THE BEHAVIOR OF SOLUTIONS TO AN ELLIPTIC EQUATION INVOLVING A $p-$LAPLACIAN AND A $q-$LAPLACIAN FOR LARGE p.

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ABSTRACT. In this paper we study the behavior as $p \to \infty$ of solutions $u_{p,q}$ to $-\Delta_p u - \Delta_q u = 0$ in a bounded smooth domain Ω with a Lipschitz Dirichlet boundary datum $u = g$ on $\partial\Omega$. We find that there is a uniform limit of a subsequence of solutions, that is, there is $p_j \to \infty$ such that $u_{p_j,q} \to u_\infty$ uniformly in $\overline{\Omega}$ and we prove that this limit u_∞ is a solution to a variational problem, that, when the Lipschitz constant of the boundary datum is less or equal than one, is given by the minimization of the L^q-norm of the gradient with a pointwise constraint on the gradient. In addition, we show that the limit is a viscosity solution to a limit PDE problem that involves the $q-$Laplacian and the $\infty-$Laplacian.

1. Introduction.

In this paper we deal with solutions to the following elliptic problem

$$\begin{cases} -\Delta_p u - \Delta_q u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

when p is large. Here $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the well-known $p-$Laplacian operator, Ω is a smooth bounded domain in $\mathbb{R}^N$, the boundary datum, g, is a Lipschitz function and we assume that $p > q$.

Existence and uniqueness of weak solutions to (**??**) can be easily obtained from a variational argument. In fact, we just have to look for the unique minimizer of the functional

$$F_{p,q}(u) = \int_\Omega \frac{|\nabla u|^p}{p} + \int_\Omega \frac{|\nabla u|^q}{q} \tag{1.2}$$

in the set $S = \{u \in W^{1,p}(\Omega) : u = g \text{ on } \partial\Omega\}$. We note that, as in [**?**, **?**], it can be proved that a continuous weak solution is also a solution in the viscosity sense (we refer to [**?**] for the definition of viscosity solutions).

Once we have existence and uniqueness of a solution, that we call $u_{p,q}$ in the sequel, we deal with our main goal in this paper, the study of the

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asymptotic behavior of $u_{p,q}$ as $p \to \infty$. We find that there is a uniform limit, u_∞, extracting a subsequence if necessary, and show that there is a variational limit problem as well as a limit PDE that are verified by u_∞. This is the content of our main result that we state below.

Theorem 1.1. *Let $u_{p,q}$ be the solution to (**??**). Then, for any fixed q there is a sequence $p_j \to \infty$ such that*

$$u_{p_j,q} \to u_\infty$$

weakly in $W^{1,r}(\Omega)$ (for any fixed $r \in (1,\infty)$) and uniformly in $\overline{\Omega}$. The limit u_∞ belongs to $W^{1,\infty}(\Omega)$ and verifies

$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le \max\{L,\ 1\}$$

where L is the Lipschitz constant of the boundary datum g.

If $L \le 1$, then u_∞ is the unique solution to the following variational problem

$$\min_{|\nabla u| \le 1,\, u|_{\partial\Omega} = g} \int_\Omega \frac{|\nabla u|^q}{q}, \tag{1.3}$$

while when $L > 1$ we have that u_∞ is a minimal Lipschitz extension, that is, u_∞ is a solution to

$$\min_{u|_{\partial\Omega} = g} \|\nabla u\|_{L^\infty(\Omega)}. \tag{1.4}$$

In addition, u_∞ is a viscosity solution to the following PDE problem

$$\begin{aligned} -\Delta_\infty u = 0, \qquad & \text{in } \Omega \cap \{|\nabla u| > 1\}, \\ -\Delta_\infty u = 0, \qquad & \text{in } \Omega \cap \{|\nabla u| = 1\}, \\ -\Delta_q u = 0, \qquad & \text{in } \Omega \cap \{|\nabla u| < 1\}. \end{aligned} \tag{1.5}$$

Remark that, due to the strict convexity of the L^q-norm, there exists a unique solution to (**??**) therefore we have existence of the limit $\lim_{p\to\infty} u_{p,q}$ in the case $L \le 1$, but we point out that uniqueness of the limit is left open for $L > 1$.

Note that the fact that u_∞ is a solution to (**??**) when $L \le 1$ does not imply that it verifies the equation $-\Delta_q u = 0$ in the whole Ω since we have the constraint $|\nabla u| \le 1$ in (**??**) (that is not necessarily fulfilled by the $q-$harmonic extension of the datum g even if it has a Lipschitz constant less than one). Also note that when $L > 1$ we don't necessarily have $-\Delta_\infty u = 0$ in the whole Ω.

Equations involving the sum of a $p-$Laplacian and a Laplacian (also known as $(p;2)$-equations) arise in mathematical physics, see, for example, [**?**] (quantum physics), [**?**] (plasma physics) and [**?**, **?**]. On the other hand, the limit of $p-$harmonic functions as $p \to \infty$, that is, solutions to $-\Delta_p u = 0$ in Ω, has been extensively studied in the literature (see [**?**] and the survey [**?**]) and leads naturally to the infinity Laplacian given by $\Delta_\infty u = \big(D^2 u\, \nabla u\big) \cdot \nabla u$.

Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) are related to the optimal Lipschitz extension problem (see [?] and the survey paper [?]) and find applications in optimal transportation, image processing and tug-of-war games (see, e.g., [?], [?], [?], [?], [?], [?] and the references therein). Also limits of the eigenvalue problem related to the p-laplacian has been exhaustively studied, see [?], [?], [?], [?], [?]. For limits in anisotropic problems like $-\sum_i (|u_{x_i}|^{p-2} u_{x_i})_{x_i} = 0$ we refer to [?], [?], [?], [?] and [?].

Regarding the ideas and methods used in the proofs we point out the following facts: the proof of the uniform convergence of $u_{p,q}$ to u_∞ is based on a priori estimates, that imply weak compactness in Sobolev spaces. After that, one can verify the passage to the limit in the viscosity sense taking care of the different cases that appear. Remark that (**??**) is not continuous as a function of the gradient and hence we have to use the upper and lower semicontinuous envelopes of the PDE in the definition of a viscosity solution.

In the next section we prove our main result, Theorem **??**; in Section **??** we present as an example that illustrates the main features of the limit problem the radial case in an annulus; and in the final section we comment briefly on possible extensions.

2. Proof of Theorem ??.

First, we show that $u_{p,q}$ is uniformly bounded in a Sobolev space. We use v the absolutely minimizing Lipschitz extension (AMLE) of g (that is, a function that extends g inside Ω and minimizes the Lipschitz constant in every subdomain, see [?] for the existence and properties of AMLE functions) as a test function in the variational problem for $u_{p,q}$, (**??**), and we get

$$\int_\Omega \frac{|\nabla u_{p,q}|^p}{p} + \int_\Omega \frac{|\nabla u_{p,q}|^q}{q} \le \int_\Omega \frac{|\nabla v|^p}{p} + \int_\Omega \frac{|\nabla v|^q}{q} \le |\Omega| \left(\frac{L^p}{p} + \frac{L^q}{q} \right).$$

Here L is the Lipschitz constant of g (note that for the AMLE extension of g we have $|\nabla v| \le L$ a.e. in Ω). Therefore, we get

$$\left(\int_\Omega \frac{|\nabla u_{p,q}|^p}{p} \right)^{1/p} \le |\Omega|^{1/p} \left(\frac{L^p}{p} + \frac{L^q}{q} \right)^{1/p}.$$

Hence, we obtain, taking $p \to \infty$,

$$\limsup_{p\to\infty} \left(\int_\Omega \frac{|\nabla u_{p,q}|^p}{p} \right)^{1/p} \le \max\{L,\ 1\}. \tag{2.1}$$

Now, we argue as follows: we fix $r \in (1, \infty)$ and for any $p > r$ large enough we obtain

$$\left(\int_\Omega |\nabla u_{p,q}|^r \right)^{1/r} \le \left(\int_\Omega |\nabla u_{p,q}|^p \right)^{1/p} |\Omega|^{(p-r)/rp} \le C.$$

Hence, extracting a subsequence $p_j \to \infty$ if necessary, we have that

$$u_{p,q} \rightharpoonup u_\infty$$

weakly in $W^{1,r}(\Omega)$ for any $1 < r < \infty$ and uniformly in $\overline{\Omega}$. From (**??**), we obtain that this weak limit verifies

$$\left(\int_\Omega |\nabla u_\infty|^r \right)^{1/r} \leq |\Omega|^{1/r} \max\{L,\ 1\}.$$

As we can assume that the above inequality holds for every r (using a diagonal argument), we get that $u_\infty \in W^{1,\infty}(\Omega)$ and moreover, taking the limit as $r \to \infty$, we obtain

$$|\nabla u_\infty| \leq \max\{L,\ 1\}, \qquad \text{a.e. } x \in \Omega.$$

Now assume that $L \leq 1$ and take v such that $|\nabla v| \leq 1$ and $v = g$ on $\partial\Omega$ (the set of such functions v is not empty since we can just consider as before the AMLE of g in Ω). From our previous arguments we get

$$\int_\Omega \frac{|\nabla u_{p,q}|^q}{q} \leq \int_\Omega \frac{|\nabla u_{p,q}|^p}{p} + \int_\Omega \frac{|\nabla u_{p,q}|^q}{q} \leq \int_\Omega \frac{|\nabla v|^p}{p} + \int_\Omega \frac{|\nabla v|^q}{q}$$
$$\leq \frac{|\Omega|}{p} + \int_\Omega \frac{|\nabla v|^q}{q}.$$

Hence, passing to the limit as $p_j \to \infty$ we obtain

$$\int_\Omega \frac{|\nabla u_\infty|^q}{q} \leq \int_\Omega \frac{|\nabla v|^q}{q}$$

and we conclude that u_∞ is a solution to the variational problem (**??**).

The fact that u_∞ is a solution to (**??**) when $L > 1$ is immediate since we have proved that $|\nabla u_\infty| \leq L$ a.e. $x \in \Omega$ in this case (note that L is the smallest value that $\|\nabla v\|_{L^\infty(\Omega)}$ can have among functions that take the boundary datum g on the boundary).

Now, we look for the equation verified by the limit u_∞ in the viscosity sense.

To this end, we first recall the definition of viscosity sub and supersolution to a nonlinear PDE problem of the form

$$\begin{cases} H(\nabla u, D^2 u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases} \tag{2.2}$$

In general the function H can be discontinuous. Then we denote by H^* and H_* the upper and lower semicontinuous envelopes of H, respectively, defined as

$$H^*(z, S) = \lim_{\varepsilon \to 0} \sup \left\{ H(z', S') \ :\ |z - z'| + |S - S'| < \varepsilon \right\}$$

for $z \in \mathbb{R}^N$ and $S \in \mathbb{S}^N$ (we denote by $\mathbb{S}^N$ the set of symmetric matrices in $\mathbb{R}^{N\times N}$) and

$$H_*(z, S) = -(-H)^*(z, S).$$

Definition 2.1. *A lower semicontinuous function u defined in Ω is a viscosity supersolution of (**??**) if, $u|_{\partial\Omega} \geq g$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that $u - \phi$ has a minimum at x_0, then*

$$H^*(\nabla\phi(x_0), D^2\phi(x_0)) \geq 0.$$

*An upper semicontinuous function u defined in Ω is a viscosity subsolution of (**??**) if, $u|_{\partial\Omega} \leq g$ and, whenever $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ are such that $u - \varphi$ has a maximum at x_0, then*

$$H_*(\nabla\varphi(x_0), D^2\varphi(x_0)) \leq 0.$$

*Finally, a continuous function u defined in Ω is a viscosity solution of (**??**) if it is both a viscosity supersolution and a viscosity subsolution.*

In what follows we will keep the notation used in the above definitions. That is, by ϕ we will denote the test functions such that $u-\phi$ has a minimum in Ω and by φ the test functions such that $u-\varphi$ has a maximum somewhere in Ω.

We refer to [**?**] for more details about general theory of viscosity solutions, and to [**?**] for viscosity solutions related to the $\infty-$Laplacian and the $p-$Laplacian operators.

In our case, to deal with (**??**), we define, for $z \in \mathbb{R}^N$ and $S \in \mathbb{S}^N$ a symmetric real matrix,

$$H(z,S) = \begin{cases} -\langle S\cdot z, z\rangle & \text{for } |z|>1, \\ -\langle S\cdot z, z\rangle & \text{for } |z|=1, \\ -|z|^{q-2}\mathrm{trace}(S) - (q-2)|z|^{q-4}\langle S\cdot z, z\rangle & \text{for } |z|<1. \end{cases} \tag{2.3}$$

As this function H is discontinuous, our first step is to characterize its upper and lower semicontinuous envelopes, H^* and H_*. The upper semicontinuous envelope of H_∞ is given by

$$H^*(z,S) = \begin{cases} -\langle S\cdot z, z\rangle \quad \text{for } |z|>1, \\ \max\Big\{ -\langle S\cdot z, z\rangle, \\ \qquad -|z|^{q-2}\mathrm{trace}(S) - (q-2)|z|^{q-4}\langle S\cdot z, z\rangle\Big\} \text{ for } |z|=1, \\ -|z|^{q-2}\mathrm{trace}(S) - (q-2)|z|^{q-4}\langle S\cdot z, z\rangle \quad \text{for } |z|<1. \end{cases}$$

The lower semicontinuous envelope has the same expression except for the case $|z|=1$ in which the max is replaced by

$$\min\Big\{ -\langle S\cdot z, z\rangle, -|z|^{q-2}\mathrm{trace}(S) - (q-2)|z|^{q-4}\langle S\cdot z, z\rangle\Big\}.$$

Now, we show that a uniform limit of $u_{p,q}$ is a viscosity solution to (**??**). We only have to check that u_∞ is a solution in the sense of Definition **??** with H given by (**??**) since the boundary condition, $u = g$ on $\partial\Omega$, is immediate from the uniform convergence in $\overline{\Omega}$.

First, we check that u_∞ is a viscosity supersolution. Let $x_0 \in \Omega$ and a test function ϕ such that $u_\infty - \phi$ has a strict minimum at x_0.

From the uniform convergence of $u_{p,q}$ to u_∞ as $p_j \to \infty$ we obtain the existence of a sequence $x_j \in \Omega$ such that $x_j \to x_0$ and $u_{p_j,q} - \phi$ has a minimum at x_j.

As $u_{p_j,q}$ is a viscosity solution to (**??**) we have

$$
\begin{aligned}
&-(p_j-2)|\nabla\phi|^{p_j-4}\Delta_\infty\phi(x_j) - |\nabla\phi|^{p_j-2}\Delta\phi(x_j) \\
&\qquad -(q-2)|\nabla\phi|^{q-4}\Delta_\infty\phi(x_j) - |\nabla\phi|^{q-2}\Delta\phi(x_j) \geq 0.
\end{aligned} \tag{2.4}
$$

First, assume that $|\nabla\phi(x_0)| > 1$. We want to show that

$$-\Delta_\infty\phi(x_0) \geq 0.$$

From (**??**), using that $|\nabla\phi(x_j)| \to |\nabla\phi(x_0)| > 1$, we get,

$$
\begin{aligned}
-\Delta_\infty\phi(x_0) = \lim_{j\to\infty}\left(-\Delta_\infty\phi(x_j)\right) &\geq \lim_{j\to\infty} \frac{|\nabla\phi|^2}{(p_j-2)}\Delta\phi(x_j) \\
&+\frac{(q-2)}{(p_j-2)}\frac{|\nabla\phi|^{q-4}}{|\nabla\phi|^{p_j-4}}\Delta_\infty\phi(x_j) + \frac{|\nabla\phi|^{q-2}}{|\nabla\phi|^{p_j-4}}\Delta\phi(x_j) = 0,
\end{aligned}
$$

as we wanted to show.

Now, assume that $|\nabla\phi(x_0)| < 1$. We want to show that

$$-\Delta_q\phi(x_0) \geq 0.$$

Using again (**??**) together with the fact that $|\nabla\phi(x_j)| \to |\nabla\phi(x_0)| < 1$, we obtain

$$
\begin{aligned}
-\Delta_q\phi(x_0) = \lim_{j\to\infty}\left(-\Delta_q\phi(x_j)\right) &\geq \lim_{j\to\infty} |\nabla\phi|^{p_j-2}\Delta\phi(x_j) \\
&+(p_j-2)|\nabla\phi|^{p_j-4}\Delta_\infty\phi(x_j) = 0,
\end{aligned}
$$

as we wanted to show.

In the case $|\nabla\phi(x_0)| = 1$ we need to show that

$$\max\Big\{-\Delta_\infty\phi(x_0),\ -\Delta_q\phi(x_0)\Big\} \geq 0.$$

Here we argue by contradiction. Assume that

$$-\Delta_\infty\phi(x_0) < 0, \qquad \text{and} \qquad -\Delta_q\phi(x_0) < 0. \tag{2.5}$$

Note that (**??**) implies that $\nabla\phi(x_0) \neq 0$ and hence $\nabla\phi(x_j) \neq 0$ for j large enough.

Suppose first that

$$(p_j-2)|\nabla\phi|^{p_j-4}(x_j) \not\to 0$$

for a subsequence. Then, we use again (**??**) to obtain

$$
\begin{aligned}
&-\Delta_\infty \phi(x_j) - \frac{|\nabla \phi|^2}{(p_j-2)} \Delta\phi(x_j) \\
&\qquad - \frac{(q-2)}{(p_j-2)} \frac{|\nabla \phi|^{q-4}}{|\nabla \phi|^{p_j-4}} \Delta_\infty \phi(x_j) - \frac{|\nabla \phi|^{q-2}}{(p_j-2)|\nabla \phi|^{p_j-4}} \Delta\phi(x_j) \geq 0.
\end{aligned}
$$

Taking limit as $p_j \to \infty$, we get $-\Delta_\infty \phi(x_0) \geq 0$, a contradiction.

When

$$
(p_j-2)|\nabla \phi|^{p_j-4}(x_j) \to 0,
$$

we just use (**??**),

$$
-(p_j-2)|\nabla \phi|^{p_j-4} \Delta_\infty \phi(x_j) - |\nabla \phi|^{p_j-2} \Delta\phi(x_j) - \Delta_q \phi(x_j) \geq 0,
$$

and we reach again a contradiction letting $j \to \infty$ since in this case we obtain $-\Delta_q \phi(x_0) \geq 0$ (note that since we have $(p_j-2)|\nabla \phi|^{p_j-4}(x_j) \to 0$ it also holds that $|\nabla \phi|^{p_j-2}(x_j) \to 0$).

Now, to prove that u_∞ is a viscosity supersolution we argue similarly. In this case, take $x_0 \in \Omega$ and a test function φ such that $u - \varphi$ has a strict minimum at x_0.

From the uniform convergence of $u_{p,q}$ to u_∞ as $p_j \to \infty$ we obtain the existence of a sequence $x_j \in \Omega$ such that $x_j \to x_0$ and $u_{p_j,q} - \varphi$ has a minimum at x_j.

Now as $u_{p_j,q}$ is a viscosity solution to (**??**) we have the reverse inequality to (**??**) for φ that is

$$
\begin{aligned}
&-(p_j-2)|\nabla \varphi|^{p_j-4} \Delta_\infty \varphi(x_j) - |\nabla \varphi|^{p_j-2} \Delta\varphi(x_j) \\
&\qquad -(q-2)|\nabla \varphi|^{q-4} \Delta_\infty \varphi(x_j) - |\nabla \varphi|^{q-2} \Delta\varphi(x_j) \leq 0.
\end{aligned} \tag{2.6}
$$

If $|\nabla \varphi(x_0)| > 1$, we aim to show that

$$
-\Delta_\infty \varphi(x_0) \leq 0.
$$

From (**??**), using the fact that $|\nabla \varphi(x_j)| \to |\nabla \varphi(x_0)| > 1$, we get

$$
\begin{aligned}
-\Delta_\infty \varphi(x_0) = \lim_{j\to\infty} -\Delta_\infty \varphi(x_j) &\leq \lim_{j\to\infty} \frac{|\nabla \varphi|^2}{(p_j-2)} \Delta\varphi(x_j) \\
&\qquad + \frac{(q-2)}{(p_j-2)} \frac{|\nabla \varphi|^{q-4}}{|\nabla \varphi|^{p_j-4}} \Delta_\infty \varphi(x_j) + \frac{|\nabla \varphi|^{q-2}}{|\nabla \varphi|^{p_j-4}} \Delta\varphi(x_j) = 0,
\end{aligned}
$$

as we wanted to show.

Now, assume that $|\nabla \varphi(x_0)| < 1$. We want to show that

$$
-\Delta_q \varphi(x_0) \leq 0.
$$

Using again (**??**) together with the fact that $|\nabla\varphi(x_j)| \to |\nabla\varphi(x_0)| < 1$, we obtain

$$-\Delta_q\varphi(x_0) = \lim_{j\to\infty} -\Delta_q\varphi(x_j) \le \lim_{j\to\infty} |\nabla\varphi|^{p_j-2}\Delta\varphi(x_j) \\ +(p_j-2)|\nabla\varphi|^{p_j-4}\Delta_\infty\varphi(x_j) = 0,$$

as we wanted to show.

Finally, for the case $|\nabla\varphi(x_0)| = 1$ we want to show that

$$\min\Big\{ -\Delta_\infty\varphi(x_0),\ -\Delta_q\varphi(x_0)\Big\} \le 0.$$

Again we argue by contradiction. Assume that

$$-\Delta_\infty\varphi(x_0) > 0, \qquad \text{and} \qquad -\Delta_q\varphi(x_0) > 0. \tag{2.7}$$

Note that this implies that $\nabla\varphi(x_j) \ne 0$ for j large enough.

Suppose first that

$$(p_j-2)|\nabla\varphi|^{p_j-4}(x_j) \not\to 0$$

for a subsequence. Then, we use again (**??**) to obtain

$$-\Delta_\infty\varphi(x_j) - \frac{|\nabla\varphi|^2}{(p_j-2)}\Delta\varphi(x_j) \\ -\frac{(q-2)}{(p_j-2)}\frac{|\nabla\varphi|^{q-4}}{|\nabla\varphi|^{p_j-4}}\Delta_\infty\varphi(x_j) - \frac{|\nabla\varphi|^{q-2}}{(p_j-2)|\nabla\varphi|^{p_j-4}}\Delta\varphi(x_j) \le 0.$$

Taking limit as $p_j \to \infty$, we get $-\Delta_\infty\phi(x_0) \le 0$, a contradiction with (**??**).

If

$$(p_j-2)|\nabla\varphi|^{p_j-4}(x_j) \to 0,$$

we use (**??**) in the form

$$-(p_j-2)|\nabla\varphi|^{p_j-4}\Delta_\infty\varphi(x_j) - |\nabla\varphi|^{p_j-2}\Delta\varphi(x_j) - \Delta_q\varphi(x_j) \le 0$$

and we reach again a contradiction letting $j \to \infty$ since in this case we obtain $-\Delta_q\varphi(x_0) \le 0$ (note that $(p_j-2)|\nabla\varphi|^{p_j-4}(x_j) \to 0$ implies that $|\nabla\varphi|^{p_j-4}(x_j) \to 0$).

The proof of Theorem **??** is thus completed.

3. An example

As an example, we consider the case in which the domain is an annulus,

$$\Omega = \{x : a < |x| < b\}$$

and the boundary datum is given by

$$\begin{aligned} g(x) &= g_a \qquad \text{for } |x| = a, \\ g(x) &= g_b \qquad \text{for } |x| = b, \end{aligned}$$

for two constants g_a, g_b. Let as assume that $g_a < g_b$. In this case we look for radial solutions to our limit problem. We have that a radial solution to the q-Laplacian is given by

$$u(r) = c_1 \int_{r_0}^{r} \frac{1}{s^\alpha} ds + c_2 \tag{3.1}$$

with

$$\alpha = \frac{n-1}{q-1} \qquad \text{and} \qquad r = |x|.$$

If $u_\infty = u_q$ then we look for c_1 and c_2 such that

$$u(a) = g_a \qquad \text{and} \qquad u(b) = g_b.$$

Taking $r_0 = a$ in (**??**) we get

$$c_2 = g_a \qquad \text{and} \qquad c_1 = \frac{g_b - g_a}{\int_a^b \frac{1}{s^\alpha} ds}.$$

Now we note that the maximum of $|\nabla u|$ is located at $r = a$ and there we have

$$u'(a) = \frac{c_1}{a^\alpha} = \frac{g_b - g_a}{\int_a^b \frac{a^\alpha}{s^\alpha} ds},$$

and we need

$$g_b - g_a \le \int_a^b \frac{a^\alpha}{s^\alpha} ds$$

to fulfill the constraint $|\nabla u| \le 1$. When this condition holds we have that our limit u_∞ is the q–harmonic extension of the boundary datum and is given by (**??**).

When this condition does not hold, then we look for a zone close to $r = a$ in which the solution u is a cone of slope one, that is

$$u(r) = g_a + r - a \qquad \text{for } a < r < r_0,$$

and a q–harmonic function, given by (**??**), for $r_0 < r < b$. In this case, continuity reasons imply

$$g_a + r_0 - a = c_2, \qquad \text{and} \qquad c_1 = r_0^\alpha.$$

Now we have to choose r_0 in such a way that

$$g_b = u(b) = \int_{r_0}^{b} \frac{r_0^\alpha}{s^\alpha} ds + g_a + r_0 - a$$

that is,

$$g_b - g_a = \int_{r_0}^{b} \frac{r_0^\alpha}{s^\alpha} ds + r_0 - a := H(r_0).$$

Note that

$$H(a) < g_b - g_a, \;\; H'(r_0) > 0 \text{ and } H(b) = b - a \ge g_b - g_a$$

when the Lipschitz constant of the boundary datum is less or equal than one. Hence, for

$$b-a > g_b - g_a > \int_a^b \frac{a^\alpha}{s^\alpha} ds,$$

we infer that there is a unique solution to $H(r_0) = g_b - g_a$ such that $a < r_0 < b$ and in this case the solution u_∞ is a cone for $a < r < r_0$ and a $q-$Harmonic function for $r_0 < r < b$.

Finally, for

$$g_b - g_a \geq b - a$$

we have that the solution is the AMLE of the boundary datum and hence it is given by the cone

$$u_\infty(r) = g_a + \frac{(g_b - g_a)(r-a)}{b-a}.$$

Therefore, we conclude that for an annulus the limit function u_∞ is given by

$$u_\infty = \begin{cases} u_q, & \text{for } g_b - g_a \leq \int_a^b \frac{a^\alpha}{s^\alpha} ds, \\ cone\chi_{(a,r_0)} + u_q \chi_{(r_0,b)}, & \text{for } b-a > g_b - g_a > \int_a^b \frac{a^\alpha}{s^\alpha} ds, \\ cone, & \text{for } g_b - g_a \geq b - a. \end{cases}$$

Here u_q stands for a $q-$harmonic function with appropriate boundary values.

Remark 3.1. Remark that the example of the annulus shows that there are boundary data g with Lipschitz constant strictly less that one such that the limit u_∞ is not given by the $q-$harmonic extension.

Remark 3.2. For a small boundary datum it holds that $u_\infty = u_q$ the $q-$harmonic extension of g in Ω. This fact holds for general domains and data not only for the annulus. In fact, if we take a fixed g and consider as boundary datum a multiple of it, $g_k = kg$, we have that the solution to $-\Delta_q u = 0$ with $u|_{\partial\Omega} = g_k$, that we denote by $u_{q,k}$, is a multiple of the solution with datum g, that is, $u_{q,k} = k u_{q,1}$ and since $\|\nabla u_{q,1}\|_{L^\infty(\Omega)}$ is finite we conclude that there exists k_0 such that for all $k < k_0$ it holds that $\|\nabla u_{q,k}\|_{L^\infty(\Omega)} = k\|\nabla u_{q,1}\|_{L^\infty(\Omega)} \leq 1$ and then for those k we get $u_\infty = u_q$.

Remark 3.3. In the case of the annulus it also holds that for large k the limit u_∞ is the AMLE of $g_k = kg$ in Ω. In fact, this phenomena is general for every datum g such that the AMLE v of g in Ω is such that there exists a positive constant c such that $|\nabla v| \geq c > 0$. Indeed, in this case, for k large enough, we have $k|\nabla v| \geq 1$ and therefore $|\nabla u_\infty| \geq 1$ a.e in Ω. Hence u_∞ is infinity harmonic in Ω with boundary values g_k and we conclude that u_∞ is the AMLE of g_k in Ω, see [**?**].

4. Extensions

We can also consider the case in which $q \to \infty$ as well as $p \to \infty$. In this case (that is simpler than the one presented here) we just obtain that there is a unique limit u_∞ that can be characterized as being the unique viscosity solution to

$$
\begin{cases} -\Delta_\infty u = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{4.1}
$$

that is, u_∞ is the AMLE extension of g in Ω. Remark that in this case we have uniqueness for the limit problem (see [?]) and hence the existence of the full limit $\lim_{p,q\to\infty} u_{p,q}$.

The results presented here can be extended to the non-homogeneous case, that is, we can consider the problem

$$
\begin{cases} -\Delta_p u - \Delta_q u = f, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega, \end{cases} \tag{4.2}
$$

and we obtain that, for a continuous right hand side f and a fixed q, the limit PDE problem is given by $-\Delta_\infty u = 0$ when $|\nabla u| > 1$ and $-\Delta_q u = f$ in Ω when $|\nabla u| < 1$ with the boundary condition $u = g$ on $\partial\Omega$, while the variational limit problem is given by

$$
\min_{|\nabla u|\le 1,\, u|_{\partial\Omega}=g} \int_\Omega \frac{|\nabla u|^q}{q} - \int_\Omega uf,
$$

when $L \le 1$. The uniform bounds needed to pass to the limit can be obtained as in [?].

Finally, let us point out that we can consider the pointwise gradient constraint to hold only in a subdomain $D \subset \Omega$, that is,

$$
\min_{\substack{|\nabla u| \le 1 \text{ in } D, \\ u|_{\partial\Omega} = g}} \int_\Omega \frac{|\nabla u|^q}{q}.
$$

In this case, we only have to consider the functional

$$
F_{p,q}(u) = \int_D \frac{|\nabla u|^p}{p} + \int_\Omega \frac{|\nabla u|^q}{q}
$$

and assume that the set

$$
\left\{ u \in W^{1,q}(\Omega) \,:\, u = g \text{ on } \partial\Omega \text{ and } |\nabla u| \le 1 \text{ in } D \right\}
$$

is not empty.

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