Complete blow-up and avalanche formation for a parabolic system with non-simultaneous blow-up

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Abstract

We study the possibility of defining a nontrivial continuation after the blow-up time for a system of two heat equations with a nonlinear coupling at the boundary. It turns out that any possible continuation that verify a maximum principle is identically infinity after the blow-up time, that is, both components blow up completely. We also analyze the propagation of the singularity to the whole space, the avalanche, when blow-up is non-simultaneous.

1 Introduction and main results

We consider solutions (u, v) to two heat equations in the half line, $\mathbb{R}_+ = (0, \infty)$,

$$\begin{cases} u_t = u_{xx}, \\ v_t = v_{xx}, \end{cases} \quad (x, t) \in \mathbb{R}_+ \times (0, T), \tag{1.1}$$

with a nonlinear flux coupling at the boundary

$$\begin{cases} -u_x(0, t) = u^{p_{11}}(0, t) v^{p_{12}}(0, t), \\ -v_x(0, t) = u^{p_{21}}(0, t) v^{p_{22}}(0, t), \end{cases} \quad t \in (0, T). \tag{1.2}$$

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The initial data

$$\begin{cases} u(x,0) = u_0(x), \\ v(x,0) = v_0(x), \end{cases} \quad x \in \mathbb{R}_+, \tag{1.3}$$

are assumed to be nonnegative, nontrivial, continuous, integrable and bounded. We will also assume that they are compatible with the boundary conditions, so that solutions may be (and will be) understood in a classical sense. In order to have a totally coupled system, we impose the condition $p_{ij} > 0$ on the nonlinearities. We also require the monotonicity in time of the solution, $u_t, v_t \geq 0$. This hypothesis, which is common in the literature, is satisfied if $u_0'' \geq 0$ and $v_0'' \geq 0$.

The time T denotes the maximal existence time for the solution (u, v). If it is infinite we say that the solution is *global*. If it is finite, we have

$$\limsup_{t \nearrow T} \left\{ \|u(\cdot, t)\|_\infty + \|v(\cdot, t)\|_\infty \right\} = \infty,$$

and we say that the solution *blows up*. There are solutions of (1.1)–(1.3) which blow up if and only if the exponents p_{ij} satisfy any of the following conditions,

$$p_{11} > 1, \quad p_{22} > 1, \quad p_{12} p_{21} > (1 - p_{11})(1 - p_{22}),$$

see [16]. The study of blow-up due to reaction at the boundary, both for scalar problems and for systems (like the one under consideration here), has attracted a lot of attention in recent years, see for example the surveys [4], [6], [11] and the references therein.

The speed at which blow up takes place (the so called *blow-up rate*), that can be obtained as in [2], implies that for any $x \neq 0$ there is a constant $K = K(x)$ such that $\sup_{t \in (0,T)} \{ |u(x,t)| + |v(x,t)| \} \leq K$. Hence, (u, v) blows up only at the origin. Thus, there may be a nontrivial extension of the solution for times $t > T$ in some weak sense. If such a continuation exists, blow-up is said to be *incomplete*; otherwise, it is called *complete*. Complete blow-up was first studied for problems where the nonlinearity occurs in the equation as a reaction term, $u_t = u_{xx} + f(u)$, see [1], [9], [10], [11], [12], [13], [18]. For the scalar version of the present problem complete blow-up is proved in [7], see also [17].

Our first aim is to study whether blow-up for problem (1.1)–(1.3) is complete or not. A natural way of obtaining a continuation consists of approximating the reaction nonlinearities in the boundary conditions by a sequence of functions that yield global in time solutions, and then pass to the limit in the approximations. Thus, we solve the heat equations (1.1) with initial data (1.3) and boundary conditions

$$\begin{cases} -u_x(0,t) = f_n^{11}(u(0,t)) f_n^{12}(v(0,t)), \\ -v_x(0,t) = f_n^{21}(u(0,t)) f_n^{22}(v(0,t)), \end{cases} \quad t > 0,$$

where

$$f_n^{ij}(s) = \min\{s^{p_{ij}}, n^{p_{ij}}\}, \tag{1.4}$$

to obtain a globally defined solution (u_n, v_n). Since the coupling functions f_n^{ij} increase with n, the same is true for u_n and v_n. Hence one may attempt to extend the solution

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after T by taking the limit

$$\lim_{n\to\infty} u_n = \overline{u}, \qquad \lim_{n\to\infty} v_n = \overline{v}.$$

The extension $(\overline{u}, \overline{v})$ obtained in this way is known in the literature as the *proper solution*, see [10], [11]. It is a minimal solution in the sense that any solution that satisfies a comparison principle must be above it. Next theorem shows that both components of the proper solution, and hence both components of any other reasonable extension, become infinite after the blow-up time; i.e., blow-up is always complete.

Theorem 1.1 *If a solution of* (1.1)–(1.3) *blows up at a finite time* T, *it blows up completely. More precisely, for all* $x \in [0, \infty)$ *the proper solution satisfies that*

$$\overline{u}(x,t) = \begin{cases} u(x,t), & 0 < t < T, \\ \lim_{t \nearrow T} u(x,t), & t = T, \\ \infty, & t > T, \end{cases} \qquad \overline{v}(x,t) = \begin{cases} v(x,t), & 0 < t < T, \\ \lim_{t \nearrow T} v(x,t), & t = T, \\ \infty, & t > T. \end{cases}$$

Observe that blow-up is complete for both components even if one of them remains bounded up to the blow-up time, a possibility that is not excluded a priori. Indeed, for certain choices of the parameters p_{ij} there are initial data for which one of the components of the system remains bounded while the other blows up. This phenomenon is commonly denoted as *non-simultaneous blow-up*. The possibility of non-simultaneous blow-up in nonlinear parabolic systems was first mentioned in [19], and has been studied more thoroughly later in [3], [15], [16] and [20]. For problem (1.1)–(1.3) this possibility was analyzed in [14], [16]: there exist solutions such that u blows up at time T while v remains bounded up to this time if and only if

$$p_{11} > p_{21} + 1. \tag{1.5}$$

Since blow-up takes place only at one point and there is complete blow-up, at $t = T$ an instantaneous propagation of the blow-up singularity to the whole spatial domain takes place, what is called an *avalanche*, see [17], [18].

The avalanche may be regarded as a discontinuity at the blow-up time between the nontrivial blow-up profiles

$$\overline{u}(x, T^-) = \lim_{t \nearrow T} u(x,t) \ \text{ and } \ \overline{v}(x, T^-) = \lim_{t \nearrow T} v(x,t),$$

and the trivial values taken afterwards,

$$\overline{u}(x, T^+) = \infty = \overline{v}(x, T^+) \qquad \text{for all } x \in [0, \infty).$$

In the case of non-simultaneous blow-up, say u blows up while v remains bounded up to time T, the discontinuity at $t = T$ between $\overline{v}(x, T^-)$ (which is finite everywhere) and $\overline{v}(x, T^+)$ (which is infinite everywhere) is even more striking. In the sequel we will

confine ourselves to this more appealing case, assuming that u blows up while v remains bounded up to $t = T^-$. In particular, (1.5) holds and we have $p_{11} > 1$.

The instantaneous propagation of the singularity to the whole domain has a counterpart when we consider finite, but large, values of n: the propagation of k-level sets for k large. Our next aim is to explain the evolution of such level sets. We will devote special attention to $k = n^\gamma$. Let us remark that the study of the approximate problems for large n is in many cases (combustion, chemistry) more realistic than the blow-up problem, which is a mathematical idealization.

Let $t_u(k)$ be the first time when $u_n(0, t) = k$ and $t_v(k)$ be the first time when $v_n(0, t) = k$. Hence $t_u(n)$ and $t_v(n)$ are the times at which truncations (1.4) start taking place. In the next result we estimate these times.

Theorem 1.2 *Let n be large enough and let $0 < \gamma \le 1$. It holds,*

$$T - t_u(n^\gamma) \quad \sim \quad n^{-2\gamma(p_{11}-1)}, \tag{1.6}$$

$$t_v(n^\gamma) - T \quad \sim \quad \begin{cases} n^{-2p_{21}}, & p_{22} > 1, \\ n^{-2p_{21}} \log n, & p_{22} = 1, \\ n^{-2p_{21}+2\gamma(1-p_{22})}, & p_{22} < 1. \end{cases} \tag{1.7}$$

By $f \sim g$ we mean that there exist constants $c_1, c_2 > 0$ such that $c_1 f \le g \le c_2 f$.

Since $p_{11} > 1$, $t_u(n) \nearrow T$ as $n \to \infty$. However, the behaviour of $t_v(n)$ depends on the involved exponents. It may happen that $t_v(n) \to \infty$ as $n \to \infty$ (this occurs if $p_{21} + p_{22} < 1$). This does not contradict the complete blow-up result for $\bar{v}$, since the size of v_n for $t > T$ can be very large for $t \approx T$ without reaching the level n before a long time after T. In fact, when $p_{22} < 1$, $t_v(n^\gamma) \to T$ as $n \to \infty$ if and only if $\gamma < p_{21}/(1 - p_{22})$.

When $p_{22} \ge 1$, the order of magnitude of the quantities $t_v(n^\gamma) - T$ does not depend on γ. Are they equal up to leading order? The answer is given in the next theorem: they are equal when $p_{22} > 1$ and differ by a constant coefficient depending on γ when $p_{22} = 1$.

Theorem 1.3 *Let n be large enough and let $0 < \gamma \le 1$. It holds,*

(i) *If $p_{22} > 1$, then $t_v(n) - t_v(n^\gamma) \sim n^{2(\gamma(p_{22}-1)-p_{21})}$.*

(ii) *If $p_{22} = 1$, then $\displaystyle\lim_{n\to\infty} \frac{n^{2p_{21}}}{\log n}(t_v(n) - t_v(n^\gamma)) = 1 - \gamma$.*

Next, we want to describe how the level n starts to propagate to the interior of the domain. To this aim, we define

$$v_T = \lim_{t \nearrow T} v(0, t)$$

and look at times which are close to $t_u(n)$, and scale u_n by a factor n so that we get something of order one. If we want the new dependent variable to be a solution to the

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heat equation, we also have to scale the space variable. The following result shows how the n-level set of u_n evolves for times $t \approx T$ close to the origin, $x \approx 0$.

Theorem 1.4 *There exists a nontrivial function* $\Phi : \mathbb{R}_+ \times \mathbb{R} \mapsto \mathbb{R}$ *such that*

$$\lim_{n \to \infty} n^{-1} u_n(n^{-(p_{11}-1)} y, t_u(n) + n^{-2(p_{11}-1)} \tau) = \Phi(y, \tau),$$

$$\lim_{n \to \infty} v_n(n^{-(p_{11}-1)} y, t_u(n) + n^{-2(p_{11}-1)} \tau) = v_T,$$

uniformly on compact subsets of $\mathbb{R}_+ \times \mathbb{R}$. *The limit* Φ *depends on the initial data only through* v_T.

Theorem 1.4 implies that the n-level set of u_n written in terms of (x, t) is given approximately by the 1-level set of the limit function Φ written in (y, τ) variables,

$$\{(x, t) \; : \; u_n(x, t) = n\} \approx \{(y, \tau) \; : \; \Phi(y, \tau) = 1\}.$$

These sets are related through the transformation

$$x = n^{-(p_{11}-1)} y, \qquad t = t_u(n) + n^{-2(p_{11}-1)} \tau.$$

A monotonicity argument shows that the 1-level set of Φ can be expressed as the graph of some increasing function g. Since $t_u(n) \approx T$, this in turn implies that the n-level set of u_n is given approximately by

$$\{(x, t) \; : \; x = n^{-(p_{11}-1)} g\big(n^{2(p_{11}-1)}(t - T)\big)\}.$$

Remark. The function that describes the onset of the avalanche for u reaches the truncation level 1 at some finite time. Hence, its long time behaviour is given by a self-similar solution of the heat equation in the half-line with inwards flux equal to $v_T^{p_{12}}$ at the boundary, i.e.

$$w(y, \tau) \sim v_T^{p_{12}} \tau^{1/2} \Phi\left(\frac{y}{\tau^{1/2}}\right),$$

where Φ solves $\Phi''(\xi) + \frac{1}{2}\xi\Phi'(\xi) - \frac{1}{2}\Phi(\xi) = 0$, $-\Phi'(0) = 1$, see [17] for the details. Using this asymptotic behaviour we see that the k-level set of u_n behaves as

$$y = \tau^{1/2} \Phi^{-1}\left(\frac{k}{v_T^{p_{12}} \tau^{1/2}}\right).$$

We have described the onset of the avalanche for the u variable. However, since the obtained profile for v is the constant v_T, there is some lack of information concerning the spatial shape of the v variable. To obtain a more precise description of the behaviour of v_n we have to look at times which are a little bit larger.

Theorem 1.5 *There exists a nontrivial function* $\Psi : \mathbb{R}_+ \times [0, \tau_0) \mapsto \mathbb{R}$ *such that*

$$\lim_{n \to \infty} v_n(n^{-p_{21}}y, T + n^{-2p_{21}}\tau) = \Psi(y, \tau), \qquad (1.8)$$

uniformly on compact subsets of $\mathbb{R}_+ \times [0, \tau_0)$. *The time* τ_0 *verifies* $0 < \tau_0 < \infty$ *if* $p_{22} > 1$, $\tau_0 = \infty$ *if* $p_{22} \le 1$. *The limit* Ψ *depends on the initial data only through* v_T.

Remark. If $p_{22} > 1$ the limit function Ψ blows up at the threshold time $\tau = \tau_0$.

Remark. The fact that τ_0 has to be finite for $p_{22} > 1$ and not for $p_{22} \le 1$ could have been guessed from Theorem 1.2.

The behaviour of the n-level set of v_n in terms of the graph of some increasing function follows arguing as we did before for u_n.

Theorems 1.4 and 1.5 provide information near the origin close to the blow-up time. To completely describe the avalanche, we also give the behaviour of both components, u_n and v_n, for fixed (x, t), with $t > T$.

Theorem 1.6 *Let*

$$\gamma_1 = p_{11} + p_{12}, \qquad \gamma_2 = p_{21} + p_{22} \qquad if \ p_{21} + p_{22} > 1,$$

$$\gamma_1 = p_{11} + \frac{p_{12}p_{21}}{1 - p_{22}}, \qquad \gamma_2 = \frac{p_{21}}{1 - p_{22}} \qquad if \ p_{21} + p_{22} \le 1.$$

There exist profiles φ *and* ψ *such that*

$$\lim_{n \to \infty} n^{-\gamma_1} u_n(x, t) = \varphi(x, t), \qquad \lim_{n \to \infty} n^{-\gamma_2} v_n(x, t) = \psi(x, t),$$

uniformly on compact subsets of $[0, \infty) \times (T, \infty)$. *The limits* φ, ψ *are independent of the initial data.*

The quantity $p_{21} + p_{22}$ measures the combined strength of the two reaction factors in the boundary condition for v. If it is above 1, reaction is big, and v_n reaches the n-level quickly, in a time $t_v(n) \to T$ as $n \to \infty$. Hence, for any time $t > T$ all truncations have taken place for n large. Thus, u_n and v_n are both solutions of the heat equation with inwards boundary fluxes given respectively by $n^{p_{11}+p_{12}}$ and $n^{p_{21}+p_{22}}$. What the theorem says is that we have to scale precisely by these powers of n if we want to obtain a non-trivial limit. Consider now $p_{21} + p_{22} < 1$. Given any fixed time t, the truncation of v_n takes place afterwards for all n large enough. In this case the exponent γ_2 which gives the size of v_n coincides with the critical value of γ below which $t_v(n^\gamma)$ goes to T. The adequate value of γ_1 follows by inserting the size of v_n in the boundary condition for u_n. In any case, the scaling exponents γ_1, γ_2 are the only ones such that $t_u(n^{\gamma_1})$, $t_v(n^{\gamma_2})$ do not go neither to T nor to infinity, see Section 3.

Our last step consists in considering the asymptotic behaviour of u_n and v_n as $t \to \infty$. Since we are considering $t > t_v(n)$, both truncations have taken place. Again, u_n and

v_n are solutions of the heat equation with inwards boundary fluxes given respectively by $n^{p_{11}+p_{12}}$ and $n^{p_{21}+p_{22}}$. As is well known, solutions to this problem with integrable initial datum converge, uniformly on compact subsets, as t goes to infinity to the solution of the same problem with zero initial data, see for instance [17]. This solution is self-similar, and coincides with the limit functions of Theorem 1.6 in the case $p_{21} + p_{22} > 1$.

Remark. The nontrivial limit profiles that appear in Theorems 1.4, 1.5, and 1.6 are solutions of explicit problems, see Sections 4 and 5.

ORGANIZATION OF THE PAPER. In Section 2 we prove complete blow-up for both components, Theorem 1.1. The estimates for $t_u(n^\gamma) - T$ and $t_v(n^\gamma) - T$ that give us the right scalings to describe the avalanche are gathered in Section 3. It includes both the case $\gamma \leq 1$, Theorem 1.2, and the case $\gamma > 1$. In order to deal with this latter case we have to study the size of u_n at time $t_v(n)$, a result that is interesting on its own. Section 4 is devoted to prove Theorems 1.4 and 1.5, which describe the onset of the avalanche. Finally, in Section 5, we prove Theorem 1.6.

Throughout the paper C, c denote constants, independent of n, which may be different in different occurrences.

2 Complete blow-up

Proof of Theorem 1.1. We can assume without loss of generality that u blows up at time $t = T$. If this not the case, then v has to blow up, and the same proof applies interchanging the roles of u and v. The component v may blow up or not, since we have not assumed a priori that blow-up is non-simultaneous.

For times before the blow-up time, $t < T$, (u, v) is bounded. Hence, if n is large enough, (u, v) solves the truncated problem up to time t and therefore $u_n(x, t) = u(x, t)$ and $v_n(x, t) = v(x, t)$.

To study what happens exactly at $t = T$ we use the monotonicity with respect to t and a comparison argument. For $t < T$ we have

$$\lim_{t \nearrow T} u(x, t) \geq u_n(x, T) \geq u_n(x, t).$$

Taking limits as $n \to \infty$, and using that $\lim_{n \to \infty} u_n(x, t) = u(x, t)$ for $t < T$, and then letting $t \nearrow T$, we get

$$\lim_{t \nearrow T} u(x, t) \geq \lim_{n \to \infty} u_n(x, T) \geq \lim_{t \nearrow T} u(x, t).$$

The same applies to v.

To end the proof we have to study the behaviour for $t > T$. From the previous step we know that $\lim_{t \nearrow T} u(0, t) = \lim_{n \to \infty} u_n(0, T)$. Since u_n is increasing in time and u blows up at the origin,

$$\infty = \lim_{t \nearrow T} u(0, t) = \lim_{n \to \infty} u_n(0, T) \leq \lim_{n \to \infty} u_n(0, t), \tag{2.1}$$

and we conclude that the proper solution, $\bar{u} = \lim_{n \to \infty} u_n$, is identically infinite at the origin for $t > T$. In order to propagate the singularity to the whole interval for both components we use the representation formula obtained from the heat kernel. Let Γ be the fundamental solution of the heat equation in $\mathbb{R}_+ \times (0, \infty)$, namely

$$\Gamma(x, t) = \frac{1}{(\pi t)^{1/2}} \exp\left(-\frac{x^2}{4t}\right).$$

For $x \in \mathbb{R}_+$ we have

$$u_n(x, t) = \int_{\mathbb{R}_+} u_n(y, 0) \Gamma(x - y, t) \, dy - \int_0^t \frac{\partial u_n}{\partial x}(0, \tau) \Gamma(x, t - \tau) \, d\tau \\ - \int_0^t u_n(0, \tau) \frac{\partial \Gamma}{\partial x}(x, t - \tau) \, d\tau. \tag{2.2}$$

Since Γ and the boundary flux, $-(u_n)_x(0, \tau)$, are both nonnegative we can bound $u_n(x, t)$ from below by

$$u_n(x, t) \geq -\int_0^t u_n(0, \tau) \frac{\partial \Gamma}{\partial x}(x, t - \tau) \, d\tau.$$

From (2.1) we have that $u_n(0, t) \geq M$ if n is large enough, how large depending on M. Hence, for any $0 < \delta < t - T$,

$$u_n(x, t) \geq M \int_T^{T+\delta} \left(-\frac{\partial \Gamma}{\partial x}(x, t - \tau)\right) d\tau.$$

We conclude that

$$\lim_{n \to \infty} u_n(x, t) = \infty, \qquad t > T, \ x \in \mathbb{R}_+.$$

This proves complete blow-up for u.

To obtain complete blow-up for v we also use the representation formula. Since v_n grows due to the influence of u_n through the boundary flux, we keep the flux term in the bound from below:

$$v_n(x, t) \geq \int_0^t f_n^{21}(u_n(0, \tau)) f_n^{22}(v_n(0, \tau)) \Gamma(x, t - \tau) \, d\tau.$$

From the monotonicity of the solutions we have that $v_n(0, t) \geq v_n(0, 0) = v(0, 0) = c > 0$. Therefore, if n is large enough and for any $0 < \delta < t - T$,

$$v_n(x, t) \geq M^{p_{21}} c^{p_{22}} \int_T^{T+\delta} \Gamma(x, t - \tau) \, d\tau.$$

This implies complete blow-up for v, that is,

$$\lim_{n \to \infty} v_n(x, t) = \infty, \qquad t > T, \ x \in \mathbb{R}_+.$$

3 Time estimates

The aim of this section is to find estimates for the differences $t_u(n^\gamma) - T$ and $t_v(n^\gamma) - T$ for different values of γ.

As blow-up is non-simultaneous, the u component touches the level n before the v component, which is bounded up to time T. Since the solutions of the truncated problems coincide with those of the non-truncated one until some component reaches the level n, we have $t_u(n) < T < t_v(n^\gamma)$ for any $\gamma > 0$ and large n. If $\gamma > 1$, $t_u(n^\gamma) > t_u(n)$. However, we cannot state at this point whether $t_u(n^\gamma)$ lies before or after T. The aim of this section is to estimate the times $t_u(n^\gamma)$ and $t_v(n^\gamma)$ for any $\gamma > 0$. In order to compute $t_u(n^\gamma)$ for $\gamma > 1$ we need to estimate the size of u_n at time $t_v(n)$, a result that is interesting on its own.

In the sequel we use the notation

$$\alpha = \frac{p_{21}}{(p_{22} - 1)}, \qquad \beta = \frac{1}{2(p_{22} - 1)}$$

for a couple of exponents that will appear frequently.

3.1 Estimates for $0 < \gamma \leq 1$

Proof of Theorem 1.2. We first obtain an estimate for $t_u(n^\gamma)$, which will then be used to estimate $t_v(n^\gamma)$.

Estimates for $t_u(n^\gamma)$. Since $t_u(n^\gamma) \leq t_u(n) < T$ for $\gamma \leq 1$, from the blow-up rate of u, which is known to be, see [8],

$$\max_x u(x,t) = u(0,t) \sim (T-t)^{-\frac{1}{2(p_{11}-1)}}, \tag{3.1}$$

we can obtain the estimate for $t_u(n^\gamma)$. Indeed, from (3.1) we get

$$n^\gamma \sim (T - t_u(n^\gamma))^{-\frac{1}{2(p_{11}-1)}},$$

which is equivalent to (1.6).

Estimates for $t_v(n^\gamma)$. We look at the problem satisfied by v_n for times larger than $t_u(n)$. For these times the nonlinearity that involves the u variable is truncated. Hence, v_n is a solution to

$$\begin{cases} (v_n)_t = (v_n)_{xx}, & (x,t) \in \mathbb{R}_+ \times (t_u(n), \infty), \\ -(v_n)_x(0,t) = n^{p_{21}} f_n^{22}(v_n(0,t)), & t \in (t_u(n), \infty), \\ v_n(x, t_u(n)) = v(x, t_u(n)), & x \in \mathbb{R}_+. \end{cases} \tag{3.2}$$

Since blow-up is non-simultaneous, v is bounded up to time T. Hence the initial datum in (3.2) is also bounded, $v(x, t_u(n)) < v(x, T)$. The idea is comparing v_n with a self-similar solution z which has an explicit formula, so that $t_z(n^\gamma)$, the first time when z reaches level n^γ, can be computed.

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$\boxed{p_{22} > 1}$ The function $z(x,t) = n^{-\alpha}(\widehat{T}-t)^{-\beta}\phi(x(\widehat{T}-t)^{-1/2})$, is a self-similar solution to (3.2) for times at which $f_n^{22}(z) = z^{p_{22}}$, i.e., for $t \leq t_z(n)$. See [8] for the existence of the profile ϕ, which has an explicit formula.

If we take $n^{-\alpha}\widehat{T}^{-\beta} \leq C$, then $z(x,0) = n^{-\alpha}\widehat{T}^{-\beta}\phi(x\widehat{T}^{-1/2}) \leq v_n(x,t_u(n))$. Using a comparison argument it follows that

$$z(x,t) \leq v_n(x,t+t_u(n)), \qquad 0 < t < t_z(n).$$

Since $\gamma \leq 1$, $t_z(n^\gamma) \leq t_z(n)$, and hence

$$t_v(n^\gamma) \leq t_u(n) + t_z(n^\gamma). \tag{3.3}$$

But we have an explicit estimate for $t_z(n^\gamma)$:

$$z(0,t_z(n^\gamma)) = n^\gamma = n^{-\alpha}(\widehat{T}-t_z(n^\gamma))^{-\beta}\phi(0).$$

Hence

$$\widehat{T} - t_z(n^\gamma) \sim n^{-2(\gamma(p_{22}-1)+p_{21})}, \tag{3.4}$$

and therefore, using that $\widehat{T} \leq Cn^{-2p_{21}} = Cn^{-2p_{21}}$, we get

$$t_z(n^\gamma) \leq -Cn^{-2(\gamma(p_{22}-1)+p_{21})} + Cn^{-2p_{21}} \leq Cn^{-2p_{21}}. \tag{3.5}$$

From (3.3) using the non-simultaneous blow-up condition on the exponents, $p_{21} < p_{11}-1$, we obtain,

$$t_v(n^\gamma) - T \leq t_u(n) - T + t_z(n^\gamma) \leq -Cn^{-2(p_{11}-1)} + Cn^{-2p_{21}} \leq Cn^{-2p_{21}}.$$

On the other hand, if we now choose C, independent of n, so that $n^{-\alpha}\widehat{T}^{-\beta} \geq C$, we get $z(x,0) \geq v_n(x,t_u(n))$ and hence

$$z(x,t) \geq v_n(x,t+t_u(n)), \qquad t > t_z(n).$$

Using the same idea we get $t_v(n^\gamma) - T \geq cn^{-2p_{21}}$, and conclude that $t_v(n^\gamma) - T \sim n^{-2p_{21}}$.

$\boxed{p_{22} < 1}$ The same argument can be carried over considering the self-similar solution $z(x,t) = n^{-\alpha}(\widehat{T}+t)^{-\beta}\phi(x(\widehat{T}+t)^{-1/2})$. The main difference between this and the previous case is that now, since $p_{22} < 1$, the estimate (3.5) becomes

$$t_z(n^\gamma) \leq -Cn^{-2(\gamma(p_{22}-1)+p_{21})} + Cn^{-2p_{21}} \leq Cn^{-2(\gamma(p_{22}-1)+p_{21})},$$

respectively $\geq$. In this case, we conclude that

$$t_v(n^\gamma) - T \sim -n^{2(p_{11}-1)} + n^{2(1-p_{22})\gamma-2p_{21}} \sim n^{2(1-p_{22})\gamma-2p_{21}}.$$

$\boxed{p_{22} = 1}$ In this critical case, if we look for a self-similar solution as before, α and β are not well defined and hence we take as self-similar solution of (3.2), $z(x,t) = Ke^{at-bx}$,

with $a = b^2 = n^{2p_{21}}$ and K a constant. Arguing as before, taking K small or big, so that $z(x,0) \sim v_n(x, t_u(n))$, we obtain $t_v(n^\gamma) - T \sim n^{-2p_{21}} \log n$. $\qquad\square$

Remark. As a byproduct of this proof we have obtained that

$$v(x, t + t_u(n)) \sim z(x, t), \qquad (3.6)$$

for times $0 < t < t_z(n) \sim t_v(n) - t_u(n)$, with z the self-similar solution defined before according to the sign of $p_{22} - 1$. We also obtain that

$$t_z(n) \sim t_v(n) - t_u(n) \sim \begin{cases} n^{-2p_{21}}, & p_{22} > 1, \\ n^{-2p_{21}+2(1-p_{22})}, & p_{22} < 1, \end{cases} \qquad (3.7)$$

and for $p_{22} = 1$ a better estimate

$$e^{n^{2p_{21}} t_z(n)} \sim e^{n^{2p_{21}}(t_v(n)-t_u(n))} \sim n. \qquad (3.8)$$

Remark. Using (3.3) and (3.4) we get $\widehat{T} - (t_v(n) - t_u(n)) \geq c(\widehat{T} - t_z(n))$. An analogous computation proves the reverse inequality. We conclude that

$$\widehat{T} - (t_v(n) - t_u(n)) \sim \widehat{T} - t_z(n) \sim n^{-2(p_{22}-1)-2p_{21}}. \qquad (3.9)$$

Proof of Theorem 1.3. We use the same technique used in the proof of Theorem 1.2, but taking as initial time $t_v(n^\gamma)$. Let us sketch it briefly for $p_{22} > 1$: v_n is a solution of

$$\begin{cases} (v_n)_t = (v_n)_{xx}, & (x,t) \in \mathbb{R}_+ \times (t_v(n), \infty), \\ -(v_n)_x(0,t) = n^{p_{21}} f_n^{22}(v_n(0,t)), & t \in (t_v(n^\gamma), \infty), \\ v_n(x, t_v(n^\gamma)) = n^\gamma, & x \in \mathbb{R}_+. \end{cases} \qquad (3.10)$$

If we take $n^{-\alpha} \widehat{T}^{-\beta} \sim n^\gamma$, then $z(x,0) = n^{-\alpha} \widehat{T}^{-\beta} \phi(x \widehat{T}^{-1/2}) \sim v_n(x, t_v(n^\gamma))$, and hence, by comparison

$$t_v(n) \sim t_v(n^\gamma) + t_z(n). \qquad (3.11)$$

Since $\widehat{T} - t_z(n) \sim n^{-\frac{1+\alpha}{\beta}}$, we conclude, using that $\gamma < 1$, that $t_z(n) \sim n^{-\frac{\gamma+\alpha}{\beta}}$ and therefore from (3.11), $t_v(n) - t_v(n^\gamma) \sim n^{-\frac{\gamma+\alpha}{\beta}}$. $\qquad\square$

3.2 Sizes

Next theorem gives an estimate of the size of u_n at time $t_v(n)$.

Theorem 3.1 *The size of u_n at time $t_v(n)$ is*

$$u_n(0, t_v(n)) \sim \begin{cases} n^{p_{11}-p_{21}+p_{12}+1-p_{22}}, & p_{22} < 1 + p_{12}, \\ n^{p_{11}-p_{21}} \log n, & p_{22} = 1 + p_{12}, \\ n^{p_{11}-p_{21}}, & p_{22} > 1 + p_{12}. \end{cases}$$

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Proof. Let $t_u(n) < t < t_v(n)$ and $d(n) = t_v(n) - t_u(n)$. For those times we have that u_n is a solution of the heat equation with boundary and initial data given by

$$\begin{cases} -(u_n)_x(0,t) \sim n^{p_{11}} v_n^{p_{12}}(0,t), \\ u_n(x, t_u(n)) \sim n. \end{cases}$$

Consider $\widehat{u}$ a solution to the heat equation in $\mathbb{R}_+ \times (0, \infty)$ with

$$\begin{cases} -\widehat{u}_x(0,t) = n^{p_{11}} v_n^{p_{12}}(0, t + t_u(n)), \\ \widehat{u}(x, 0) = n. \end{cases}$$

A comparison argument gives that $u_n(x, t + t_u(n)) \sim \widehat{u}(x, t)$.

Using formula (2.2) we can estimate $\widehat{u}(0, t)$ at $d(n)$. Indeed,

$$\widehat{u}(0, d(n)) = \int_{\mathbb{R}_+} \widehat{u}(y, d(n)) \Gamma(y, d(n)) \, dy + \int_0^{d(n)} n^{p_{11}} v_n^{p_{12}}(0, \tau + t_u(n)) \Gamma(0, d(n) - \tau) \, d\tau$$

$$\sim n + n^{p_{11}} \int_0^{d(n)} v_n^{p_{12}}(0, \tau)(d(n) - \tau)^{-1/2} \, d\tau = n + n^{p_{11}} I.$$

Hence, our aim is estimating I according to the estimates we have for v_n, see (3.6), in order to get the size of u_n at $t_v(n)$.

$\boxed{p_{22} < 1}$ Using (3.6) for $0 < t < t_v(n) - t_u(n)$, we have $v_n(0, t + t_u(n)) \sim n^{-\alpha}(\widehat{T} + t)^{-\beta}$. Hence,

$$I \sim \int_0^{d(n)} n^{-\alpha p_{12}}(\widehat{T} + \tau)^{-\beta p_{12}}(d(n) - \tau)^{-1/2} \, d\tau$$

$$= n^{-\alpha p_{12}} \int_0^{d(n)} (\widehat{T} + \tau)^{-\beta p_{12}}(d(n) - \tau)^{-1/2} \, d\tau$$

$$\leq C n^{-\alpha p_{12}} \left(d(n)^{-1/2} \int_0^{d(n)/2} (\widehat{T} + \tau)^{-\beta p_{12}} \, d\tau \right.$$

$$\left. + (\widehat{T} + d(n))^{-\beta p_{12}} \int_{d(n)/2}^{d(n)} (d(n) - \tau)^{-1/2} \, d\tau \right)$$

$$\leq C n^{-\alpha p_{12}} \left(d(n)^{-1/2}(d(n) + \widehat{T})^{-\beta p_{12}+1} + (\widehat{T} + d(n))^{-\beta p_{12}} d(n)^{1/2} \right)$$

$$\leq C n^{-\alpha p_{12}} d(n)^{-\beta p_{12}+1/2},$$

where we have used that $d(n) \sim n^{-2p_{21}+2(1-p_{22})} > \widehat{T} \sim n^{-2p_{21}}$, see (3.7). On the other hand,

$$I \sim \int_0^{d(n)} n^{-\alpha p_{12}}(\widehat{T} + \tau)^{-\beta p_{12}}(d(n) - \tau)^{-1/2} \, d\tau$$

$$\geq n^{-\alpha p_{12}}(\widehat{T} + d(n))^{-\beta p_{12}} \int_0^{d(n)} (d(n) - \tau)^{-1/2} \geq C n^{-\alpha p_{12}} d(n)^{-\beta p_{12}+1/2}.$$

We conclude, using again (3.7), that

$$\widehat{u}(0, t_v(n) - t_u(n)) \sim n + n^{p_{11}} n^{-\alpha p_{12}} n^{-2(p_{21}+p_{22}-1)(-\beta p_{12}+1/2)},$$

and hence $u_n(0, t_v(n)) \sim n + n^{p_{11}-p_{21}+p_{12}+1-p_{22}} \sim n^{p_{11}-p_{21}+p_{12}+1-p_{22}}$.

$\boxed{p_{22} = 1}$ According to (3.6) we set $v_n(x, t + t_u(n)) \sim K e^{at-bx}$, so that

$$I \sim \int_0^{d(n)} e^{ap_{12}\tau} (d(n) - \tau)^{-1/2} \, d\tau$$

$$\geq e^{ap_{12}(d(n)-d(n)/\log n)} \int_{d(n)-d(n)/\log n}^{d(n)} (d(n) - \tau)^{-1/2} \, d\tau \geq C n^{p_{12}} n^{p_{21}},$$

where we use that $d(n) \sim n^{2p_{21}} \log(n)$ and estimate (3.8).

Following the idea of the case $p_{22} < 1$, we split I into two integrals that are treated in a different way, in order to obtain a bound from above,

$$I \sim \int_0^{d(n)} e^{ap_{12}\tau} (d(n) - \tau)^{-1/2} \, d\tau$$

$$= \int_0^{d(n)-d(n)/\log n} e^{ap_{12}\tau} (d(n) - \tau)^{-1/2} \, d\tau$$

$$+ \int_{d(n)-d(n)/\log n}^{d(n)} e^{ap_{12}\tau} (d(n) - \tau)^{-1/2} d\tau$$

$$\leq n^{p_{21}} \int_0^{d(n)-d(n)/\log n} e^{ap_{12}\tau} \, d\tau$$

$$+ n^{p_{12}} \int_{d(n)-d(n)/\log n}^{d(n)} (d(n) - \tau)^{-1/2} d\tau \leq C n^{p_{12}-p_{21}}.$$

We conclude that $u(0, t_v(n)) \sim n^{p_{11}+p_{12}-p_{21}}$.

$\boxed{1 < p_{22} < 1 + p_{12}}$ As $p_{22} > 1$, using again (3.6) we get $v_n(0, t+t_u(n)) \sim n^{-\alpha}(\widehat{T}-t)^{-\beta}$.

In order to find a bound from above for $\widehat{u}$ we observe that similar computations as the ones done in the case $p_{22} < 1$ do not yield a "reasonable" bound, since we obtain $\widehat{u} \leq \infty$. Hence, instead of using the representation formula, we compare $\widehat{u}$ with

$$U(x, t) = n^{-\lambda}(\widehat{T} - t)^{-\mu} G(x(\widehat{T} - t)^{-1/2}), \tag{3.12}$$

which is a solution to

$$\begin{cases} U_t = U_{xx}, \\ -U_x(0, t) = n^{p_{11}-\alpha p_{12}}(\widehat{T} - t)^{-\beta p_{12}} G'(0), \\ U(x, 0) = n^{p_{11}-p_{21}} G(x\widehat{T}^{-1/2}), \end{cases}$$

if $\mu = \beta p_{12} - 1/2$ and $\lambda = \alpha p_{12} - p_{11}$. Due to the assumption of non-simultaneous blow-up we have that $p_{11} - p_{21} > 1$, hence $U(x, 0) > n = \widehat{u}(x, 0)$ and a comparison argument yields $U(0, d(n)) \geq \widehat{u}(0, d(n))$. Hence, using (3.9), we conclude that

$$\widehat{u}(0, d(n)) \leq C n^{p_{11}-\alpha p_{12}}(\widehat{T} - d(n))^{-\mu} \leq C n^{p_{12}-p_{22}+1-p_{21}+p_{11}}.$$

In order to obtain the lower bound we observe that $-\beta p_{12} + 1/2 < 0$ and hence, since $\widehat{T} > d(n)$,

$$I \sim n^{-\alpha p_{12}} \int_0^{d(n)} (\widehat{T} - \tau)^{-\beta p_{12}} (d(n) - \tau)^{-1/2} \, d\tau$$

$$= n^{-\alpha p_{12}} \int_0^{d(n)} (\widehat{T} - \tau)^{-\beta p_{12} - 1/2} \frac{(\widehat{T} - \tau)^{1/2}}{(d(n) - \tau)^{1/2}} \, d\tau$$

$$\geq C n^{-\alpha p_{12}} ((\widehat{T} - d(n))^{-\beta p_{12} + 1/2} - \widehat{T}^{-\beta p_{12} + 1/2})$$

$$\geq C n^{-\alpha p_{12}} ((\widehat{T} - d(n))^{-\beta p_{12} + 1/2} \Big(1 - \frac{\widehat{T}^{-\beta p_{12} + 1/2}}{(\widehat{T} - d(n))^{-\beta p_{12} + 1/2}} \Big)$$

$$\geq C n^{-\alpha p_{12}} (\widehat{T} - d(n))^{-\beta p_{12} + 1/2} \geq C n^{p_{12} - p_{22} + 1 - p_{21}}.$$

Summing up, $u_n(0, t_v(n)) \sim n + n^{p_{11} - p_{21} + p_{12} - p_{22} + 1} \sim n^{p_{11} - p_{21} + p_{12} - p_{22} + 1}$.

$\boxed{p_{22} = 1 + p_{12}}$ In this critical case, we compute I explicitly and obtain

$$I \sim n^{-\alpha p_{12}} \int_0^{d(n)} (\widehat{T} - \tau)^{-1/2} (d(n) - \tau)^{-1/2} \, d\tau = n^{-\alpha p_{12}} \log \left| \frac{d(n) - \widehat{T}}{2\sqrt{\widehat{T} d(n)} - \widehat{T} - d(n)} \right|.$$

But,

$$\left| \frac{d(n) - \widehat{T}}{2\sqrt{\widehat{T} d(n)} - \widehat{T} - d(n)} \right| = \frac{\widehat{T} - d(n)}{\widehat{T} \left(1 - \sqrt{\frac{d(n)}{\widehat{T}}} \right)^2}.$$

Since $d(n) < \widehat{T}$ and $\widehat{T} - d(n) \sim n^{-2(p_{12} - p_{21})}$, see (3.9), we conclude that $I \sim n^{-\alpha p_{12}} \log n$, which yields $u_n(0, t_v(n)) \sim n^{p_{11} - \alpha p_{12}} \log n$.

$\boxed{p_{22} > 1 + p_{12}}$ Although $p_{22} > 1$, and hence $v_n(0, t + t_u(n)) \sim n^{-\alpha} (\widehat{T} - t)^{-\beta}$, the main difference between this case and the case $1 < p_{22} < 1 + p_{12}$ is that now $-\beta p_{12} + 1/2 > 0$. Thus, the estimates for I are done in a different way:

$$I \sim n^{-\alpha p_{12}} \int_0^{d(n)} (\widehat{T} - \tau)^{-\beta p_{12}} (d(n) - \tau)^{-1/2} \, d\tau \leq n^{-\alpha p_{12}} \int_0^{d(n)} (d(n) - \tau)^{-\beta p_{12} - 1/2}$$

$$\leq C n^{-\alpha p_{12}} d(n)^{-\beta p_{12} + 1/2}.$$

On the other hand,

$$I \sim n^{-\alpha p_{12}} \int_0^{d(n)} (\widehat{T} - \tau)^{-\beta p_{12}} (d(n) - \tau)^{-1/2} \, d\tau$$

$$= n^{-\alpha p_{12}} \int_0^{d(n)} (\widehat{T} - \tau)^{-\beta p_{12} - 1/2} \frac{(\widehat{T} - \tau)^{1/2}}{(d(n) - \tau)^{1/2}} \, d\tau$$

$$\geq C n^{-\alpha p_{12}} (\widehat{T}^{-\beta p_{12} + 1/2} - (\widehat{T} - d(n))^{-\beta p_{12} + 1/2}) \geq C n^{-\alpha p_{12}} \widehat{T}^{-\beta p_{12} + 1/2}.$$

Hence, since $\widehat{T} \sim d(n)$, we get $I \sim n^{-\alpha p_{12}} d(n)^{-\beta p_{12} + 1/2}$, and we conclude using (3.7) that $u_n(0, t_v(n)) \sim n + n^{p_{11} - p_{21}} \sim n^{p_{11} - p_{21}}$. $\qquad\square$

3.3 Estimates for $\gamma > 1$

Estimates for $t_v(n^\gamma)$. Since $t_v(n^\gamma) - T = t_v(n^\gamma) - t_v(n) + t_v(n) - T$, we can use (1.7) with $\gamma = 1$, so that it only remains to compute $t_v(n^\gamma) - t_v(n)$.

Theorem 3.2 *Let n be large enough and $\gamma > 1$. It holds,*

$$t_v(n^\gamma) - T \sim \begin{cases} n^{-2p_{21}}, & 1 < \gamma \le p_{22}, \\ n^{2(\gamma - (p_{21} + p_{22}))}, & \gamma > p_{22}. \end{cases} \tag{3.13}$$

Proof. For times $t > t_v(n)$ the truncations have taken place; hence v_n is a solution of the heat equation with boundary data $-(v_n)_x(0,t) = n^{p_{21}+p_{22}}$, and initial data $v_n(x, t_v(n)) \sim n$.

Using the representation formula for the heat equation, (2.2), we get that

$$v_n(0, t_v(n^\gamma)) = \int_{\mathbb{R}_+} v_n(y, t_v(n))\Gamma(y, t_v(n^\gamma) - t_v(n))\, dy$$

$$+ \int_0^{t_v(n^\gamma) - t_v(n)} n^{p_{21}+p_{22}}\Gamma(0, t_v(n^\gamma) - t_v(n)) - \tau)\, d\tau$$

$$- \int_0^{t_v(n^\gamma) - t_v(n)} v_n(0, \tau)\frac{\partial \Gamma}{\partial x}(0, t_v(n^\gamma) - t_v(n) - \tau)\, d\tau$$

$$\sim n + n^{p_{21}+p_{22}} \int_0^{t_v(n^\gamma) - t_v(n)} \Gamma(0, t_v(n^\gamma) - t_v(n) - \tau)\, d\tau$$

$$= n + Cn^{p_{21}+p_{22}}(t_v(n^\gamma) - t_v(n))^{1/2}.$$

Since $v_n(0, t_v(n^\gamma)) = n^\gamma$ and $\gamma > 1$ we get

$$t_v(n^\gamma) - t_v(n) \sim n^{2(\gamma - (p_{21} + p_{22}))}.$$

This expression together with (1.7) yields (3.13). □

Estimates for $t_u(n^\gamma)$. The idea is to use the estimates of the size of $u_n(0,t)$ at $t = t_v(n)$, that we have just obtained, to see if it has reached or not the level n^γ. If the size of u_n at $t_v(n)$ is smaller than n^γ, then $t_u(n^\gamma) > t_v(n)$, and we estimate the difference $t_u(n^\gamma) - T$ taking into account the size of $u_n(x, t_v(n))$ and that for $t > t_v(n)$ both truncations have taken place. If the size of u_n at $t_v(n)$ is bigger than n^γ, we use that u_n is a solution of the heat equation, with boundary flux $-(u_n)_x(0,t) \sim n^{p_{11}}v^{p_{12}}(0,t)$, and initial data $u(x, t_u(n)) \sim n$. If $u_n(0, t_v(n)) = n^\gamma$, then $t_u(n^\gamma) = t_v(n)$ and the result follows from (1.7). In the critical cases $\gamma = p_{11} - p_{21}, p_{11} - p_{21} + p_{12} - p_{22} + 1$, the size estimates do not imply an ordering between $t_u(n^\gamma)$ and $t_v(n)$; we will have to take into account the two possibilities separately.

If $\gamma < \max\{p_{11} - p_{21}, p_{11} - p_{21} + p_{12} - p_{22} + 1\}$, which implies $t_u(n^\gamma) < t_v(n)$, we have

$$n^\gamma = u_n(0, t_u(n^\gamma)) \sim \widehat{u}(0, t_u(n^\gamma) - t_u(n))$$

$$\sim n + n^{p_{11}} \int_0^{t_u(n^\gamma) - t_u(n)} v^{p_{12}}(0, \tau + t_u(n))(t_u(n^\gamma) - t_u(n) - \tau)^{-\frac{1}{2}} d\tau$$

$$= n + n^{p_{11}}\mathcal{I},$$

and since $\gamma > 1$ we get

$$\mathcal{I} \sim n^{\gamma - p_{11}}. \tag{3.14}$$

Let us sketch briefly how to bound $\mathcal{I}$. As before, $t_u(n^\gamma) - T = t_u(n^\gamma) - t_u(n) + t_u(n) - T$, so that we only have to estimate $t_u(n^\gamma) - t_u(n)$, (remember that $T - t_u(n) \sim n^{-2(p_{11}-1)}$, see (1.6)). We follow the proof of Theorem 3.1, replacing $d(n)$ by $t_u(n^\gamma) - t_u(n)$. Note that, since $t_v(n) > t_u(n^\gamma)$, we have that $d(n) > t_u(n^\gamma) - t_u(n)$.

Theorem 3.3 *Let n be large enough and let $1 < \gamma < p_{11} - p_{21}$. It holds,*

$$t_u(n^\gamma) - T \sim n^{-2(p_{11}-\gamma)}. \tag{3.15}$$

Proof. The main difference between this proof and that of Theorem 3.1 is that we do not know the relation between $t_u(n^\gamma) - t_u(n)$ and $\widehat{T}$.

$\boxed{p_{22} < 1}$ Assume first that $t_u(n^\gamma) - t_u(n) > \widehat{T}$, then

$$\mathcal{I} \sim n^{-\alpha p_{12}} \int_0^{t_u(n^\gamma) - t_u(n)} (\widehat{T} + \tau)^{-\beta p_{12}}(t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} d\tau$$

$$\sim n^{-\alpha p_{12}}(t_u(n^\gamma) - t_u(n))^{-\beta p_{12} + 1/2}.$$

Hence, using (3.14), $t_u(n^\gamma) - t_u(n) \sim n^{\frac{\gamma - p_{11} + \alpha p_{12}}{-\beta p_{12} + 1/2}}$. Thus, if $\gamma < p_{11} - p_{21}$ then

$$\frac{\gamma - p_{11} + \alpha p_{12}}{-\beta p_{12} + 1/2} < -2p_{21}.$$

Since $\widehat{T} \sim n^{-2p_{21}}$ this means that $t_u(n^\gamma) - t_u(n) < \widehat{T}$ and therefore, in this range of values for γ, we can not have $t_u(n^\gamma) - t_u(n) > \widehat{T}$. In other words, the region where $t_u(n^\gamma) - t_u(n) > \widehat{T}$ is at most $p_{11} - p_{21} + p_{12} - p_{22} + 1 > \gamma > p_{11} - p_{21}$, and in this region we get

$$t_u(n^\gamma) - T \sim n^{\frac{\gamma - p_{11} - \alpha p_{12}}{-\beta p_{12} + 1/2}} - (T - t_u(n)) \sim n^{\frac{\gamma - p_{11} + \alpha p_{12}}{-\beta p_{12} + 1/2}} - n^{-2(p_{11}-1)} \sim n^{\frac{\gamma - p_{11} + \alpha p_{12}}{-\beta p_{12} + 1/2}}. \tag{3.16}$$

On the other hand, if $t_u(n^\gamma) - t_u(n) < \widehat{T}$, then

$$\mathcal{I} \sim n^{-\alpha p_{12}} \int_0^{t_u(n^\gamma) - t_u(n)} (\widehat{T} + \tau)^{-\beta p_{12}}(t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} d\tau$$

$$\leq Cn^{-\alpha p_{12}}\widehat{T}^{-\beta p_{12}} \int_0^{t_u(n^\gamma) - t_u(n)} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} d\tau$$

$$\leq Cn^{-\alpha p_{12}}\widehat{T}^{-\beta p_{12}}(t_u(n^\gamma) - t_u(n))^{1/2}.$$

For the upper bound we have

$$\mathcal{I} \sim n^{-\alpha p_{12}} \int_0^{t_u(n^\gamma) - t_u(n)} (\widehat{T} + \tau)^{-\beta p_{12}} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau$$

$$\geq n^{-\alpha p_{12}} (\widehat{T} + d(n))^{-\beta p_{12}} \int_0^{d(n)} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2}$$

$$\geq C n^{-\alpha p_{12}} (\widehat{T} + d(n))^{-\beta p_{12}} (t_u(n^\gamma) - t_u(n))^{1/2} \geq C n^{-\alpha p_{12}} \widehat{T}^{-\beta p_{12}} (t_u(n^\gamma) - t_u(n))^{1/2}.$$

Hence, $t_u(n^\gamma) - t_u(n) \sim n^{2(\gamma - p_{11})}$. As before, we conclude that the values of γ for which $t_u(n^\gamma) - t_u(n) < \widehat{T}$ is a subset of $1 < \gamma < p_{11} - p_{21}$, where we have

$$t_u(n^\gamma) - T \sim n^{2(\gamma - p_{11})} - (T - t_u(n)) \sim n^{2(\gamma - p_{11})} - n^{-2(p_{11} - 1)} \sim n^{2(\gamma - p_{11})}.$$

Summing up

$$t_u(n^\gamma) - T \sim \begin{cases} n^{\frac{\gamma - p_{11} + \alpha p_{12}}{-\beta p_{12} + 1/2}}, & \gamma > p_{11} - p_{21}, \\ n^{2(\gamma - p_{11})}, & \gamma < p_{11} - p_{21}. \end{cases} \tag{3.17}$$

$\boxed{p_{22} = 1}$ We bound $\mathcal{I}$ from above in the following way:

$$\mathcal{I} \sim \int_0^{t_u(n^\gamma) - t_u(n)} e^{\alpha p_{12} \tau} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau$$

$$\geq \int_0^{t_u(n^\gamma) - t_u(n)} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau \geq C(t_u(n^\gamma) - t_u(n))^{1/2}.$$

Hence

$$t_u(n^\gamma) - t_u(n) \leq C n^{2(\gamma - p_{11})} \tag{3.18}$$

On the other hand,

$$\mathcal{I} \sim \int_0^{t_u(n^\gamma) - t_u(n)} e^{\alpha p_{12} \tau} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau$$

$$\leq e^{p_{12} a (t_u(n^\gamma) - t_u(n))} \int_0^{t_u(n^\gamma) - t_u(n)} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} d\tau.$$

Inequality (3.18) and the fact that $\gamma < p_{11} - p_{21}$ imply that $a(t_u(n^\gamma) - t_u(n)) \leq C$. Hence, $t_u(n^\gamma) - t_u(n) \geq n^{2(\gamma - p_{11})}$. It follows that $t_u(n^\gamma) - t_u(n) \sim n^{2(\gamma - p_{11})}$, which yields (3.15).

$\boxed{p_{22} > 1}$ The idea is the same as in the case $p_{22} = 1$, we first bound $t_u(n^\gamma) - t_u(n)$ from above and then use this estimate to obtain the lower bound,

$$\mathcal{I} \sim n^{-\alpha p_{12}} \int_0^{t_u(n^\gamma) - t_u(n)} (\widehat{T} - \tau)^{-\beta p_{12}} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau$$

$$\geq C n^{-\alpha p_{12}} \widehat{T}^{-\beta p_{12}} \int_0^{t_u(n^\gamma) - t_u(n)} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau$$

$$\geq C n^{-\alpha p_{12}} \widehat{T}^{-\beta p_{12}} (t_u(n^\gamma) - t_u(n))^{1/2}.$$

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Hence
$$t_u(n^\gamma) - t_u(n) \le Cn^{2(\gamma - p_{11})}. \tag{3.19}$$

On the other hand,
$$\mathcal{I} \sim n^{-\alpha p_{12}} \int_0^{t_u(n^\gamma) - t_u(n)} (\widehat{T} - \tau)^{-\beta p_{12}} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau$$

$$\le n^{-\alpha p_{12}} (\widehat{T} - (t_u(n^\gamma) - t_u(n)))^{-\beta p_{12}} \int_0^{t_u(n^\gamma) - t_u(n)} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau$$

$$\le Cn^{-\alpha p_{12}} (\widehat{T} - (t_u(n^\gamma) - t_u(n)))^{-\beta p_{12}} (t_u(n^\gamma) - t_u(n))^{1/2}.$$

By (3.19), we have that $\widehat{T} \sim n^{-2p_{21}} > t_u(n^\gamma) - t_u(n)$. Therefore,
$$\mathcal{I} \le Cn^{-\alpha p_{12}} \widehat{T}^{-\beta p_{12}} (t_u(n^\gamma) - t_u(n))^{1/2}.$$

We conclude again that $t_u(n^\gamma) - t_u(n) \sim n^{2(\gamma - p_{11})}$. $\qquad\square$

Theorem 3.4 *Let n be large enough and let $p_{11} - p_{21} < \gamma < p_{11} - p_{21} + p_{12} - p_{22} + 1$. It holds,*
$$t_u(n^\gamma) - T \sim \begin{cases} n^{\frac{\gamma - p_{11} + \alpha p_{12}}{-\beta p_{12} + 1/2}}, & p_{22} < 1, \\ n^{-2p_{21}} \log n, & p_{22} = 1, \\ n^{-2p_{21}}, & 1 < p_{22} < 1 + p_{12}. \end{cases}$$

Remark. In the range $p_{11} - p_{21} < \gamma < p_{11} - p_{21} + p_{12} - p_{22} + 1$, we have $p_{22} < 1 + p_{12}$.

Proof. $\boxed{p_{22} < 1}$ See the proof of Theorem 3.3, formulas (3.16) and (3.17).

$\boxed{p_{22} = 1}$ In this case, since $\gamma > p_{11} - p_{21}$, inequality (3.18) does not imply that $t_u(n^\gamma) - t_u(n) < n^{-2p_{21}}$, and we have to find a bound for $\mathcal{I}$ in a different way:
$$\mathcal{I} \ge e^{ap_{12}(t_u(n^\gamma) - t_u(n))} \int_{t_u(n^\gamma) - t_u(n) - d(n)/\log n}^{t_u(n^\gamma) - t_u(n)} ((t_u(n^\gamma) - t_u(n)) - \tau)^{-1/2} \, d\tau$$
$$\ge Cn^{-p_{21}} e^{p_{12} n^{2p_{21}} (t_u(n^\gamma) - t_u(n))}.$$

Following the proof of Theorem 3.1, we split $\mathcal{I}$ into two integrals,
$$\mathcal{I} \sim \int_0^{t_u(n^\gamma) - t_u(n) - d(n)/\log n} e^{ap_{12}\tau} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} \, d\tau$$
$$+ \int_{t_u(n^\gamma) - t_u(n) - d(n)/\log n}^{t_u(n^\gamma) - t_u(n)} e^{ap_{12}\tau} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} d\tau$$
$$\le n^{p_{21}} \int_0^{t_u(n^\gamma) - t_u(n) - d(n)/\log n} e^{ap_{12}\tau} \, d\tau$$
$$+ e^{ap_{12}(t_u(n^\gamma) - t_u(n))} \int_{t_u(n^\gamma) - t_u(n) - d(n)/\log n}^{t_u(n^\gamma) - t_u(n)} (t_u(n^\gamma) - t_u(n) - \tau)^{-1/2} d\tau$$
$$\le Cn^{-p_{21}} e^{p_{12} n^{2p_{21}} (t_u(n^\gamma) - t_u(n))}.$$

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We conclude that

$$Ce^{p_{12}n^{2p_{21}}(t_u(n^\gamma)-t_u(n))} \le n^{\gamma-p_{11}+p_{21}} \le Ce^{p_{12}n^{2p_{21}}(t_u(n^\gamma)-t_u(n))},$$

and hence, since $\gamma > p_{11} - p_{21}$,

$$C(\gamma - p_{11} + p_{21})n^{-2p_{21}}\log n \le t_u(n^\gamma) - t_u(n) \le (\gamma - p_{11} + p_{21})n^{-2p_{21}}\log n.$$

Therefore, using (1.6), $t_u(n^\gamma) - T \sim (\gamma - p_{11} + p_{21})n^{-2p_{21}}\log n$.

$\boxed{1 < p_{22} < 1 + p_{12}}$　　Comparison with the self-similar solution (3.12) yields

$$n^\gamma = \widehat{u}(0, t_u(n^\gamma) - t_u(n)) \le n^{p_{11}-\alpha p_{12}}(\widehat{T} - (t_u(n^\gamma) - t_u(n)))^{-\beta p_{12}+1/2}.$$

On the other hand,

$$\mathcal{I} \ge n^{-\alpha p_{12}} \int_0^{t_u(n^\gamma)-t_u(n)} (\widehat{T} - \tau)^{-\beta p_{12}-1/2} \frac{(\widehat{T}-\tau)^{1/2}}{(t_u(n^\gamma) - t_u(n) - \tau)^{1/2}}\, d\tau$$

$$\ge n^{-\alpha p_{12}}(\widehat{T} - (t_u(n^\gamma) - t_u(n)))^{-\beta p_{12}+1/2}.$$

Therefore,

$$\widehat{T} - n^{\frac{\gamma-p_{11}+\alpha p_{12}}{-\beta p_{12}+1/2}} \le t_u(n^\gamma) - t_u(n) \le \widehat{T} - n^{\frac{\gamma-p_{11}+\alpha p_{12}}{-\beta p_{12}+1/2}}.$$

The fact that $\gamma > p_{11} - p_{21}$ implies that $t_u(n^\gamma) - t_u(n) \sim \widehat{T} \sim n^{-2p_{21}}$, from where the Theorem follows. $\qquad\square$

We consider now $\gamma > \max\{p_{11} - p_{21}, p_{11} - p_{21} + p_{12} - p_{22} + 1\}$. We have that $t_u(n^\gamma) > t_v(n)$ and hence u_n satisfies the heat equation in $(t_v(n), \infty)$ with boundary flux $-(u_n)_x(0,t) \sim n^{p_{11}+p_{12}}$. The initial data is the size of u_n at $t_v(n)$; i.e, $u_n(x, t_v(n)) \sim n^\eta$, with $\eta = \max\{p_{11} - p_{21}, p_{11} - p_{21} + p_{12} - p_{22} + 1\}$, see Theorem 3.1.

Theorem 3.5 *Let n be large enough and let $\gamma > \max\{p_{11}-p_{21}, p_{11}-p_{21}+p_{12}-p_{22}+1\}$. It holds,*

$$t_u(n^\gamma) - T \sim \begin{cases} n^{-2p_{21}}, & \gamma \le p_{11} + p_{12} - p_{21}, \\ n^{2(\gamma-p_{11}-p_{12})}, & \gamma > p_{11} + p_{12} - p_{21}. \end{cases} \tag{3.20}$$

Proof. Using the representation formula,

$$n^\gamma = \widehat{u}(0, t_u(n^\gamma) - t_v(n)) \sim n^\eta + n^{p_{11}+p_{12}} \int_0^{t_u(n^\gamma)-t_v(n)} (t_u(n^\gamma) - t_v(n) - \tau)^{-1/2}\, d\tau$$

$$\sim n^\eta + n^{p_{11}+p_{12}}(t_u(n^\gamma) - t_v(n))^{1/2}.$$

Since $\gamma > \eta$ we get $t_u(n^\gamma) - t_v(n) \sim n^{2(\gamma-p_{11}-p_{12})}$; hence, using see (1.7), (3.20) follows. $\square$

Finally, we consider the critical cases.

Theorem 3.6 *Let n be large enough.*

(i) *If $\gamma = p_{11} - p_{21}$, then $t_u(n^\gamma) - T \sim n^{-2p_{21}}$.*

(ii) *If $\gamma = p_{11} - p_{21} + p_{12} - p_{22} + 1$, then*

$$t_u(n^\gamma) - T \sim \begin{cases} n^{2(-p_{21}+1-p_{22})}, & p_{22} < 1, \\ n^{-2p_{21}} \log n, & p_{22} = 1, \\ n^{-2p_{21}}, & p_{22} > 1. \end{cases}$$

Proof. Assume first that $t_u(n^\gamma) < t_v(n)$. If $\gamma = p_{11} - p_{21}$, then the proof of Theorem 3.3 yields $t_u(n^\gamma) - t_u(n) \sim n^{-2p_{21}}$. In the case $p_{22} > 1$ we have to use that $\widehat{T} > t_u(n^\gamma) - t_u(n)$, which follows from $\widehat{T} > t_v(n) - t_u(n)$. If $\gamma = p_{11} - p_{21} + p_{12} - p_{22} + 1$ the proof of Theorem 3.4 can extended to include this case.

If $t_u(n^\gamma) > t_v(n)$ the proof of Theorem 3.5 still applies. Notice that when $\gamma = \eta$, though $u_n(0, t_v(n)) \sim n^\gamma$, we still have (3.20), since $u_n(0, t_v(n)) < n^\gamma$. $\qquad\square$

We summarize the results of this subsection in Figure 1. It is worth noticing that the order of magnitude of $t_u(n^\gamma) - T$ depends in a continuous manner on the parameters except across the segment $p_{22} = 1$, $p_{11} - p_{21} < \gamma \le p_{11} - p_{21} + p_{12}$, where it appears a logarithmic correction.

Figure 1: $t_u(n^\gamma) - T$

4   Onset of the avalanche

The aim of this section is to prove theorems 1.4 and 1.5. In order to study the onset of the avalanche for the component u we first need to know its behaviour at time $t = T^-$. It turns out that it is given by a self-similar profile.

Lemma 4.1 *Let $p > 1$ and let u be a solution to*

$$\begin{cases} u_t = u_{xx}, & (x,t) \in \mathbb{R}_+ \times (0,T), \\ -u_x(0,t) = g(t)\,u^p(0,t), & t \in (0,T), \\ u(x,0) = u_0(x), & x \in \mathbb{R}_+, \end{cases}$$

where g is monotone increasing and bounded and u blows up at time T. Then

$$\lim_{t \nearrow T}(T-t)^{\frac{1}{2(p-1)}} u(y(T-t)^{1/2}, t) = g(T)^{\frac{1}{p-1}}\zeta(y), \tag{4.1}$$

uniformly on compact intervals $0 \le y \le C$, where

$$\zeta(y) = \frac{1}{\sqrt{\pi}\,\Gamma\big(\frac{1}{2(p-1)}\big)} \left(\frac{\Gamma\big(\frac{p}{2(p-1)}\big)}{2(p-1)\Gamma\big(\frac{2p-1}{2(p-1)}\big)} \right)^{\frac{1}{p-1}} \int_0^\infty e^{-\frac{y^2}{4t}} t^{\frac{3-2p}{2(p-1)}} (1+t)^{-\frac{p}{2(p-1)}}\, dt.$$

Proof. The blow-up rate is given by

$$c(T-t)^{\frac{-1}{2(p-1)}} \le \|u(\cdot,t)\|_\infty \le C(T-t)^{\frac{-1}{2(p-1)}},$$

see [2], [16]. If we write the function u in self-similar variables we get that the rescaled function

$$w(y,\tau) = (T-t)^{\frac{1}{2(p-1)}} u(y(T-t)^{1/2}, t), \qquad \tau = -\log(T-t),$$

is a bounded solution of

$$\begin{cases} w_\tau = w_{yy} - \dfrac{1}{2}y w_y - \dfrac{1}{2(p-1)} w, & (y,\tau) \in \mathbb{R}_+ \times (-\log T, \infty), \\ -w_y(0,\tau) = g(T - e^{-\tau}) w^p(0,\tau), & t \in (-\log T, \infty), \\ w(y, -\log T) = T^{\frac{1}{2(p-1)}} u_0(yT^{1/2}), & y \in \mathbb{R}_+. \end{cases}$$

If g is a constant, $g = k$, then w converges uniformly on sets of the form $0 \le y \le C$ to a steady state, which is a multiple of the profile ζ,

$$\lim_{\tau \to \infty} w(y,\tau) = k^{\frac{1}{p-1}}\zeta(y),$$

see [8]. For a general monotone g a comparison argument yields that for every $\eta > 0$

$$(g(T-\eta))^{\frac{1}{p-1}}\zeta(y) \le \liminf_{\tau \to \infty} w(y,\tau) \le \limsup_{\tau \to \infty} w(y,\tau) \le (g(T))^{\frac{1}{p-1}}\zeta(y),$$

from where (4.1) follows. $\qquad\square$

Proof of Theorem 1.4. Since $u_n(0, t_u(n)) = n$ and $v_n(0, t_u(n)) \le K$, the change of variables,

$$w_n(y,\tau) = n^{-1} u_n(n^{-(p_{11}-1)}y, t_u(n) + n^{-2(p_{11}-1)}\tau),$$

$$z_n(y,\tau) = v_n(n^{-(p_{11}-1)}y, t_u(n) + n^{-2(p_{11}-1)}\tau),$$

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scales both components to order one for $\tau = 0$. We obtain the system

$$\begin{cases} (w_n)_\tau = (w_n)_{yy}, \\ (z_n)_\tau = (z_n)_{yy}, \end{cases} \quad (y, \tau) \in \mathbb{R}_+ \times (-n^{2(p_{11}-1)} t_u(n), \infty),$$

coupled with the boundary conditions,

$$\begin{cases} -(w_n)_y(0, \tau) = f_1^{11}(w_n(0, \tau)) f_n^{12}(z_n(0, \tau)), \\ -(z_n)_y(0, \tau) = n^{-p_{11}+p_{21}} f_1^{21}(w_n(0, \tau)) f_n^{22}(z_n(0, \tau)), \end{cases} \quad \tau \in (-n^{2(p_{11}-1)} t_u(n), \infty),$$

and initial data

$$\begin{cases} w_n(y, 0) = n^{-1} u_n(n^{-(p_{11}-1)} y, t_u(n)), \\ z_n(y, 0) = v_n(n^{-(p_{11}-1)} y, t_u(n)), \end{cases} \quad y \in \mathbb{R}_+.$$

The same super and subsolutions used in the the proof of Theorem 1.2 guarantee that w_n and z_n are uniformly bounded (independently of n) in compact subsets of (y, τ). Therefore, by standard regularity theory, we get uniform bounds in $C^{2+\epsilon, 1+\epsilon/2}$. Hence, we can pass to the limit as $n \to \infty$. Let

$$z(y, \tau) = \lim_{n \to \infty} z_n(y, \tau).$$

For any $\tau \leq 0$, we get that $z(y, \tau) = v_T$. If $\tau \geq 0$, passing to the limit in the equation for z_n, we obtain that the limit (uniform on compact subsets), z, satisfies

$$\begin{cases} z_\tau = z_{yy}, & (y, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ -z_y(0, \tau) = 0, & \tau \in \mathbb{R}_+, \\ z(y, 0) = v_T, & y \in \mathbb{R}_+. \end{cases}$$

Here we have used that, since $p_{21} < p_{11} - 1$, then $p_{11} - p_{21} > 0$, and that

$$\lim_{n \to \infty} -n^{2(p_{11}-1)} t_u(n) = -\infty.$$

We conclude that $z \equiv v_T$ for all y and τ.

Let

$$\Phi(y, \tau) = \lim_{n \to \infty} w_n(y, \tau).$$

Passing to the limit in the equation we get

$$\begin{cases} \Phi_\tau = \Phi_{yy}, & (y, \tau) \in \mathbb{R}_+ \times \mathbb{R}, \\ -\Phi_y(0, \tau) = f_1^{11}(\Phi(0, \tau)) v_T^{p_{12}}, & \tau \in \mathbb{R}. \end{cases} \tag{4.2}$$

This problem does not have uniqueness because it is invariant under time translations. In order to characterize the limit completely we need some extra information. This information comes from the behaviour of the component u as $t \nearrow T$. Actually, from Lemma 4.1 we know that the blow-up profile is self-similar,

$$\lim_{t \nearrow T} (T - t)^{\frac{1}{2(p_{11}-1)}} u(y(T - t)^{1/2}, t) = v_T^{\frac{p_{12}}{p_{11}-1}} \zeta(y),$$

uniformly on compact intervals $0 \leq y \leq C$. The unique solution of (4.2) giving the asymptotic behaviour is the only one that coincides for τ negative and large enough with the self-similar function

$$w(y, \tau) = v_T^{\frac{p_{12}}{p_{11}-1}} (-\tau)^{-\frac{1}{2(p_{11}-1)}} \zeta(y(-\tau)^{-1/2}),$$

see [17] for the details. □

Proof of Theorem 1.5. Let $z_n(y, \tau) = v_n(n^{-p_{21}}y, T + n^{-2p_{21}}\tau)$. The function z_n satisfies

$$\begin{cases} (z_n)_\tau = (z_n)_{yy}, & (y, \tau) \in \mathbb{R}_+ \times \mathbb{R}_+, \\ -(z_n)_y(0, \tau) = f_n^{22}(z_n(0, \tau)), & \tau \in \mathbb{R}_+, \\ z_n(y, 0) = v_n(n^{-p_{21}}y, T), & y \in \mathbb{R}_+. \end{cases}$$

Since z_n is uniformly bounded for τ in compact subsets of $[0, \widetilde{\tau})$ for some $\widetilde{\tau} > 0$ and $v_n(n^{-p_{21}}y, T)$ goes to v_T as $n \to \infty$,

$$\lim_{n \to \infty} z_n(y, \tau) = \lim_{n \to \infty} v_n(n^{-p_{21}}y, T + n^{-2p_{21}}\tau) = \Psi(y, \tau),$$

where Ψ is the solution of the heat equation with initial datum $\Psi(y, 0) = v_T$ and boundary flux $-\Psi_y(0, \tau) = \Psi^{p_{22}}(0, \tau)$. This limit Ψ blows up in a finite time τ_0 when $p_{22} > 1$. With this fact in mind one can extend the convergence to compact subsets of $\overline{\mathbb{R}_+} \times [0, \tau_0)$. □

5 Avalanche

Proof of Theorem 1.6. Let $z_n(x, t) = n^{-\gamma_2}v_n(x, t)$. For times $\tau > t_u(n)$, the truncation for u_n has already taken place. Hence z_n is a solution to the heat equation in $\mathbb{R}_+ \times (\tau, \infty)$ with initial data $z_n(x, \tau) = n^{-\gamma_2}v_n(x, \tau)$, and boundary flux

$$-(z_n)_x(0, t) = n^{p_{21}-\gamma_2}f_n^{22}(v_n(0, t)), \quad t \in (\tau, \infty). \tag{5.1}$$

The idea is to choose γ_2 so that in the limit we get a nontrivial boundary flux condition. This choice depends on the value of $f_n^{22}(v_n(0, t))$; i.e. on the behaviour of $t_v(n)$ as n tends to infinity.

$\boxed{p_{21} + p_{22} < 1}$ In this range of parameters $t_v(n) \to \infty$, see Theorem 1.2. Hence, for any fixed time $\bar{t} > T$ and n large, $t_v(n) > \bar{t}$. Thus, v_n never reaches level n and equation (5.1) becomes, taking $\tau = t_u(n)$ and $\gamma_2 = \frac{p_{21}}{1-p_{22}}$,

$$-(z_n)_x(0, t) = n^{p_{21}-\gamma_2+\gamma_2 p_{22}}z_n^{p_{22}}(0, t) = z_n^{p_{22}}(0, t), \quad t \in (t_u(n), \bar{t}).$$

Using a comparison argument we have, for $t \in [t_u(n), \bar{t}]$ and $x \in \overline{\mathbb{R}_+}$,

$$\psi_n(x, t) \leq z_n(x, t) \leq \varepsilon + \psi_n(x, t),$$

where ψ_n is the unique positive solution of the heat equation in $\mathbb{R}_+ \times (t_u(n), \infty)$ with boundary flux $-(\psi_n)_x(0,t) = \psi_n^{p_{22}}(0,t)$ and initial datum $\psi_n(x, t_u(n)) = 0$. Letting first $n \to \infty$ and then $\varepsilon \to 0$, z_n converges uniformly on compact subsets of $[0, L] \times (T, \infty)$ (notice that $\bar{t}$ is arbitrary) to ψ, the unique positive solution to

$$\begin{cases} \psi_t = \psi_{xx}, & (x,t) \in \mathbb{R}_+ \times (T, \infty), \\ \psi(x, T) = 0, & x \in \mathbb{R}_+, \end{cases} \tag{5.2}$$

with boundary flux

$$-\psi_x(0,t) = \psi^{p_{22}}(0,t), \quad t \in (T, \infty).$$

This function has a self similar form $\psi(x,t) = (t-T)^{1/(2(p_{22}-1))}\phi(x(t-T)^{-1/2})$, see [5].

$\boxed{p_{21} + p_{22} = 1}$ In this case $t_v(n) \sim T$, see Theorem 1.2, and we cannot determine if the truncation takes place or not before $\bar{t}$. In any case, we take $\gamma_2 = 1$, so that (5.1) is independent of n. Hence the limit function $\psi(x,t)$ is now a positive solution to (5.2), but with boundary flux given by

$$-\psi_x(0,t) = f_1^{22}(\psi(0,t)), \quad t \in (T, \infty).$$

Since $p_{22} < 1$ this problem has a positive solution (that can be obtained as in [5]). As before ψ has a self-similar form until the time $\tilde{t}$ where it reaches level one, $\psi(0, \tilde{t}) = 1$. From that time it behaves like the solution of the heat equation with boundary flux equal to one and initial datum $\psi(x, \tilde{t})$.

$\boxed{p_{21} + p_{22} > 1}$ Let $\bar{t} > T$ be a fixed time. Since $t_v(n) \to T$, for large values of n, we haver that $t_v(n) < \bar{t}$, the truncation for v_n takes place, and (5.1) becomes

$$-(z_n)_x(0,t) = n^{p_{21}-\gamma_2+p_{22}} = 1, \quad t \in (t_v(n), \bar{t}),$$

if $\tau = t_v(n)$ and $\gamma_2 = p_{21} + p_{22}$. This choice of γ_2 implies that $z(x, t_v(n)) \leq n^{1-\gamma_2} \to 0$ as $n \to \infty$. Using a comparison argument as before, we have that z_n converges uniformly on compact subsets of $[0, \infty) \times (T, \infty)$ to ψ, the unique positive solution of (5.2) with boundary data

$$-\psi_x(0,t) = 1, \quad t \in (T, \infty).$$

Again in this case ψ has a self-similar structure, $\psi(x,t) = (t-T)^{1/2}\phi\left(x(t-T)^{-1/2}\right)$. The profile ϕ can be computed explicitly.

To conclude the proof we have to deal with the limit of $u_n(x,t)$. To this aim we define $w_n(x,t) = n^{-\gamma_1}u_n(x,t)$, which is a solution to

$$\begin{cases} (w_n)_t = (w_n)_{xx}, & (x,t) \in \mathbb{R}_+ \times (t_u(n), \infty), \\ -(w_n)_x(0,t) = n^{p_{11}-\gamma_1}f_n^{12}(n^{\gamma_2}z_n(0,t)), & t \in (t_u(n), \infty), \\ w_n(x, t_u(n)) = n^{-\gamma_1}u_n(x, t_u(n)) \leq n^{1-\gamma_1}, & x \in \mathbb{R}_+, \end{cases}$$

with γ_1 a constant. As we already know that $z_n(0,t) \to \psi(0,t)$ uniformly on $t \in (T, \infty)$, we pass to the limit and obtain that there exists a nontrivial profile φ such that

$w_n(x,t) \to \varphi(x,t)$, as $n \to \infty$. This limit profile is a solution to the heat equation in $\mathbb{R}_+ \times (T, \infty)$ and if $\gamma_1 > 1$ (which will be shown to be true) the initial datum becomes $\varphi(x, T) = 0$. The constant γ_1 is chosen so that the limit boundary condition

$$-\varphi_x(0,t) = n^{p_{11}-\gamma_1} f_n^{12}(n^{\gamma_2}\psi(0,t)) = n^{p_{11}-\gamma_1+\gamma_2 p_{12}} f_{n^{1-\gamma_2}}^{12}(\psi(0,t)) \qquad (5.3)$$

is independent of n.

$\boxed{p_{21} + p_{22} < 1}$ Since ψ is bounded, we have that $f_{n^{1-\gamma_2}}^{12}(\psi(0,t)) = \psi^{p_{12}}(0,t)$, for n large enough. Hence, taking $\gamma_1 = p_{11} + \gamma_2 p_{12}$ we get $-\varphi_x(0,t) = \psi^{p_{12}}(0,t)$.

$\boxed{p_{21} + p_{22} = 1}$ We have $\gamma_2 = 1$, therefore the boundary condition (5.3) becomes $-\varphi_x(0,t) = f_1^{12}(\psi(0,t))$ if $\gamma_1 = p_{11} + p_{12}$.

$\boxed{p_{21} + p_{22} > 1}$ Since $\gamma_2 > 1$, we take $\gamma_1 = p_{11} + p_{12}$ and (5.3) turns out to be $-\varphi_x(0,t) = 1$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

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