

$$u_\varepsilon \to v$$

en $L^1(\mathbb{R}\times[0,T))$ for any $T>0$ as $\varepsilon\to 0$ as we wanted to prove. $\square$

We end this note with the following remark: the extra hypothesis $u_\varepsilon(x,0)=v(x,0)=w_0(x)+\delta$ with $\delta>0$ in Theorem 1.2 has to be replaced by $u_\varepsilon(x,0)=v(x,0)=w_0(x)$ with $w_0\in L^1(\mathbb{R})$. We hope we can do this in the near future.

Once this is done an interesting question arises: Assume that w_0 is even with respect to the origin and supp $w_0=[-a,a]$. Then it is known that there exist non decreasing functions, called the free boundaries, $s:[0,\infty)\to[0,\infty)$ and $s_\varepsilon:[0,\infty)\to[0,\infty)$ such that supp $v(\cdot,t)=[-s(t),s(t)]$ and supp $u_\varepsilon(\cdot,t)=[-s_\varepsilon(t),s_\varepsilon(t)]$. A natural question to ask is whether and in which sense $\lim\limits_{\varepsilon\to 0}s_\varepsilon = s$.

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Carmen Cortázar and Manuel Elgueta, Facultad de Matemáticas, P. Universidad Catolica de Chile, Vicuna Mackenna 4860, Santiago, Chile,

Salomé Martínez, Departamento de Ingeniería Matemática, and Centro de Modelamiento Matemático, UMI 2807 CNRS-UChile, Universidad de Chile, Casilla 170 Correo 3, Santiago, Chile.

Julio D Rossi, Depto. Matematica, FCEyN,, Universidad de Buenos Aires, Ciudad Universitaria, Pab 1, (1428),, Buenos Aires, Argentina.