# THE LIMIT AS $p \to \infty$ IN THE EIGENVALUE PROBLEM FOR A SYSTEM OF $p-$LAPLACIANS.

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Abstract. In this paper we study the behavior as $p \to \infty$ of eigenvalues and eigenfunctions of a system of $p-$Laplacians, that is

$$\begin{cases} -\Delta_p u = \lambda \alpha u^{\alpha-1} v^{\beta} & \Omega, \\ -\Delta_p v = \lambda \beta u^{\alpha} v^{\beta-1} & \Omega, \\ u = v = 0, & \partial\Omega, \end{cases}$$

in a bounded smooth domain $\Omega$. Here $\alpha+\beta = p$. We assume that $\frac{\alpha}{p} \to \Gamma$ and $\frac{\beta}{p} \to 1-\Gamma$ as $p \to \infty$ and we prove that for the first eigenvalue $\lambda_{1,p}$ we have

$$(\lambda_{1,p})^{1/p} \to \lambda_\infty = \frac{1}{\max_{x\in\Omega} dist(x, \partial\Omega)}.$$

Concerning the eigenfunctions $(u_p, v_p)$ associated with $\lambda_{1,p}$ normalized by $\int_\Omega |u_p|^\alpha |v_p|^\beta = 1$, there is a uniform limit $(u_\infty, v_\infty)$ that is a solution to a limit minimization problem as well as a viscosity solution to

$$\begin{cases} \min\{-\Delta_\infty u_\infty,\ |\nabla u_\infty| - \lambda_\infty u_\infty^{\Gamma} v_\infty^{1-\Gamma}\} = 0, \\ \min\{-\Delta_\infty v_\infty,\ |\nabla v_\infty| - \lambda_\infty u_\infty^{\Gamma} v_\infty^{1-\Gamma}\} = 0. \end{cases}$$

In addition, we also analyze the limit PDE when we consider higher eigenvalues.

## 1. Introduction.

In this paper we deal with non-negative solutions to the following elliptic problem

$$\begin{cases} -\Delta_p u = \lambda \alpha u^{\alpha-1} v^{\beta} & \Omega, \\ -\Delta_p v = \lambda \beta u^{\alpha} v^{\beta-1} & \Omega, \\ u = v = 0, & \partial\Omega, \end{cases} \tag{1.1}$$

when $p$ is large. Here $p > 1$, $\Delta_p u = \text{div}(|\nabla u|^{p-2}\nabla u)$ is the well-known $p-$Laplacian operator, $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$ and $\alpha$ and $\beta$ are real numbers greater or equal than one and verify

$$\alpha + \beta = p. \tag{1.2}$$

Existence of weak solutions to (1.1) can be easily obtained from a variational argument, see [16]. In fact, we just have to look for a minimizer of

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the quotient

$$(1.3)\quad \lambda_{1,p} = \min_{(u,v)\in S_p} Q_p(u,v) \quad \text{where} \quad Q_p(u,v) = \frac{\displaystyle\int_\Omega \frac{|\nabla u|^p}{p} + \int_\Omega \frac{|\nabla v|^p}{p}}{\displaystyle\int_\Omega |u|^\alpha |v|^\beta}$$

in $S_p := W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ to obtain the first eigenvalue $\lambda_{1,p}$ whose associated a pair of eigenfunctions $(u_p, v_p)$ is non-negative. Note that, up to our knowledge, except in the symmetric case $\alpha=\beta$ where we recover the first eigenfunction of the $p$-Laplacian, it is not known that the first eigenvalue (1.3) is simple as it happens for a single equation.

**Theorem 1.1.** *Let $(u_p, v_p)$ be a minimizer in* (1.3) *normalized by*

$$(1.4)\quad \int_\Omega |u_p|^\alpha |v_p|^\beta = 1,$$

*Assume that*

$$\frac{\alpha}{p} \to \Gamma \qquad \text{as } p\to\infty$$

*with $0<\Gamma<1$ (in view of (1.2), this implies that $\frac{\beta}{p}\to 1-\Gamma$ as $p\to\infty$). Then, there exist functions $u_\infty, v_\infty \in C(\overline{\Omega})$ and a sequence $p_j\to\infty$ such that*

$$u_{p_j}\to u_\infty, \qquad \text{and} \qquad v_{p_j}\to v_\infty,$$

*uniformly in $\overline{\Omega}$. In addition,*

$$(\lambda_{1,p})^{1/p} \to \lambda_\infty = \frac{1}{R}$$

*where $R$ is the radius of the largest ball included in $\Omega$ that is*

$$R = \max_{x\in\Omega} dist(x,\partial\Omega).$$

*The limit pair of functions $(u_\infty, v_\infty)$ belongs to $S_\infty = W_0^{1,\infty}(\Omega)\times W_0^{1,\infty}(\Omega)$ and is a minimizer for the limit variational problem defined by*

$$(1.5)\quad \min_{(u,v)\in S_\infty} Q(u,v) = \min_{(u,v)\in S_\infty} \frac{\max\left\{\|\nabla u\|_{L^\infty(\Omega)};\ \|\nabla v\|_{L^\infty(\Omega)}\right\}}{\||u|^\Gamma |v|^{1-\Gamma}\|_{L^\infty(\Omega)}}.$$

*In addition, $(u_\infty, v_\infty)$ is a viscosity solution to the following limit eigenvalue problem*

$$(1.6)\quad \begin{cases} \min\{-\Delta_\infty u_\infty,\ |\nabla u_\infty| - \lambda_\infty u_\infty^\Gamma v_\infty^{1-\Gamma}\} = 0, \\ \min\{-\Delta_\infty v_\infty,\ |\nabla v_\infty| - \lambda_\infty u_\infty^\Gamma v_\infty^{1-\Gamma}\} = 0. \end{cases}$$

*where $\Delta_\infty u = \sum_{i,j=1}^n \partial_{ij}u\partial_i u\partial_j u$ is the $\infty$-Laplacian of $u$.*

Remark that the limit of $(\lambda_{1,p})^{1/p}$ as $p \to \infty$ is given by $\lambda_\infty = \frac{1}{R}$. This is the same limit as the one for the first eigenvalue for the usual $p-$Laplacian (that is, for a *single* equation not for a system) and is known as the first eigenvalue for the $\infty-$Laplacian, see [13]. Hence, we have the surprising (except in the symmetric case $\alpha = \beta$) fact that the first eigenvalue for the system converges to the same limit as for a single equation.

In addition, when $\Omega$ is a ball of radius $R$ we have that there is a unique minimizer of $\lambda_\infty = \inf_{u \in W_0^{1,\infty}(\Omega)} \frac{\|\nabla u\|_{L^\infty(\Omega)}}{\|u\|_{L^\infty(\Omega)}}$ that is given by the cone $c(x) = R - |x|$. Therefore, in this case it can be proved that the limit of $u_p$ and $v_p$ coincide and is given exactly by the same cone $c(x)$. Hence, we conclude that for the ball the first eigenvalue is associated with a pair of eigenfunctions that are quite close to each other for $p$ large.

Next, we show that the limits of the eigenfunctions of the first eigenvalue verify an uncoupled problem. To show this fact we use ideas from optimal mass transportation, see [6], [19] for similar ideas and [20] for basic concepts and definitions.

**Theorem 1.2.** *Under the same conditions of Theorem 1.1, consider the measures $f_p = u_p^{\alpha-1} v_p^{\beta}\, dx$ and $g_p = u_p^{\alpha} v_p^{\beta-1}\, dx$. Then, there exists $f_\infty, g_\infty \in P(\overline{\Omega})$ (the space of probability measures on $\overline{\Omega}$) such that up to a subsequence,*

$$f_p\, dx \rightharpoonup f_\infty \qquad \text{and} \qquad g_p\, dx \rightharpoonup g_\infty.$$

*In addition, we have that $((u_\infty, f_\infty), (v_\infty, g_\infty))$ is a minimizer of the functional $G_\infty$ given by*

$$G_\infty((u,\sigma),(v,\tau)) = \begin{cases} -\displaystyle\int_{\overline{\Omega}} u\sigma - \int_{\overline{\Omega}} v\tau & \text{if } \begin{array}{l} u, v \in W_0^{1,\infty}(\Omega), \\ \|\nabla u\|_{L^\infty(\Omega)} \le \lambda_\infty, \\ \|\nabla u\|_{L^\infty(\Omega)} \le \lambda_\infty, \\ \sigma, \tau \in M(\overline{\Omega}), \\ \int |\sigma| \le 1, \int |\tau| \le 1 \end{array} \\ +\infty & \text{otherwise.} \end{cases}$$

Concerning higher eigenvalues we have the following result: for (1.1) with fixed $p$, $\alpha$, $\beta$ it can be proved using topological arguments that there is a sequence of eigenvalues $\lambda_{n,p} \to \infty$ with eigenfunctions $(u_p, v_p)$ that change sign in $\Omega$. Note that since solutions change sign we have to write $u^\alpha$ as $|u|^{\alpha-1}u$ and analogously for $v^\beta$ in (1.1). The next result find the associated limit PDE as $p \to \infty$.

**Theorem 1.3.** *Let $\lambda_{n,p}$ be a sequence of eigenvalues with eigenfunctions $(u_p, v_p)$ normalized by*

$$\int_\Omega |u_p|^\alpha |v_p|^\beta = 1,$$

*and assume that*

$$\frac{\alpha}{p} \to \Gamma \qquad \text{as } p \to \infty$$

*with* $0 < \Gamma < 1$ *(note that* (1.2) *implies that* $\frac{\beta}{p} \to 1 - \Gamma$ *as* $p \to \infty$*). If there is a constant* $C$ *independent of* $p$ *such that*

$$(\lambda_{n,p})^{1/p} \leq C,$$

*then, there exists a sequence* $p_j \to \infty$ *such that*

$$(\lambda_{n,p_j})^{1/p_j} \to \Lambda$$

*and*

$$u_{p_j} \to u_\infty, \qquad \text{and} \qquad v_{p_j} \to v_\infty,$$

*uniformly in* $\overline{\Omega}$*. The limit pair of functions* $(u_\infty, v_\infty)$ *belongs to* $S_\infty = W_0^{1,\infty}(\Omega) \times W_0^{1,\infty}(\Omega)$ *and is a viscosity solution to the following limit eigenvalue problem*

$$(1.7) \quad \begin{cases} \min\{-\Delta_\infty u_\infty,\, |\nabla u_\infty| - \Lambda u_\infty^\Gamma |v_\infty|^{1-\Gamma}\} = 0, & \text{if } u_\infty > 0, \\ -\Delta_\infty u_\infty = 0, & \text{if } u_\infty v_\infty = 0, \\ \max\{-\Delta_\infty u_\infty,\, -|\nabla u_\infty| - \Lambda u_\infty^\Gamma |v_\infty|^{1-\Gamma}\} = 0, & \text{if } u_\infty v_\infty < 0, \end{cases}$$

*together with the analogous equation that holds for* $v_\infty$*.*

The condition $(\lambda_{n,p})^{1/p} \leq C$ holds, for example, for the eigenvalues constructed using topological arguments in [16]. We remark that it is not known if this set of eigenvalues exhaust the whole spectrum. Therefore, we prefer to state our result assuming $(\lambda_{n,p})^{1/p} \leq C$ and let $\lambda_{n,p}$ be any possible eigenvalue.

Let us finish the introduction giving some references and motivation for the analysis of this problem. The limit of $p-$harmonic functions, that is, of solutions to $-\Delta_p u = -\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$, as $p \to \infty$ has been extensively studied in the literature (see [3] and the survey [2]) and leads naturally to solutions of the infinity Laplacian, given by $-\Delta_\infty u = -\nabla u D^2 u (\nabla u)^t = 0$. Infinity harmonic functions (solutions to $-\Delta_\infty u = 0$) are related to the optimal Lipschitz extension problem (see the survey [2]) and find applications in optimal transportation, image processing and tug-of-war games (see, e.g., [5], [9], [17], [18] and the references therein). Also limits of the eigenvalue problem related to the $p$-Laplacian have been exhaustively examined, see [10], [12] and [13] and lead naturally to the infinity Laplacian eigenvalue problem

$$(1.8) \qquad \min\left\{|\nabla u|(x) - \Lambda_\infty u(x),\; -\Delta_\infty u(x)\right\} = 0.$$

In fact, it is proved in [12] and [13] that the limit as $p \to \infty$ exists both for the eigenfunctions, $u_p \to u_\infty$ uniformly, and for the eigenvalues $(\Lambda_p)^{1/p} \to \Lambda_\infty = 1/R$, where the pair $u_\infty$, $\Lambda_\infty$ is a nontrivial solution to (1.8).

Concerning eigenvalues for systems of $p-$Laplacian type there is a rich recent literature, we refer to [4], [16], [21] and references therein.

The paper is organized as follows: in Section 2 we prove Theorem 1.1, in Section 3 we collect some extra remarks concerning the limit problem for the first eigenvalue and we prove Theorem 1.2, and finally, in Section 4 we deal with higher eigenvalues and prove Theorem 1.3.

## 2. Proof of Theorem 1.1.

We first look for a uniform bound for $\lambda_{1,p}^{1/p}$. To this end, let us consider a Lipschitz function $w \in W^{1,\infty}(\Omega)$ that is a first eigenfunction for the $\infty-$Laplacian normalized according to $\|w\|_{L^\infty(\Omega)} = 1$. This function verifies

$$\|\nabla w\|_{L^\infty(\Omega)} = \frac{1}{R}.$$

Using the pair $(w, w) \in S$ as a test-function in (1.3) to estimate $\lambda_{1,p}$, we obtain

$$\limsup_{p\to\infty}(\lambda_{1,p})^{1/p} \le \limsup_{p\to\infty}\left(\frac{2}{p}\right)^{1/p}\frac{\|\nabla w\|_{L^p(\Omega)}}{\|w\|_{L^p(\Omega)}} = \frac{\|\nabla w\|_{L^\infty(\Omega)}}{\|w\|_{L^\infty(\Omega)}} \le \frac{1}{R}. \tag{2.1}$$

Therefore, there is a constant, $C$, independent of $p$ such that, for $p$ large,

$$(\lambda_{1,p})^{1/p} \le C.$$

Recalling that $(u_p, v_p)$ are minimizer for $\lambda_{1,p}$ normalized by (1.4), we have that

$$\int_\Omega |\nabla u_p|^p + \int_\Omega |\nabla v_p|^p = p\lambda_{1,p},$$

from which we deduce with (2.1) that

$$\limsup_{p\to+\infty}\|\nabla u_p\|_{L^p(\Omega)} \le \frac{1}{R} \qquad \text{and} \qquad \limsup_{p\to+\infty}\|\nabla v_p\|_{L^p(\Omega)} \le \frac{1}{R}. \tag{2.2}$$

Now, we argue as follows. We fix $r \in (1, \infty)$. Using Holder's inequality, we obtain for $p > r$ large enough that

$$\left(\int_\Omega |\nabla u_p|^r\right)^{1/r} \le \left(\int_\Omega |\nabla u_p|^p\right)^{1/p} |\Omega|^{\frac{1}{r}-\frac{1}{p}} \le C.$$

Hence, extracting a subsequence $p_j \to \infty$ if necessary, we have that

$$u_p \rightharpoonup u_\infty$$

weakly in $W^{1,r}(\Omega)$ for any $1 < r < \infty$ and uniformly in $\overline{\Omega}$. From (2.2), we obtain that this weak limit verifies

$$\left(\int_\Omega |\nabla u_\infty|^r\right)^{1/r} \le \frac{|\Omega|^{1/r}}{R}.$$

As we can assume that the above inequality holds for every $r$ (using a diagonal argument), we get that $u_\infty \in W^{1,\infty}(\Omega)$ and moreover, taking the limit as $r \to \infty$, we obtain

$$|\nabla u_\infty| \le \frac{1}{R}, \qquad \text{a.e. } x \in \Omega.$$

Analogously, we obtain the existence of a function $v_\infty \in W^{1,\infty}(\Omega)$ such satisfying

$$v_p \to v_\infty$$

weakly in $W^{1,r}(\Omega)$ for any $1 < r < \infty$ and uniformly in $\overline{\Omega}$, with

$$|\nabla v_\infty| \le \frac{1}{R}, \qquad \text{a.e. } x \in \Omega.$$

From the uniform convergence and the normalization condition (1.4), we obtain that

$$\| |u_\infty|^\Gamma |v_\infty|^{1-\Gamma} \|_{L^\infty(\Omega)} = 1.$$

Therefore, we get

$$\frac{\max\Big\{ \|\nabla u_\infty\|_{L^\infty(\Omega)};\ \|\nabla v_\infty\|_{L^\infty(\Omega)} \Big\}}{\| |u_\infty|^\Gamma |v_\infty|^{1-\Gamma} \|_{L^\infty(\Omega)}} \le \frac{1}{R}.$$

Now, let us point out that the limit for the first eigenvalue stated in Theorem 1.1 can be also characterized as follows:

$$\lambda_{1,p}^{1/p} \to \lambda_\infty := \inf \max \{ \|\nabla u\|_\infty, \|\nabla v\|_\infty \} = \frac{1}{R}$$

where the inf is taken over all pairs $(u, v) \in W_0^{1,\infty}(\Omega) \times W_0^{1,\infty}(\Omega)$ such that $\| |u|^\Gamma |v|^{1-\Gamma} \|_{L^\infty(\Omega)} = 1$. Indeed, to prove that

$$\inf \max \{ \|\nabla u\|_\infty, \|\nabla v\|_\infty \} = \frac{1}{R}$$

we argue as follows. First, taking $u = v$ we obtain that $\lambda_\infty$ is less or equal than the first Dirichlet eigenvalue of $-\Delta_\infty$ that is equal to $1/R$. On the other hand if $(u, v)$ satisfies $\| |u|^\Gamma |v|^{1-\Gamma} \|_{L^\infty(\Omega)} = 1$ then $\|u\|_{L^\infty(\Omega)} \ge 1$ or $\|v\|_{L^\infty(\Omega)} \ge 1$. If e.g. $\|u\|_{L^\infty(\Omega)} \ge 1$ then $\|\nabla u\|_{L^\infty(\Omega)} \ge 1/R$ so that $\lambda_\infty \ge 1/R$.

To prove the convergence of $\lambda_{1,p}^{1/p}$ to $\lambda_\infty$, we use the fact that for $u, v \in L^\infty(\Omega)$ (independent of $p$),

$$\Big( \int_\Omega |u|^\alpha |v|^\beta \, dx \Big)^{1/p} \to \| |u|^\Gamma |v|^{1-\Gamma} \|_{L^\infty(\Omega)}$$

as $p \to \infty$ and argue as before.

In order to identify the limit PDE problem satisfied by any limit $(u_\infty, v_\infty)$, we introduce the concept of viscosity solutions to each of the equations in

(1.1). Assuming that $u_p$ is smooth enough, we can rewrite the first equation in (1.1) as

$$-|\nabla u_p|^{p-4}\left(|\nabla u_p|^2\Delta u_p+(p-2)\Delta_\infty u_p\right)=\alpha\lambda_{1,p}u_p^{\alpha-1}v_p^\beta. \tag{2.3}$$

This equation is nonlinear but elliptic (degenerate), thus it makes sense to consider viscosity subsolutions and supersolutions of it. Let $x, y \in \mathbb{R}$, $z \in \mathbb{R}^N$, and $S$ a real symmetric matrix. We define the following continuous function

$$\begin{aligned} H_p(x,y,z,S)= \quad & -|z|^{p-4}\Big(|z|^2\text{trace}(S)+(p-2)\langle S\cdot z,z\rangle\Big) \\ & -\alpha\lambda_{1,p}|y|^{\alpha-2}yv_p(x)^\beta. \end{aligned} \tag{2.4}$$

Observe that $H_p$ is elliptic in the sense that $H_p(x,y,z,S) \geq H_p(x,y,z,S')$ if $S \leq S'$ in the sense of bilinear forms, and also that (2.3) can then be written as $H_p(x,u_p,\nabla u_p,D^2u_p)=0$. We are thus interested in viscosity super and subsolutions of the partial differential equation

$$\begin{cases} H_p(x,u,\nabla u,D^2u)=0, & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega. \end{cases} \tag{2.5}$$

**Definition 2.1.** *An upper semicontinuous function $u$ defined in $\Omega$ is a viscosity subsolution of* (2.5) *if, $u|_{\partial\Omega} \leq 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that*

i) $u(x_0)=\phi(x_0)$,
ii) $u(x)<\phi(x)$, *if* $x \neq x_0$,

*then*

$$H_p(x_0,\phi(x_0),\nabla\phi(x_0),D^2\phi(x_0)) \leq 0.$$

**Definition 2.2.** *A lower semicontinuous function $u$ defined in $\Omega$ is a viscosity supersolution of* (2.5) *if, $u|_{\partial\Omega} \geq 0$ and, whenever $x_0 \in \Omega$ and $\phi \in C^2(\Omega)$ are such that*

i) $u(x_0)=\phi(x_0)$,
ii) $u(x)>\phi(x)$, *if* $x \neq x_0$,

*then*

$$H_p(x_0,\phi(x_0),\nabla\phi(x_0),D^2\phi(x_0)) \geq 0.$$

We observe that in both of the above definitions the second condition is required just in a neigbourhood of $x_0$ and the strict inequality can be relaxed. We refer to [7] for more details about general theory of viscosity solutions, and to [14] for viscosity solutions related to the $\infty-$Laplacian and the $p-$Laplacian operators. The following result can be shown as in [15][Proposition 2.4].

**Lemma 2.3.** *A continuous weak solution to the equation*

$$\begin{cases} -\Delta_p u=\lambda\alpha|u|^{\alpha-2u}v^\beta & \Omega, \\ u=0, & \partial\Omega, \end{cases} \tag{2.6}$$

*is a viscosity solution to* (2.5)*.*

Now, we have all the ingredients to compute the limit of the equation

$$H_p(x, u_p, \nabla u_p, D^2 u_p) = 0$$

as $p \to \infty$ in the viscosity sense, that is to identify the limit equation verified by any limit $u_\infty$. For $x, y \in \mathbb{R}$ $z \in \mathbb{R}^N$ and $S$ a symmetric real matrix, we define the limit operator $H_\infty$ by

$$H_\infty(x, y, z, S) = \min\{-\langle S \cdot z, z\rangle, |z| - \lambda_\infty |y|^{\Gamma-2} y v_\infty(x)^{1-\Gamma}\}. \tag{2.7}$$

Note that $H_\infty(x, u, \nabla u, D^2 u) = 0$ is the first equation in the system that we are looking for.

**Theorem 2.4.** *A function $u_\infty$ obtained as a limit of a subsequence of $\{u_p\}$ is a viscosity solution of the equation*

$$H_\infty(x, u, \nabla u, D^2 u) = 0, \tag{2.8}$$

*with $H_\infty$ defined in* (2.7)*, and $v_\infty$ a uniform limit of $v_p$.*

*Proof.* In the sequel we assume that we have a subsequence $p_n \to \infty$ such that

$$\lim_{n\to\infty} u_{p_n} = u_\infty$$

uniformly in $\Omega$ and $(\lambda_{p_n})^{1/p_n} \to \lambda_\infty$. In what follows we omit the subscript $n$ and denote as $u_p$ and $\lambda_p$ such subsequences for simplicity.

We first check that $u_\infty$ is a supersolution of (2.8). To this end, we consider a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) > \phi(x)$ for every $x \in B(x_0, R)$, $x \neq x_0$, with $R > 0$ fixed and verifying that $B(x_0, 2R) \subset \Omega$. We must show that

$$H_\infty(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \geq 0. \tag{2.9}$$

Let $x_p$ be a minimum point of $u_p - \phi$ in $\bar{B}(x_0, R)$. Up to a subsequence the $x_p$ converge to some point $x_\infty \in \bar{B}(x_0, R)$. Recalling that $u_p \to u_\infty$ uniformly in $\bar{B}(x_0, R)$, we see that $x_\infty$ is a minimum point of $u_\infty - \phi$ so that $x_\infty = x_0$.

In view of lemma 2.3, $u_p$ is a viscosity supersolution of (2.5) so that

$$\begin{aligned} -|\nabla\phi(x_p)|^{p-4}\Big(|\nabla\phi(x_p)|^2\Delta\phi(x_p) + (p-2)\Delta_\infty\phi(x_p)\Big) \\ \geq \alpha\lambda_{1,p}|\phi(x_p)|^{\alpha-2}\phi(x_p)v_p^\beta(x_p). \end{aligned} \tag{2.10}$$

Assume that $\phi(x_0) = u_\infty(x_0) > 0$ and $v_\infty(x_0) > 0$. Then for $p$ large, $\phi(x_p) > 0$ and $v_p(x_p) > 0$ so that the right hand side of (2.10) is positive.

It follows that $|\nabla\phi(x_p)| > 0$ and then that

$$
\begin{aligned}
&-\Big(\frac{|\nabla\phi(x_p)|^2\Delta\phi(x_p)}{(p-2)} + \Delta_\infty\phi(x_p)\Big) \\
&\geq \left(\frac{\alpha^{\frac{1}{p}}}{(p-2)^{\frac{1}{p}}}(\lambda_{1,p})^{\frac{1}{p}}|\phi(x_p)|^{\frac{\alpha-2}{p}}\phi^{\frac{1}{p}}(x_p)v_p^{\frac{\beta}{p}}(x_p)|\nabla\phi(x_p)|^{-1+\frac{4}{p}}\right)^p.
\end{aligned} \tag{2.11}
$$

Note that we have

$$
\lim_{p\to\infty} -\Big(\frac{|\nabla\phi(x_p)|^2\Delta\phi(x_p)}{(p-2)} + \Delta_\infty\phi(x_p)\Big) = -\Delta_\infty\phi(x_0) < \infty. \tag{2.12}
$$

Hence

$$
\limsup_{p\to\infty} \frac{\alpha^{\frac{1}{p}}}{(p-2)^{\frac{1}{p}}}(\lambda_{1,p})^{\frac{1}{p}}\phi^{\frac{\alpha-1}{p}}(x_p)v_p^{\frac{\beta}{p}}(x_p)|\nabla\phi(x_p)|^{-1+\frac{4}{p}} \leq 1.
$$

Recalling that by assumptions $\frac{\alpha}{p} \to \Gamma$ as $p \to +\infty$, we obtain

$$
\lambda_\infty\phi(x_0)^\Gamma v_\infty^{1-\Gamma}(x_0) \leq |\nabla\phi(x_0)| \tag{2.13}
$$

and

$$
-\Delta_\infty\phi(x_0) \geq 0, \tag{2.14}
$$

which is (2.9).

Assume now that either $u_\infty(x_0) = 0$ or $v_\infty(x_0) = 0$. In particular, (2.13) holds. Note first that if $\nabla\phi(x_0) = 0$ then $\Delta_\infty\phi(x_0) = 0$ by definition so that (2.14) holds. We now assume that $|\nabla\phi(x_0)| > 0$ and write (2.11). The parentesis in the right hand side goes to 0 as $p \to +\infty$ so that the right hand side goes to 0 and (2.14) follows.

To complete the proof it just remains to see that $u_\infty$ is a viscosity subsolution. Let us consider a point $x_0 \in \Omega$ and a function $\phi \in C^2(\Omega)$ such that $u_\infty(x_0) = \phi(x_0)$ and $u_\infty(x) < \phi(x)$ for every $x$ in a neighbourhood of $x_0$. We want to show that

$$
H_\infty(x_0, \phi(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \leq 0.
$$

We first observe that if $\nabla\phi(x_0) = 0$ the previous inequality trivially holds. Hence, let us assume that $\nabla\phi(x_0) \neq 0$. Now, we argue as follows: assuming that

$$
|\nabla\phi(x_0)| - \lambda_\infty\phi(x_0)^\Gamma v_\infty^{1-\Gamma}(x_0) > 0, \tag{2.15}
$$

we will show that

$$
-\Delta_\infty\phi(x_0) \leq 0. \tag{2.16}
$$

As before, using that $u_p$ is a viscosity subsolution of (2.5), we get a sequence of points $x_p \to x_0$ such that

$$\begin{aligned} &-\Big(\frac{|\nabla\phi|^2\Delta\phi(x_p)}{(p-2)}+\Delta_\infty\phi(x_p)\Big) \\ &\qquad\le\left(\frac{\alpha^{1/p}}{(p-2)}(\lambda_{1,p})^{1/p}|\phi(x_p)|^{\alpha/p}v_p^{\beta/p}(x_p)|\nabla\phi(x_p)|^{-1+4/p}\right)^p. \end{aligned} \tag{2.17}$$

Using (2.15) we get

$$\limsup_{p\to\infty}\left(\frac{\alpha^{1/p}}{(p-2)}(\lambda_{1,p})^{1/p}|\phi(x_n)|^{\alpha/p}v_p^{\beta/p}(x_n)|\nabla\phi(x_n)|^{-1+4/p}\right)^p=0.$$

Hence, we conclude (2.16) taking limits in (2.17) and we obtain that

$$\min\{-\Delta_\infty\phi(x_0),\, |\nabla\phi(x_0)|-\lambda_\infty\phi(x_0)^\Gamma v_\infty^{1-\Gamma}(x_0)\}\le 0. \tag{2.18}$$

Since we have obtained (2.9) and (2.18), the proof is now complete. □

In a complete analogous way we can prove that $v_\infty$ is a viscosity solution to

$$G_\infty(x,v,\nabla v,D^2v)=0$$

with

$$G_\infty(x,y,z,S)=\min\{-\langle S\cdot z,z\rangle,|z|-\lambda_\infty u_\infty^\Gamma(x)|y|^{-\Gamma}y\}.$$

## 3. A mass transport approach. Proof of Theorem 1.2.

Now we want to put our limit for the first eigenvalue in the context of optimal mass transportation. We find the interesting fact that, from this point of view, the system completely decouples in the limit.

**Lemma 3.1.** *Consider the measures*

$$f_p=u_p^{\alpha-1}v_p^\beta\,dx \qquad \text{and} \qquad g_p=u_p^\alpha v_p^{\beta-1}\,dx.$$

*Then $f_p,g_p\in L^{\frac{p}{p-1}}(\Omega)$ and there exists $f_\infty,g_\infty\in P(\overline{\Omega})$ (the space of probability measures on $\overline{\Omega}$) such that up to a subsequence,*

$$f_p\rightharpoonup f_\infty \qquad \text{and} \qquad g_p\rightharpoonup g_\infty.$$

*Proof.* We have

$$\begin{aligned} \int_\Omega f_p &= \int_\Omega u_p^{\alpha-1}v_p^\beta\,dx \\ &\le\Big(\int_\Omega u_p^\alpha v_p^\beta\,dx\Big)^{\frac{\alpha-1}{\alpha}}\Big(\int_\Omega v_p^{p-\alpha}\Big)^{\frac{1}{\alpha}} \\ &\le\Big(\int_\Omega v_p^p\Big)^{\frac{p-\alpha}{\alpha p}}|\Omega|^{\frac{1}{p}} \end{aligned}$$

with
$$\int_\Omega v_p^p \le \frac{1}{\lambda_{p,D}} \int_\Omega |\nabla v_p|^p \le p\frac{\lambda_{1,p}}{\lambda_{p,D}}.$$
Then
$$\limsup_p \left(\int_\Omega v_p^p\right)^{\frac{p-\alpha}{\alpha p}} \le \limsup_p \left(p\frac{\lambda_{1,p}}{\lambda_{p,D}}\right)^{\frac{p-\alpha}{\alpha p}} \le 1.$$
Hence
$$\limsup_p \int_\Omega f_p \le 1.$$
In an analogous way we obtain
$$\limsup_p \int_\Omega g_p \le 1,$$
and therefore we can extract a subsequence such that
$$f_p \rightharpoonup f_\infty \qquad \text{and} \qquad g_p \rightharpoonup g_\infty,$$
with $f_\infty$ and $g_\infty$ nonnegative measures with total mass less or equal than one. Moreover, we have
$$\int_\Omega f_p u_p = \int_\Omega u_p^\alpha v_p^\beta \, dx = 1,$$
whence
$$\int_\Omega u_\infty f_\infty = 1. \tag{3.1}$$
Now, we observe that, since we have
$$\|\nabla u_\infty\|_{L^\infty(\Omega)} \le \frac{1}{R},$$
we get
$$\frac{1}{R} = \lambda_\infty \le \frac{\|\nabla u_\infty\|_{L^\infty(\Omega)}}{\|u_\infty\|_{L^\infty(\Omega)}} \le \frac{1/R}{\|u_\infty\|_{L^\infty(\Omega)}}$$
and we conclude that
$$\|u_\infty\|_{L^\infty(\Omega)} \le 1,$$
and therefore we conclude from (3.1) that the total mass of $f_\infty$ is equal to one.

In an analogous way we obtain that $g_\infty$ is also a nonnegative probability measure on $\overline{\Omega}$. $\square$

Let us consider the functional $F_p : C(\overline{\Omega}) \times C(\overline{\Omega}) \to \mathbb{R} \cup \{+\infty\}$ defined by
$$F_p(u,v) = \begin{cases} \displaystyle\int_\Omega \frac{|\nabla u|^p}{p\lambda_{1,p}\alpha} + \frac{|\nabla v|^p}{p\lambda_{1,p}\beta} - (f_p, u) - (g_p, v) & \text{if } u, v \in W_0^{1,p}(U) \\ +\infty & \text{otherwise.} \end{cases}$$
We have that $(u_p, v_p)$ is a minimizer of $F_p$ with
$$\lim_{p\to+\infty} F_p(u_p, v_p) = -2.$$

In addition, using ideas as in [6] we can show that $F_p$ $\Gamma$-converge to the functional $F_\infty$ given by

$$F_\infty(u,v) = \begin{cases} -(f_\infty, u) - (g_\infty, v) & \text{if } u, v \in W_0^{1,\infty}(\Omega), \\ & \text{and } \|\nabla u\|_{L^\infty(\Omega)}, \|\nabla v\|_{L^\infty(\Omega)} \le \lambda_\infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Then, $(u_\infty, v_\infty)$ is a minimizer of $F_\infty$ with

$$F_\infty(u_\infty, v_\infty) = -2.$$

Now let $X = C(\overline{\Omega}) \times M(\overline{\Omega})$ and we consider the functional $G_\infty : X \times X \to \mathbb{R} \cup \{+\infty\}$ defined by

$$G_\infty((u,\sigma),(v,\tau)) = \begin{cases} & u, v \in W_0^{1,\infty}(\Omega), \\ & \|\nabla u\|_{L^\infty(\Omega)} \le \lambda_\infty, \\ -\displaystyle\int_{\overline{\Omega}} u\sigma - \int_{\overline{\Omega}} v\tau & \text{if } \|\nabla u\|_{L^\infty(\Omega)} \le \lambda_\infty, \\ & \sigma, \tau \in M(\overline{\Omega}), \\ & \int |\sigma| \le 1, \int |\tau| \le 1 \\ +\infty & \text{otherwise.} \end{cases}$$

Since $(u_\infty, v_\infty)$ is a minimizer of $F_\infty$ and we have $(\mu, u) \le 1$ for any pair $(u, \mu) \in X$ such that $\|\nabla u\|_{L^\infty(\Omega)} \le \lambda_\infty = 1/R$ (note that this fact implies that $\|u\|_{L^\infty(\Omega)} \le 1$) and $\int |\mu| \le 1$, we obtain that $((u_\infty, f_\infty), (v_\infty, g_\infty))$ is a minimizer of $G_\infty$ and

$$\begin{aligned} 2 &= \max -G_\infty((u_\infty, f_\infty), (v_\infty, g_\infty)) \\ &= \max_{\sigma,\tau \in P(\bar{U})} \sup_{\|\nabla u\|_\infty, \|\nabla u\|_\infty \le \lambda_\infty} (\sigma, u) - \chi_C(u) + (\tau, v) - \chi_C(v) \end{aligned}$$

where $\chi_C(u) = 0$ if $u = 0$ on $\partial\Omega$, and $+\infty$ otherwise. We then infer that

$$\begin{aligned} \frac{2}{\lambda_\infty} &= 2 \max_{\sigma \in P(\bar{U})} \sup_{\|\nabla u\|_{L^\infty(\Omega)} \le \lambda_\infty} (\sigma, u) - \chi_C(u) \\ &= 2 \max_{\sigma \in P(\bar{U})} W_1(\sigma, P(\partial U)) = \frac{2}{\lambda_{\infty,D}} \end{aligned}$$

using the computations in [6] to justify the two last equalities. Here $W_1(\cdot,\cdot)$ stands for the the Monge-Kantorovich distance, see [20] for its definition and properties. We thus recover from these computations, as expected, that the limit of $(\lambda_{1,p})^{1/p}$, $\lambda_\infty$, is the first eigenvalue of $\Delta_\infty$ with Dirichlet boundary conditions.

We want to highlight the fact that the limit pair $(u_\infty, v_\infty)$ together with the limit pair of measures $(f_\infty, g_\infty)$ give a solution to a variational problem (minimize the functional $G_\infty$) that is clearly uncoupled.

## 4. Higher eigenvalues. Proof of Theorem 1.3.

We have assumed that there is a constant, $C$, independent of $p$ such that, for $p$ large,

$$(\lambda_{n,p})^{1/p} \le C.$$

Recall also that we have normalized the eigenvalues according to

$$\int_\Omega |u_p|^\alpha |v_p|^\beta = 1.$$

This implies

$$\left( \int_\Omega |\nabla u_p|^p \right)^{1/p} = (\lambda_{n,p})^{1/p} \alpha^{1/p} \le C$$

and analogously

$$\left( \int_\Omega |\nabla v_p|^p \right)^{1/p} = (\lambda_{n,p})^{1/p} (\beta)^{1/p} \le C$$

for large $p$. Hence, for $p$ large, we have

$$\max \left\{ \|\nabla u_p\|_{L^p(\Omega)} ; \|\nabla v_p\|_{L^p(\Omega)} \right\} \le C,$$

with $C$ independent of $p$.

Hence, arguing as in the proof of Theorem 1.1, we can extract a subsequence $p_j \to \infty$ if necessary, such that

$$u_p \rightharpoonup u_\infty$$

weakly in $W^{1,r}(\Omega)$ for any $1 < r < \infty$ and uniformly in $\overline{\Omega}$. In addition, we get that $u_\infty \in W^{1,\infty}(\Omega)$. Analogously, we obtain that

$$v_p \rightharpoonup v_\infty$$

weakly in $W^{1,r}(\Omega)$ for any $1 < r < \infty$ and uniformly in $\overline{\Omega}$, with $v_\infty \in W^{1,\infty}(\Omega)$.

Now our aim is to show that $u_\infty$ is a viscosity solution to (1.7). Fix $x_0 \in \Omega$. First we consider the case $u_\infty(x_0) > 0$. Then there exits $\rho > 0$ such that $u_{p_j} > 0$ in $B_\rho(x_0)$ for all $p_j$ sufficiently large, and we may proceed as in the case of the first eigenvalue, to conclude that

$$\min\{-\Delta_\infty u_\infty, |\nabla u_\infty| - \Lambda u_\infty^\Gamma |v_\infty|^{1-\Gamma}\} = 0.$$

The case $u_\infty(x_0) < 0$ is similar but we have to reverse the inequalities.

Finally for the case $u_\infty(x_0) = 0$ we argue as follows. Let $\phi$ be such that $u_\infty - \phi$ has a strict local maximum at $x_0$. Since $u_{p_j} \to u_\infty$ uniformly, there exists a sequence $x_j \to x_0$ such that $u_{p_j} - \phi$ has a local maximum at $x_j$.

Hence, assuming that $\nabla\phi(x_0) \neq 0$, we get

$$
\begin{aligned}
&-\Big(\frac{|\nabla\phi|^2\Delta\phi(x_n)}{(p-2)} + \Delta_\infty\phi(x_n)\Big) \\
&\quad\le \left(\frac{\alpha^{1/p}}{(p-2)}(\lambda_{1,p})^{1/p}|\phi(x_n)|^{\alpha/p}v_p^{\beta/p}(x_n)|\nabla\phi(x_n)|^{-1+4/p}\right)^p.
\end{aligned}
\tag{4.1}
$$

Now we observe that

$$
\frac{\alpha^{1/p}}{(p-2)}(\lambda_{1,p})^{1/p}|\phi(x_n)|^{\alpha/p}v_p^{\beta/p}(x_n)|\nabla\phi(x_n)|^{-1+4/p} \to 0
$$

as $p\to\infty$ and we conclude that

$$
-\Delta_\infty\phi(x_0) \le 0.
$$

Note that this inequality holds trivially when $\nabla\phi(x_0) = 0$. This shows that $u_\infty$ is a viscosity sub solution to $-\Delta_\infty u = 0$.

The fact that it is also a supersolution can be deduced considering $-u_\infty$ and repeating the previous argument.

**Remark 4.1.** The condition $(\lambda_{n,p})^{1/p} \le C$ holds, for example, for the eigenvalues constructed using topological arguments in [16]. In fact, let us consider

$$
\lambda_{m,p} = \inf_{K\in K_m}\sup_{(u,v)\in K} Q_p(u,v)
$$

where $K_m$ is the class of compact symmetric ($K = -K$) subsets of $W_0^{1,p}(\Omega)\times W_0^{1,p}(\Omega)$ of genus greater or equal than $m$. For such an eigenvalue $\lambda_{m,p}$ it holds that there exists a constant $C$ independent of $p$ such that $(\lambda_{m,p})^{1/p} \le C$. To see this fact it is enough to consider the union of $m$ disjoints balls of radius $r$, $B_i$, inside $\Omega$ and as $K$ the set $\{span(\phi_1,...,\phi_m)\cap S_1 \times \{\sum_i \phi_i\}\}$, where $\phi_i$ is an eigenfunction of the $p-$Laplacian in the ball $B_i \subset \Omega$ and $S_1$ denotes the unit ball in $W_0^{1,p}(\Omega)$. Such set $K$ has genus $m$ and we have

$$
\sup_{(u,v)\in K} Q_p(u,v) = \sup_{(u,v)\in K}\frac{\displaystyle\int_\Omega \frac{|\nabla u|^p}{p} + \int_\Omega\frac{|\nabla v|^p}{p}}{\displaystyle\int_\Omega |u|^\alpha|v|^\beta} \le \frac{2m}{p}\lambda_1(B_i),
$$

where $\lambda_1(B_i)$ is the first eigenvalue of the $p-$Laplacian in $B_i$. Now we just note that from the results in [13] it follows that $(\lambda_1(B_i))^{1/p}$ is bounded independently of $p$ and we obtain the desired uniform in $p$ bound for the eigenvalues constructed using the genus argument at level $m$, $(\lambda_{m,p})^{1/p} \le C$.

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