NONLINEAR EVOLUTION EQUATIONS THAT ARE NON-LOCAL IN SPACE AND TIME

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Abstract. We deal with a nonlocal nonlinear evolution problem of the form

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)|\overline{v}(y,s)-v(x,t)|^{p-2}(\overline{v}(y,s)-v(x,t))\,dy\,ds=0$$

for $(x,t)\in\mathbb{R}^n\times[0,\infty)$. Here $p\geq 2$, $J:\mathbb{R}^{n+1}\to\mathbb{R}$ is a nonnegative kernel, compactly supported inside the set $\{(x,t)\in\mathbb{R}^{n+1}: t\geq 0\}$ with $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)\,dx\,dt=1$ and $\overline{v}$ stands for an extension of a given initial value f, that is,

$$\overline{v}(x,t)=\begin{cases} v(x,t) & t\geq 0,\\ f(x,t) & t<0.\end{cases}$$

For this problem we prove existence and uniqueness of a solution. In addition, we show that the solutions approximate viscosity solutions to the local nonlinear PDE $\|\nabla u\|^{p-2}u_t=\Delta_p u$ when the kernel is rescaled in a suitable way.

1. Introduction

Our main goal in this paper is the study of nonlinear evolution problems that are nonlocal both in space and time. Let $F(z)=|z|^{p-2}z$ be a power type nonlinearity and let $J:\mathbb{R}^{n+1}\to\mathbb{R}$, a nonnegative, continuous kernel, compactly supported in the set $\{(x,t)\in\mathbb{R}^{n+1}: t\geq 0\}$ with $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)dxdt=1$. We fix an initial condition $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$. Our aim is to look for solutions to the nonlocal nonlinear evolution problem

$$(P(J,f))\qquad \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{v}(y,s)-v(x,t))\,dyds=0$$

for $(x,t)\in\mathbb{R}^n\times[0,\infty)$ where we denoted by $\overline{v}$ the extension by f for $t<0$ of a function v defined for $t\geq 0$, that is,

$$\overline{v}(x,t)=\begin{cases} v(x,t) & t\geq 0,\\ f(x,t) & t<0.\end{cases}$$

This paper can be viewed as a natural continuation of [1] where the linear case $p=2$ was considered. Notice that here a solution u verifies a nonlinear mean value formula given by $P(J,f)$.

Our first result deals with existence and uniqueness of solutions. We denote by $\overline{\mathcal{C}}$ the set of uniform continuous functions, and $L^\infty(f)$ stands for the set of bounded functions with norm less or equal than $\|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$.

Theorem 1. *Let $J:\mathbb{R}^{n+1}\to\mathbb{R}$ be, nonnegative, continuous and compactly supported in the set $\{(x,t)\in\mathbb{R}^{n+1}: t\geq 0\}$, with $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)dxdt=1$. Let $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$. Then, there exists a unique $u\in\overline{\mathcal{C}}\cap L^\infty(f)(\mathbb{R}^n\times[0,\infty))$ that solves $P(J,f)$.*

We will use the notation u for a solution with initial datum f and we will say that u solves the problem $P(J,f)$. Note that here we assumed that the kernel is nonnegative and integrable (singular kernels are out of the scope of this paper). This fact together with the choice of $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$, makes the space $\overline{\mathcal{C}}\cap L^\infty(f)(\mathbb{R}^n\times[0,\infty))$ a natural choice to look for solutions (remark that the integral that appears in $P(J,f)$ is finite under these conditions). Notice that there is a regularizing effect, for $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$ we obtain a uniformly continuous solution, $u\in\overline{\mathcal{C}}\cap L^\infty(f)(\mathbb{R}^n\times[0,\infty))$. This is due to the fact that we assumed continuity of the kernel J.

We have two different proofs of this existence and uniqueness result. The first one is simpler. We just prove first the result for a class of kernels that are compactly supported in the set $\left\{(x,t)\in\mathbb{R}^{n+1}:\delta\leq t\leq\delta+\gamma\right\}$, where δ and γ are positive numbers, (this allows us to easily obtain existence and uniqueness of solutions in the strip $t\in[0,\delta)$ and then in $t\in[\delta,2\delta)$, etc.). After that we obtain the result for a general kernel by approximating it with kernels in the previously mentioned class. The second proof is more involved technically and is based on a fixed point argument (we include this proof here since we believe that it has independent interest). This fixed point strategy was used for the

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linear case in [1], but there the arguments are much simpler due to linearity. The main difficulty here comes from the fact that the derivative of the nonlinearity $F(z) = |z|^{p-2}z$ vanishes at $z = 0$.

Concerning properties of solutions we have the following result:

Theorem 2. *Under the hypothesis of Theorem 1 the following properties hold:*

(1) [Comparison principle] *Let f and g in $L^\infty(\mathbb{R}^n \times (-\infty, 0))$ with $f \le g$. If u and v are solutions of $P(J, f)$ and $P(J, g)$ respectively, then $u \le v$.*
(2) [Continuity with respect to the initial condition] *Let f and g in $L^\infty(\mathbb{R}^n \times (-\infty, 0))$, u and v solutions of $P(J, f)$ and $P(J, g)$, then $\|u - v\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le \|f - g\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$.*
(3) [Lipschitz continuity in space] *Let $f \in L^\infty(\mathbb{R}^n\times(-\infty, 0))$ be such that $f(\cdot, t)$ is Lipschitz with Lipschitz constant K for every $t \in (-\infty, 0)$ and let u be the solution to $P(J, f)$. Then $u(\cdot, t)$ is Lipschitz with the same constant K for every $t \in [0, \infty)$.*
(4) [Radial symmetry] *Let $f \in L^\infty(\mathbb{R}^n \times (-\infty, 0))$, a nonnegative function such that $f(\cdot, t)$ is radial for every $t \in (-\infty, 0)$. Assume that $J(\cdot, t)$ is a radial function. Then, if u solves $P(J, f)$, $u(\cdot, t)$ is radial for every $t \ge 0$. If we furthermore assume that $J(\cdot, t)$ and $f(\cdot, s)$ are radially decreasing for every $t \in \mathbb{R}$ and $s \in (-\infty, 0)$, then $u(\cdot, t)$ is radially decreasing for every $t \ge 0$.*
(5) [Scaling invariance] *Let $r > 0$ and $J_r(x, t) = \frac{1}{r^{n+2}} J\left(\frac{x}{r}, \frac{t}{r^2}\right)$. If u is the solution to $P(J_r, f)$ and v solves $P(J, r^{n+2}f(rx, r^2t))$, then*
$$u(x, t) = r^{-n-2}v\left(\frac{x}{r}, \frac{t}{r^2}\right).$$

The last point (the scaling invariance) in the previous result suggests that we have to rescale the kernel according to $J_r(x, t) = \frac{1}{r^{n+2}} J\left(\frac{x}{r}, \frac{t}{r^2}\right)$. We will show that then the corresponding solutions of $P(J_r, f)$ converge (along subsequences) to a viscosity solution to the nonlinear degenerate parabolic PDE
$$C(J)\|\nabla u\|^{p-2}\frac{\partial u}{\partial t} = \triangle_p u.$$
Here $C(J)$ is a constant that depends on the kernel J and $\triangle_p u = \operatorname{div}(\|\nabla u\|^{p-2}\nabla u)$ is the well known $p-$Laplacian operator. We refer to [10] for definitions and general properties of viscosity solutions (note that the spatial operator in this equation is not in divergence form and therefore viscosity theory gives the natural notion of weak solutions). This equation recently deserved attention in the literature due to the presence of challenging regularity problems (note that we run into troubles when ∇u vanishes). Holder regularity for the gradient of solutions was recently proved in [17], see also [6, 7, 8, 15]. This equation also appears in connection with probability (game theory). It is a natural limit of value functions of Tug-of-War games with noise when one also considers the number of plays, see [19, 20, 23]. We just mention here that mean value properties and PDEs are closely related. The fundamental works of Doob, Hunt, Kakutani, Kolmogorov and many others have shown the close connection between the classical linear potential theory and the corresponding probability theory. The idea behind the classical interplay is that harmonic functions and martingales share a common origin in mean value properties. This connection turns out to be useful in the nonlinear theory as well. In fact, our next result shows that solutions to the mean value formula $P(J_r, f)$ with $r \to 0$ and solutions to the local PDE $\|\nabla u\|^{p-2}u_t = \Delta_p u$ are related.

To state our convergence result we remark that the problem $P(J, f)$ can be considered with an initial condition $f \in L^\infty(\mathbb{R}^n)$. In this case we say that u solve the problem $P(J, f)$ if u solves $P(J, \overline{f})$ where $\overline{f}(x, t) = f(x)$ for all $t \in (-\infty, 0)$.

Our last result is the following:

Theorem 3. *Let $J : \mathbb{R}^{n+1} \to \mathbb{R}$, $J \ge 0$, compactly supported in the set $\{(x, t) \in \mathbb{R}^{n+1} : \delta \le t \le \delta + \gamma\}$ where δ and γ are positive constants, $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x, t)dxdt = 1$, and assume that $J(\cdot, t)$ is radially symmetric. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function, such that f is C^2 with bounded derivatives. Then, the solutions to $P(J_r, f)$ converge along subsequence uniformly as $r \to 0$ on compact sets to a viscosity solution to the problem*
$$\begin{cases} A\,\|\nabla u\|^{p-2}\,\dfrac{\partial u}{\partial t} = B\left(\|\nabla u\|^{p-2}\triangle u + (p-2)\,\|\nabla u\|^{p-4}\,\nabla u D^2 u \nabla u\right) & (x, t) \in \mathbb{R}^n \times [0, \infty) \\ u(x, 0) = f(x) & x \in \mathbb{R}^n \end{cases}$$
where A, B are two constants that depend on J that are given by
$$A = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x, w)|z_1|^{p-2}w\,dz\,dw \qquad \text{and} \qquad B = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}} J(z, w)|z_1|^p\,dz\,dw.$$

For the linear case $p = 2$ the analogous result (approximating the linear heat equation, $u_t = C(J)\Delta u$) was proved in [1] using a completely different technique.

Concerning approximations of local problems by nonlocal ones, we mention that there are nonlocal (but only in space) nonlinear problems that approximate the classical $p-$Laplacian evolution equation, $u_t = \mathrm{div}(\|\nabla u\|^{p-2}\nabla u)$. We refer to the book [5] and [2, 3, 4, 16]. For equations with a singular kernel in space we quote [11, 21, 22, 24] (this equation is known as the fractional $p-$Laplacian in the literature).

The paper is organized as follows: in Section 2 we deal with existence and uniqueness of solutions and collect some properties of them, proving Theorems 1 and 2; while in Section 3 we prove our limit result when rescaling the kernel, Theorem 3.

2. Existence and uniqueness of solutions to the nonlocal evolution problem.

2.1. **Existence and uniqueness for some particular kernels** J**.** First, we consider a kernel $J : \mathbb{R}^{n+1} \to \mathbb{R}$, $J \geq 0$, compactly supported in the set $\{(x,t) \in \mathbb{R}^{n+1} : \delta \leq t \leq \delta + \gamma\}$ where δ and γ are positive constants, and $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)\,dx\,dt = 1$. The fact that the support of the kernel is δ away from zero in time allows us to prove existence of solutions to our nonlocal problem defined in $\mathbb{R}^n \times [0,\delta)$. Next, we find a solution to our problem in the whole $\mathbb{R}^n \times [0,\infty)$ iterating the previous construction.

Let us start with the following simple lemma:

Lemma 4. *Let $F(z) = |z|^{p-2}z$, a and b real numbers with $b < a$. Let $G_{a,b} : \mathbb{R} \to \mathbb{R}$ be given by*

$$G_{a,b}(z) := F(z-b) - F(z-a).$$

Then, it holds that

$$G_{a,b}(z) \geq G\left(\frac{a+b}{2}\right) = \frac{1}{2^{p-2}}(a-b)^{p-1}$$

for every $z \in \mathbb{R}$.

Proof. When we differentiate $G_{a,b}(z)$ we obtain

$$G'_{a,b}(z) = F'(z-b) - F'(z-a) = (p-1)\left(|z-b|^{p-2} - |z-a|^{p-2}\right).$$

Hence $G'(z) = 0$ if and only if $z = (a+b)/2$. Moreover, on the left of $(a+b)/2$, $G'_{a,b} < 0$, and on the right of $(a+b)/2$, $G'_{a,b} > 0$, then we get that at $z = (a+b)/2$ there is a minimum and hence

$$\begin{aligned} G_{a,b}\left(\frac{a+b}{2}\right) &= F\left(\frac{a+b}{2} - b\right) - F\left(\frac{a+b}{2} - a\right) = F\left(\frac{a-b}{2}\right) - F\left(\frac{-a+b}{2}\right) \\ &= \left|\frac{a-b}{2}\right|^{p-2}\frac{a-b}{2} - \left|\frac{-a+b}{2}\right|^{p-2}\frac{-a+b}{2} = \frac{1}{2^{p-2}}(a-b)^{p-1}. \end{aligned}$$

□

Now we consider the space

$$\overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [0,\delta)),$$

where $\overline{\mathcal{C}}$ is the set of uniformly continuous functions and $L^\infty(f)$ stands for the set of bounded functions with norm less or equal than $\|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$. The space $\overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [0,\delta))$ is a Banach space equipped with the L^∞-norm.

Theorem 5. *Let $J : \mathbb{R}^{n+1} \to \mathbb{R}$, $J \geq 0$, compactly supported inside the set $\{(x,t) \in \mathbb{R}^{n+1} : \delta \leq t \leq \delta + \gamma\}$ and uniformly continuous verifying $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)dxdt = 1$. Let $f \in L^\infty(\mathbb{R}^n \times (-\infty,0))$. Then, there exists a unique function $u_1 \in \overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [0,\delta))$ such that u_1 solves $P(J,f)$ in the strip $\mathbb{R}^n \times [0,\delta)$.*

Proof. Let $(x,t) \in \mathbb{R}^n \times [0,\delta)$, we define $u_1(x,t)$ as the unique value such that

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(f(y,s) - u_1(x,t))dyds = 0.$$

The integral above is well defined since $J(x-y,t-s)$, as function of (y,s), has support in the set $\{t-(\delta+\gamma) \leq s \leq t-\delta\} \subset \{t-(\delta+\gamma) \leq s \leq 0\}$. The value $u_1(x,t)$ is unique because $J \geq 0$ and F is an increasing function. Then, $u_1(x,t)$ satisfies

$$0 = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u_1}(y,s) - u_1(x,t))\,dy\,ds$$

where

$$\overline{u_1}(y,s) = \begin{cases} f(y,s) & s<0 \\ u_1(y,s) & s\in[0,\delta). \end{cases}$$

Therefore, u_1 is the unique function that resolves $P(J,f)$.

Let see that $u_1 \in L^\infty(\mathbb{R}^n \times [0,\delta))$. To this end, fix $(x,t)\in\mathbb{R}^n\times[0,\delta)$. If $u_1(x,t) < -\|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$ then

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(f(y,s)-u_1(x,t))\,dyds > 0$$

and if $u_1(x,t) > \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$ then

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(f(y,s)-u_1(x,t))\,dyds < 0,$$

in any of the two cases we have a contradiction with the definition of $u_1(x,t)$. Hence, we conclude that

$$|u_1(x,t)| \le \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$$

for all $(x,t)\in\mathbb{R}^n\times[0,\delta)$.

Now we have to show that $u_1\in\overline{\mathcal{C}}(\mathbb{R}^n\times[0,\delta))$. To this end, let (x,t) and $(x+h,t+k)$ in $\mathbb{R}^n\times[0,\delta)$, and we assume without loss of generality that $u_1(x+h,t+k)<u_1(x,t)$, then we have

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)F(f(y,s)-u_1(x+h,t+k))\,dy\,ds \\
&\quad - \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(f(y,s)-u_1(x,t))\,dy\,ds \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)\left[F(f(y,s)-u_1(x+h,t+k))-F(f(y,s)-u_1(x,t))\right]dy\,ds \\
&\quad + \iint_{\mathbb{R}^n\times\mathbb{R}} \left[J(x+h-y,t+k-s)-J(x-y,t-s)\right]F(f(y,s)-u_1(x,t))\,dy\,ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)\left[F(f(y,s)-u_1(x+h,t+k))-F(f(y,s)-u_1(x,t))\right]dy\,ds \\
&\le \left|\iint_{\mathbb{R}^n\times\mathbb{R}} \left[J(x+h-y,t+k-s)-J(x-y,t-s)\right]F(f(y,s)-u_1(x,t))dy\,ds\right| \\
&\le \iint_{\mathbb{R}^n\times\mathbb{R}} |J(x+h-y,t+k-s)-J(x-y,t-s)|\,|F(f(y,s)-u_1(x,t))|\,dy\,ds.
\end{aligned}$$

Using Lemma 4 we get

$$\begin{aligned}
&\frac{1}{2^{p-2}}(u_1(x,t)-u_1(x+h,t+k))^{p-1} \\
&\le \iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)\left[F(f(y,s)-u_1(x+h,t+k))-F(f(y,s)-u_1(x,t))\right]dy\,ds \\
&\le \iint_{\mathbb{R}^n\times\mathbb{R}} |J(x+h-y,t+k-s)-J(x-y,t-s)|\,|F(f(y,s)-u_1(x,t))|\,dy\,ds \\
&\le \omega_1(h,k)F(2\|f\|_{L^\infty})
\end{aligned}$$

where ω_1 is the modulus of continuity of J.

Hence, we conclude that $u_1\in\overline{\mathcal{C}}\cap L^\infty(f)(\mathbb{R}^n\times[0,\delta))$, with a modulus of continuity of u_1 depending only on the continuity of J, f and on the nonlinearity F (but independent of δ). □

Now, we just iterate the previous result to obtain existence and uniqueness of a solution with $t\in[0,\infty)$.

Theorem 6. *Let $J:\mathbb{R}^{n+1}\to\mathbb{R}$, $J\ge 0$, compactly supported inside $\{(x,t)\in\mathbb{R}^{n+1} : \delta\le t\le\delta+\gamma\}$ and uniformly continuous with $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)dxdt = 1$. Let $f\in L^\infty(\mathbb{R}^n\times(-\infty,0))$. Then, there exists a unique $u\in\overline{\mathcal{C}}\cap L^\infty(f)(\mathbb{R}^n\times[0,\infty))$ that solves $P(J,f)$.*

Proof. We construct the solution of $P(J,f)$ in time strips. First, we build a solution in $\mathbb{R}^n \times [0,\delta)$ and next we extend it in sets of the form $\mathbb{R}^n \times [(m-1)\delta, m\delta)$ with $m \in \mathbb{N}$. Let $u_1 \in \overline{\mathcal{C}} \cap L^\infty(f)$ be the unique solution of $P(J,f)$ in $\mathbb{R}^n \times [0,\delta)$. Now, let $u_2 \in \overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [\delta, 2\delta))$ the unique function that solves $P(J,\overline{u}_1)$, where $\overline{u}_1 = f\mathcal{X}_{t<0} + u_1\mathcal{X}_{t\in[0,\delta)}$, on the strip $\mathbb{R}^n \times [\delta, 2\delta)$. Inductively, we can take $u_m \in \overline{\mathcal{C}} \cap L^\infty(f)$ that solves $P(J,\overline{u}_{m-1})$, where $\overline{u}_{m-1} = f\mathcal{X}_{t<0} + \sum_{i=1}^{m-1} u_i \mathcal{X}_{[(i-1)\delta, i\delta)}$, on the strip $[(m-1)\delta, m\delta)$. Then $u(x,t) = \sum_{m=1}^\infty u_m(x,t)\mathcal{X}_{[(m-1)\delta,m\delta)}(t)$ solves $P(J,f)$ in $\mathbb{R}^n \times [0,\infty)$. Also $\|u\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$. The uniquines of u follows from the fact that it is unique on every strip. With the same argument we have used in Theorem 5 we obtain that $u \in \overline{\mathcal{C}}$. □

2.2. **Properties of the solutions.** Now we show some properties of the solutions that will be used in what follows. In the next lemmas will be prove the results only for solutions on the strip $\mathbb{R}^n \times [0,\delta)$. Later we will see that we can extend the results to solutions on $\mathbb{R}^n \times [0,\infty)$. We still assume that $J : \mathbb{R}^{n+1} \to \mathbb{R}$, $J \ge 0$, is uniformly continuous and compactly supported inside the set $\left\{(x,t) \in \mathbb{R}^{n+1} : \delta \le t \le \delta+\gamma\right\}$ with $\iint_{\mathbb{R}^n\times\mathbb{R}} J(x,t)\,dx\,dt = 1$ (up to this point we have proved existence and uniqueness only for this particular class of kernels).

Lemma 7. *If u solves $P(J,f)$ and $k \in \mathbb{R}$, then ku solves $P(J,\epsilon f)$ and $u+k$ solves $P(J,f+k)$.*

Proof. Immediate. □

Lemma 8. *Let f and g in $L^\infty(\mathbb{R}^n \times (-\infty,0))$ with $f \le g$, with u and v solutions of $P(J,f)$ and $P(J,g)$ respectively, then $u \le v$.*

Proof. Let $(x,t) \in \mathbb{R}^n \times [0,\delta)$, then
$$0 = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F\left(f(y,s)-u(x,t)\right)dy\,ds \le \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F\left(g(y,s)-u(x,t)\right)dy\,ds.$$
So, from the definition of $u(x,t)$ and $v(x,t)$, we have that $u(x,t) \le v(x,t)$. □

Lemma 9 (Continuity with respect to the initial datum)**.** *Let f and g in $L^\infty(\mathbb{R}^n \times (-\infty,0))$, u and v solutions with initial conditions f and g respectively. Then $\|u-v\|_{L^\infty(\mathbb{R}^n\times[0,\delta))} \le \|f-g\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$.*

Proof. Let $\epsilon = \|f-g\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$. By hypothesis we have
$$-\epsilon + f \le g \le f + \epsilon$$
and then by Lemmas 7 and 8 we obtain that
$$-\epsilon + u \le v \le u + \epsilon.$$
Therefore
$$\|u-v\|_{L^\infty(\mathbb{R}^n\times[0,\delta))} \le \epsilon,$$
as we wanted to prove. □

Now, let us show that for initial data f with a prescribed Lipschitz smoothness, this Lipschitz smoothness is preserved.

Lemma 10. *Let $f \in L^\infty(\mathbb{R}^n\times(-\infty,0))$ be such that $f(\cdot,t)$ is Lipschitz with Lipschitz constant K for every $t \in (-\infty,0)$ and u is the solution to $P(J,f)$ then $u(\cdot,t)$ is Lipschitz with the same constant K for every $t \in [0,\delta)$.*

Proof. We fix $t \in [0,\delta)$ and x_1 and x_2 in $\mathbb{R}^n$, then we have
$$\begin{aligned}
0 &= \iint_{\mathbb{R}^n\times\mathbb{R}} J(x_1-y,t-s)F(f(y,s)-u(x_1,t))\,dy\,ds \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,t-s)F(f(x_1-z,s)-u(x_1,t))\,dy\,ds \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,t-s)F(f(x_1-z,s)-f(x_2-z,s)-(u(x_1,t)-f(x_2-z,s)))\,dy\,ds \\
&\le \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,t-s)F(K|x_1-x_2|-(u(x_1,t)-f(x_2-z,s)))\,dy\,ds \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}} J(x_2-y,t-s)F(f(y,s)-(u(x_1,t)-K|x_1-x_2|))\,dy\,ds.
\end{aligned}$$
Hence, $u(x_1,t)-K|x_1-x_2| \le u(x_2,t)$, and then we get $u(x_1,t)-u(x_2,t) \le K|x_1-x_2|$. In the same way we can show that $u(x_2,t)-u(x_1,t) \le K|x_1-x_2|$ and we conclude that $u(\cdot,t)$ is Lipschitz with constant K for every $t \in [0,\delta)$. □

Lemma 11. *Let $f \in L^\infty(\mathbb{R}^n \times (-\infty, 0))$, a nonnegative function such that $f(\cdot, t)$ is radial for every $t \in (-\infty, 0)$. Suppose also that $J(\cdot, t)$ is radial for every $t \in \mathbb{R}$. Then, if u solves $P(J, f)$, $u(\cdot, t)$ is radial for every $t \in [0, \delta)$. Furthermore, if we assume that $J(\cdot, t)$ and $f(\cdot, s)$ are radially decreasing for every $t \in \mathbb{R}$ and $s \in (-\infty, 0)$, then $u(\cdot, t)$ is radially decreasing for every $t \in [0, \delta)$.*

Proof. First, we show that u is radial. Consider u restricted to the strip $\mathbb{R}^n \times [0, \delta)$. For (x, t) in that set we have

$$0 = \iint_{\mathbb{R}^n \times \mathbb{R}} J(x - y, t - s) F\left(f(y, s) - u(x, t)\right) dy\, ds.$$

Hence, if ρ is a rotation, we get

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^n \times \mathbb{R}} J(\rho x - y, t - s) F\left(f(y, s) - u(\rho x, t)\right) dy\, ds \\
&= \iint_{\mathbb{R}^n \times \mathbb{R}} J(\rho(x - \rho^{-1} y), t - s) F\left(f(\rho^{-1} y, s) - u(\rho x, t)\right) dy\, ds \\
&= \iint_{\mathbb{R}^n \times \mathbb{R}} J(x - \rho^{-1} y, t - s) F\left(f(\rho^{-1} y, s) - u(\rho x, t)\right) dy\, ds \\
&= \iint_{\mathbb{R}^n \times \mathbb{R}} J(x - y, t - s) F\left(f(y, s) - u(\rho x, t)\right) dy\, ds.
\end{aligned}$$

Therefore, $u(\rho x, t)$ solves $P(J, f)$ in $\mathbb{R}^n \times [0, \delta)$, and, by uniqueness, we get $u(\rho x, t) = u(x, t)$ for $t \in [0, \delta)$.

Assume now that $J(\cdot, t)$ and $f(\cdot, s)$ are radially decreasing for every $t \in \mathbb{R}$ and $s \in (-\infty, 0)$, and let us show that $u(\cdot, t)$ is radially decreasing for every $t \in [0, \delta)$. Fix x_1 and x_2 in $\mathbb{R}^n$ with $x_1 = \mu x_2$, $\mu > 1$, then our aim is to show that $u(x_2, t) \le u(x_1, t)$. Let us first show that

$$\iint_{\mathbb{R}^n \times \mathbb{R}} J(x_1 - y, t - s) F\left(f(y, s) - L\right) dy\, ds - \iint_{\mathbb{R}^n \times \mathbb{R}} J(x_2 - y, t - s) F\left(f(y, s) - L\right) dy\, ds \ge 0 \tag{1}$$

for every $L \in \mathbb{R}$. As $x_1 = \mu x_2$ with $\mu > 1$, there exists a rotation ρ such that $\rho x_1 = z_1 e_1$ and $\rho x_2 = z_2 e_1$, where $e_1 = (1, 0, \dots, 0)$ and $0 \le z_1 < z_2$. Then, as $J(\cdot, t - s)$ and $f(\cdot, s)$ are radial for every $t \in [0, \delta)$ and $s < 0$, the inequality (1) is equivalent to

$$\iint_{\mathbb{R}^n \times \mathbb{R}} J(z_1 e_1 - y, t - s) F\left(f(y, s) - L\right) dy\, ds - \iint_{\mathbb{R}^n \times \mathbb{R}} J(z_2 e_1 - y, t - s) F\left(f(y, s) - L\right) dy\, ds \ge 0$$

and to

$$\begin{aligned}
&\int_{s \le 0} \int_{\overline{y} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}} J(z_1 - y_1, -\overline{y}, t - s) F\left(f(y_1, \overline{y}, s) - L\right) dy_1\, d\overline{y}\, ds \\
&\quad - \int_{s \le 0} \int_{\overline{y} \in \mathbb{R}^{n-1}} \int_{\mathbb{R}} J(z_2 - y_1, -\overline{y}, t - s) F\left(f(y_1, \overline{y}, s) - L\right) dy_1\, d\overline{y}\, ds \\
&= \int_{s \le 0} \int_{\overline{y} \in \mathbb{R}^{n-1}} \left[\int_{\mathbb{R}} J(z_1 - y_1, -\overline{y}, t - s) F\left(f(y_1, \overline{y}, s) - L\right) dy_1 \right. \\
&\quad \left. - \int_{\mathbb{R}} J(z_2 - y_1, -\overline{y}, t - s) F\left(f(y_1, \overline{y}, s) - L\right) dy_1 \right] d\overline{y}\, ds \ge 0.
\end{aligned}$$

Since $J(\cdot, -\overline{y}, t - s)$ and $f(\cdot, \overline{y}, s)$ are radially decreasing for every $\overline{y} \in \mathbb{R}^{n-1}$, $t \in [0, \delta)$ and $s < 0$, we only have to show that if $J : \mathbb{R} \to \mathbb{R}$ (supported in $[-R, R]$ for some $R > 0$) and $f : \mathbb{R} \to \mathbb{R}$ are nonnegative and radially decreasing functions, then

$$\int_{\mathbb{R}} J(z_1 - y) F(f(y) - L) dy - \int_{\mathbb{R}} J(z_2 - y) F(f(y) - L) dy \ge 0 \tag{2}$$

for $0 \le z_1 < z_2$ and every $L \in \mathbb{R}$. To prove this fact we observe that

$$\begin{aligned}
&\int_{\mathbb{R}} J(z_1 - y)F(f(y) - L)dy - \int_{\mathbb{R}} J(z_2 - y)F(f(y) - L)dy \\
&= \int_{\mathbb{R}} J(y) \left[F(f(z_1 - y) - L) - F(f(z_2 - y) - L)\right] dy \\
&= \int_{\mathbb{R}} J\left(\frac{z_1 + z_2}{2} + z\right) \left[F\left(f\left(z_1 - \frac{z_1 + z_2}{2} - z\right) - L\right) - F\left(f\left(z_2 - \frac{z_1 + z_2}{2} - z\right) - L\right)\right] dz \\
&= \int_{\mathbb{R}} J\left(\frac{z_1 + z_2}{2} + z\right) \left[F\left(f\left(\frac{z_1 - z_2}{2} - z\right) - L\right) - F\left(f\left(\frac{z_2 - z_1}{2} - z\right) - L\right)\right] dz \\
&= \int_{\mathbb{R}} J\left(\frac{z_1 + z_2}{2} + z\right) \left[F\left(f\left(\frac{z_2 - z_1}{2} + z\right) - L\right) - F\left(f\left(\frac{z_2 - z_1}{2} - z\right) - L\right)\right] dz \\
&= \int_{\mathbb{R}} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz
\end{aligned}$$

where $g(z) = F\left(f\left(\frac{z_2 - z_1}{2} + z\right) - L\right) - F\left(f\left(\frac{z_2 - z_1}{2} - z\right) - L\right)$, that is a odd function in z with $g(z) \ge 0$ for $z < 0$, since $f\left(\frac{z_2 - z_1}{2} + z\right) \ge f\left(\frac{z_2 - z_1}{2} - z\right)$ for $z < 0$ (recall that f is radially decreasing and that $\frac{z_2 - z_1}{2} > 0$). Then, let us show that

$$\int_{\mathbb{R}} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz \ge 0. \tag{3}$$

As $g(z)$ is odd we have

$$\begin{aligned}
&\int_{\mathbb{R}} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz \\
&= \int_{-R - \frac{z_1 + z_2}{2}}^{-\left|R - \frac{z_1 + z_2}{2}\right|} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz + \int_{-\left|R - \frac{z_1 + z_2}{2}\right|}^{0} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz \\
&\quad + \int_0^{\left|R - \frac{z_1 + z_2}{2}\right|} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz \\
&= \int_{-R - \frac{z_1 + z_2}{2}}^{-\left|R - \frac{z_1 + z_2}{2}\right|} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz - \int_0^{\left|R - \frac{z_1 + z_2}{2}\right|} J\left(\frac{z_1 + z_2}{2} - z\right) g(z)dz \\
&\quad + \int_0^{\left|R - \frac{z_1 + z_2}{2}\right|} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz \\
&= \int_{-R - \frac{z_1 + z_2}{2}}^{-\left|R - \frac{z_1 + z_2}{2}\right|} J\left(\frac{z_1 + z_2}{2} + z\right) g(z)dz + \int_0^{\left|R - \frac{z_1 + z_2}{2}\right|} \left(-J\left(\frac{z_1 + z_2}{2} - z\right) + J\left(\frac{z_1 + z_2}{2} + z\right)\right) g(z)dz.
\end{aligned}$$

The first integral is nonnegative since J and g are greater or equal than zero in the set $[-R - \frac{z_1 + z_2}{2}, -\left|R - \frac{z_1 + z_2}{2}\right|]$. The second integral is also nonnegative since $-J\left(\frac{z_1 + z_2}{2} - z\right) + J\left(\frac{z_1 + z_2}{2} + z\right) \le 0$ for $z \ge 0$ (since J is radially decreasing and $\frac{z_1 + z_2}{2} > 0$) and $g(z) \le 0$ for $z \ge 0$). Then we conclude that (3), and hence (1) hold, for $t \in [0, \delta)$. If we take $L = u(x_1, t)$ in (1), we obtain

$$\iint_{\mathbb{R}^n \times \mathbb{R}} J(x_2 - y, t - s)F\left(f(y, s) - u(x_1, t)\right) dy\, ds \le 0$$

and this inequality implies that $u(x_2, t) \le u(x_1, t)$ for $t \in [0, \delta)$. □

In the next lemma we assume that the solutions are defined on $\mathbb{R}^n \times [0, \infty)$.

Lemma 12. (Scaling invariance) *Let $r > 0$ and $J_r(x, t) = \frac{1}{r^{n+2}} J\left(\frac{x}{r}, \frac{t}{r^2}\right)$. If u is the solution to $P(J_r, f)$ and v solves $P(J, r^{n+2} f(rx, r^2 t))$, then*

$$u(x, t) = r^{-n-2} v\left(\frac{x}{r}, \frac{t}{r^2}\right).$$

Proof. As v solves $P(J, r^{n+2} f(rx, r^2 t))$ then we have

$$\iint_{\mathbb{R}^n \times \mathbb{R}} J(x - y, t - s)F(\overline{v}(y, s) - v(x, t))\, dy\, ds = 0$$

for $(x,t) \in \mathbb{R}^n \times [0,\infty)$ where

$$\overline{v}(y,s) = \begin{cases} r^{n+2} f(ry, r^2 s) & t<0, \\ v(y,s) & t \geq 0. \end{cases}$$

Therefore,

$$0 = \iint_{\mathbb{R}^n \times \mathbb{R}} J\left(\frac{x}{r} - y, \frac{t}{r^2} - s\right) F\left(\overline{v}(y,s) - v\left(\frac{x}{r}, \frac{t}{r^2}\right)\right) dy\, ds$$

and then

$$\begin{aligned} 0 &= (r^{-n-2})^{p-1} \iint_{\mathbb{R}^n \times \mathbb{R}} J\left(\frac{x}{r} - y, \frac{t}{r^2} - s\right) F\left(\overline{v}(y,s) - v\left(\frac{x}{r}, \frac{t}{r^2}\right)\right) dy\, ds \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}} J\left(\frac{x}{r} - y, \frac{t}{r^2} - s\right) F\left(r^{-n-2}\overline{v}(y,s) - r^{-n-2} v\left(\frac{x}{r}, \frac{t}{r^2}\right)\right) dy\, ds \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}} \frac{1}{r^{n+2}} J\left(\frac{x-y}{r}, \frac{t-s}{r^2}\right) F\left(r^{-n-2}\overline{v}\left(\frac{y}{r}, \frac{s}{r^2}\right) - r^{-n-2} v\left(\frac{x}{r}, \frac{t}{r^2}\right)\right) dy\, ds \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}} J_r(x-y, t-s) F\left(r^{-n-2}\overline{v}\left(\frac{y}{r}, \frac{s}{r^2}\right) - r^{-n-2} v\left(\frac{x}{r}, \frac{t}{r^2}\right)\right) dy\, ds \end{aligned}$$

where

$$r^{-n-2}\overline{v}\left(\frac{y}{r}, \frac{s}{r^2}\right) = \begin{cases} f(y,s) & t<0 \\ r^{-n-2} v\left(\frac{y}{r}, \frac{s}{r^2}\right) & t \geq 0. \end{cases}$$

We have obtained that $r^{-n-2} v\left(\frac{x}{r}, \frac{t}{r^2}\right)$ solves $P(J,f)$, and then, by uniqueness, it coincides with $u(x,t)$. $\square$

2.3. **Existence and uniqueness for general kernels** J. Our aim is to get rid of the hypothesis that the kernel is compactly supported in the set $\left\{(x,t) \in \mathbb{R}^{n+1} : \delta \leq t \leq \delta + \gamma\right\}$.

Theorem 13. *Let $J : \mathbb{R}^{n+1} \to \mathbb{R}$, $J \geq 0$, uniformly continuous and compactly supported inside the set $\{(x,t) \in \mathbb{R}^{n+1} : t \geq 0\}$, with $\iint J(x,t)dxdt = 1$. Let $f \in L^\infty(\mathbb{R}^n \times (-\infty, 0))$. Let $(h_m, k_m) \in \mathbb{R}^n \times (0,\infty)$ be such that $(h_m, k_m) \to (0,0)$ as $m \to \infty$. Consider $J_m(x,t) := J(x - h_m, t - k_m)$. Then, if u_m are the solutions of $P(J_m, f)$, there exists a subsequence u_{m_l} that converges uniformly, in every compact subset K of $\mathbb{R}^n \times [0,\infty)$, to a limit $u \in \overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [0,\infty))$ that solves $P(J,f)$.*

Proof. As u_m are uniformly continuity functions and their modulus of continuity is the same for every m, the functions u_m are equicontinuous. Also the functions u_m are uniformly bounded. Hence, by Arzela-Ascoli's theorem and using a diagonal procedure, there is a subsequence u_{m_l} that converges uniformly to a function $u \in \overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [0,\infty))$ in compact subsets K in $\mathbb{R}^n \times [0,\infty)$. Let us show that this function u is a solution to $P(J,f)$. As u_{m_l} solves $P(J_{m_l}, f)$ we have that

$$\begin{aligned} 0 &= \iint_{\mathbb{R}^n \times \mathbb{R}} J_{m_l}(x-y, t-s) F(\overline{u_{m_l}}(y,s) - u_{m_l}(x,t))\, dy\, ds \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}} J(x - h_{m_l} - y, t - k_{m_l} - s) F(\overline{u_{m_l}}(y,s) - u_{m_l}(x,t))\, dy\, ds. \end{aligned}$$

Since the functions $u_{m_l}(y,s) - u_{m_l}(x,t)$ are uniformly bounded we get that

$$|J(x - h_{m_l} - y, t - k_{m_l} - s) F\left(\overline{u_{m_l}}(y,s) - u_{m_l}(x,t)\right)| \leq J(x - h_{m_l} - y, t - k_{m_l} - s)\, F(2\|f\|_{L^\infty}).$$

So, from the dominated convergence theorem, we obtain

$$\begin{aligned} 0 &= \lim_{l \to \infty} \iint_{\mathbb{R}^n \times \mathbb{R}} J(x - h_{m_l} - y, t - k_{m_l} - s) F(\overline{u_{m_l}}(y,s) - u_{m_l}(x,t))\, dy\, ds \\ &= \iint_{\mathbb{R}^n \times \mathbb{R}} J(x-y, t-s) F(\overline{u}(y,s) - u(x,t))\, dy\, ds, \end{aligned}$$

and we conclude that u solves $P(J,f)$. $\square$

Theorem 14. *Let $J : \mathbb{R}^{n+1} \to \mathbb{R}$, $J \geq 0$, uniformly continuous and compactly supported in the set $\{(x,t) \in \mathbb{R}^{n+1} : t \geq 0\}$, and $\iint J(x,t)dxdt = 1$. Let $f \in L^\infty(\mathbb{R}^n \times (-\infty, 0))$. Then there exists a unique function $u \in \overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [0,\infty))$ that solves $P(J,f)$.*

Proof. By Theorem 13 there exists a function $u \in \overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [0,\infty))$ that solves $P(J,f)$.

Let see that such solution u is unique. Let $\alpha = \sup\left\{\gamma : \iint_{t\le\gamma} J(x,t)dxdt < 1\right\}$. It is enough to show that u is unique in $\mathbb{R}^n \times [0,\alpha]$. Suppose, arguing by contradiction, that there exists $u, v \in \overline{\mathcal{C}} \cap L^\infty(f)(\mathbb{R}^n \times [0,\alpha])$ that solves $P(J,f)$ and $u \neq v$. Then, there exist $\nu > 0$ and a point $(x,t) \in \mathbb{R}^n \times [0,\alpha]$ such that $|u(x,t) - v(x,t)| > \nu$. Let

$$t_0 = \sup\left\{t \in [0,\alpha] : \|u - v\|_{L^\infty(\mathbb{R}^n \times [0,t])} \le \nu\right\}.$$

This value is well defined since $u(\cdot,0) = v(\cdot,0)$. Note that $t_0 < \alpha$. Then, there exists a secuence $\{t_m\}_{m\in\mathbb{N}}$ with $t_m \downarrow t_0$ and there are $x_m \in \mathbb{R}^n$ such that $|u(x_m,t_m) - v(x_m,t_m)| > \nu$. We can suppose whitout loss of generality that $u(x_m,t_m) + \nu < v(x_m,t_m)$. Then

$$\begin{aligned} 0 &= \iint_{\mathbb{R}^n\times\mathbb{R}} J(x_m - y, t_m - s)F(\overline{v}(y,s) - v(x_m,t_m))dyds - \iint_{\mathbb{R}^n\times\mathbb{R}} J(x_m - y, t_m - s)F(\overline{u}(y,s) - u(x_m,t_m))dy\,ds \\ &= I_m + II_m + III_m, \end{aligned}$$

where

$$\begin{aligned} I_m &= \iint_{t_m-\alpha\le s\le 0} J(x_m - y, t_m - s)\left(F(f(y,s) - v(x_m,t_m)) - F(f(y,s) - u(x_m,t_m))\right)dy\,ds, \\ II_m &= \iint_{0\le s\le t_0} J(x_m - y, t_m - s)\left(F(v(y,s) - v(x_m,t_m)) - F(u(y,s) - u(x_m,t_m))\right)dy\,ds, \\ III_m &= \iint_{t_0\le s\le t_m} J(x_m - y, t_m - s)\left(F(v(y,s) - v(x_m,t_m)) - F(u(y,s) - u(x_m,t_m))\right)dy\,ds. \end{aligned}$$

From Lemma 4 we have that

$$\begin{aligned} -I_m &\ge \frac{1}{2^{p-2}}(v(x_m,t_m) - u(x_m,t_m))^{p-1} \iint_{t_m-\alpha\le s\le 0} J(x_m - y, t_m - s)dy\,ds \\ &= \frac{1}{2^{p-2}}\nu^{p-1} \iint_{t_m\le w\le\alpha} J(z,w)dz\,dw. \end{aligned}$$

Now, from the definition of t_0 we have that

$$\begin{aligned} II_m &\le \iint_{0\le s\le t_0} J(x_m - y, t_m - s)\left(F(u(y,s) + \nu - v(x_m,t_m)) - F(u(y,s) - u(x_m,t_m))\right)dy\,ds \\ &\le \iint_{0\le s\le t_0} J(x_m - y, t_m - s)\left(F(u(y,s) - u(x_m,t_m)) - F(u(y,s) - u(x_m,t_m))\right)dy\,ds = 0. \end{aligned}$$

Then,

$$\begin{aligned} &\frac{1}{2^{p-2}}\nu^{p-1} \iint_{t_m\le s\le\alpha} J(y,s)dy\,ds \\ &\le \iint_{t_0\le s\le t_m} J(x_m - y, t_m - s)\left(F(v(y,s) - v(x_m,t_m)) - F(u(y,s) - u(x_m,t_m))\right)dy\,ds \\ &\le 2F(2\|f\|_{L^\infty}) \iint_{t_0\le s\le t_m} J(x_m - y, t_m - s)dy\,ds \\ &= 2F(2\|f\|_{L^\infty}) \iint_{0\le s\le t_m-t_0} J(y,s)dy\,ds. \end{aligned}$$

The first term of the chain of inequalities goes to

$$\frac{\nu^{p-1}}{2^{p-2}} \iint_{t_0\le s\le\alpha} J(y,s)\,dy\,ds > 0,$$

when m goes to $+\infty$, and the last term vanishes when m goes to $+\infty$. So we obtained a contradiction. □

Remark 1. The properties of the solutions to $P(J,f)$ that we proved in Lemmas 7 to 12 can be extended for solutions to $P(J,f)$ with a general kernel J. We only have to argue as follows: first (considering kernels compactly supported in $\left\{(x,t) \in \mathbb{R}^{n+1} : \delta \le t \le \delta + \gamma\right\}$) we extend the results to $t > 0$ iterating the results that we proved in the strip $0 \le t < \delta$ and finally we extend the results for a general kernel by an approximation procedure.

2.4. **Existence and uniqueness via Banach's fixed point theorem.** Now, we want to provide an alternative proof for existence and uniqueness of solutions. Here, as fro the previous proof, we assume that the kernel J is continuous. Let

$$\alpha = \sup\left\{\gamma : \iint_{t\leq\gamma} J(x,t)dxdt < 1\right\},$$

and let us consider the space

$$B_1 := \overline{C} \cap L^\infty(f)(\mathbb{R}^n \times [0,R]),$$

with $0 < R < \alpha$, as before, $\overline{C}$ denote the set of uniform continuous funcions and $L^\infty(f)$ stands for the set of bounded functions with norm less or equal than $\|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$. The space $\overline{C} \cap L^\infty(f)(\mathbb{R}^n \times [0,R])$ is a Banach space with the L^∞-norm.

Let us show that our nonlocal problem has a unique solution in this space. To see this fact we define an operator $T : B_1 \to B_1$ and, using a fixed point argument (Banach's fixed point theorem), we will prove existence and uniqueness of solutions. This task requires extra conditions on the datum f that will be removed latter using a delicate approximation argument. Let us define $T : B_1 \to B_1$. Given $u \in B_1$ we set

$$Tu(x,t) = \lambda_u(x,t)$$

where $\lambda_u(x,t)$ is the unique value that satisfies the equation

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s) - \lambda_u(x,t))\,dy\,ds = 0.$$

Note that $\lambda_u(x,t)$ is unique due to the fact that F is increasing and $J \geq 0$.

Lemma 15. *The operator T is well defined from B_1 to B_1.*

Proof. The argument is similar to the one used in the proof of Theorem 5. Let $u \in B_1$ and $(x,t) \in \mathbb{R}^n \times [0,R]$. If $\lambda_u(x,t) < -\|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$ then

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s) - \lambda_u(x,t))\,dy\,ds > 0$$

and if $\lambda_u(x,t) > \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$ then

$$\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s) - \lambda_u(x,t))\,dy\,ds < 0,$$

in any of the two cases we have a contradiction with the definition of T. Hence, we conclude that

$$|\lambda_u(x,t)| \leq \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}.$$

Now we have to show that $\lambda_u(x,t) \in \overline{C}(\mathbb{R}^n \times [0,R])$. To this end, let (x,t) and $(x+h,t+k)$ in $\mathbb{R}^n \times [0,R]$, and we assume without loss of generality that $\lambda_u(x+h,t+k) < \lambda_u(x,t)$, then we have

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)F(\overline{u}(y,s) - \lambda_u(x+h,t+k))\,dy\,ds \\
&\quad - \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s) - \lambda_u(x,t))\,dy\,ds \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)\left[F(\overline{u}(y,s) - \lambda_u(x+h,t+k)) - F(\overline{u}(y,s) - \lambda_u(x,t))\right]dy\,ds \\
&\quad + \iint_{\mathbb{R}^n\times\mathbb{R}} \left[J(x+h-y,t+k-s) - J(x-y,t-s)\right]F(\overline{u}(y,s) - \lambda_u(x,t))\,dy\,ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\left|-\iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)\left[F(\overline{u}(y,s) - \lambda_u(x+h,t+k)) - F(\overline{u}(y,s) - \lambda_u(x,t))\right]dy\,ds\right| \\
&= \left|\iint_{\mathbb{R}^n\times\mathbb{R}} \left[J(x+h-y,t+k-s) - J(x-y,t-s)\right]F(\overline{u}(y,s) - \lambda_u(x,t))dy\,ds\right|
\end{aligned}$$

and then

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)\left[F(\overline{u}(y,s)-\lambda_u(x+h,t+k))-F(\overline{u}(y,s)-\lambda_u(x,t))\right]dy\,ds\\ &\quad\le \iint_{\mathbb{R}^n\times\mathbb{R}} \left[J(x+h-y,t+k-s)-J(x-y,t-s)\right]\left|F(\overline{u}(y,s)-\lambda_u(x,t))\right|dy\,ds.\end{aligned}$$

Using Lemma 4 we get

$$\begin{aligned}&\frac{1}{2^{p-2}}(\lambda_u(x,t)-\lambda_u(x+h,t+k))^{p-1}\\ &\le \iint_{\mathbb{R}^n\times\mathbb{R}} J(x+h-y,t+k-s)\left[F(\overline{u}(y,s)-\lambda_u(x+h,t+k))-F(\overline{u}(y,s)-\lambda_u(x,t))\right]dy\,ds\\ &\le \iint_{\mathbb{R}^n\times\mathbb{R}} \left[J(x+h-y,t+k-s)-J(x-y,t-s)\right]\left|F(\overline{u}(y,s)-\lambda_u(x,t))\right|dy\,ds\\ &\le \omega_1(h,k)F(2\left\|f\right\|_{L^\infty}).\end{aligned}$$

where ω_1 depends on J. Therefore, we have that $\lambda_u\in\overline{C}$. □

Now we prove an auxiliary result.

Lemma 16. *Let $\{u_m\}_{m\in\mathbb{N}}$ be a sequence of functions in B_1, then, if $u_m\to u$ uniformly in $\mathbb{R}^n\times[0,R]$ then $Tu_m\to Tu$ uniformly in $\mathbb{R}^n\times[0,R]$.*

Proof. Fix $\epsilon>0$, as $u_m\to u$ uniformly then there exists $N_\epsilon\in\mathbb{N}$ such that

$$-\epsilon\le\overline{u_m}(y,s)-\overline{u}(y,s)\le\epsilon$$

for every $(y,s)\in\mathbb{R}^n\times(-\infty,R]$ and $m\ge N_\epsilon$. Given $(x,t)\in\mathbb{R}^n\times[0,R]$, we have

$$\overline{u}(y,s)-\epsilon-\lambda_{u_m}(x,t)\le\overline{u_m}(y,s)-\lambda_{u_m}(x,t)\le\overline{u}(y,s)+\epsilon-\lambda_{u_m}(x,t).$$

As F is increasing and J is nonnegative we obtain

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s)-(\epsilon+\lambda_{u_m}(x,t)))\,dy\,ds\\ &\le\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u_m}(y,s)-\lambda_{u_m}(x,t))\,dy\,ds\\ &\le\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s)-(-\epsilon+\lambda_{u_m}(x,t)))dy\,ds.\end{aligned}$$

Hence,

$$\begin{aligned}&\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s)-(\epsilon+\lambda_{u_m}(x,t)))dy\,ds\le 0\\ &\qquad\le\iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s)-(\lambda_{u_m}(x,t)-\epsilon))dy\,ds\end{aligned}$$

and we obtain that

$$\lambda_{u_m}(x,t)-\epsilon\le\lambda_u(x,t)\le\epsilon+\lambda_{u_m}(x,t)$$

holds for every $(x,t)\in\mathbb{R}^n\times[0,R]$. In this way we have proved that

$$-\epsilon\le\lambda_u(x,t)-\lambda_{u_m}(x,t)\le\epsilon$$

for every $(x,t)\in\mathbb{R}^n\times[0,R]$ and $m\ge N_\epsilon$. This implies that $\lambda_{u_m}\to\lambda_u$ uniformly in $\mathbb{R}^n\times[0,R]$. □

Lemma 17. *If u and v belong to $L^\infty(\mathbb{R}^n\times[0,R])$, then*

$$\left\|Tu-Tv\right\|_{L^\infty(\mathbb{R}^n\times[0,R])}\le\left\|u-v\right\|_{L^\infty(\mathbb{R}^n\times[0,R])}.$$

Proof. The proof follows from the same arguments used in the proof of the previous lemma taking u_m as v and $\epsilon=\left\|u-v\right\|_{L^\infty(\mathbb{R}^n\times[0,R])}$. □

Now we are ready to show that T is contractive. To this end we need to assume some extra conditions on the datum f that will be removed latter by an approximation argument.

Lemma 18 (T is contractive)**.** *If f verifies that for every $(x,t)\in\mathbb{R}^n\times[0,R]$*

i) *there exists* $a(x,t)$ *and* $b(x,t)$ *such that*
$$\inf_{(y,s)\in C(x,t)} f(y,s) < a(x,t) < b(x,t) < \sup_{(y,s)\in C(x,t)} f(y,s)$$
where $C(x,t) := \{(y,s) \in \mathbb{R}^{n+1} : (y,s) \in supp J(x-\cdot, t-\cdot) \text{ and } s<0\}$;

ii) *there are sets* $A(x,t)$ *and* $B(x,t)$ *with* $A(x,t) \subset \{(y,s) \in C(x,t) : f(y,s) \le a(x,t)\}$ *and* $B(x,t) \subset \{(y,s) \in C(x,t) : f(y,s) \ge b(x,t)\}$ *such that* $\iint_{A(x,t)} J(x-y,t-s)dyds > d_1 > 0$ *and* $\iint_{B(x,t)} J(x-y,t-s)dyds > d_2 > 0$;

iii) *there exists* $\varrho > 0$ *such that* $\varrho(x,t) = \frac{b(x,t)-a(x,t)}{4} \ge \varrho$.

Then, the operator $T : B_1 \to B_1$ *is a strict contraction.*

Proof. Fix u and v in B_1, and let $(x,t) \in \mathbb{R}^n \times [0,R]$. Assume, without loss of generality that $\lambda_v(x,t) < \lambda_u(x,t)$, then
$$0 = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s)-\lambda_u(x,t))dy\,ds - \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{v}(y,s)-\lambda_v(x,t))\,dy\,ds,$$
equivalently,
$$\begin{aligned}0 = &\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F(f(y,s)-\lambda_u(x,t))dy\,ds + \iint_{0\le s\le t} J(x-y,t-s)F(u(y,s)-\lambda_u(x,t))dy\,ds\\ &-\left(\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F(f(y,s)-\lambda_v(x,t))dy\,ds + \iint_{0\le s\le t} J(x-y,t-s)F(v(y,s)-\lambda_v(x,t))dy\,ds\right).\end{aligned}$$
Hence, we have
$$\begin{aligned}&\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F(f(y,s)-\lambda_v(x,t))dy\,ds - \iint_{t-\alpha\le s\le 0} J(x-y,t-s)F(f(y,s)-\lambda_u(x,t))\,dy\,ds\\ &= \iint_{0\le s\le t} J(x-y,t-s)F(u(y,s)-\lambda_u(x,t))dy\,ds - \iint_{0\le s\le t} J(x-y,t-s)F(v(y,s)-\lambda_v(x,t))\,dy\,ds.\end{aligned}$$
From the mean value theorem we conclude that
$$\begin{aligned}&\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F'(C(f(y,s),\lambda_u(x,t),\lambda_v(x,t)))[\lambda_u(x,t)-\lambda_v(x,t)]dy\,ds\\ &= \iint_{0\le s\le t} J(x-y,t-s)F'(C(u(y,s),v(y,s),\lambda_u(x,t),\lambda_v(x,t)))[u(y,s)-v(y,s)-\lambda_u(x,t)+\lambda_v(x,t)]dy\,ds\end{aligned}$$
where the mean value $C(f(y,s),\lambda_u(x,t),\lambda_v(x,t))$ is between $f(y,s)-\lambda_u(x,t)$ and $f(y,s)-\lambda_v(x,t)$, and the mean value $|C(u(y,s),v(y,s),\lambda_u(x,t),\lambda_v(x,t))|$ is bounded by $2\|f\|_{L^\infty}$. To simplify the notation we call
$$C_1 := C(f(y,s),\lambda_u(x,t),\lambda_v(x,t)) \qquad \text{and} \qquad C_2 := C(u(y,s),v(y,s),\lambda_u(x,t),\lambda_v(x,t)).$$
From the previous identity we get
$$\begin{aligned}&[\lambda_u(x,t)-\lambda_v(x,t)]\\ &\le \frac{\displaystyle\iint_{0\le s\le t} J(x-y,t-s)F'(C_2)dyds}{\displaystyle\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F'(C_1)dyds + \iint_{0\le s\le t} J(x-y,t-s)F'(C_2)dyds}\|u-v\|_{L^\infty(\mathbb{R}^n\times[0,R])}\\ &\le \frac{\displaystyle F'(2\|f\|_{L^\infty})\iint_{0\le s\le R} J(y,s)dyds}{\displaystyle\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F'(C_1)dyds + F'(2\|f\|_{L^\infty})\iint_{0\le s\le R} J(y,s)dyds}\|u-v\|_{L^\infty(\mathbb{R}^n\times[0,R])}.\end{aligned}\tag{4}$$
To obtain the second inequality we have used the facts that F' is positive and increasing, that J is nonnegative and that the function $h(x) = \frac{x}{c+x}$ is non-decreasing on $[0,\infty)$ for every $c \ge 0$.

Now we observe that, if we bound $|C_1|$ from below by a positive number (recall that this constant is between $f(y,s)-\lambda_u(x,t)$ and $f(y,s)-\lambda_v(x,t)$), using that the previous inequalities hold for every $(x,t) \in \mathbb{R}^n \times [0,R]$, we obtain that
$$\|\lambda_u-\lambda_v\|_{L^\infty(\mathbb{R}^n\times[0,R])} \le \tau\|u-v\|_{L^\infty(\mathbb{R}^n\times[0,R])}$$
with $0<\tau<1$.

Then our goal is to find a lower bound for $|C_1|$. First, let us assume that $\|u-v\|_{L^\infty(\mathbb{R}^n\times[0,R])} \le \varrho$. From Lemma 17, we have that $\lambda_u(x,t)-\lambda_v(x,t) \le \varrho$. Now we analyze three different cases.

1) Assume that $\lambda_u(x,t) \le \frac{a(x,t)+b(x,t)}{2}$ and $\lambda_v(x,t) \le \frac{a(x,t)+b(x,t)}{2}$. Then, for every $(y,s) \in B(x,t)$, we have

$$f(y,s) - \lambda_u(x,t) \ge b(x,t) - \frac{a(x,t)+b(x,t)}{2} = \frac{b(x,t)-a(x,t)}{2} > \frac{b(x,t)-a(x,t)}{4} \ge \varrho$$
$$f(y,s) - \lambda_v(x,t) \ge b(x,t) - \frac{a(x,t)+b(x,t)}{2} = \frac{b(x,t)-a(x,t)}{2} > \frac{b(x,t)-a(x,t)}{4} \ge \varrho.$$

2) Assume that $\lambda_u(x,t) \ge \frac{a(x,t)+b(x,t)}{2}$ and $\lambda_v(x,t) \ge \frac{a(x,t)+b(x,t)}{2}$. Then, for every $(y,s) \in A(x,t)$, it holds that

$$\lambda_u(x,t) - f(y,s) \ge \frac{a(x,t)+b(x,t)}{2} - a(x,t) = \frac{b(x,t)-a(x,t)}{2} > \frac{b(x,t)-a(x,t)}{4} \ge \varrho$$
$$\lambda_v(x,t) - f(y,s) \ge \frac{a(x,t)+b(x,t)}{2} - a(x,t) = \frac{b(x,t)-a(x,t)}{2} > \frac{b(x,t)-a(x,t)}{4} \ge \varrho.$$

3) Assume that $\lambda_u(x,t) > \frac{a(x,t)+b(x,t)}{2} > \lambda_v(x,t)$. As we have $\lambda_u(x,t) - \lambda_v(x,t) \le \varrho \le \frac{b(x,t)-a(x,t)}{4}$ for every $(x,t) \in \mathbb{R}^n \times [0,R]$, we get

$$\lambda_v(x,t) + \frac{b(x,t)-a(x,t)}{4} > \frac{a(x,t)+b(x,t)}{2}.$$

Hence,

$$\lambda_v(x,t) > \frac{3}{4}a(x,t) + \frac{1}{4}b(x,t)$$

and then, for $(y,s) \in A(x,t)$ we obtain

$$\lambda_v(x,t) - f(y,s) > \frac{3}{4}a(x,t) + \frac{1}{4}b(x,t) - a(x,t) = \frac{b(x,t)-a(x,t)}{4} \ge \varrho$$
$$\lambda_u(x,t) - f(y,s) > \frac{3}{4}a(x,t) + \frac{1}{4}b(x,t) - a(x,t) = \frac{b(x,t)-a(x,t)}{4} \ge \varrho.$$

In any of the three cases we have that $|C_1| \ge \varrho$ in $A(x,t)$ or in $B(x,t)$, therefore, we get

$$\begin{aligned}\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F'(C_1)dyds &\ge F'(\varrho)\iint_{A(x,t)} J(x-y,t-s)dyds \\ &= F'(\varrho)\iint_{A(x,t)} J(y_1,s_1)dy_1ds_1 \\ &\ge F'(\varrho)d_1 > 0\end{aligned}$$

or

$$\begin{aligned}\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F'(C_1)dyds &\ge F'(\varrho)\iint_{B(x,t)} J(x-y,t-s)dyds \\ &= F'(\varrho)\iint_{B(x,t)} J(y_1,s_1)dy_1ds_1 \\ &\ge F'(\varrho)d_2 > 0.\end{aligned}$$

Hence,

$$\iint_{t-\alpha\le s\le 0} J(x-y,t-s)F'(C_1)dyds \ge F'(\varrho)\min\{d_1,d_2\} > 0.$$

Then, from (4), we get that

$$\lambda_u(x,t) - \lambda_v(x,t) \le \tau \|u-v\|_{L^\infty(\mathbb{R}^n\times[0,R])}$$

for some $\tau < 1$. As the previous inequality holds for every $(x,t) \in \mathbb{R}^n \times [0,R]$, we conclude that

$$\|\lambda_u - \lambda_v\|_{L^\infty(\mathbb{R}^n\times[0,R])} \le \tau \|u-v\|_{L^\infty(\mathbb{R}^n\times[0,R])}$$

with $0 < \tau < 1$.

Assume now that $\|u-v\|_{L^\infty(\mathbb{R}^n\times[0,R])} \ge \varrho$. In this case we argue by contradiction. Assume that there is no $0 < \tau < 1$ such that

$$\|Tu - Tv\| \le \tau \|u-v\|$$

for every u and v with $\|u-v\|_{L^\infty(\mathbb{R}^n\times[0,R])} \ge \varrho$. Then, there exist $\tau_l > 0$, u_l, v_l in B_1, with $\|u_l-v_l\|_{L^\infty(\mathbb{R}^n\times[0,R])} \ge \varrho$, $\tau_l \to 1$ as $l \to \infty$, and such that

$$\|Tu_l - Tv_l\| \ge \tau_l \|u_l - v_l\|.$$

As the functions u_l are uniformly bounded and are equicontinuous, by Arzela-Ascoli's theorem, there exists a subsequence $\{u_{l_i}\}_{i\in\mathbb{N}}$ that converges uniformly to a function $u \in B_1$. By the same argument we can extract of $\{u_{l_i}\}_{i\in\mathbb{N}}$ a converging subsequence v_{l_i} to some $v \in B_1$. Then, there exist $\{u_m\}_{m\in\mathbb{N}}$ and $\{v_m\}_{m\in\mathbb{N}}$ with $u_m \to u$, $v_m \to v$ uniformly and such that

$$\|Tu_m - Tv_m\| \geq \tau_m \|u_m - v_m\|$$

with $\tau_m \to 1$. If we take the limit as $m \to \infty$, by Lemma 17 and Lemma 16, we get

$$\|u - v\| \leq \|Tu - Tv\| \leq \|u - v\|.$$

Take $\nu := \|u - v\|_{L^\infty(\mathbb{R}^n\times[0,R])}$, so we have $\nu \geq \varrho$. From the previous inequality, we can suppose without loss of generality that there exist points $(x_k, t_k) \in \mathbb{R}^n \times [0, R]$ such that $\lambda_v(x_k, t_k) - \lambda_u(x_k, t_k) \to \nu$ as $k \to \infty$. Hence, for $(y, s) \in \mathbb{R}^n \times [0, R]$, we have

$$-\nu + v(y, s) - \lambda_u(x_k, t_k) \leq u(y, s) - \lambda_u(x_k, t_k).$$

Then, using again that F is increasing and that $J \geq 0$, we get

$$\begin{aligned}
&\iint_{t_k-\alpha\leq s\leq 0} J(x_k - y, t_k - s)F(f(y, s) - \lambda_u(x_k, t_k))\,dy\,ds\\
&\qquad + \iint_{0\leq s\leq t_k} J(x_k - y, t_k - s)F(-\nu + v(y, s) - \lambda_u(x_k, t_k))\,dy\,ds\\
&\leq \iint_{t_k-\alpha\leq s\leq 0} J(x_k - y, t_k - s)F(f(y, s) - \lambda_u(x_k, t_k))\,dy\,ds\\
&\qquad + \iint_{0\leq s\leq t_k} J(x_k - y, t_k - s)F(u(y, s) - \lambda_u(x_k, t_k))\,dy\,ds.
\end{aligned}$$

Now we observe that the second term in the previus inequalitie vanishes, and hence we have

$$\begin{aligned}
&\iint_{t_k-\alpha\leq s\leq 0} J(x_k - y, t_k - s)F(f(y, s) - \lambda_u(x_k, t_k))\,dy\,ds\\
&\qquad + \iint_{0\leq s\leq t_k} J(x_k - y, t_k - s)F(v(y, s) - (\lambda_u(x_k, t_k) + \nu))\,dy\,ds \leq 0.
\end{aligned} \tag{5}$$

The first term is equal to $I + II + III$, where

$$\begin{aligned}
I = &\iint_{t_k-\alpha\leq s\leq 0} J(x_k - y, t_k - s)F(f(y, s) - \lambda_u(x_k, t_k))\,dy\,ds\\
&- \iint_{t_k-\alpha\leq s\leq 0} J(x_k - y, t_k - s)F(f(y, s) - \lambda_v(x_k, t_k))\,dy\,ds\\
II = &\iint_{t_k-\alpha\leq s\leq 0} J(x_k - y, t_k - s)F(f(y, s) - \lambda_v(x_k, t_k))\,dy\,ds\\
&+ \iint_{0\leq s\leq t_k} J(x_k - y, t_k - s)F(v(y, s) - \lambda_v(x_k, t_k))\,dy\,ds\\
III = &- \iint_{0\leq s\leq t_k} J(x_k - y, t_k - s)F(v(y, s) - \lambda_v(x_k, t_k))\,dy\,ds\\
&+ \iint_{0\leq s\leq t_k} J(x_k - y, t_k - s)F(v(y, s) - (\lambda_u(x_k, t_k) + \nu))\,dy\,ds.
\end{aligned}$$

From the definition of $\lambda_v(x, t)$ we have that $II = 0$. Now, if we show that I tends to $c\nu^{p-1}$ (with $c > 0$) as $k \to \infty$ and III goes to zero as $k \to \infty$, we obtain that $c\nu^{p-1} \leq 0$, a contradiction.

Therefore we have to prove the two previously mentioned limits.

First, by Lemma 4 we have

$$\begin{aligned}
I = &\iint_{t_k-\alpha\le s\le 0} J(x_k-y,t_k-s)F(f(y,s)-\lambda_u(x_k,t_k))dyds \\
&\quad -\iint_{t_k-\alpha\le s\le 0} J(x_k-y,t_k-s)F(f(y,s)-\lambda_v(x_k,t_k))dyds \\
= &\iint_{t_k-\alpha\le s\le 0} J(x_k-y,t_k-s)\left[F(f(y,s)-\lambda_u(x_k,t_k))-F(f(y,s)-\lambda_v(x_k,t_k))\right]dyds \\
\ge &\iint_{t_k-\alpha\le s\le 0} J(x_k-y,t_k-s)\frac{1}{2^{p-2}}(\lambda_v(x_k,t_k)-\lambda_u(x_k,t_k))^{p-1}dyds \\
\ge &\iint_{R\le s_1\le \alpha} J(y_1,s_1)dy_1ds_1\frac{1}{2^{p-2}}(\lambda_v(x_k,t_k)-\lambda_u(x_k,t_k))^{p-1}\underset{k\to\infty}{\to} C\nu^{p-1}.
\end{aligned}$$

Finally, as F is locally Lipschitz, we get

$$\begin{aligned}
|III| = &\left|-\iint_{0\le s\le t_k} J(x_k-y,t_k-s)F(v(y,s)-\lambda_v(x_k,t_k))dyds\right. \\
&\quad \left.+\iint_{0\le s\le t_k} J(x_k-y,t_k-s)F(v(y,s)-(\lambda_u(x_k,t_k)+\nu))dyds\right| \\
\le &\; C\iint_{0\le s\le t_k} J(x_k-y,t_k-s)dyds\,|-\lambda_v(x_k,t_k)+\lambda_u(x_k,t_k)+\nu|\underset{k\to\infty}{\to} 0
\end{aligned}$$

This ends the proof. □

As an immediate corollary we have existence and uniqueness of local in time solutions to our evolution problem when f satisfies the hypotheses of Theorem 18.

Theorem 19. *If f verifies the hypothesis of Theorem 18 then for all $0<R<\alpha$ there exists a unique function $u_R\in B_1=\overline{C}\cap L^\infty(f)(\mathbb{R}^n\times[0,R])$ that solves $P(J,f)$ in $\mathbb{R}^n\times[0,R]$. Even more, if $0<R_1<R_2<\alpha$ then $u_{R_1}(x,t)=u_{R_2}(x,t)$ for all $(x,t)\in\mathbb{R}^n\times[0,R_1]$.*

Proof. Since B_1 is a Banach space the existence of u_R is a direct consequence of the Banach fixed point theorem using Lemmas 15 and 18.

Now, if u_{R_2} solves $P(J,f)$ in $\mathbb{R}^n\times[0,R_2]$ we have that for every $(x,t)\in\mathbb{R}^n\times[0,R_1]$

$$\iint J(x-y,t-s)F(\overline{u_{R_2}}(y,s)-u_{R_2}(x,t))dyds=0.$$

If $(x,t)\in\mathbb{R}^{n+1}\times[0,R_1]$, since $suppJ\subset\{(x,t)\in\mathbb{R}^{n+1}:t\ge 0\}$, then the values (y,s) involves in the above integral lives in $\mathbb{R}^n\times[0,R_1]$. So u_{R_2} solves $P(J,f)$ in $\mathbb{R}^n\times[0,R_1]$, then by the uniqueness of solutions $u_{R_1}\equiv u_{R_2}$ in $\mathbb{R}^n\times[0,R_1]$. □

Now our aim is to extend this existence and uniqueness result to uniformly continuous initial data f in the same spaces. That is, we look for a solution to $P(J,f)$ in B_1 defined in $\mathbb{R}^n\times[0,R]$. Before proving this extension we need to introduce some notations. For $m\in\mathbb{N}$ we denote by V_m the set of dyadic cubes of the form $I_{m,k}=2^{-m}[0,1)^n+2^{-m}k$ where $k\in\mathbb{Z}^n$. We call $V_{m,2l}$ the set of cubes $I_{m,k}$ that belong to V_m and are such that $\sum_{i=1}^n k_i$ is an even number, where $k=(k_1,k_2,\dots,k_n)$. Analogously, we call $V_{m,2l+1}$ the set of cubes $I_{m,k}$ that belong to V_m and such that $\sum_{i=1}^n k_i$ is odd. We also denote W_m the set of intervals $\widetilde{I}_{m,k}=2^{-m}(-1,0]-2^{-m}k$ where $k\in\mathbb{N}$, note that $\widetilde{I}_{m,k}\subset(-\infty,0]$.

Lemma 20. *Assume that $f:\mathbb{R}^n\times(-\infty,0)\to\mathbb{R}$ is uniformly continuous, and J is a continuous function, then there exist functions f_m such that*

1) *f_m satisfy the conditions i) ii) and iii) assumed in Lemma 18.*
2) *$f_m\ge f$ (or $f_m\le f$) for every $m\in\mathbb{N}$.*
3) *$f_m\to f$ uniformly as $m\to\infty$.*

Proof. First, we prove 1). As J is continuous there exist an open set $\Omega\subset suppJ\cap\{t\ge R\}$ such that $J\ge\theta>0$ in Ω. Hence, for N large enough we have that, for every $m\ge N$ and $(x,t)\in\mathbb{R}^n\times[0,R]$ there exist $I_{m,2l}(x,t)\in V_{m,2l}$, $I_{m,2l+1}(x,t)\in V_{m,2l+1}$ with $\overline{I}_{m,2l}(x,t)\cap\overline{I}_{m,2l+1}(x,t)\ne\emptyset$, and $\widetilde{I}_{m,k}(x,t)\in W_m$ such that

$$\begin{aligned}
I_{m,2l}(x,t)\times\widetilde{I}_{m,k}(x,t)&\subset(x,t)-\Omega\subset suppJ(x-y,t-s)\cap\{s\le 0\}\\
I_{m,2l+1}(x,t)\times\widetilde{I}_{m,k}(x,t)&\subset(x,t)-\Omega\subset suppJ(x-y,t-s)\cap\{s\le 0\}.
\end{aligned}$$

For $m \geq N$ we choose $\kappa(m) \in \mathbb{N}$ such that when $|(z,w)-(y,s)| < 2^{-\kappa(m)+1}\sqrt{n+1}$, then $|f(z,w)-f(y,s)| < \frac{1}{2m}$. Let us define $f_m^+ : \mathbb{R}^n \times (-\infty, 0)$ in the following way

$$f_m^+(y,s) := \sum_{I \in V_{\kappa(m),2l}} f(y,s)\chi_I(y,s) + \sum_{I \in V_{\kappa(m),2l+1}} \Big(f(y,s) + \frac{1}{m}\Big)\chi_I(y,s).$$

Hence, for every $(x,t) \in \mathbb{R}^n \times [0,R]$ we can choose sets $A_m(x,t) := I_{\kappa(m),2l}(x,t) \times \widetilde{I}_{\kappa(m),k}(x,t)$ and $B_m(x,t) := I_{\kappa(m),2l+1}(x,t) \times \widetilde{I}_{\kappa(m),k}(x,t)$ contained in $(x,t) - \Omega \subset suppJ(x-y,t-s) \cap \{s \leq 0\}$. Now, if $(y_1,s_1) \in A_m(x,t)$ and $(y_2,s_2) \in B_m(x,t)$ we have that

$$f_m^+(y_2,s_2) - f_m^+(y_1,s_1) = f(y_2,s_2) + \frac{1}{m} - f(y_1,s_1) \geq \frac{1}{2m}.$$

If we set

$$a_m(x,t) = \sup_{(y,s) \in A_m(x,t)} f(y,s)$$

and

$$b_m(x,t) = \inf_{(y,s) \in B_m(x,t)} f(y,s)$$

we have

$$a(x,t) - b(x,t) \geq \frac{1}{2m}$$

independently of (x,t). From our choice of the sets $A_m(x,t)$ and $B_m(x,t)$ we have that

$$\iint_{A_m(x,t)} J(x-y,t-s)dyds > \theta 2^{-\kappa(m)(n+1)}$$

and

$$\iint_{B_m(x,t)} J(x-y,t-s)dyds > \theta 2^{-\kappa(m)(n+1)}.$$

Hence, f_m satisfy all the necessary conditions that appear in $i)$ $ii)$ and $iii)$ in Lemma 18. In addition, $f_m^+ \geq f$ for every m and $f_m^+ \to f$ uniformly, hence these functions also satisfy 2) and 3).

Finally, if we define

$$f_m^-(y,s) := \sum_{I \in V_{\kappa(m),2l}} \left(f(y,s) - \frac{1}{m}\right)\chi_I(y,s) + \sum_{I \in V_{\kappa(m),2l+1}} f(y,s)\chi_I(y,s)$$

with an analogous argument as the one used with the functions f_m^+ one can show that they verify 1), and $f_m^- \leq f$ for every m with $f_m^- \to f$ uniformly. □

Theorem 21. *Let $J : \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous function, $J \geq 0$, compactly supported in the set $\{(x,t) \in \mathbb{R}^{n+1} : t \geq 0\}$ and $\iint_{\mathbb{R}^n \times \mathbb{R}} J(x,t)dxdt = 1$. Let $f : \mathbb{R}^n \times (-\infty,0) \to \mathbb{R}$ uniformly continuous and bounded functions. Then there exists a unique $u \in B_1$ that solves $P(J,f)$.*

Proof. We consider the sequence of functions f_m^+ constructed in the previous lemma. Then, as they verify $f_m^+ \to f$ uniformly, they are a Cauchy sequence in $\mathbb{R}^n \times (-\infty,0)$. Then, by Theorem 19 there exist u_m, solutions of $P(J,f_m^+)$, and by Lemma 9 they are a Cauchy sequence with the infinity norm in $\mathbb{R}^n \times [0,R]$ (note that $\|f_m\|_{L^\infty(\mathbb{R}^n \times (-\infty,0))} \leq \|f\|_{L^\infty(\mathbb{R}^n \times (-\infty,0))} + \frac{1}{m}$ and so $\|u_m\|_{L^\infty(\mathbb{R}^n \times [0,\infty))} \leq \|f\|_{L^\infty(\mathbb{R}^n \times (-\infty,0))} + \frac{1}{m}$). Hence, there exists a function $u \in B_1$ such that $u_m \to u$ uniformly in $\mathbb{R}^n \times [0,R]$. Then, as

$$\begin{aligned}
0 &= \iint_{\mathbb{R}^n \times \mathbb{R}} J(x-y,t-s)F(\overline{u_m}(y,s) - u_m(x,t))\,dy\,ds \\
&= \iint_{t-\alpha \leq s \leq 0} J(x-y,t-s)F(f_m^+(y,s) - u_m(x,t))\,dy\,ds \\
&\quad + \iint_{0 \leq s \leq t} J(x-y,t-s)F(u_m(y,s) - u_m(x,t))\,dy\,ds
\end{aligned}$$

for every $m \in \mathbb{N}$ and every $(x,t) \in \mathbb{R}^n \times [0,R]$, and as the functions f_m^+ and u_m are uniformly bounded and $J(x-\cdot, t-\cdot)$ is in L^1, by the dominated convergence theorem, taking limit as $m \to \infty$, we obtain

$$\begin{aligned} 0 &= \iint_{\mathbb{R}^n \times \mathbb{R}} J(x-y,t-s) F(\overline{u}(y,s) - u(x,t))\,dy\,ds \\ &= \iint_{t-\alpha \le s \le 0} J(x-y,t-s) F(f(y,s) - u(x,t))\,dy\,ds \\ &\quad + \iint_{0 \le s \le t} J(x-y,t-s) F(u(y,s) - u(x,t))\,dy\,ds \end{aligned}$$

and therefore u solves $P(J,f)$ in $\mathbb{R}^n \times [0,R]$. Hence, we have existence of solutions.

Let us show now uniqueness of the solution. We argue by contradiction. Assume that v is another function in B_1 that solves our problem with the same initial condition f. Then

$$0 = \iint_{\mathbb{R}^n \times \mathbb{R}} J(x-y,t-s) F(\overline{v}(y,s) - v(x,t))\,dy\,ds.$$

For $m \ge 1$, we take f_m^+ and f_m^- as in Lemma 20. Let u_m^+ and u_m^- the solutions to the problems $P(J, f_m^+)$ and $P(J, f_m^-)$ respectively. Let us see that $u_m^- \le v \le u_m^+$ in $\mathbb{R}^n \times [0,R]$ for every $m \ge 1$, and therefore we conclude that $u = v$.

From the construction of u_m^+, this function is a uniform limit of functions $u_{m,i}^+$ that satisfy $Tu_{m,i}^+ = u_{m,i+1}^+$, and where $u_{m,0}^+$ can be chosen as any function in B_1. In particular we can take $u_{m,0}^+ = v$. Then, we have that $u_{m,1}^+$ satisfies

$$0 = \iint_{\mathbb{R}^n \times \mathbb{R}} J(x-y,t-s) F(\overline{u_{m,0}^+}(y,s) - u_{m,1}^+(x,t))\,dy\,ds$$

for every $(x,t) \in \mathbb{R}^n \times [0,R]$. Where

$$\overline{u_{m,0}^+}(x,t) = \begin{cases} f_m^+(x,t) & t < 0, \\ v(x,t) & t \ge 0. \end{cases}$$

On the other hand v satisfies

$$0 = \iint_{\mathbb{R}^n \times \mathbb{R}} J(x-y,t-s) F(\overline{v}(y,s) - v(x,t))\,dy\,ds$$

and, since

$$\overline{v}(x,t) = \begin{cases} f(x,t) & t < 0, \\ v(x,t) & t \ge 0, \end{cases}$$

we have that $\overline{v} \le \overline{u_{m,0}^+}$. Hence $v(x,t) \le u_{m,1}^+(x,t)$ for every $(x,t) \in \mathbb{R}^n \times [0,R]$. In the same way, given now that $\overline{v} \le \overline{u_{m,1}^+}$ we obtain $v(x,t) \le u_{m,2}^+(x,t)$ for every $(x,t) \in \mathbb{R}^n \times [0,R]$. With this kind of argument we can show that $v(x,t) \le u_{m,i}^+(x,t)$ for every $i \in \mathbb{N}$, and as $u_{m,i}^+ \rightrightarrows u_m^+$ this implies $v(x,t) \le u_m^+(x,t)$ for every $(x,t) \in \mathbb{R}^n \times [0,R]$. In an analogous way we can show that $u_m^-(x,t) \le v(x,t)$ for every $(x,t) \in \mathbb{R}^n \times [0,R]$.

As the previous argument works for every $m \ge 1$ we have that $u_m^- \le v \le u_m^+$ in $\mathbb{R}^n \times [0,R]$ for every $m \ge 1$, as we wanted to prove.

Let us prove that $u \in B_1$. As we have that $u_m^- \le u \le u_m$ and from Theorem 19 we know that $\inf f_m^- \le u_m^-$ and that $u_m^+ \le f_m^+$, then we get $\inf f_m^- \le u \le \sup f_m$. Taking the limit as $m \to \infty$ we obtain that $\inf f \le u \le \sup f$. The proof that $u \in \overline{C}$ follows as in Theorem 5. Therefore, we conclude that $u \in B_1$. □

Note that in the previous lemma the hypothesis f is uniformly continuous can be assumed only on the strip $\mathbb{R}^n \times (-\alpha, 0)$, where $\alpha = \sup\left\{\gamma : \iint_{\gamma \le t} J(x,t)dxdt < 1\right\}$.

Theorem 22. *Let $J : \mathbb{R}^{n+1} \to \mathbb{R}$ be a continuous function such that $J \ge 0$, compactly supported in the set $\{(x,t) \in \mathbb{R}^{n+1} : t \ge 0\}$ and $\iint_{\mathbb{R}^n \times \mathbb{R}} J(x,t)dsdt = 1$. Let $f : \mathbb{R}^n \times (-\infty, 0) \to \mathbb{R}$ be a uniformly continuous and bounded function. Then, there exists a unique $u \in B := \overline{C} \cap L^\infty(f)(\mathbb{R}^n \times [0,\infty))$ that solves $P(J,f)$.*

Proof. Let $\alpha = \sup\left\{\gamma : \iint_{\gamma \le t} J(x,t)dxdt < 1\right\}$. We construct the solution of $P(J,f)$ in time strips. First, we build a solution in $\mathbb{R}^n \times [0,\alpha)$ and next we extend it in sets of the form $\mathbb{R}^n \times [(m-1)\alpha, m\alpha)$ with $m \in \mathbb{N}$.

By Theorem 21 we have that there exists a unique function v, solution to $P(J,f)$, that belongs to the space $B_1 = \overline{C} \cap L^\infty(f)(\mathbb{R}^n \times [0,R])$ where R is any positive number less than α. Taking the limit $R \to \alpha$ we obtain that there is a unique function u_1 that belongs to the space $\overline{C} \cap L^\infty(f)(\mathbb{R}^n \times [0,\alpha))$ and solves $P(J,f)$ in $\mathbb{R}^n \times [0,\alpha)$, that is,

$$0 = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u_1}(y,s) - u_1(x,t))\,dy\,ds$$

for every $(x,t) \in \mathbb{R}^n \times [0,\alpha)$, and where

$$\overline{u_1}(y,s) = \begin{cases} f(y,s) & s < 0, \\ u_1(y,s) & s \in [0,\alpha). \end{cases}$$

As $u_1 \in \overline{C} \cap L^\infty(f)(\mathbb{R}^n \times [0,\alpha))$ we have

$$\|u_1\|_{L^\infty(\mathbb{R}^n\times[0,\alpha))} \le \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}.$$

Now we consider this function u_1 as initial datum (recall that it is uniformly continuous) and then with the same argument used before we can construct a function u_2 in the space $\overline{C} \cap L^\infty(\overline{u_1})(\mathbb{R}^n \times [\alpha,2\alpha))$ that solves $P(J,\overline{u_1})$, that is,

$$0 = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u_2}(y,s) - u_2(x,t))\,dy\,ds$$

for every $(x,t) \in \mathbb{R}^n \times [\alpha,2\alpha)$, where

$$\overline{u_2}(y,s) = \begin{cases} \overline{u_1}(y,s) & s < \alpha, \\ u_2(y,s) & s \in [\alpha,2\alpha) \end{cases}$$

Moreover, we get

$$\|u_2\|_{L^\infty(\mathbb{R}^n\times[\alpha,2\alpha))} \le \|\overline{u_1}\|_{L^\infty(\mathbb{R}^n\times[0,\alpha))} = \|u_1\|_{L^\infty(\mathbb{R}^n\times[0,\alpha))} \le \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}.$$

Now we assume that we have proved the existence of functions u_m $(m = 1,...K)$ in the space $\overline{C} \cap L^\infty(\overline{u_{m-1}})(\mathbb{R}^n \times [(m-1)\alpha, m\alpha))$ such that

$$0 = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u_m}(y,s) - u_m(x,t))\,dy\,ds$$

for every $(x,t) \in \mathbb{R}^n \times [(m-1)\alpha, m\alpha)$, with

$$\overline{u_m}(y,s) = \begin{cases} \overline{u_{m-1}}(y,s) & s < (m-1)\alpha, \\ u_m(y,s) & s \in [(m-1)\alpha, m\alpha) \end{cases}$$

that verify

$$\|u_m\|_{L^\infty(\mathbb{R}^n\times[(m-1)\alpha,m\alpha))} \le \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}.$$

Arguing as we did for $m = 2$, we can continue and construct a function u_{K+1} in the space $\overline{C} \cap L^\infty(\overline{u_m})(\mathbb{R}^n \times [m\alpha,(m+1)\alpha))$ such that

$$0 = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u_{K+1}}(y,s) - u_{K+1}(x,t))\,dy\,ds$$

for every $(x,t) \in \mathbb{R}^n \times [K\alpha,(K+1)\alpha)$, where

$$\overline{u_{K+1}}(y,s) = \begin{cases} \overline{u_K}(y,s) & s < K\alpha, \\ u_{K+1}(y,s) & s \in [K\alpha,(K+1)\alpha). \end{cases}$$

Moreover,

$$\|u_{K+1}\|_{L^\infty(\mathbb{R}^n\times[K\alpha,(K+1)\alpha))} \le \|\overline{u_K}\|_{L^\infty(\mathbb{R}^n\times[(K-1)\alpha,K\alpha))} = \|u_K\|_{L^\infty(\mathbb{R}^n\times[(K-1)\alpha,K\alpha))} \le \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}.$$

Then, if we take $u(x,t) = u_m(x,t)$ where m is such that $t \in [(m-1)\alpha, m\alpha)$, we have that u verifies

$$0 = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x-y,t-s)F(\overline{u}(y,s) - u(x,t))\,dy\,ds$$

for every $(x,t) \in \mathbb{R}^n \times [0,\infty)$, and

$$\overline{u}(y,s) = \begin{cases} f(y,s) & s < 0, \\ u(y,s) & s \ge 0. \end{cases}$$

Therefore u solves $P(J,f)$. In addition,

$$\|u\|_{L^\infty(\mathbb{R}^n\times[0,\infty))} \le \|f\|_{L^\infty(\mathbb{R}^n\times(-\infty,0))}$$

and $u \in \overline{C}$ (this fact follows as in Theorem 5).

Let us show that this solution is unique. Assume that there exist two solutions u and v to $P(J,f)$ in $\mathbb{R}^n \times [0,\infty)$. Then, they are solutions to $P(J,f)$ in $\mathbb{R}^n \times [0,R]$ for every $R < \alpha$, but from Theorem 21 we have uniqueness of such solution and we conclude that $u \equiv v$ in $\mathbb{R}^n \times [0, \alpha)$. Now we consider u and v restricted to $\mathbb{R}^n \times [\alpha, 2\alpha)$. They solve $P(J, \overline{u_1})$. Using again Theorem 21 we obtain $u \equiv v$ in $\mathbb{R}^n \times [\alpha, 2\alpha)$. In this way we conclude that $u \equiv v$ in $\mathbb{R}^n \times [(m-1)\alpha, m\alpha)$ for every $m \in \mathbb{N}$, and uniqueness follows. □

3. Rescaling the kernel. Approximations to a local PDE problem

Our idea to obtain convergence along subsequences of the rescaled problems is to apply the following variant of the Arzela-Ascoli Lemma. For its proof we refer the reader to [20] Lemma 4.2.

Lemma 23. *Fix $\sigma > 0$. Let $\{u_r : \overline{\Omega} \to \mathbb{R},\ \sigma \geq r > 0\}$ be a set of functions such that*

(1) *there exists $C > 0$ so that $|u_r(x)| < C$ for every $\sigma \geq r > 0$ and every $x \in \overline{\Omega}$,*
(2) *given $\nu > 0$ there are constants r_0 and r_1 such that for every $r < r_0$ and any $x, y \in \overline{\Omega}$ with $|x - y| < r_1$ it holds*
$$|u_r(x) - u_r(y)| < \nu.$$

Then, there exists a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ and a subsequence denoted by $\{u_{r_j}\}$ such that
$$u_{r_j} \to u \qquad \text{uniformly in} \quad \overline{\Omega},$$
as $r_j \to 0$.

Now we prove that the solutions of the rescaled problem $P(J_r, f)$, with $J_r(x,t) = \frac{1}{r^{n+2}} J\left(\frac{x}{r}, \frac{t}{r^2}\right)$ satisfy the hypotheses of the Arzela-Ascoli type lemma.

Theorem 24. *Let $J : \mathbb{R}^{n+1} \to \mathbb{R}$, $J \geq 0$, be compactly supported in the set $\{(x,t) \in \mathbb{R}^{n+1} \delta \leq t \leq \delta + \gamma\}$ where δ and γ are positive constants, $\iint_{\mathbb{R}^n \times \mathbb{R}} J(x,t)dxdt = 1$, and $J(\cdot, t)$ is radially symmetric. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function, such that f is C^2 with bounded derivatives. If u_r is the solution to $P(J_r, f)$, with $J_r(x,t) = \frac{1}{r^{n+2}} J\left(\frac{x}{r}, \frac{t}{r^2}\right)$, then, for all compact set contained in $\mathbb{R}^n \times [0, \infty)$, the functions u_r satisfies the hypothesis of Lemma 23.*

Proof. Note that $\|u_r\|_{L^\infty(\mathbb{R}^n \times [0,\infty))} \leq \|f\|_{L^\infty(\mathbb{R}^n)}$ for all $r > 0$. So, we only need to prove that the functions u_r satisfies (2) in Lemma 23. Let us show that

1) For all $x \in \mathbb{R}^n$ and $t \in [0, \delta r^2]$, $|u_r(x,t) - f(x)| \leq Cr^2$, with C independent of r.
2) For all $x \in \mathbb{R}^n$, $|u_r(x, t_1) - u_r(x, t_2)| \leq Cr^2$ for $|t_1 - t_2| \leq \delta r^2$, where t_1 and t_2 are nonnegatives, and $C > 0$ do not depends of r.
3) If $|t_1 - t_2| > \delta r^2$ then $|u_r(x, t_1) - u_r(x, t_2)| \leq C|t_1 - t_2|$, where C is independent of r.

Note that 1) and 2) are enough to satisfies the hypothesis of Lema 23. The condition 3) guarantees that the uniform limit u in Lemma 23 is a Lipschitz function.

Let us prove 1). Let $(x,t) \in \mathbb{R}^n \times [0, \delta r^2]$. Since $suppJ_r \subset \{\delta r^2 \leq t \leq (\delta + \gamma) r^2\}$ we have that
$$\begin{aligned} 0 &= \iint_{\mathbb{R}^n \times \mathbb{R}} J_r(x-y, t-s) F(\overline{u_r}(y,s) - u_r(x,t))\, dy\, ds = \iint_{\mathbb{R}^n \times \mathbb{R}} J_r(x-y, t-s) F(f(y) - u_r(x,t))\, dy\, ds \\ &= \int_{\mathbb{R}^n} \int_{\mathbb{R}} J(z,w) dw F(f(x - rz) - u_r(x,t)) dz = \int_{\mathbb{R}^n} G(z) F(f(x-rz) - u_r(x,t)) dz \end{aligned}$$
where $G(x) = \int_{\mathbb{R}} J(x,t)dt$ (remark that G is radial). Now we claim that
$$\int_{\mathbb{R}^n} G(z) F(f(x - rz) - (f(x) - Cr^2))\, dz \geq 0 \tag{6}$$
and
$$\int_{\mathbb{R}^n} G(z) F(f(x - rz) - (f(x) + Cr^2))\, dz \leq 0 \tag{7}$$
hold, being $C > 0$ a constant independent of r. Assuming the claim we get
$$f(x) - Cr^2 \leq u(x,t) \leq f(x) + Cr^2$$
and hence
$$|u_r(x,t) - f(x)| \leq Cr^2$$
for every $t \in [0, \delta r^2]$.

Let $C = \left\|D^2 f\right\|_{L^\infty} [R(J)]^2$, where $R(J)$ is such that $supp(G) \subset B(0, R(J))$. Let us prove (6). First, we have that
$$f(x - rz) - (f(x) - Cr^2) = \nabla f(x) \cdot (-rz) + \frac{1}{2} rz D^2 f(\xi) rz + Cr^2 \tag{8}$$

where ξ is in the segment that joins x with $x-rz$. On the other hand, for a and b real numbers, we have that

$$F(a+b)=F(a)+\int_0^1 F'(a+tb)dtb. \tag{9}$$

Taking in the previous formula $a=\nabla f(x)\cdot(-rz)$ and $b=\frac{1}{2}rzD^2f(\xi)rz+Cr^2$, combining (8) and (9) we have that

$$\begin{aligned}&F(f(x-rz)-(f(x)-Cr^2))\\&=-r^{p-1}\left|\nabla f(x)\cdot z\right|^{p-2}\nabla f(x)\cdot z+(p-1)\int_0^1\left|-\nabla f(x)\cdot rz+t\left(\frac{r^2}{2}zD^2f(\xi)z+Cr^2\right)\right|^{p-2}dt\left(\frac{r^2}{2}zD^2f(\xi)z+Cr^2\right).\end{aligned}$$

Then, since $-r^{p-1}\left|\nabla f(x)\cdot z\right|^{p-2}\nabla f(x)\cdot z$ is a odd function, the integral in (6) is equal to

$$(p-1)\int_{\mathbb{R}}G(z)\int_0^1\left|-\nabla f(x)\cdot rz+t\left(\frac{r^2}{2}zD^2f(\xi)z+Cr^2\right)\right|^{p-2}dt\left(\frac{r^2}{2}zD^2f(\xi)z+Cr^2\right)dz$$

and, from the choice of C, this integral is greater or equal than 0. Then (6) is true.

Now let us prove (7). Reasoning analogously to the previuos case, we obtain that the integral in (7) is equal to

$$(p-1)\int_{\mathbb{R}}G(z)\int_0^1\left|-\nabla f(x)\cdot rz+t\left(\frac{r^2}{2}zD^2f(\xi)z-Cr^2\right)\right|^{p-2}dt\left(\frac{r^2}{2}zD^2f(\xi)z-Cr^2\right)dz$$

and again, using the choice of C, this integral is less or equal than 0. So we colnclude that (7) holds.

Let us prove 2). We shall proceed inductively, covering $\mathbb{R}^n\times[0,\infty)$ with strips of the form $\mathbb{R}^n\times[0,i\delta r^2]$ for $i=1,2,\ldots$. The case $i=1$ is done and follows from item 1). Also note if $t_1\in(-\infty,\delta r^2]$, $t_2\in(-\infty,\delta r^2]$ then

$$|\overline{u_r}(x,t_1)-\overline{u_r}(x,t_2)|\le Cr^2 \tag{10}$$

for all $x\in\mathbb{R}^n$ and C do not depends of r.

Now suppose that 2) holds for $i=1,2,\ldots,k$ and see that it is true for $i=k+1$. Let $t_1\in[0,(k+1)\delta r^2]$ and $t_2\in[0,(k+1)\delta r^2]$ such that $|t_1-t_2|\le\delta r^2$, then

$$\begin{aligned}0&=\iint_{\mathbb{R}^n\times\mathbb{R}}J_r(x-y,t_1-s)F(\overline{u_r}(y,s)-u_r(x,t_1))\,dy\,ds\\&=\iint_{\mathbb{R}^n\times\mathbb{R}}J_r(x-y,w)F(\overline{u_r}(y,t_1-w)-u_r(x,t_1))\,dy\,dw\\&=\iint_{\mathbb{R}^n\times\mathbb{R}}J_r(x-y,w)F(\overline{u_r}(y,t_2-w)+(\overline{u_r}(y,t_1-w)-\overline{u_r}(y,t_2-w))-u_r(x,t_1))\,dy\,dw\end{aligned}$$

if t_1-w and t_2-w are in $(-\infty,\delta r^2]$, from (10) we have that

$$\overline{u_r}(y,t_1-w)-\overline{u_r}(y,t_2-w)\le Cr^2,$$

and, if t_1-w and t_2-w are in $[0,k\delta r^2]$, for the inductive hypothesis, we get

$$\overline{u_r}(y,t_1-w)-\overline{u_r}(y,t_2-w)=u_r(y,t_1-w)-u_r(y,t_2-w)\le Cr^2.$$

Then, as J is no negative and F is increasing, we have

$$\begin{aligned}0&\le\iint_{\mathbb{R}^n\times\mathbb{R}}J_r(x-y,w)F(\overline{u_r}(y,t_2-w)-(u_r(x,t_1)-Cr^2))\,dy\,dw\\&\le\iint_{\mathbb{R}^n\times\mathbb{R}}J_r(x-y,t_2-s)F(\overline{u_r}(y,s)-(u_r(x,t_1)-Cr^2))\,dy\,ds.\end{aligned}$$

Hence

$$u_r(x,t_1)-u_r(x,t_2)\le Cr^2,$$

similarly we can show

$$u_r(x,t_2)-u_r(x,t_1)\le Cr^2,$$

and then

$$|u_r(x,t_1)-u_r(x,t_2)|\le Cr^2,$$

where the constant C does not depend on r.

Let us now prove 3). Suppose now that t_1 and t_2 are nonnegatives and such that $|t_1 - t_2| \geq \delta r^2$. Then t_1 belongs to $[m_1\delta r^2, (m_1+1)\delta r^2]$ and t_2 to $[m_2\delta r^2, (m_2+1)\delta r^2]$. Assume that $t_1 < t_2$, then $m_2\delta r^2 - (m_1+1)\delta r^2 \leq t_2 - t_1$, that implies that $m_2 - m_1 \leq \frac{t_2-t_1}{\delta r^2} + 1 \leq 2\frac{t_2-t_1}{\delta r^2}$. Hence, if $k = m_2 - m_1$, we obtain

$$|u_r(x,t_2) - u_r(x,t_1)| \leq \sum_{j=0}^{k} \left|u_r(x,\widetilde{t}_{j+1}) - u_r(x,\widetilde{t}_j)\right|$$

where $\widetilde{t}_0 = t_1$ and $\widetilde{t}_k = t_2$ and $\widetilde{t}_j = (m_1 + j)\delta r^2$. Therefore,

$$|u_r(x,t_2) - u_r(x,t_1)| \leq Cr^2(k+1) \leq Cr^2 3\frac{t_2-t_1}{\delta r^2} = \frac{3C}{\delta}(t_2 - t_1)$$

and we get 3). □

From Theorem 24 and the Arzela-Ascoli type lemma there exists a sequence of functions u_{r_m} that are solutions to $P(J_{r_m}, f)$ such that u_{r_m} converges uniformly to a continuous function u as $m \to \infty$ on every compact of $\mathbb{R}^n \times [0,\infty)$. Our aim now is to show that u is a viscosity solution to

$$A\left\|\nabla u\right\|^{p-2}\frac{\partial u}{\partial t} = B\left(\left\|\nabla u\right\|^{p-2}\triangle u + (p-2)\left\|\nabla u\right\|^{p-4}\nabla u D^2 u \nabla u\right)$$

where A and B are constants that depend on J.

Before starting to prove this fact we state a result that will be use later.

Lemma 25. *Let $K : \mathbb{R}^n \to \mathbb{R}$ be a radially symmetric function with compact support, and let $2 \leq p < \infty$. Then*

$$\int_{\mathbb{R}^n} K(z)|z_1|^p\,dz = (p-1)\int_{\mathbb{R}^n} K(z)|z_1|^{p-2}z_l^2\,dz$$

for all $l = 2, 3, \ldots, n$.

Proof. By Fubini's theorem it is enough prove the Lemma in the case $n = 2$. Then, if $X = (x,y) \in \mathbb{R}^2$ and $R > 0$ such that $suppK \subset B(0,R)$, we have that

$$\int_{\mathbb{R}^2} K(x,y)|x|^p dxdy = \int_0^R r^{p+1}K(r)\int_0^{2\pi} |cos(\theta)|^p d\theta = 2\int_0^R r^{p+1}K(r)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^p(\theta)d\theta$$

and

$$\int_{\mathbb{R}^2} K(x,y)|x|^{p-2}y^2 dX = \int_0^R r^{p+1}K(r)\int_0^{2\pi} |cos(\theta)|^{p-2}sin^2(\theta)d\theta = 2\int_0^R r^{p+1}K(r)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{p-2}(\theta)sin^2(\theta)d\theta.$$

Let us see that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^p(\theta)d\theta = (p-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{p-2}(\theta)sin^2(\theta)d\theta,$$

in fact, integrating by parts que have that

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^p(\theta)d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{p-1}(\theta)cos(\theta)d\theta = (p-1)\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} cos^{p-2}(\theta)sin^2(\theta)d\theta.$$

□

Theorem 26. *Let $J : \mathbb{R}^{n+1} \to \mathbb{R}$, $J \geq 0$, compactly supported in the set $\{(x,t) \in \mathbb{R}^{n+1} : \delta \leq t \leq \delta + \gamma\}$ where δ and γ are positive constants, $\iint J(x,t)dxdt = 1$, and $J(\cdot,t)$ is radially symmetric. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function, such that f is C^2 with bounded derivatives. Then, there exists a sequence of solutions to $P(J_{r_m}, f)$, u_{r_m}, that converge uniformly on compact sets to a viscosity solution to the problem*

$$\begin{cases} A\left\|\nabla u\right\|^{p-2}\dfrac{\partial u}{\partial t} = B\left(\left\|\nabla u\right\|^{p-2}\triangle u + (p-2)\left\|\nabla u\right\|^{p-4}\nabla u D^2 u \nabla u\right) & (x,t) \in \mathbb{R}^n \times [0,\infty) \\ u(x,0) = f(x) & x \in \mathbb{R}^n \end{cases}$$

where $A = \iint_{\mathbb{R}^n\times\mathbb{R}} J(x,w)|z_1|^{p-2}w\,dz\,dw$ and $B = \frac{1}{2}\iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w)|z_1|^p\,dz\,dw$.

Proof. For Theorem 24 and Lemma 23 there exists functions u_{r_m} solutions of the problem $P(J_{r_m}, f)$ that converges uniformly to a funtion u when m goes to infty.

Let's see that u is a viscocity solutions of the local problem. Let $\varphi(x,t)$ be a $C^2(\mathbb{R}^{n+1})$, and assume that $\overline{u} - \varphi$ has an strict minimum at the point $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and $\nabla\varphi(x,t) \neq 0$. Then, as $u_{r_m} \to u$ uniformly in a neighborhood of (x,t), there exist $(x_{r_m}, t_{r_m}) \in \mathbb{R}^n \times (0,\infty)$ such that $\overline{u_{r_m}} - \varphi$ has an absolute minimum at (x_{r_m}, t_{r_m}), with $(x_{r_m}, t_{r_m}) \to (x,t)$. Then, we have

$$\overline{u_{r_m}}(y,s) - \varphi(y,s) \geq u_{r_m}(x_{r_m}, t_{r_m}) - \varphi(x_{r_m}, t_{r_m}),$$

that is,

$$\overline{u_{r_m}}(y,s) - u_{r_m}(x_{r_m}, t_{r_m}) \geq \varphi(y,s) - \varphi(x_{r_m}, t_{r_m}).$$

Hence,

$$F\left(\overline{u_{r_m}}(y,s) - u_{r_m}(x_{r_m}, t_{r_m})\right) \geq F\left(\varphi(y,s) - \varphi(x_{r_m}, t_{r_m})\right).$$

This implies that

$$\begin{aligned} 0 &= \frac{1}{r_m^p} \iint_{\mathbb{R}^n\times\mathbb{R}} J_{r_m}((x_{r_m}-y),(t_{r_m}-s)) F(\overline{u_{r_m}}(y,s) - u_{r_m}(x_{r_m}, t_{r_m}))\, dy\, ds \\ &\geq \frac{1}{r_m^p} \iint_{\mathbb{R}^n\times\mathbb{R}} J_{r_m}((x_{r_m}-y),(t_{r_m}-s)) F(\varphi(y,s) - \varphi(x_{r_m}, t_{r_m}))\, dy\, ds \\ &= \frac{1}{r_m^p} \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w) F(\varphi(x_{r_m} - r_m z, t_{r_m} - r_m^2 w) - \varphi(x_{r_m}, t_{r_m}))\, dz\, dw. \end{aligned} \tag{11}$$

Our target will be obtain, taking limit when r_m tends to 0, a local diferential operator acting over φ on the right term of the last inequality.

$$\begin{aligned} &\frac{1}{r_m^p} \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w) F(\varphi(x_{r_m} - r_m z, t_{r_m} - r_m^2 w) - \varphi(x_{r_m}, t_{r_m}))\, dz\, dw \\ &= \frac{1}{r_m^p} \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w) F(-r_m \nabla\varphi(x_{r_m}, t_{r_m})\cdot z)\, dz\, dw \\ &\quad + \frac{1}{r_m^p} \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w) F(\varphi(x_{r_m} - r_m z, t_{r_m} - r_m^2 w) - \varphi(x_{r_m}, t_{r_m})) - F(-r_m \nabla\varphi(x_{r_m}, t_{r_m})\cdot z)\, dz\, dw \\ &= I_{r_m} + II_{r_m}. \end{aligned}$$

First, let us look at I_{r_m}. Let O be a rotation matrix such that $O\frac{\nabla\varphi(x_{r_m},t_{r_m})}{\|\nabla\varphi(x_{r_m},t_{r_m})\|} = e_1$ (note that the first row of O is $\frac{\nabla\varphi(x_{r_m},t_{r_m})}{\|\nabla\varphi(x_{r_m},t_{r_m})\|}$). If we make the changing of variables $O^t\bar{z} = z$, taking account that $J(\cdot,t)$ is radially symmetric for every t, we have that

$$\begin{aligned} &\frac{1}{r_m^p} \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w) F(-r_m \nabla\varphi(x_{r_m}, t_{r_m})\cdot z)\, dz\, dw \\ &= -r_m^{-1} \|\nabla\varphi(x_{r_m}, t_{r_m})\|^{p-1} \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w) \left| \frac{\nabla\varphi(x_{r_m}, t_{r_m})}{\|\nabla\varphi(x_{r_m}, t_{r_m})\|} \cdot z \right|^{p-2} \frac{\nabla\varphi(x_{r_m}, t_{r_m})}{\|\nabla\varphi(x_{r_m}, t_{r_m})\|} \cdot z\, dz\, dw \\ &= -r_m^{-1} \|\nabla\varphi(x_{r_m}, t_{r_m})\|^{p-1} \iint_{\mathbb{R}^n\times\mathbb{R}} J(O^t\bar{z},w) \left| \frac{\nabla\varphi(x_{r_m}, t_{r_m})}{\|\nabla\varphi(x_{r_m}, t_{r_m})\|} \cdot O^t\bar{z} \right|^{p-2} \frac{\nabla\varphi(x_{r_m}, t_{r_m})}{\|\nabla\varphi(x_{r_m}, t_{r_m})\|} \cdot O^t\bar{z}\, d\bar{z}\, dw \\ &= -r_m^{-1} \|\nabla\varphi(x_{r_m}, t_{r_m})\|^{p-1} \iint_{\mathbb{R}^n\times\mathbb{R}} J(\bar{z},w)\, |\bar{z}_1|^{p-2}\, \bar{z}_1\, d\bar{z}\, dw = 0. \end{aligned}$$

Let us deal with II_{r_m}. By Taylor's theorem we have that if $|b| < |a|$ then

$$|a+b|^{p-2}(a+b) = |a|^{p-2}a + (p-1)|a|^{p-2}b + (p-1)(p-2)|\xi|^{p-4}\xi b^2$$

where ξ is in the segment that joins 0 with b, and

$$\varphi(x_{r_m} - r_m z, t_{r_m} - r_m^2 w) - \varphi(x_{r_m}, t_{r_m}) = -r_m \nabla\varphi(x_{r_m}, t_{r_m})\cdot z - r_m^2 \frac{\partial\varphi(x_{r_m}, t_{r_m})}{\partial t} w + r_m^2 \frac{1}{2} \sum_{ij=1}^n \frac{\partial^2\varphi(x_{r_m}, t_{r_m})}{\partial x_i \partial x_j} z_i z_j + o(r_m^2).$$

If we take $a = -r_m \nabla\varphi(x_{r_m}, t_{r_m}) \cdot z$ and $b = -r_m^2 \frac{\partial\varphi(x_{r_m},t_{r_m})}{\partial t} w + r_m^2 \frac{1}{2}\sum_{ij=1}^n \frac{\partial^2\varphi(x_{r_m},t_{r_m})}{\partial x_i \partial x_j} z_i z_j + o(r_m^2)$ we have that for r_m small enought $|b| < |a|$, then

$$\frac{1}{r_m^p}\left(F(\varphi(x_{r_m} - r_m z, t_{r_m} - r_m^2 w) - \varphi(x_{r_m}, t_{r_m})) - F(-r_m\nabla\varphi(x_{r_m}, t_{r_m}) \cdot z)\right)$$

$$= -(p-1)|\nabla\varphi(x_{r_m}, t_{r_m}) \cdot z|^{p-2}\frac{\partial\varphi(x_{r_m}, t_{r_m})}{\partial t} w + (p-1)|\nabla\varphi(x_{r_m}, t_{r_m}) \cdot z|^{p-2}\left(\frac{1}{2}\sum_{ij=1}^n \frac{\partial^2\varphi(x_{r_m}, t_{r_m})}{\partial x_i\partial x_j} z_i z_j + \frac{o(r_m^p)}{r_m^p}\right).$$

So, by the dominated convergence theorem

$$\begin{aligned}
\lim_{r_m\to 0} II_{r_m} &= \lim_{r_m\to 0}\frac{1}{r_m^p}\iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w) F(\varphi(x_{r_m} - r_m z, t_{r_m} - r_m^2 w) - \varphi(x_{r_m}, t_{r_m})) - F(-r_m\nabla\varphi(x_{r_m}, t_{r_m}) \cdot z)\, dz\, dw \\
&= \iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w) \lim_{r_m\to 0}\frac{1}{r_m^p}\left(F(\varphi(x_{r_m} - r_m z, t_{r_m} - r_m^2 w) - \varphi(x_{r_m}, t_{r_m})) - F(-r_m\nabla\varphi(x_{r_m}, t_{r_m}) \cdot z)\right) dz\, dw \\
&= -(p-1)\iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w)\, |\nabla\varphi(x,t)\cdot z|^{p-2}\, \frac{\partial\varphi(x,t)}{\partial t} w\, dz\, dw \\
&\quad + (p-1)\iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w)\, |\nabla\varphi(x,t)\cdot z|^{p-2}\, \frac{1}{2}\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j} z_i z_j\, dz\, dw \\
&= II' + II''.
\end{aligned}$$

Now

$$\begin{aligned}
II' &= -\left\|\nabla\varphi(x,t)\right\|^{p-2}(p-1)\iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w)\left|\frac{\nabla\varphi(x,t)}{\|\nabla\varphi(x,t)\|}\cdot z\right|^{p-2}\frac{\partial\varphi(x,t)}{\partial t} w\, dz\, dw \\
&= -\left\|\nabla\varphi(x,t)\right\|^{p-2}\frac{\partial\varphi(x,t)}{\partial t}(p-1)\iint_{\mathbb{R}^n\times\mathbb{R}} J(O^t\bar{z}, w)\left|\frac{\nabla\varphi(x,t)}{\|\nabla\varphi(x,t)\|}\cdot O^t\bar{z}\right|^{p-2} w\, d\bar{z}\, dw \\
&= -\left\|\nabla\varphi(x,t)\right\|^{p-2}\frac{\partial\varphi(x,t)}{\partial t}(p-1)\iint_{\mathbb{R}^n\times\mathbb{R}} J(\bar{z}, w)|\bar{z}_1|^{p-2} w\, d\bar{z}\, dw.
\end{aligned}$$

Let $\bar{B} := \frac{1}{2}\iint J(z,w)|z_1|^{p-2} z_2 dz dw$ (for Lemma 25, $\bar{B} = \frac{1}{2(p-1)}\iint J(z,w)|z_1|^p dz dw$), then

$$\begin{aligned}
II'' &= \frac{p-1}{2}\left\|\nabla\varphi(x,t)\right\|^{p-2}\iint_{\mathbb{R}^n\times\mathbb{R}} J(z,w)\left|\frac{\nabla\varphi(x,t)}{\|\nabla\varphi(x,t)\|}\cdot z\right|^{p-2}\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j} z_i z_j\, dz\, dw \\
&= \frac{p-1}{2}\left\|\nabla\varphi(x,t)\right\|^{p-2}\iint_{\mathbb{R}^n\times\mathbb{R}} J(O^t\bar{z}, w)\left|\frac{\nabla\varphi(x,t)}{\|\nabla\varphi(x,t)\|}\cdot O^t\bar{z}\right|^{p-2}\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j}(O^t\bar{z})_i(O^t\bar{z})_j\, d\bar{z}\, dw \\
&= \frac{p-1}{2}\left\|\nabla\varphi(x,t)\right\|^{p-2}\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j}\iint_{\mathbb{R}^n\times\mathbb{R}} J(\bar{z}, w)|\bar{z}_1|^{p-2}\sum_{kl=1}^n O^t_{ik}O^t_{jl}\bar{z}_k\bar{z}_l\, d\bar{z}\, dw \\
&= \frac{p-1}{2}\left\|\nabla\varphi(x,t)\right\|^{p-2}\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j}\iint_{\mathbb{R}^n\times\mathbb{R}} J(\bar{z}, w)|\bar{z}_1|^{p-2}\sum_{l=1}^n O^t_{il}O^t_{jl}(\bar{z}_l)^2\, d\bar{z}\, dw \\
&= \frac{p-1}{2}\left\|\nabla\varphi(x,t)\right\|^{p-2}\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j}\left(2\bar{B}\sum_{l=2}^n O^t_{il}O^t_{jl} + (p-1)2\bar{B}O^t_{i1}O^t_{j1}\right) \\
&= (p-1)\bar{B}\left\|\nabla\varphi(x,t)\right\|^{p-2}\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j}\left(\sum_{l=1}^n O^t_{il}O^t_{jl} + (p-2)O^t_{i1}O^t_{j1}\right) \\
&= (p-1)\bar{B}\left\|\nabla\varphi(x,t)\right\|^{p-2}\left(\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j}\left(\sum_{l=1}^n O^t_{il}O^t_{jl}\right) + (p-2)\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j}O^t_{i1}O^t_{j1}\right) \\
&= (p-1)\bar{B}\left\|\nabla\varphi(x,t)\right\|^{p-2}\left(\triangle\varphi(x,t) + (p-2)\left\|\nabla\varphi(x,t)\right\|^{-2}\sum_{ij=1}^n \frac{\partial^2\varphi(x,t)}{\partial x_i\partial x_j}\frac{\partial\varphi(x,t)}{\partial x_i}\frac{\partial\varphi(x,t)}{\partial x_j}\right) \\
&= (p-1)\bar{B}\left\|\nabla\varphi(x,t)\right\|^{p-2}\left(\triangle\varphi(x,t) + (p-2)\left\|\nabla\varphi(x,t)\right\|^{-2}\nabla\varphi(x,t)D^2\varphi(x,t)\nabla\varphi(x,t)\right).
\end{aligned}$$

Therefore, from (11) we obtain

$$
\begin{aligned}
0 \geq & -\left(\iint J(z,w) z_1^{p-2} w dz dw\right) \|\nabla\varphi(x,t)\|^{p-2} \frac{\partial \varphi(x,t)}{\partial t} \\
& + \left(\frac{1}{2}\iint J(z,w) z_1^{p} dz dw\right) \|\nabla\varphi(x,t)\|^{p-2} \triangle\varphi(x,t) \\
& + \left(\frac{p-2}{2}\iint J(z,w) z_1^{p} dz dw\right) \|\nabla\varphi(x,t)\|^{p-4} \nabla\varphi(x,t) D^2\varphi(x,t)\nabla\varphi(x,t) \\
= & -A \|\nabla\varphi(x,t)\|^{p-2} \frac{\partial \varphi(x,t)}{\partial t} \\
& + B \|\nabla\varphi(x,t)\|^{p-2} \triangle\varphi(x,t) + B(p-2) \|\nabla\varphi(x,t)\|^{p-4} \nabla\varphi(x,t) D^2\varphi(x,t)\nabla\varphi(x,t).
\end{aligned} \tag{12}
$$

On the other hand if $\psi(x,t)$ is a $C^2(\mathbb{R}^{n+1})$ function such that $\overline{u} - \psi$ has a strict maximum at $(x,t) \in \mathbb{R}^n \times (0,\infty)$ and $\nabla\psi(x,t) \neq 0$, arguing as before we obtain

$$
0 \leq \frac{1}{r_m^p} \iint_{\mathbb{R}^n \times \mathbb{R}} J(z,w) F(\psi(x_{r_m} - r_m z, t_{r_m} - r_m^2 w) - \psi(x,t))\, dz\, dw.
$$

Using again Taylor expansion, we get

$$
0 \leq -A \|\nabla\psi(x,t)\|^{p-2} \frac{\partial \psi(x,t)}{\partial t} + B \|\nabla\psi(x,t)\|^{p-2} \triangle\psi(x,t) + (p-2) B \|\nabla\psi(x,t)\|^{p-4} \nabla\psi(x,t) D^2\psi(x,t)\nabla\psi(x,t). \tag{13}
$$

From (12), (13), and the fact that $\lim_{m\to\infty} u_{r_m}(x,0) = f(x)$, we conclude that u is a viscosity solution to the problem

$$
\begin{cases}
A \|\nabla u\|^{p-2} \dfrac{\partial u}{\partial t} = B\left(\|\nabla u\|^{p-2} \triangle u + (p-2)\|\nabla u\|^{p-4} \nabla u D^2 u \nabla u\right) & (x,t) \in \mathbb{R}^n \times [0,\infty) \\
u(x,0) = f(x) & x \in \mathbb{R}^n
\end{cases}
$$

as we wanted to show. $\square$

References

[1] H. Aimar, G. Beltritti and I. Gomez, *Continuous time random walks and the Cauchy problem for the heat equation.* To appear in J. d'Analyse Math.

[2] F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo, *A nonlocal p-Laplacian evolution equation with nonhomogeneous Dirichlet boundary conditions.* SIAM J. Math. Anal. **40** (2008/09), 1815–1851.

[3] F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo, *A nonlocal p-Laplacian evolution equation with Neumann boundary conditions.* J. Math. Pures Appl. **90** (2008), 201–227.

[4] F. Andreu, J. M. Mazón, J. D. Rossi and J. Toledo, *The limit as $p \to \infty$ in a nonlocal p-Laplacian evolution equation: a nonlocal approximation of a model for sandpiles.* Cal. Var **35** (2009), 279–316.

[5] F. Andreu-Vaillo, J. M. Mazón, J. D. Rossi and J. Toledo, Nonlocal Diffusion Problems. Mathematical Surveys and Monographs, vol. 165. AMS, 2010.

[6] A. Banerjee and N. Garofalo. *Gradient bounds and monotonicity of the energy for some nonlinear singular diffusion equations.* Indiana Univ. Math. J., 62(2) (2013), 699–736.

[7] A. Banerjee and N. Garofalo. *Modica type gradient estimates for an inhomogeneous variant of the normalized p-Laplacian evolution.* Nonlinear Anal., 121, (2015), 458–468.

[8] A. Banerjee and N. Garofalo. *On the Dirichlet boundary value problem for the normalized p-Laplacian evolution.* Commun. Pure Appl. Anal., 14(1) (2015), 1–21.

[9] L. A. Caffarelli and X. Cabre. Fully nonlinear elliptic equations, volume 43 of American Mathematical Society Colloquium Publications. American Mathematical Society, Providence, RI, 1995.

[10] M. G. Crandall, H. Ishii and P. L. Lions. *User'ĂŹs guide to viscosity solutions of second order partial differential equations.* Bull. Amer. Math. Soc., 27, (1992), 1–67.

[11] L. Del Pezzo and A. Quaas, *Global Bifurcation for Fractional p-Laplacian and an Application.* Z. Anal. Anwend. 35 (2016), no. 4, 411–447.

[12] E. DiBenedetto. *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations.* Nonlinear Anal., 7(8), (1983), 827–850.

[13] E. DiBenedetto. Degenerate parabolic equations. Universitext. Springer-Verlag, New York, 1993.

[14] E. DiBenedetto and A. Friedman. *Holder estimates for nonlinear degenerate parabolic systems.* J. Reine Angew. Math., 357, (1985), 1–22.

[15] K. Does. *An evolution equation involving the normalized $p-$Laplacian.* Commun. Pure Appl. Anal., 10(1), (2011), 361–396.

[16] B. Gess and M. Tolle. *Ergodicity and local limits for stochastic local and nonlocal p-Laplace equations.* Preprint, to appear in SIAM J. Math. Anal.

[17] T. Jin and L. Silvestre. *Holder gradient estimates for parabolic homogeneous p-Laplacian equations.* Preprint to appear in J. Math. Pures Appl.

[18] P. Juutinen and B. Kawohl. *On the evolution governed by the infinity Laplacian.* Math. Ann., 335(4), (2006), 819–851.

[19] M. Lewicka and J. J. Manfredi. *Game theoretical methods in PDEs.* Boll. Unione Mat. Ital., 7(3), (2014), 211–216.

[20] J. J. Manfredi, M. Parviainen and J. D. Rossi. *An asymptotic mean value characterization for a class of nonlinear parabolic equations related to tug-of-war games.* SIAM J. Math. Anal., 42(5), (2010), 2058–2081.
[21] J. M. Mazon, J. D. Rossi and J. Toledo, *Fractional $p-$Laplacian evolution equations.* J. Math. Pures Appl. (9) 105 (2016), no. 6, 810–844.
[22] D. Puhst, *On the evolutionary fractional $p-$Laplacian.* Appl. Math. Res. Express. AMRX 2015, no. 2, 253–273.
[23] J. D. Rossi. *Tug-of-war games and PDEs.* Proc. Roy. Soc. Edinburgh Sect. A, 141(2), (2011), 319–369.
[24] J. L. Vazquez, *The Dirichlet problem for the fractional $p-$Laplacian evolution equation.* J. Differential Equations 260 (2016), no. 7, 6038–6056.

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