ON THE BEST LIPSCHITZ EXTENSION PROBLEM FOR A DISCRETE DISTANCE AND THE DISCRETE ∞-LAPLACIAN

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Abstract. This paper is concerned with the best Lipschitz extension problem for a discrete distance that counts the number of steps. We relate this absolutely minimizing Lipschitz extension with a discrete ∞-Laplacian problem, which arise as the dynamic programming formula for the value function of some ε-tug-of-war games. As in the classical case, we obtain the absolutely minimizing Lipschitz extension of a datum f by taking the limit as $p \to \infty$ in a nonlocal p–Laplacian problem.

1. Introduction

Since the classical work of Arosson [6], in which he introduced the concept of absolutely minimizing Lipschitz extension and showed its relation with the infinity Laplace equation, a large amount of literature has appeared in this direction. For a systematic treatment of the theory of absolute minimizers see the recent survey [7] by Aronson, Cradall and Juutinen, and the references therein. A new insight has come in with the work of Peres, Schramm, Sheffield and Wilson [20] where it has been shown an interesting connection between absolutely minimizing Lipschitz extension and Game Theory. More precisely, the authors of [20] proved that if u_ε is the value function for a certain ε-tug-of-war game with final payoff function f, then the uniform limit u of u_ε, as ε goes to zero, is the absolutely minimizing Lipschitz extension of f.

In this work our aim is twofold, first we characterize the value function u_ε as the absolutely minimizing Lipschitz extension with respect to a discrete distance in a proper way, and next we show that u_ε can be obtained by taking the limit as $p \to \infty$ in a nonlocal p–Laplacian Dirichlet problem with boundary data f.

Let (X, d) be an arbitrary metric space and let $f : A \subset X \to \mathbb{R}$. We denote by $L_d(f, A)$ the smallest Lipschitz constant of f in A, i.e.,

$$L_d(f, A) := \sup_{x,y \in A} \frac{|f(x) - f(y)|}{d(x, y)}.$$

If we are given a Lipschitz function $f : A \subset X \to \mathbb{R}$, i.e., $L_d(f, A) < +\infty$, then it is well-known that there exists a *minimal Lipschitz extension* (MLE for short) of f to X, that is, a function $h : X \to \mathbb{R}$ such that $h_{|A} = f$ and $L_d(h, X) = L_d(f, A)$. We will denote the space of such extensions as $\mathrm{MLE}(f, X)$.

Extremal extensions were explicitly constructed by McShane [17] and Whitney [21],

$$\Psi(f)(x) := \inf_{y \in A} \left(f(y) + L_d(f, A) d(x, y) \right), \quad x \in X,$$

Key words and phrases. Lipschitz extension, Nonlocal p–Laplacian problem, Infinity Laplacian, Tug-of-war games.

2000 *Mathematics Subject Classification.* 49J20, 49J45, 45G10.

and

$$\Lambda(f)(x) := \sup_{y\in A} \left(f(y) - L_d(f,A)d(x,y)\right), \quad x \in X,$$

belong to MLE(f,X), and if $u \in$ MLE(f,X) then $\Lambda(f) \leq u \leq \Psi(f)$.

The notion of minimal Lipschitz extension is not completely satisfactory since it involves only the global Lipschitz constant of the extension and ignore what may happen locally. To solve this problem, in the particular case of the euclidean space $\mathbb{R}^N$, Arosson [6] introduced the concept of *absolutely minimizing Lipschitz extension* (AMLE for short) and proved the existence of AMLE by means of a variant of the Perron's method. An extension of this concept to the case of a general metric space is due to Juutinen [11] (see also [18]). In [11], Juutinen gave the following definition.

Definition 1.1. Let A be any nonempty subset of the metric spaces (X,d) and let $f : A \subset X \to \mathbb{R}$ be a Lipschitz function. A function $h : X \to \mathbb{R}$ is an *absolutely minimizing Lipschitz extension* of f to X if

(i) $h \in$ MLE(f,X),
(ii) whenever $B \subset X$ and $g \in$ MLE(f,X) such that $g = h$ in $X \setminus B$, then $L_d(h,B) \leq L_d(g,B)$.

Also in [11] it is proved the existence of an AMLE under the assumption that the metric space (X,d) is a separable length space.

Aronsson's original definition in $\mathbb{R}^N$ was formulated in a slightly different way. He assumed that A is a compact set and required that $L_d(h,D) = L_d(h,\partial D)$ for every bounded open set D in $\mathbb{R}^N$. As remarked by Juutinen in [11], for a general metric space "this kind of definition would be somewhat ambiguous because the boundary of an open subset of a metric space may very well be empty", and the issue of [11] was to find a right way to interpret the "boundary condition".

Moreover, in [6], Aronsson proposed an approach to obtain the AMLE extension of a datum f by taking the limit as $p \to \infty$ in the p–Laplacian problem

$$\begin{cases} -\Delta_p u_p = 0 & \text{in } \Omega, \\ u_p = f & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

This approach was made completely rigorous by Jensen in [10] (see also [9]). In [7] you can find the following result: the limit as $p \to \infty$ of u_p, u_∞, is the best Lipschitz extension (AMLE) of f in Ω and moreover it is characterized as the unique viscosity solution to

$$\begin{cases} -\Delta_\infty u_\infty = 0, & \Omega, \\ u_\infty = f, & \partial\Omega, \end{cases} \tag{1.2}$$

where Δ_∞ is the infinity Laplace operator, that is, the degenerate elliptic operator given by

$$\Delta_\infty u := \sum_{i,j=1}^N u_{x_i} u_{x_j} u_{x_i x_j}.$$

Recently, Peres, Schramm, Sheffield and Wilson [20] have shown that the infinity Laplace equation (1.2) is solved by the continuous value function for a random turn tug-of-war game, in which the players, at each step, flip a fair coin to determine which player plays.

Given a bounded domain Ω in $\mathbb{R}^N$ and a function defined outside Ω (this will be properly stated afterward), our aim is to study the Lipschitz extension problem to Ω respect to the discrete distance that counts the number of steps,

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } x = y, \\ \varepsilon\left(\left[\!\left[\frac{|x-y|}{\varepsilon}\right]\!\right] + 1\right) & \text{if } x \neq y, \end{cases} \tag{1.3}$$

where $|.|$ is the Euclidean norm and $[\![r]\!]$ is defined for $r > 0$ by $[\![r]\!] := n$, if $n < r \leq n+1$, $n = 0, 1, 2, \dots$, that is,

$$d_\varepsilon(x,y) = \begin{cases} 0 & \text{if } \ x = y, \\ \varepsilon & \text{if } \ 0 < |x-y| \leq \varepsilon, \\ 2\varepsilon & \text{if } \ \varepsilon < |x-y| \leq 2\varepsilon, \\ \vdots & \end{cases}$$

The distance d_ε was used in [20] in relation with ε-tug-of-war games. It was also used in [2] to give a mass transport interpretation of a nonlocal model of sandpiles.

1.1. **Description of the main results.** Since $(\mathbb{R}^N, d_\varepsilon)$ is not a separable length space, the general concept of AMLE due to Juutinen does not work on it. We give a concept of AMLE respect to the distance d_ε, which we name as $\mathrm{AMLE}_\varepsilon(f,\Omega)$, in an slight different way that finds the right manner to interpret the "boundary condition" (observe that for the metric d_ε the boundary of Ω is empty).

In addition, we relate this absolutely minimizing Lipschitz extension problem with a discrete ∞-Laplacian problem, which arise as the dynamic programming formula for the value function of some ε-tug-of-war game. More precisely, we characterize the value function for the ε-tug-of-war game with payoff function f as the $\mathrm{AMLE}_\varepsilon(f,\Omega)$. Therefore, as consequence of the results in [20] we have existence and uniqueness of $\mathrm{AMLE}_\varepsilon(f,\Omega)$.

Finally, we also obtain the nonlocal version of the approximation by the p–Laplacian, that is, we get the $\mathrm{AMLE}_\varepsilon(f,\Omega)$ by taking the limit as $p \to \infty$ in a nonlocal p–Laplacian problem.

2. Definition and characterization of $\mathrm{AMLE}_\varepsilon$

Given a set $A \subset \mathbb{R}^N$ and $\varepsilon > 0$, we denote

$$A_\varepsilon := \left\{ x \in \mathbb{R}^N \ : \ \operatorname{dist}(x,A) := \inf_{y\in A} |x-y| < \varepsilon \right\}.$$

The euclidean open ball centred at x with radius r will be denoted by $B_r(x)$, and with $\overline{B}_r(x)$ its closure. Throughout the paper, we assume that Ω is a bounded domain of $\mathbb{R}^N$.

Given $u : \Omega_\varepsilon \to \mathbb{R}$ and $D \subset \Omega$, we define

$$L_\varepsilon(u,D) := \sup_{\substack{x\,\in\,D,\ y\,\in\,D_\varepsilon \\ |x-y|\,\leq\,\varepsilon}} \frac{|u(x)-u(y)|}{\varepsilon}.$$

Observe that

$$L_\varepsilon(u,D) = \sup_{\substack{x\,\in\,D,\ y\,\in\,D_\varepsilon \\ |x-y|\,\leq\,\varepsilon}} \frac{|u(x)-u(y)|}{d_\varepsilon(x,y)} \leq \sup_{x\in D,\, y\in D_\varepsilon} \frac{|u(x)-u(y)|}{d_\varepsilon(x,y)}.$$

And that, if D is convex, the above inequality is an equality. Indeed, for $x_0 = x \in D,\ x_n = y \in D_\varepsilon$ and $x_1, x_2, ..., x_{n-1}$ in the segment between x and y such that $|x_i - x_{i-1}| = \varepsilon$ for $i = 1, ..., n-1$, and $|x_n - x_{n-1}| \le \varepsilon$, we have that

$$|u(x) - u(y)| \le \sum_{i=1}^{n} |u(x_i) - u(x_{i-1})| \le \varepsilon \sum_{i=1}^{n} L_\varepsilon(u, D) = \varepsilon n L_\varepsilon(u, D) = d_\varepsilon(x, y) L_\varepsilon(u, D).$$

Also, for any convex D,

$$L_{d_\varepsilon}(u, D) = \sup_{x,y\in D, |x-y|\le\varepsilon} \frac{|u(x) - u(y)|}{\varepsilon}.$$

Therefore, the constant $L_\varepsilon(u, D)$ is not genuinely the Lipschitz constant $L_{d_\varepsilon}(u, D)$, even if D is convex, but, as we will see, it is the right one to treat the absolutely minimizing Lipschitz extensions when d_ε is considered.

Definition 2.1. Let $f : \Omega_\varepsilon \setminus \Omega \to \mathbb{R}$ be bounded. We say that a function $u : \Omega_\varepsilon \to \mathbb{R}$ is an *Absolutely Minimizing Lipschitz Extension for L_ε of f into Ω* (u is $\mathrm{AMLE}_\varepsilon(f, \Omega)$ for shortness) if

(i) $u = f$ in $\Omega_\varepsilon \setminus \Omega$,

(ii) for every $D \subset \Omega$ and $v : \Omega_\varepsilon \to \mathbb{R}$ with $v = u$ in $\Omega_\varepsilon \setminus D$, then $L_\varepsilon(u, D) \le L_\varepsilon(v, D)$.

Lemma 2.2. *When Ω is convex the above definition is equivalent to the following two conditions, that match better the idea of Definition* 1.1*,*

(i') $u \in \mathrm{MLE}(f, \Omega_\varepsilon)$,

(ii') *for every $D \subset \Omega$ and $v \in \mathrm{MLE}(f, \Omega_\varepsilon)$ with $v = u$ in $\Omega_\varepsilon \setminus D$, then $L_\varepsilon(u, D) \le L_\varepsilon(v, D)$.*

Proof. Let us first see that (i), (ii) implies (i') ((ii') is immediate): take $v \in \mathrm{MLE}(f, \Omega_\varepsilon)$, then

$$L_\varepsilon(u, \Omega) \le L_\varepsilon(v, \Omega) \le L_{d_\varepsilon}(v, \Omega_\varepsilon) = L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega).$$

Therefore, since Ω is convex,

$$\sup_{x\in\Omega,\, y\in\Omega_\varepsilon} \frac{|u(x) - u(y)|}{d_\varepsilon(x, y)} = L_\varepsilon(u, \Omega) \le L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega). \tag{2.1}$$

On the other hand, since $u = f$ in $\Omega_\varepsilon \setminus \Omega$,

$$L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega) \le L_{d_\varepsilon}(u, \Omega_\varepsilon) = \sup_{x\in\Omega_\varepsilon,\, y\in\Omega_\varepsilon} \frac{|u(x) - u(y)|}{d_\varepsilon(x, y)}. \tag{2.2}$$

Consequently, from (2.1) and (2.2), $L_{d_\varepsilon}(u, \Omega_\varepsilon) = L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega)$.

Let us now see that (i'), (ii') implies (ii) ((i') is immediate). Let us argue by contradiction and suppose that there exist D and v such that $v = u$ in $\Omega_\varepsilon \setminus D$ and $L_\varepsilon(v, D) < L_\varepsilon(u, D)$. Then, by (ii'), v can not be in $\mathrm{MLE}(f, \Omega_\varepsilon)$. But also, the above strict inequality implies that, on account that Ω_ε is convex,

$$L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega) \le L_{d_\varepsilon}(v, \Omega_\varepsilon) = \sup_{\substack{x \notin D,\, y \notin D \\ |x-y| \le \varepsilon}} \frac{|v(x) - v(y)|}{\varepsilon}$$

$$= \sup_{\substack{x \notin D,\, y \notin D \\ |x-y| \le \varepsilon}} \frac{|u(x) - u(y)|}{\varepsilon} \le L_{d_\varepsilon}(u, \Omega_\varepsilon) = L_{d_\varepsilon}(f, \Omega_\varepsilon \setminus \Omega)$$

and consequently $v \in \mathrm{MLE}(f, \Omega_\varepsilon)$, which is a contradiction. $\square$

Remark that independently of the convexity of Ω, if u is $\mathrm{AMLE}_\varepsilon(f,\Omega)$, it always holds that

$$L_\varepsilon(u,\Omega) \le L_{d_\varepsilon}(f,\Omega_\varepsilon\setminus\Omega).$$

In the next result we obtain the characterization of the $\mathrm{AMLE}_\varepsilon(f,\Omega)$ by means of a discrete ∞-Laplacian problem.

Theorem 2.3. *Let $f:\Omega_\varepsilon\setminus\Omega\to\mathbb{R}$ be bounded. Then, $u:\Omega_\varepsilon\to\mathbb{R}$ is* $\mathrm{AMLE}_\varepsilon(f,\Omega)$ *if and only if u is a solution of*

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } \ \Omega, \\ u = f & \text{on } \ \Omega_\varepsilon\setminus\Omega, \end{cases} \tag{2.3}$$

where

$$\Delta_\infty^\varepsilon u(x) := \sup_{y\in\overline{B}_\varepsilon(x)} u(y) + \inf_{y\in\overline{B}_\varepsilon(x)} u(y) - 2u(x) \tag{2.4}$$

is the discrete infinity Laplace operator.

Proof. Without loss of generality we will take $\varepsilon=1$ along the proof. Let us first take u a solution of (2.3) and suppose that u is not $\mathrm{AMLE}_1(f,\Omega)$. Then, there exists $D\subset\Omega$ and $v:\Omega_1\to\mathbb{R}$, $v=u$ in $\Omega_1\setminus D$, such that $L_1(v,D)<L_1(u,D)$. Set $\delta:=L_1(u,D)-L_1(v,D)>0$, and let $n\in\mathbb{N}$, $n>3$, such that

$$\sup_D u - \inf_D u \le (n-1)L_1(u,D). \tag{2.5}$$

Take $(x_0,y_0)\in D\times\Omega_1$, $|x_0-y_0|\le 1$, such that

$$L_1(u,D)-\frac{\delta}{n} \le |u(x_0)-u(y_0)| \le L_1(u,D).$$

We have that $\Delta_\infty^1 u(x_0)=0$ and $\Delta_\infty^1 u(y_0)=0$ if $y_0\in\Omega$. Let us suppose that $u(y_0)\ge u(x_0)$ (the other case being similar), which implies

$$L_1(u,D)-\frac{\delta}{n} \le u(y_0)-u(x_0) \le L_1(u,D). \tag{2.6}$$

If $y_0\notin D$, set $y_1=y_0$. If $y_0\in D$, since $\Delta_\infty^1 u(y_0)=0$ and $x_0\in\overline{B}_1(y_0)$, we have

$$\sup_{y\in\overline{B}_1(y_0)} u(y)-u(y_0) = u(y_0)-\inf_{y\in\overline{B}_1(y_0)} u(y) \ge u(y_0)-u(x_0) \ge L_1(u,D)-\frac{\delta}{n}.$$

Hence, there exists $y_1\in\overline{B}_1(y_0)$ such that

$$u(y_1)-u(y_0)\ge L_1(u,D)-\frac{2\delta}{n}.$$

Also, since $\Delta_\infty^1 u(x_0)=0$, we have

$$u(x_0)-\inf_{x\in\overline{B}_1(x_0)} u(x) = \sup_{x\in\overline{B}_1(x_0)} u(x)-u(x_0) \ge u(y_0)-u(x_0) \ge L_1(u,D)-\frac{\delta}{n},$$

and consequently, there exists $x_1\in\overline{B}_1(x_0)$ such that

$$u(x_0)-u(x_1)\ge L_1(u,D)-\frac{2\delta}{n}.$$

Following this construction, and with the rule that in the case $x_j \notin D$ or $y_j \notin D$, then $x_i = x_j$ or $y_i = y_j$ for all $i \geq j$, we claim that there exists $m \leq n$ for which $x_m \notin D$ and $y_m \notin D$. In fact, if not, then either $\{x_i\}_{i=1,\ldots,n} \subset D$, either $\{y_i\}_{i=1,\ldots,n} \subset D$, with $\{x_i\}_{i=1,\ldots,n}$ and $\{y_i\}_{i=1,\ldots,n}$ satisfying

$$u(y_i) - u(y_{i-1}) \geq L_1(u,D) - \frac{2\delta}{n}, \quad y_i \in \overline{B}_1(y_{i-1}),\ i = 1,\ldots,n, \tag{2.7}$$

and

$$u(x_i) - u(x_{i-1}) \geq L_1(u,D) - \frac{2\delta}{n}, \quad x_i \in \overline{B}_1(x_{i-1}),\ i = 1,\ldots,n. \tag{2.8}$$

Let us suppose the first of these two possibilities, that is, $\{x_i\}_{i=1,\ldots,n} \subset D$. Then, having in mind (2.5), (2.6) and (2.8), we get

$$\begin{aligned}
&(n-1)L_1(u,D) \geq u(y_0) - u(x_n) \\
&= u(y_0) - u(x_0) + u(x_0) - u(x_1) + \cdots + u(x_{n-1}) - u(x_n) \\
&\geq L_1(u,D) - \tfrac{\delta}{n} + (n+1)(L_1(u,D) - \tfrac{2\delta}{n}),
\end{aligned}$$

from where it follows that

$$\frac{2n+3}{n}\delta \geq 3L_1(u,D) \geq 3\delta,$$

which is a contradiction since $n > 3$. Now, for $\{x_i, y_i\}_{i=1,\ldots,m}$, we have

$$v(y_m) - v(x_m) = u(y_m) - u(x_m) \geq 2m\left(L_1(u,D) - \frac{2\delta}{n}\right) + L_1(u,D) - \frac{\delta}{n},$$

$$v(y_m) - v(x_m) \leq (2m+1)L_1(v,D),$$

and therefore,

$$(2m+1)L_1(u,D) - (4m+1)\frac{\delta}{n} \leq (2m+1)L_1(v,D),$$

that is

$$\delta = L_1(u,D) - L_1(v,D) \leq \frac{4m+1}{2m+1}\frac{\delta}{n},$$

which implies $n \leq \frac{4m+1}{2m+1} \leq 2$, which is a contradiction since $n > 3$.

Let us now consider u an $\mathrm{AMLE}_1(f,\Omega)$ and suppose that u is not a solution of (2.3). Then, $\{x \in \Omega : \Delta^1_\infty u(x) \neq 0\} \neq \emptyset$. Let us suppose without loss of generality, that,

$$\left\{x \in \Omega \ : \ \sup_{y \in \overline{B}_1(x)} u(y) - u(x) > u(x) - \inf_{y \in \overline{B}_1(x)} u(y)\right\} \neq \emptyset.$$

Then, there exists $\delta > 0$ and a nonempty set $D \subset \Omega$ such that

$$\sup_{y \in \overline{B}_1(x)} u(y) - u(x) > u(x) - \inf_{y \in \overline{B}_1(x)} u(y) + \delta \quad \text{for all } x \in D. \tag{2.9}$$

Consider the function $v : \Omega_1 \to \mathbb{R}$ defined by

$$v(x) = \begin{cases} u(x) & \text{if } x \in \Omega_1 \setminus D, \\ u(x) + \frac{\delta}{2} & \text{if } x \in D. \end{cases}$$

Then, since u is an $\text{AMLE}_1(f,\Omega)$, we have $L_1(u,D) \le L_1(v,D)$. Now, there exists $x_0 \in D$ and $y_0 \in \overline{B}_1(x_0)$ such that

$$L_1(v,D) \le \frac{\delta}{4} + |v(x_0) - v(y_0)|.$$

Therefore, if $v(x_0) \ge v(y_0)$, by (2.9),

$$\begin{aligned} L_1(v,D) &\le \frac{\delta}{4} + v(x_0) - v(y_0) \le \frac{3\delta}{4} + u(x_0) - u(y_0) \\ &\le \frac{3\delta}{4} + u(x_0) - \inf_{x\in\overline{B}_1(x_0)} u(x) < -\frac{\delta}{4} + \sup_{x\in\overline{B}_1(x_0)} u(x) - u(x_0) < L_1(u,D), \end{aligned}$$

which is a contradiction, and, if $v(x_0) < v(y_0)$,

$$L_1(v,D) \le \frac{\delta}{4} + v(y_0) - v(x_0) = -\frac{\delta}{4} + v(y_0) - u(x_0),$$

so, if $y_0 \notin D$,

$$L_1(v,D) \le -\frac{\delta}{4} + u(y_0) - u(x_0) < L_1(u,D),$$

also a contradiction, and if $y_0 \in D$, since also $x_0 \in \overline{B}_1(y_0)$, by (2.9),

$$\begin{aligned} L_1(v,D) &\le \frac{\delta}{4} + u(y_0) - u(x_0) \le \frac{\delta}{4} + u(y_0) - \inf_{y\in\overline{B}_1(y_0)} u(y) \\ &< -\frac{3\delta}{4} + \sup_{y\in\overline{B}_1(y_0)} u(y) - u(y_0) < L_1(u,D), \end{aligned}$$

again a contradiction. Then, in any case we arrive to a contradiction and consequently u is a solution of (2.3). $\square$

The first analysis of the interesting functional equation $-\Delta_\infty^\varepsilon u = 0$ appeared in the article by Le Gruyer and Archer [13], but also arise as the dynamic programming formula for the value function of some tug-of-war games (see for instance [8, 15, 16, 20]). Let us briefly review the *ε-tug-of-war game* introduced by Peres, Schramm, Sheffield and Wilson in [20]. Fix a number $\varepsilon > 0$. The dynamic of the game is as follows. There are two players moving a token inside a set E_Ω containing Ω, a bounded domain in $\mathbb{R}^N$. The token is placed at an initial position $x_0 \in \Omega$. At the kth stage of the game, player I and player II select points x_k^I and x_k^{II}, respectively, both belonging to $\overline{B}_\varepsilon(x_{k-1}) \cap E_\Omega$. The token is then moved to x_k, where x_k is chosen randomly so that $x_k = x_k^I$ or $x_k = x_k^{II}$, depending who was the winner of a flip of a fair coin. After the kth stage of the game, if $x_k \in \Omega$ then the game continues to stage $k+1$. Otherwise, if $x_k \in E_\Omega \setminus \Omega$, the game ends and player II pays player I the amount $f(x_k)$, where $f : E_\Omega \setminus \Omega \to \mathbb{R}$ is a final payoff function of the game. Of course, player I attempts to maximize the payoff, while player II attempts to minimize it.

Given a strategy for player I, that is a mapping S_I from the set of all possible partially played games $(x_0, x_1, \ldots, x_{k-1})$ to possible positions $x_k \in \overline{B}_\varepsilon(x_{k-1})$, and a strategy S_{II} for player II, we denote by $E_I^{x_0}(S_I, S_{II})$ and $E_{II}^{x_0}(S_I, S_{II})$ the expected value of $f(x_k)$ ($\mathbb{E}_{S_I,S_{II}}^{x_0}[f(x_k)]$), if the game terminates a.s., $-\infty$ and $+\infty$, respectively, otherwise (there is a severe penalization for both players if the game never ends). The value of the game for player I is the quantity

$$\inf_{S_{II}} \sup_{S_I} E_I^{x_0}(S_I, S_{II}),$$

where the supremum is taken over all possible strategies for player I and the infimum over all strategies of player II. Similarly, the value of the game for player II is

$$\sup_{S_I}\inf_{S_{II}} E_{II}^{x_0}(S_I, S_{II}).$$

We denote the value for player I as a function of the starting point $x_0 \in \Omega$ by $u_I^\varepsilon(x_0)$, and similarly the value for player II by $u_{II}^\varepsilon(x_0)$. The game is said to have a value if $u_I^\varepsilon = u_{II}^\varepsilon =: u^\varepsilon$. According with the Dynamic Programming Principle, see [20], there is a value function for the ε-tug-of-war game, u^ε, that satisfies the functional equation

$$u^\varepsilon(x) = \frac{1}{2}\left(\sup_{y\in \overline{B}_\varepsilon(x)\cap E_\Omega} u^\varepsilon(y) + \inf_{y\in \overline{B}_\varepsilon(x)\cap E_\Omega} u^\varepsilon(y)\right) \quad \text{in } \Omega,$$

with $u^\varepsilon = f$ in $E_\Omega \setminus \Omega$. Observe that this is (2.3) when $E_\Omega = \Omega_\varepsilon$ (see [15, 16] for this problem). In [20], using martingale methods, it is proved that problem (2.3) has a unique solution; then, by Theorem 2.3, we get the following existence and uniqueness result.

Theorem 2.4. *Let $f : \Omega_\varepsilon \setminus \Omega \to \mathbb{R}$ be bounded. Then, there is a unique* $\mathrm{AMLE}_\varepsilon(f, \Omega)$.

Some of the difficulties in the analysis of the ε-tug-of-war game are due to the fact that the value function u^ε can be discontinuous. When the limit $u := \lim_{\varepsilon\to 0} u^\varepsilon$ exists pointwise, the function u is called the *continuum value of the game*. In [20], Peres, Schramm, Sheffield and Wilson proved that if $E_\Omega = \overline{\Omega}$ and the terminal payoff function of the game f is Lipschitz continuous on $\partial\Omega$ then the continuum value u exists and $u^\varepsilon \to u$ uniformly in $\overline{\Omega}$ as $\varepsilon \to 0$. Moreover, u is the unique AMLE extension of f to Ω and the unique viscosity solution of the boundary value problem

$$\begin{cases} -\Delta_\infty u = 0 & \text{in } \Omega, \\ u = f & \text{on } \partial\Omega. \end{cases}$$

Our Theorem 2.3 gives this characterization in the case of the discrete distance. We will see in the next section that we can also obtain the $\mathrm{AMLE}_\varepsilon$ extension by taking the limit as $p \to \infty$ in a nonlocal p–Laplacian problem, which represents the nonlocal version of the approximation of the local problem with the p–Laplacian.

3. Existence of $\mathrm{AMLE}_\varepsilon$ by a nonlocal L^p-variational approach

First, let us introduce some notation. Given $f : \Omega_\varepsilon \setminus \Omega \to \mathbb{R}$ and $u : \Omega \to \mathbb{R}$, we will denote

$$u_f(x) := \begin{cases} u(x) & \text{if } x \in \Omega, \\ f(x) & \text{if } x \in \Omega_\varepsilon \setminus \Omega. \end{cases}$$

Given a convex set $K \subset L^2(\Omega)$, we denote by $\mathbb{I}_K$ to the indicator function of K, that is, the function defined as

$$\mathbb{I}_K(u) := \begin{cases} 0 & \text{if } u \in K, \\ +\infty & \text{if } u \notin K. \end{cases}$$

Let $J : \mathbb{R}^N \to \mathbb{R}$ be a nonnegative, radial, continuous function, strictly positive in $B_1(0)$, vanishing in $\mathbb{R}^N \setminus B_1(0)$ and such that $\int_{\mathbb{R}^N} J(z)\, dz = 1$. For $1 < p < +\infty$ and $f : \Omega_1 \setminus \overline{\Omega} \to \mathbb{R}$

such that $|f|^{p-1} \in L^1(\Omega_1 \setminus \overline{\Omega})$, we define in $L^1(\Omega)$ the operator $\mathrm{B}^J_{p,f}$ by

$$\begin{aligned} \mathrm{B}^J_{p,f}(u)(x) \ &:= -\int_\Omega J(x-y)|u(y)-u(x)|^{p-2}(u(y)-u(x))\,dy \\ &\qquad -\int_{\Omega_1\setminus\overline{\Omega}} J(x-y)|f(y)-u(x)|^{p-2}(f(y)-u(x))\,dy, \qquad x\in\Omega, \end{aligned}$$

that is,

$$\mathrm{B}^J_{p,f}(u)(x) = -\int_{\Omega_1} J(x-y)|u_f(y)-u(x)|^{p-2}(u_f(y)-u(x))\,dy, \qquad x\in\Omega.$$

In [1] (see also [3]) we have seen that the nonlocal version of the Dirichlet problem (1.1), with boundary value f, can be written as

$$\mathrm{B}^J_{p,f}(u) = 0. \tag{3.1}$$

We have also established the following Poincaré's type inequality for such kind of integral operators.

Proposition 3.1 ([1])**.** *Given $J : \mathbb{R}^N \to \mathbb{R}$ as above, $p \geq 1$ and $f \in L^p(\Omega_1 \setminus \overline{\Omega})$, there exists $\lambda = \lambda(J,\Omega,p) > 0$ such that*

$$\lambda \int_\Omega |u(x)|^p\,dx \leq \int_\Omega\int_{\Omega_1} J(x-y)|u_f(y)-u(x)|^p\,dy\,dx + \int_{\Omega_1\setminus\overline{\Omega}} |f(y)|^p\,dy \tag{3.2}$$

for all $u \in L^p(\Omega)$.

We say that u is a supersolution (resp. subsolution) of the nonlocal Dirichlet problem (3.1) with boundary value f if $\mathrm{B}^J_{p,f}(u) \geq 0$ (resp. $\mathrm{B}^J_{p,f}(u) \leq 0$). We have the following comparison principle.

Lemma 3.2. *Let $J : \mathbb{R}^N \to \mathbb{R}$ as above, $p > 1$ and $\overline{f}, \underline{f} \in L^p(\Omega_1 \setminus \overline{\Omega})$, with $\overline{f} \geq \underline{f}$. If $\overline{u}$ is a supersolution of the Dirichlet problem* (3.1) *with boundary value $\overline{f}$ and $\underline{u}$ is a subsolution of the Dirichlet problem* (3.1) *with boundary value $\underline{f}$ then $\overline{u} \geq \underline{u}$.*

Proof. By assumption we have

$$0 \geq -\int_{\Omega_1} J(x-y)|\underline{u}_{\underline{f}}(y)-\underline{u}(x)|^{p-2}(\underline{u}_{\underline{f}}(y)-\underline{u}(x))\,dy, \qquad \forall\, x\in\Omega$$

and

$$0 \leq -\int_{\Omega_1} J(x-y)|\overline{u}_{\overline{f}}(y)-\overline{u}(x)|^{p-2}(\overline{u}_{\overline{f}}(y)-\overline{u}(x))\,dy, \qquad \forall\, x\in\Omega.$$

Then, multiplying by $(\underline{u}-\overline{u})^+(x)$, integrating and having in mind that

$$(\underline{u}_{\underline{f}}(x)-\overline{u}_{\overline{f}}(x))^+ = (\underline{f}(x)-\overline{f}(x))^+ = 0$$

if $x \in \Omega_1 \setminus \Omega$, we get

$$0 \geq -\int_{\Omega_1\times\Omega_1} J(x-y)|\underline{u}_{\underline{f}}(y)-\underline{u}_{\underline{f}}(x)|^{p-2}(\underline{u}_{\underline{f}}(y)-\underline{u}_{\underline{f}}(x))(\underline{u}_{\underline{f}}(x)-\overline{u}_{\overline{f}}(x))^+\,dydx$$

$$= \frac{1}{2}\int_{\Omega_1\times\Omega_1} J(x-y)|\underline{u}_{\underline{f}}(y)-\underline{u}_{\underline{f}}(x)|^{p-2}(\underline{u}_{\underline{f}}(y)-\underline{u}_{\underline{f}}(x))\left[(\underline{u}_{\underline{f}}(y)-\overline{u}_{\overline{f}}(y))^+ - (\underline{u}_{\underline{f}}(x)-\overline{u}_{\overline{f}}(x))^+\right]dydx$$

and

$$\begin{aligned} 0 &\geq \int_{\Omega_1\times\Omega_1} J(x-y)|\overline{u}_{\overline{f}}(y)-\overline{u}_{\overline{f}}(x)|^{p-2}(\overline{u}_{\overline{f}}(y)-\overline{u}_{\overline{f}}(x))(\underline{u}_{\underline{f}}(x)-\overline{u}_{\overline{f}}(x))^+\,dydx \\ &= -\int_{\Omega_1\times\Omega_1} \frac{J(x-y)}{2}|\overline{u}_{\overline{f}}(y)-\overline{u}_{\overline{f}}(x)|^{p-2}(\overline{u}_{\overline{f}}(y)-\overline{u}_{\overline{f}}(x)) \\ &\qquad \times \Big[(\underline{u}_{\underline{f}}(y)-\overline{u}_{\overline{f}}(y))^+-(\underline{u}_{\underline{f}}(x)-\overline{u}_{\overline{f}}(x))^+\Big]\,dydx. \end{aligned}$$

Then, adding the last two inequalities we obtain that

$$\begin{aligned} &\int_{\Omega_1\times\Omega_1} J(x-y)\left(|\underline{u}_{\underline{f}}(y)-\underline{u}_{\underline{f}}(x)|^{p-2}(\underline{u}_{\underline{f}}(y)-\underline{u}_{\underline{f}}(x))-|\overline{u}_{\overline{f}}(y)-\overline{u}_{\overline{f}}(x)|^{p-2}(u_{\overline{f}}(y)-\overline{u}_{\overline{f}}(x))\right) \\ &\qquad\qquad \times\left((\underline{u}_{\underline{f}}(y)-\overline{u}_{\overline{f}}(y))^+-(\underline{u}_{\underline{f}}(x)-\overline{u}_{\overline{f}}(x))^+\right)\ dydx\leq 0. \end{aligned}$$

Therefore, since $(|r|^{p-2}r-|s|^{p-2}s)(r^+-s^+)\geq 0$, we get that

$$\begin{aligned} &J(x-y)\left(|\underline{u}_{\underline{f}}(y)-\underline{u}_{\underline{f}}(x)|^{p-2}(\underline{u}_{\underline{f}}(y)-\underline{u}_{\underline{f}}(x))-|\overline{u}_{\overline{f}}(y)-\overline{u}_{\overline{f}}(x)|^{p-2}(u_{\overline{f}}(y)-\overline{u}_{\overline{f}}(x))\right) \\ &\qquad\qquad \times\left((\underline{u}_{\underline{f}}(y)-\overline{u}_{\overline{f}}(y))^+-(\underline{u}_{\underline{f}}(x)-\overline{u}_{\overline{f}}(x))^+\right)=0 \end{aligned} \tag{3.3}$$

for a.e. $(x,y)\in\Omega_1\times\Omega_1$.

Let $\tilde\Omega:=\{x\in\Omega:\underline{u}(x)>\overline{u}(x)\}$. Since $(|r|^{p-2}r-|s|^{p-2}s)(r-s)\geq C|r-s|^p$, from (3.3) we obtain that, for a.e. $(x,y)\in\tilde\Omega\times\tilde\Omega$,

$$\begin{aligned} 0 &= J(x-y)\left(|\underline{u}(y)-\underline{u}(x)|^{p-2}(\underline{u}(y)-\underline{u}(x))-|\overline{u}(y)-\overline{u}(x)|^{p-2}(\overline{u}(y)-\overline{u}(x))\right) \\ &\qquad\qquad \times(\underline{u}(y)-\underline{u}(x)-(\overline{u}(y)-\overline{u}(x))) \\ &\geq CJ(x-y)|\underline{u}(y)-\underline{u}(x)-(\overline{u}(y)-\overline{u}(x))|^p \end{aligned}$$

that is,

$$J(x-y)|\underline{u}(y)-\overline{u}(y)-(\underline{u}(x)-\overline{u}(x))|^p=0\quad\text{for a.e. }(x,y)\in\tilde\Omega\times\tilde\Omega. \tag{3.4}$$

Therefore, if

$$|\tilde\Omega|>0, \tag{3.5}$$

from (3.4), we get

$$\underline{u}(x)-\overline{u}(x)=\lambda>0\quad\text{a.e. in }\tilde\Omega.$$

Then, taking the above conclusion in (3.3) we have that, for a.e. $(x,y)\in(\Omega_1\setminus\tilde\Omega)\times\tilde\Omega$,

$$0=J(x-y)\left(|\underline{u}(y)-\underline{u}_{\underline{f}}(x)|^{p-2}(\underline{u}(y)-\underline{u}_{\underline{f}}(x))-|\overline{u}(y)-\overline{u}_{\overline{f}}(x)|^{p-2}(\overline{u}(y)-\overline{u}_{\overline{f}}(x))\right)(\underline{u}(y)-\overline{u}(y))$$

$$=J(x-y)\left(|\overline{u}(y)-(\underline{u}_{\underline{f}}(x)-\lambda)|^{p-2}(\overline{u}(y)-(\underline{u}_{\underline{f}}(x)-\lambda))-|\overline{u}(y)-\overline{u}_{\overline{f}}(x)|^{p-2}(\overline{u}(y)-\overline{u}_{\overline{f}}(x))\right)\lambda\ .$$

Now, since $|r|^{p-2}r-|s|^{p-2}s=0$ if and only if $r=s$ we conclude that

$$\underline{u}_{\underline{f}}(x)-\lambda=\overline{u}_{\overline{f}}(x)\qquad\text{a.e. in }\Omega_1\setminus\tilde\Omega,$$

which contradicts that $\Omega_1\setminus\tilde\Omega$ contains the non-null set $\Omega_1\setminus\Omega$ (since $\overline{f}\geq\underline{f}$). Therefore (3.5) is false, and then $\underline{u}\leq\overline{u}$ a.e. in Ω. $\square$

For the energy functional

$$G^J_{p,f}(u)=\frac{1}{2p}\int_\Omega\int_\Omega J(x-y)|u(y)-u(x)|^p\,dy\,dx+\frac{1}{p}\int_\Omega\int_{\Omega_1\setminus\Omega} J(x-y)|f(y)-u(x)|^p\,dy\,dx,$$

we have the following result:

Theorem 3.3. *Assume that $p\geq 2$. Then, there exists a unique $u_p\in L^p(\Omega)$ such that*

$$G^J_{p,f}(u_p)=\min\{G^J_{p,f}(u)\ :\ u\in L^p(\Omega)\}. \tag{3.6}$$

Moreover, u_p is the solution of the nonlocal Euler–Lagrange equation $\mathrm{B}^J_{p,f}(u_p)=0$, and it has a continuous representative in Ω.

Proof. Let $v_n\in L^p(\Omega)$ a minimizing sequence, that is,

$$m:=\inf\{G^J_{p,f}(u)\ :\ u\in L^p(\Omega)\}=\lim_{n\to+\infty}G^J_{p,f}(v_n).$$

Then, by the Poincaré inequality (3.2), we have

$$\|v_n\|_p\leq\left(\frac{1}{\lambda}\left(m+1+\int_{\Omega_1\setminus\overline{\Omega}}|f(y)|^p\,dy\right)\right)^{\frac{1}{p}}.$$

Therefore, we can assume that $v_n\rightharpoonup u_p$ weakly in $L^2(\Omega)$. Hence, since the functional $G^J_{p,f}$ is weakly lower semi-continuous in $L^2(\Omega)$, we get

$$G^J_{p,f}(u_p)\leq\liminf_{n\to\infty}G^J_{p,f}(v_n)=m,$$

consequently, $m=G^J_{p,f}(u_p)$ and (3.6) holds.

By results in [1], we know that the operator $\mathrm{B}^J_{p,f}$ is completely accretive and verifies the range condition $L^p(\Omega)\subset\mathrm{Ran}(I+\mathrm{B}^J_{p,f})$. Let us see that

$$\partial G^J_{p,f}=\overline{\mathrm{B}^J_{p,f}}^{L^2(\Omega)}. \tag{3.7}$$

Since $G^J_{p,f}$ is convex an lower semi-continuous in $L^2(\Omega)$, to prove (3.7) it is enough to show that

$$\mathrm{B}^J_{p,f}\subset\partial G^J_{p,f}. \tag{3.8}$$

Let us see that (3.8) holds. Set $v=\mathrm{B}^J_{p,f}(u)$ and let $w\in D(G^J_{p,f})$. Then

$$\begin{aligned}\int_\Omega v(x)(w(x)-u(x))\,dx&=-\int_\Omega\int_{\Omega_1}J(x-y)|u_f(y)-u(x)|^{p-2}(u_f(y)-u(x))\,dy(w(x)-u(x))\,dx\\&=-\int_{\Omega_1}\int_{\Omega_1}J(x-y)|u_f(y)-u_f(x)|^{p-2}(u_f(y)-u_f(x))\,dy(w_f(x)-u_f(x))\,dx\\&=\frac{1}{2}\int_{\Omega_1}\int_{\Omega_1}J(x-y)|u_f(y)-u_f(x)|^{p-2}(u_f(y)-u_f(x))\,(w_f(y)-w_f(x)-(u_f(x)-u_f(y)))\,dydx.\end{aligned}$$

From here, using the numerical inequality

$$\frac{1}{2}|r|^{p-2}r(s-r)\leq\frac{1}{2p}\left(|s|^p-|r|^p\right),$$

we obtain that

$$G^J_{p,f}(w) - G^J_{p,f}(u) = \frac{1}{2p}\int_{\Omega_1}\int_{\Omega_1} J(x-y)|w_f(y)-w_f(x)|^p\,dy\,dx$$
$$-\frac{1}{2p}\int_{\Omega_1}\int_{\Omega_1} J(x-y)|u_f(y)-u_f(x)|^p\,dy\,dx \geq \int_\Omega v(x)(w(x)-u(x))\,dx,$$

from where it follows (3.8). The second part of the theorem is a consequence of (3.7).

Now, $\mathrm{B}^J_{p,f}(u_p)=0$ can be written as

$$\int_{\Omega_1} J(x-y)\varphi_p((u_p)_f(y)-u_p(x))dy = 0 \quad \forall x\in\Omega\setminus N,\ \mathrm{meas}(N)=0,$$

for $\varphi_p(r):=|r|^{p-2}r$. Then, the continuity of u_p in Ω follows by the above conclusion and the Implicit Function Theorem ([12]): since J is continuous and φ_p is continuous and increasing,

$$F(x,\alpha):=\int_{\Omega_1} J(x-y)\varphi_p((u_p)_f(y)-\alpha)dy$$

is continuous in $\Omega\times\mathbb{R}$ and for fixed $x\in\Omega$ it is increasing in α. Therefore, by [12, Theorem 1.1] $F(x,\alpha)=0$ has a unique solution $\alpha(x)$ continuous in Ω. In fact, this can be proved in a direct way as follows. Since $\lim_{\alpha\to-\infty}F(x,\alpha)=+\infty$, $\lim_{\alpha\to+\infty}F(x,\alpha)=-\infty$ and $F(x,\cdot)$ is continuous and increasing, there exists a unique $\alpha(x)$ such that $F(x,\alpha(x))=0$. Now $\alpha(x)$ is l.s.c. at $x_0\in\Omega$; indeed, take $\alpha<\alpha(x_0)$, therefore

$$\int_{\Omega_1} J(x-y)\varphi_p((u_p)_f(y)-\alpha)dy > 0.$$

Since J is continuous, there exists $r>0$ such that

$$\int_{\Omega_1} J(x-y)\varphi_p((u_p)_f(y)-\alpha)dy > 0 \qquad \forall x\in B_r(x_0).$$

Therefore

$$\alpha<\alpha(x) \qquad \forall x\in B_r(x_0).$$

Similarly $\alpha(x)$ is u.s.c. Finally, since $\alpha(x)=u_p(x)$ in $\Omega\setminus N$ we conclude that u_p has a continuous representative. □

From now on, we will suppose that minimizers u_p of $G^J_{p,f}$ are continuous and satisfy the Euler–Lagrange equation $\mathrm{B}^J_{p,f}(u_p)=0$ everywhere.

At this step we also rescale de kernel J in order to deal with d_ε instead of with d_1. So, let

$$J_\varepsilon(z)=\frac{1}{\varepsilon^N}J\left(\frac{z}{\varepsilon}\right).$$

We want to study the limit as $p\to\infty$ of the minimizers u_p^ε of $G^{J_\varepsilon}_{p,f}$. From now on, we assume that $f\in L^\infty(\Omega_\varepsilon\setminus\overline{\Omega})$.

In [1], we have proved that

$$\lim_{p\to+\infty} G^{J_\varepsilon}_{p,f} = G^\varepsilon_{\infty,f} \quad \text{in the sense of Mosco,} \tag{3.9}$$

where

$$G^{\varepsilon}_{\infty,f}(u)=\begin{cases}0 & \text{if } \ |u(x)-u(y)|\le\varepsilon, \ \text{for } x,y\in\Omega, \ |x-y|\le\varepsilon, \ \text{and} \\ & |f(y)-u(x)|\le\varepsilon, \ \text{for } x\in\Omega, \ y\in\Omega_\varepsilon\setminus\Omega, \ |x-y|\le\varepsilon, \\ +\infty & \text{in other case.}\end{cases}$$

Now, by Hölder's and Poincaré's inequality (3.2), we have $\|u^{\varepsilon}_p\|_2 \le C\|f\|_\infty$ for every $p\ge 2$. Therefore, we can assume that

$$u^{\varepsilon}_p \rightharpoonup v_\infty \quad \text{weakly in } L^2(\Omega) \text{ as } p\to+\infty. \tag{3.10}$$

Then, by (3.9), we have

$$G^{\varepsilon}_{\infty,f}(v_\infty)\le \liminf_{p\to\infty} G^{J_\varepsilon}_{p,f}(u^{\varepsilon}_p). \tag{3.11}$$

Given $v\in D(G^{\varepsilon}_{\infty,f})$, by the definition of Mosco convergence, there exists $v_p\in D(G^{J}_{p,f})$, such that $v_p\to v$ in $L^2(\Omega)$, and such that

$$G^{\varepsilon}_{\infty,f}(v)\ge \limsup_{p\to\infty} G^{J_\varepsilon}_{p,f}(v_p). \tag{3.12}$$

Now, by (3.6), $G^{J_\varepsilon}_{p,f}(u^{\varepsilon}_p)\le G^{J_\varepsilon}_{p,f}(v_p)$ for all $p\ge 2$, and therefore, by (3.11) and (3.12), we obtain that $G^{\varepsilon}_{\infty,f}(v_\infty)\le G^{\varepsilon}_{\infty,f}(v)$, and consequently

$$G^{\varepsilon}_{\infty,f}(v_\infty)=\min\{G^{\varepsilon}_{\infty,f}(u) \ : \ u\in D(G^{\varepsilon}_{\infty,f})\}.$$

Therefore,

$$0\in\partial G^{\varepsilon}_{\infty,f}(v_\infty). \tag{3.13}$$

If we define

$$K^{\varepsilon}_{\infty,f}:=\left\{u\in L^2(\Omega) \ : \ \begin{array}{l}|u(x)-u(y)|\le\varepsilon \text{ for } x,y\in\Omega, \ |x-y|\le\varepsilon, \ \text{and} \\ |f(y)-u(x)|\le\varepsilon \text{ for } x\in\Omega, \ y\in\Omega_\varepsilon\setminus\overline{\Omega}, \ |x-y|\le\varepsilon\end{array}\right\},$$

we have that the functional $G^{\varepsilon}_{\infty,f}$ is given by the indicator function of $K^{\varepsilon}_{\infty,f}$, that is, $G^{\varepsilon}_{\infty,f}=\mathbb{I}_{K^{\varepsilon}_{\infty,f}}$. Therefore, the Euler-Lagrange equation (3.13) can be written as

$$0\in\partial\mathbb{I}_{K^{\varepsilon}_{\infty,f}}(v_\infty). \tag{3.14}$$

Observe that $K^{\varepsilon}_{\infty,f}$ is not empty if we assume that $L_{d_\varepsilon}(f,\Omega_\varepsilon\setminus\Omega)\le 1$. In this case it is not difficult to see that

$$\Psi(f),\Lambda(f)\in\big\{u\in L^2(\Omega) \ : \ |u_f(x)-u_f(y)|\le d_\varepsilon(x,y), \ x,y\in\Omega_\varepsilon\big\}\subset K^{\varepsilon}_{\infty,f}, \tag{3.15}$$

being $\Psi(f)$ and $\Lambda(f)$ the McShane-Whitney extensions.

Now, to study the Lipschitz extension problem we can always assume that f satisfies

$$L_{d_\varepsilon}(f,\Omega_\varepsilon\setminus\Omega)=1. \tag{3.16}$$

In fact: given $f:\Omega_\varepsilon\setminus\Omega\to\mathbb{R}$ Lipschitz continuous respect to the distance d_ε, consider $\tilde f(x):=\frac{f(x)}{k}$, $k:=L_{d_\varepsilon}(f,\Omega_\varepsilon\setminus\Omega)$. Now, if $v:\Omega_\varepsilon\to\mathbb{R}$ is $\mathrm{AMLE}_\varepsilon(\tilde f,\Omega)$, then, $u(x):=kv(x)$ is $\mathrm{AMLE}_\varepsilon(f,\Omega)$.

Consequently, if f satisfies (3.16), on account of (3.14) and (3.15), $(v_\infty)_f\in K^{\varepsilon}_{\infty,f}$.

Remark that for Ω convex, if f satisfies (3.16) then $\mathrm{MLE}(f,\Omega_\varepsilon)=K^{\varepsilon}_{\infty,f}$, and we conclude, directly, that $(v_\infty)_f\in\mathrm{MLE}(f,\Omega_\varepsilon)$.

Our aim is to see that $(v_\infty)_f$ is $\mathrm{AMLE}_\varepsilon(f,\Omega)$. To this aim we need the following result.

Lemma 3.4. *Let $\delta>0$. There exists a unique solution $u_{\infty,\delta}$ of*

$$\begin{cases} -\Delta_\infty^\varepsilon u = \delta & \text{in } \Omega, \\ u = f+\delta & \text{in } \Omega_\varepsilon\setminus\Omega, \end{cases} \tag{3.17}$$

where

$$\Delta_\infty^\varepsilon u(x) := \sup_{y\in\overline{B}_\varepsilon(x)} u(y) + \inf_{y\in\overline{B}_\varepsilon(x)} u(y) - 2u(x), \tag{3.18}$$

and we have a bound of the form

$$u_{\infty,\delta}(x) - C(\Omega,\varepsilon)\delta \le u_\infty(x) \le u_{\infty,\delta}(x),$$

where u_∞ is the solution of Problem (2.3).

Analogously, there is a unique solution $u_{\infty,-\delta}$ of

$$\begin{cases} -\Delta_\infty^\varepsilon u = -\delta & \text{in } \Omega, \\ u = f-\delta & \text{in } \Omega_\varepsilon\setminus\Omega, \end{cases} \tag{3.19}$$

and we have a bound of the form

$$u_{\infty,-\delta}(x) + C(\Omega,\varepsilon)\delta \ge u_\infty(x) \ge u_{\infty,-\delta}(x).$$

Proof. We use probabilistic arguments. The existence and uniqueness of $u_{\infty,\delta}$ comes from the fact that it can be obtained as the value of the tug-of-war game with running payoff δ and final payoff $f(x)+\delta$, see [20], [16]. In fact, the equation verified by $u_{\infty,\delta}$ is just the dynamic programming principle that holds for the value function of this game, see [15].

Hence we are left with the proof of the bounds. The fact that $u_\infty(x)\le u_{\infty,\delta}(x)$ is almost immediate since both functions can be seen as values of the same tug-of-war game in which the running payoff and the final payoff for u_∞ are strictly below than those for $u_{\infty,\delta}$. See [16] for a detailed proof of a comparison principle. To see the other bound,

$$u_{\infty,\delta}(x) - C(\Omega,\varepsilon)\delta \le u_\infty(x),$$

we argue as follows. Fix $\eta>0$. Player II follows any strategy and Player I follows a strategy S_I^0 such that at $x_{k-1}\in\Omega$ he chooses to step to a point that almost maximizes $u_{\infty,\delta}$, that is, to a point $x_k\in\overline{B}_\varepsilon(x_{k-1})$ such that

$$u_{\infty,\delta}(x_k) \ge \sup_{\overline{B}_\varepsilon(x_{k-1})} u_{\infty,\delta} - \eta 2^{-k}.$$

We start from the point x_0. The following inequality for the expectation holds:

$$\begin{aligned} &\mathbb{E}^{x_0}_{S_I^0,S_{II}}[u_{\infty,\delta}(x_k)+k\delta-\eta2^{-k}\,|\,x_0,\dots,x_{k-1}] \\ &\quad\ge \frac12\left\{\inf_{\overline{B}_\varepsilon(x_{k-1})} u_{\infty,\delta}+k\delta-\eta2^{-k} + \sup_{\overline{B}_\varepsilon(x_{k-1})} u_{\infty,\delta}-\eta2^{-k}+k\delta-\eta2^{-k}\right\} \\ &\quad\ge u_{\infty,\delta}(x_{k-1})+(k-1)\delta-\eta2^{-(k-1)}, \end{aligned}$$

where we have estimated the strategy of Player II by inf and used the fact that $u_{\infty,\delta}$ verifies (3.17). Thus $M_k = u_{\infty,\delta}(x_k) + k\delta - \eta 2^{-k}$ is a submartingale and consequently, if τ is the stopping time of the game, and S_{II}^0 is a quasioptimal strategy for Player II, that is a strategy such that

$$\inf_{S_{II}} \mathbb{E}_{S_I^0,S_{II}}^{x_0}[f(x_\tau)] \geq \mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[f(x_\tau)] - \eta,$$

we deduce that

$$\begin{aligned}
u_\infty(x_0) &= \inf_{S_{II}} \sup_{S_I} \mathbb{E}_{S_I,S_{II}}^{x_0}[f(x_\tau)] \\
&\geq \mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[f(x_\tau)] - \eta \\
&\geq \mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[f(x_\tau) + \delta + \delta\tau - \eta 2^{-\tau} - \delta(\tau+1)] - \eta \\
&= \mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[M_\tau - \delta(\tau+1)] - \eta \\
&\geq \limsup_{k\to\infty} \mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[M_{\tau\wedge k}] - \mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[\delta(\tau+1)] - \eta \\
&\geq \mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[M_0] - \delta\mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[\tau+1] - \eta \\
&= u_{\infty,\delta}(x_0) - 2\eta - \delta\mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[\tau+1],
\end{aligned}$$

where we have used Fatou's Lemma and the Optional Stopping Theorem for the submartingale M_k. Now, we just observe that, under strategies S_I^0, S_{II}^0, the game finishes if, in some moment, Player II obtains $n = n(\Omega, \varepsilon)$ consecutive victories. Now, the expected number of tosses to get n consecutive victories of Player II is a finite number $N = N(n)$. Therefore

$$\mathbb{E}_{S_I^0,S_{II}^0}^{x_0}[\tau] \leq N = c(\Omega, \varepsilon).$$

Consequently, we have

$$u_\infty(x_0) \geq u_{\infty,\delta}(x_0) - 2\eta - \delta C(\Omega, \varepsilon),$$

and, since η was arbitrary, this implies the desired estimate. □

Remark 3.5. From [19, Proposition 7.1] and [14, Theorem 1.9] (see also [5]), the expected value for the stopping time for a standard ε-tug-of-war game, is $O(\varepsilon^{-2})$ (see also [16]). Since we are looking at this problem with a fixed $\epsilon > 0$ we don't need this more precise estimate.

Lemma 3.6. *Let $u \in L^\infty(\Omega)$. Then,*

$$\lim_{p\to+\infty} \left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} = L_\varepsilon(u_f, \Omega). \tag{3.20}$$

Proof. We have

$$\begin{aligned}
&\left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} \\
&\quad = \left(\frac{1}{2p}\int_\Omega\int_\Omega J_\varepsilon(x-y)|u(y)-u(x)|^p\,dy\,dx + \frac{1}{p}\int_\Omega\int_{\Omega_\varepsilon\setminus\Omega} J_\varepsilon(x-y)|u_f(y)-u(x)|^p\,dy\,dx\right)^{\frac{1}{p}} \\
&\quad \le \left((L_\varepsilon(u_f,\Omega))^p\,\frac{1}{2p}\int_\Omega\int_\Omega J_\varepsilon(x-y)\,dy\,dx + (L_\varepsilon(u_f,\Omega))^p\,\frac{1}{p}\int_\Omega\int_{\Omega_\varepsilon\setminus\Omega} J_\varepsilon(x-y)\,dy\,dx\right)^{\frac{1}{p}} \\
&\quad = L_\varepsilon(u_f,\Omega)\left(\frac{1}{2p}\int_\Omega\int_\Omega J_\varepsilon(x-y)\,dy\,dx + \frac{1}{p}\int_\Omega\int_{\Omega_\varepsilon\setminus\Omega} J(x-y)\,dy\,dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Hence,

$$\limsup_{p\to+\infty}\left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} \le L_\varepsilon(u_f,\Omega).$$

On the other hand, suppose that

$$\alpha := \liminf_{p\to+\infty}\left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} < L_\varepsilon(u_f,\Omega).$$

Let $\tilde\alpha$ be such that $\alpha < \tilde\alpha < L_\varepsilon(u_f,\Omega)$. Then, there exists a set $A \subset \{(x,y)\in\Omega\times\Omega_\varepsilon : |x-y|\le\varepsilon\}$, with positive measure, such that $|u_f(x)-u_f(y)| > \tilde\alpha$ if $(x,y)\in A$. Consequently,

$$\left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} \ge \left(\frac{1}{2p}\int_A J_\varepsilon(x-y)|u_f(y)-u_f(x)|^p\,dy\,dx\right)^{\frac{1}{p}} > \tilde\alpha\left(\frac{1}{2p}\int_A J_\varepsilon(x-y)\,dy\,dx\right)^{\frac{1}{p}},$$

from where it follows the contradiction

$$\alpha = \liminf_{p\to+\infty}\left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}} \ge \tilde\alpha > \alpha.$$

Therefore,

$$L_\varepsilon(u_f,\Omega) \le \liminf_{p\to+\infty}\left(G_{p,f}^{J_\varepsilon}(u)\right)^{\frac{1}{p}},$$

and we have concluded the proof. □

In the next result we denote $M_p^\varepsilon := G_{p,f}^{J_\varepsilon}(u_p^\varepsilon) = \min\{G_{p,f}^{J_\varepsilon}(u) \ : \ u\in L^p(\Omega)\}$.

Theorem 3.7. *Let $f\in L^\infty(\Omega_\varepsilon\setminus\overline{\Omega})$, $L_{d_\varepsilon}(f,\Omega_\varepsilon\setminus\Omega)=1$, and let u_p^ε a minimizer of $G_{p,f}^{J_\varepsilon}$. Then, there exists a sequence $p_i\to+\infty$, as $i\to+\infty$, such that*

$$u_{p_i}^\varepsilon \rightharpoonup v_\infty \in L^\infty(\Omega) \quad \text{in } L^q(\Omega) \quad \text{as } i\to+\infty, \tag{3.21}$$

$$(M_p^\varepsilon)^{1/p} \to \inf_{u\in L^\infty(\Omega)} L_\varepsilon(u_f,\Omega) \quad \text{as } p\to+\infty, \tag{3.22}$$

$$\inf_{u\in L^\infty(\Omega)} L_\varepsilon(u_f,\Omega) = L_\varepsilon((v_\infty)_f,\Omega), \tag{3.23}$$

and $(v_\infty)_f$ is $\mathrm{AMLE}_\varepsilon(f,\Omega)$. *Moreover, $u_p^\varepsilon \to v_\infty$ pointwise and hence strongly in any $L^q(\Omega)$.*

Proof. Set $M^\varepsilon_\infty := \inf_{u\in L^\infty(\Omega)} L_\varepsilon(u_f,\Omega)$. Let $v \in L^\infty(\Omega)$ such that $L_\varepsilon(v_f,\Omega) \le M^\varepsilon_\infty + \delta$. Then, for p large enough,

$$(M^\varepsilon_p)^{1/p} \le G^{J_\varepsilon}_{p,f}(v)^{1/p} \le L_\varepsilon(v_f,\Omega)\left(\frac{1}{2p}\int_\Omega\int_\Omega J_\varepsilon(x-y)\,dy\,dx + \frac{1}{p}\int_\Omega\int_{\Omega_\varepsilon\setminus\Omega} J_\varepsilon(x-y)\,dy\,dx\right)^{\frac{1}{p}}$$

$$\le (M^\varepsilon_\infty+\delta)\left(\frac{1}{p}\int_{\Omega_\varepsilon\times\Omega_\varepsilon} J(x-y)dxdy\right)^{1/p} \le M^\varepsilon_\infty + 2\delta,$$

and consequently

$$\limsup_p M_p^{1/p} \le M_\infty.$$

Fix now $q \ge 2$. For $p > q$, by Holder inequality,

$$q^{1/q} G^{J_\varepsilon}_{q,f}(u^\varepsilon_p)^{1/q} \le (2p)^{1/p}(M^\varepsilon_p)^{1/p}\left(\int_{\Omega_\varepsilon\times\Omega_\varepsilon} J_\varepsilon(x-y)dxdy\right)^{1/q-1/p}.$$

Therefore, by Poincare's inequality (3.2), there exist a subsequence p_i such that, for any $q \ge 2$,

$$u^\varepsilon_{p_i} \rightharpoonup v_\infty \quad \text{in } L^q(\Omega) \ \text{ as } i \to +\infty, \tag{3.24}$$

$v_\infty \in L^\infty(\Omega)$. Moreover, by the lower semicontinuity of $G^{J_\varepsilon}_{q,f}$,

$$G^{J_\varepsilon}_{q,f}(v_\infty)^{1/q} \le \limsup_p (M^\varepsilon_p)^{1/p}\left(\frac{1}{q}\int_{\Omega_\varepsilon\times\Omega_\varepsilon} J_\varepsilon(x-y)dxdy\right)^{1/q}.$$

Letting now q to $+\infty$, and having in mind Lemma 3.6, we get

$$L_\varepsilon((v_\infty)_f,\Omega) \le \limsup_p (M^\varepsilon_p)^{1/p} \le M^\varepsilon_\infty,$$

and we have proved (3.21), (3.22) and (3.23).

Let us prove now that $(v_\infty)_f$ is AMLE$_\varepsilon(f,\Omega)$. By Theorem 2.3 we need to prove that v_∞ coincide with the unique solution u_∞ of Problem (2.3). To this end we want to use comparison arguments. Take $u_{\infty,\delta}$ as in Lemma 3.4 and regularize it as follows:

$$u^\theta_{\infty,\delta}(x) = u_{\infty,\delta} * \rho_\theta(x),$$

where ρ_θ is a usual mollifier. Here the convolution is taken in the whole Ω_ε. As $u_{\infty,\delta}$ is a solution of (3.17), we get that $u^\theta_{\infty,\delta}$ is a continuous function that, for θ small, verifies pointwise

$$\begin{cases} -\Delta^\varepsilon_\infty u \ge \frac{\delta}{2} & \text{in } \overline{\Omega}, \\ u \ge f + \frac{\delta}{2} & \text{in } \Omega_\varepsilon\setminus\Omega. \end{cases} \tag{3.25}$$

Now, we claim that there exists $p_{\delta,\theta}$, $p_{\delta,\theta} \to +\infty$ as $\delta,\theta \to 0$, such that for every $p \ge p_{\delta,\theta}$ the following inequality holds:

$$\int_{\Omega_\varepsilon} J_\varepsilon(x-y)\varphi_p((u^\theta_{\infty,\delta})_f(y) - u^\theta_{\infty,\delta}(x))dy \le 0 \quad \forall x\in\Omega,$$

being $\varphi_p(r) := |r|^{p-2}r$. To see this fact we argue by contradiction. Assume that there exists $p_n \to \infty$ and $x_n \in \Omega$ such that

$$\int_{\Omega_\varepsilon} J_\varepsilon(x_n - y)\varphi_{p_n}((u^\theta_{\infty,\delta})_f(y) - u^\theta_{\infty,\delta}(x_n))dy > 0.$$

We rewrite this as

$$\int_{\Omega_\varepsilon \cap \{(u^\theta_{\infty,\delta})_f(y) > (u^\theta_{\infty,\delta})_f(x_n)\}} J_\varepsilon(x_n - y)\varphi_{p_n}((u^\theta_{\infty,\delta})_f(y) - u^\theta_{\infty,\delta}(x_n))dy$$
$$> \int_{\Omega_\varepsilon \cap \{(u^\theta_{\infty,\delta})_f(y) < (u^\theta_{\infty,\delta})_f(x_n)\}} J_\varepsilon(x_n - y)\varphi_{p_n}((u^\theta_{\infty,\delta})_f(x_n) - u^\theta_{\infty,\delta}(y))dy.$$

Thus,

$$\left(\int_{\Omega_\varepsilon \cap \{(u^\theta_{\infty,\delta})_f(y) > (u^\theta_{\infty,\delta})_f(x_n)\}} J_\varepsilon(x_n - y)\varphi_{p_n}((u^\theta_{\infty,\delta})_f(y) - (u^\theta_{\infty,\delta}(x_n))dy\right)^{\frac{1}{p_n-1}}$$
$$> \left(\int_{\Omega_\varepsilon \cap \{(u^\theta_{\infty,\delta})_f(y) < (u^\theta_{\infty,\delta})_f(x_n)\}} J_\varepsilon(x_n - y)\varphi_{p_n}(u^\theta_{\infty,\delta}(x_n) - (u^\theta_{\infty,\delta})_f(y))dy\right)^{\frac{1}{p_n-1}}.$$

Then, passing to the limit, using that $\overline{\Omega}$ is compact (hence we can assume that $x_n \to x_0$) and that $u^\theta_{\infty,\delta}$ is a uniformly continuous function that does not depend on n, we obtain

$$\sup_{y \in \overline{B}_\varepsilon(x_0)} u^\theta_{\infty,\delta}(y) + \inf_{y \in \overline{B}_\varepsilon(x_0)} u^\theta_{\infty,\delta}(y) - 2u^\theta_{\infty,\delta}(x_0) \geq 0,$$

a contradiction with the fact that $u^\theta_{\infty,\delta}$ verifies (3.25).

Therefore $u^\theta_{\infty,\delta}$ is a supersolution of the problem for every $p \geq p_{\delta,\theta}$ and, using the comparison principle given in Lemma 3.2, we have $u^\varepsilon_p \leq u^\theta_{\infty,\delta}$ for every $p \geq p_{\delta,\theta}$. Therefore, letting $p \to \infty$, we get $v_\infty \leq u^\theta_{\infty,\delta}$. Now, we let $\theta \to 0$ and use the bounds in Lemma 3.4 to obtain

$$v_\infty \leq u_\infty(x) + C\delta.$$

Finally, we take $\delta \to 0$ and conclude that

$$v_\infty \leq u_\infty.$$

A symmetric argument using a regularization of $u_{\infty,-\delta}$ as subsolution proves the reverse inequality. Hence we have that

$$v_\infty = u_\infty.$$

In addition, since

$$u^\theta_{\infty,-\delta} \leq u^\varepsilon_p \leq u^\theta_{\infty,\delta} \qquad \forall p \geq p_{\delta,\theta},$$

and $u^\theta_{\infty,-\delta}$, $u^\theta_{\infty,\delta} \to u_\infty$ pointwise as $\theta, \delta \to 0$, we have

$$u^\varepsilon_p \to v_\infty \quad \text{pointwise as } p \to +\infty.$$

□

4. Viscosity solutions

The solutions of problem (1.2) are usually understood in the viscosity sense, nevertheless, in Theorem 2.3, we have understood the solution of Problem (2.3) in the pointwise sense: $u \in L^\infty(\Omega_\varepsilon)$ is a solution of (2.3) if

$$\begin{cases} \sup_{y\in\overline{B}_\varepsilon(x)} u(y) + \inf_{y\in\overline{B}_\varepsilon(x)} u(y) - 2u(x) = 0 & \text{for all } \ x \in \Omega, \\ u(x) = f(x) & \text{for all } \ x \in \Omega_\varepsilon \setminus \Omega. \end{cases}$$

In this section we will see that this concept implies also the viscosity one.

Since the solutions of Problem (2.3) are discontinuous in general (see the Appendix), to work with viscosity solutions we need to use the generalized definition of discontinuous viscosity solutions. Let us consider the upper and lower semi-continuous envelopes of u in Ω_ε defined as

$$u^*(x) := \limsup_{y\in\Omega_\varepsilon, y\to x} u(y) \qquad \text{and} \qquad u_*(x) := \liminf_{y\in\Omega_\varepsilon, y\to x} u(y),$$

respectively. Then, we say that $u \in L^\infty(\Omega_\varepsilon)$ is a *viscosity subsolution* of problem (2.3) if $u(x) = f(x)$ for almost all $x \in \Omega_\varepsilon \setminus \Omega$ and $-\Delta_\infty^\varepsilon \phi(x_0) \le 0$ when $\phi \in C(\Omega_\varepsilon)$, $\phi(x_0) = u^*(x_0)$ and $u^* - \phi$ achieves a maximum at $x_0 \in \Omega$. Likewise, $u \in L^\infty(\Omega_\varepsilon)$ is a *viscosity supersolution* of problem (2.3) if $u(x) = f(x)$ for almost all $x \in \Omega_\varepsilon \setminus \Omega$ and $-\Delta_\infty^\varepsilon \phi(x_0) \ge 0$ when $\phi \in C(\Omega_\varepsilon)$, $\phi(x_0) = u^*(x_0)$ and $u_* - \phi$ achieves a minimum at $x_0 \in \Omega$. We say that u is a *viscosity solution* of problem (2.3) if u is both a viscosity subsolution and a viscosity supersolution.

Proposition 4.1. *Let $u \in L^\infty(\Omega_\varepsilon)$. We have*

(i) *If $-\Delta_\infty^\varepsilon u(x) \le 0$ for all $x \in \Omega$, then $-\Delta_\infty^\varepsilon u^*(x) \le 0$ for all $x \in \Omega$ and consequently u is a viscosity subsolution of Problem* (2.3).

(ii) *If $-\Delta_\infty^\varepsilon u(x) \ge 0$ for all $x \in \Omega$, then $-\Delta_\infty^\varepsilon u_*(x) \ge 0$ for all $x \in \Omega$ and consequently u is a viscosity supersolution of Problem* (2.3).

(iii) *If $-\Delta_\infty^\varepsilon u(x) = 0$ for all $x \in \Omega$, then u is a viscosity solution of Problem* (2.3).

Proof. We are going to prove (i), the proof of (ii) is similar, and (iii) is a consequence of (i) and (ii).

Fix $x_0 \in \Omega$, and let $x_k \in \Omega$ such that $x_k \to x_0$ and $u^*(x_0) = \lim_{k\to\infty} u(x_k)$. Fix $0 < \delta < \frac{\varepsilon}{2}$, and select for each $k \in \mathbb{N}$, point $y_k, z_k \in B_\varepsilon(x_k)$ such that

$$\sup_{y\in B_\varepsilon(x_k)} u(y) \le u(y_k) + \delta, \qquad \inf_{z\in B_\varepsilon(x_k)} u(z) \ge u(z_k) - \delta. \tag{4.1}$$

By taking subsequences, we may assume that $y_k \to y \in \overline{B}_\varepsilon(x_0)$ and $z_k \to z \in \overline{B}_\varepsilon(x_0)$. Then,

$$\begin{aligned} \sup_{x\in\overline{B}_\varepsilon(x_0)} u^*(x) - u^*(x_0) &\ge u^*(y) - u^*(x_0) \ge \limsup_{k\to+\infty} (u(y_k) - u(x_k)) \\ &\ge \limsup_{k\to+\infty} \left(\sup_{y\in\overline{B}_\varepsilon(x_k)} u(y) - \delta - u(x_k) \right). \end{aligned}$$

Sending $\delta \to 0^+$, we get

$$\sup_{x\in\overline{B}_\varepsilon(x_0)} u^*(x) - u^*(x_0) \ge \limsup_{k\to+\infty} \left(\sup_{y\in\overline{B}_\varepsilon(x_k)} u(y) - u(x_k) \right). \tag{4.2}$$

On the other hand,

$$\begin{aligned} u^*(x_0) - \inf_{x\in \overline{B}_\varepsilon(x_0)} u^*(x) &\le u^*(x_0) - u^*(z) \le \liminf_{k\to+\infty}(u(x_k)-u(z_k)) \\ &\le \liminf_{k\to+\infty}\left(u(x_k) - \inf_{z\in\overline{B}_\varepsilon(x_k)} u(z) + \delta \right). \end{aligned}$$

Sending $\delta \to 0^+$, we get

$$u^*(x_0) - \inf_{x\in \overline{B}_\varepsilon(x_0)} u^*(x) \le \liminf_{k\to+\infty}\left(u(x_k) - \inf_{z\in\overline{B}_\varepsilon(x_k)} u(z) \right). \tag{4.3}$$

From (4.2) and (4.3), and having in ind that by hypothesis we have $-\Delta_\infty^\varepsilon u \le 0$, we obtain that

$$\begin{aligned} -\Delta_\infty^\varepsilon u_*(x_0) &= -\left(\sup_{x\in\overline{B}_\varepsilon(x_0)} u^*(x) - u^*(x_0)\right) - \left(\inf_{x\in\overline{B}_\varepsilon(x_0)} u^*(x) - u^*(x_0)\right) \\ &\le \liminf_{k\to+\infty}\left(u(x_k) - \sup_{y\in\overline{B}_\varepsilon(x_k)} u(y) + u(x_k) - \inf_{z\in\overline{B}_\varepsilon(x_k)} u(z)\right) = \liminf_{k\to+\infty}\left(-\Delta_\infty^\varepsilon u(x_k)\right) \le 0. \end{aligned}$$

This ends the proof. □

Problem (2.3) has not continuous solutions even for continuous boundary data, however assuming the continuity of the data and the continuity of the solution at the boundary, adapting an argument due to Le Gruyer and J. C. Archer [13] (see also [4]), we obtain the following result.

Proposition 4.2. *Let $f:\Omega_\varepsilon\setminus\Omega\to\mathbb{R}$ be a continuous function. If $u:\Omega_\varepsilon\to\mathbb{R}$ is a solution of*

$$\begin{cases} -\Delta_\infty^\varepsilon u = 0 & \text{in } \ \Omega, \\ u = f & \text{on } \ \Omega_\varepsilon\setminus\Omega, \end{cases}$$

and we assume that $u^*(x) = u_*(x) = f(x)$ for all $x\in\Omega_\varepsilon\setminus\Omega$, then u is continuous in Ω_ε.

Proof. By Lemma 4.1, we have

$$-\Delta_\infty^\varepsilon u^*(x) \le 0 \le -\Delta_\infty^\varepsilon u_*(x) \quad \text{for all } x\in\Omega. \tag{4.4}$$

Set $\alpha := \sup\{u^*(x) - u_*(x) \ : \ x\in\overline{\Omega}\}$. To prove the result it is enough to show that $\alpha = 0$. Arguing by contradiction, we suppose $\alpha > 0$. By the upper semi-continuity of the function $u^* - u_*$ and having in mind that $u^*(x) = u_*(x) = f(x)$ for all $x\in\Omega_\varepsilon\setminus\Omega$, we have the set

$$A := \{x\in\Omega_\varepsilon \ : \ (u^*-u_*)(x) = \alpha\}$$

is nonempty, closed and contained in Ω. Define $B := \{x\in A \ : \ u^*(x) = \max_A u^*\}$. By the upper semi-continuity of the function u^*, B is nonempty. Then, take $x_0\in\partial B$. Since $x_0\in A$, we have

$$(u^*-u_*)(x_0) \ge \sup_{x\in\overline{B_\varepsilon(x_0)}} (u^*-u_*)(x) \ge \inf_{x\in\overline{B_\varepsilon(x_0)}} u^*(x) - \inf_{x\in\overline{B_\varepsilon(x_0)}} u_*(x). \tag{4.5}$$

First suppose that

$$u^*(x_0) = \sup_{x\in\overline{B_\varepsilon(x_0)}} u^*(x).$$

Then, since $-\Delta_\infty^\varepsilon u^*(x_0) \le 0$, we have

$$u^*(x_0) = \inf_{x \in \overline{B_\varepsilon(x_0)}} u^*(x),$$

and by (4.5) we deduce that

$$u_*(x_0) = \inf_{x \in \overline{B_\varepsilon(x_0)}} u_*(x).$$

Then, since $-\Delta_\infty^\varepsilon u_*(x_0) \ge 0$, we obtain that

$$u_*(x_0) = \sup_{x \in \overline{B_\varepsilon(x_0)}} u_*(x).$$

Therefore, u^* and u_* are constant in $\overline{B_\varepsilon(x_0)}$, contradicting our assumption that $x_0 \in \partial B$.

It remains to arrive to a contradiction in the case

$$u^*(x_0) < \sup_{x \in \overline{B_\varepsilon(x_0)}} u^*(x).$$

By the upper semi-continuity of the function u^* there is $y_0 \in \overline{B_\varepsilon(x_0)}$ such that

$$u^*(y_0) = \sup_{x \in \overline{B_\varepsilon(x_0)}} u^*(x).$$

Since $u^*(y_0) > u^*(x_0)$ and $x_0 \in B$, we see that $y_0 \notin A$. Then,

$$u^*(y_0) - u_*(y_0) < \alpha = u^*(x_0) - u_*(x_0).$$

Hence,

$$\sup_{x \in \overline{B_\varepsilon(x_0)}} u_*(x) - u_*(x_0) \ge u_*(y_0) - u_*(x_0) > u^*(y_0) - u^*(x_0) = \sup_{x \in \overline{B_\varepsilon(x_0)}} u^*(x) - u^*(x_0). \tag{4.6}$$

Combining (4.5) and (4.6), we obtain $-\Delta_\infty^\varepsilon u_*(x_0) < -\Delta_\infty^\varepsilon u^*(x_0)$, which contradicts (4.4), and the proposition follows. $\square$

5. Appendix: Examples

In this appendix we collect some concrete examples that are illustrative of the difficulties of the problem. In the first example we see that there exists f for which the AMLE_1 of f is not AMLE of f in the sense of Definition 1.1 (in fact, there is no AMLE of f in that sense).

Example 5.1. For $\varepsilon = 1$, $\Omega = (0, \frac{1}{2})$ and $f = 0\chi_{(-1,0]} + 1\chi_{[\frac{1}{2}, \frac{3}{2})}$, for any z defined in $(0, \frac{1}{2})$ such that $z(x) \in [0,1]$, $f + z\chi_{(0,\frac{1}{2})} \in \mathrm{MLE}(f, \Omega_1)$. Between all of them, $u = f + \frac{1}{2}\chi_{(0,\frac{1}{2})}$ is the unique $\mathrm{AMLE}_1(f, \Omega)$ (it is very easy to prove that it is solution of (2.3)). On the other hand, there is not AMLE of f in the sense of Definition 1.1. In fact, if u is AMLE of f, then if $B = (-\frac{1}{2}, \frac{1}{2})$, the function $g = 0\chi_{(-1,\frac{1}{2})} + 1\chi_{(\frac{1}{2},\frac{3}{2})} \in \mathrm{MLE}(f, \Omega_1)$ and $g = u$ in $\Omega_1 \setminus B$, therefore $L_{d_1}(u, B) \le L_{d_1}(g, B) = 0$, and, hence, u is constant in B, that is, $u = 0$ in $(0, \frac{1}{2})$. Similarly, we can prove that $u = 1$ in $(0, \frac{1}{2})$ by taking $B = (0, 1)$ and $g = 0\chi_{(-1,0)} + 1\chi_{(0,\frac{3}{2})}$, which gives a contradiction.

Example 5.2. For $\varepsilon = 1$, $\Omega = (0,2)$ and $f = x\chi_{(-1,0]} + 2\chi_{[2,3)}$, the unique solution u of (2.3) can be explicitly found as follows. First, we observe that u is increasing in x. Indeed, Since $L_{d_1}(f, \Omega_1 \setminus \Omega) = 1$, it is easy to see that the McShane-Whitney extensions are given in Ω by

$$\Psi(f)(x) = x, \qquad \Lambda(f)(x) = 0\chi_{(0,1)}(x) + 1\chi_{[1,2)}(x).$$

Then, if u is the solution of (2.3), since Ω is convex, by Theorem 2.3, $u \in \mathrm{MLE}(f, \Omega_1)$ and therefore

$$0\chi_{(0,1)}(x) + 1\chi_{[1,2)}(x) \le u(x) \le x \quad \forall x \in (0,2). \tag{5.1}$$

By (5.1), for any $x \in (0,1)$ we have

$$u(x) = \frac{1}{2}x + \frac{1}{2}\sup_{y \in [1,x+1]} u(y),$$

so it is nondecreasing in this interval. For any $x \in (1,2)$ we have

$$u(x) = \frac{1}{2}\inf_{y \in [x-1,1]} u(y) + 1,$$

so it also is nondecreasing in this interval. So, taking into account again (5.1), u is nondecreasing in all $\Omega = (0,2)$. Therefore, for any $x \in (0,1)$ we have

$$u(x+1) = 2u(x) + 1 - x$$

and for any $z \in (1,2)$ we have

$$2 = u(z+1) = 2u(z) - u(z-1)$$

but taking $z - 1 = x$ we get,

$$2 = 2u(x+1) - u(x) = 3u(x) + 2 - 2x$$

and we conclude that

$$u(x) = \frac{2}{3}x, \quad x \in (0,1).$$

This implies

$$u(x) = 1 + \frac{1}{3}(x-1), \quad x \in (1,2).$$

Finally, $u(1) = 1$.

Note that $u_*(2) = \frac{4}{3} < 2 = u^*(2) = f(2)$ and u is discontinuous at $x = 1$, therefore, the assumption $u^*(x) = u_*(x) = f(x)$ for all $x \in \Omega_\varepsilon \setminus \Omega$ in Proposition 4.2 is necessary for the continuity of u on Ω.

Example 5.3. For $\varepsilon = 3/2$, $\Omega = (0,2)$ and $f = x\chi_{(-\frac{3}{2},0]} + 2\chi_{[2,\frac{7}{2})}$, the unique solution u of (2.3) can also be explicitly found as follows. Since $L_{d_{\frac{3}{2}}}(f, \Omega_{\frac{3}{2}} \setminus \Omega) = 1$, it is easy to see that the McShane-Whitney extensions are given in Ω by

$$\Psi(f)(x) = x, \quad \Lambda(f)(x) = -1\chi_{(0,\frac{1}{2})}(x) + \frac{1}{2}\chi_{[\frac{1}{2},2)}(x).$$

Then, if u is the solution of (2.3), since Ω is convex, by Theorem 2.3, $u \in \mathrm{MLE}(f, \Omega_{\frac{3}{2}})$ and therefore

$$-1\chi_{(0,\frac{1}{2})}(x) + \frac{1}{2}\chi_{(\frac{1}{2},2)}(x) \le u(x) \le x \quad \forall x \in (0,2). \tag{5.2}$$

By (5.2), for any $x \in (0, 1/2)$ we have

$$u(x) = \frac{1}{2}\left(x - \frac{3}{2}\right) + \frac{1}{2} \sup_{y \in [\frac{1}{2}, x+\frac{3}{2}]} u(y),$$

so it is nondecreasing in this interval. For any $x \in (1/2, 2)$ we have

$$u(x) = \frac{1}{2} \inf_{y \in [x-\frac{3}{2}, \frac{1}{2}]} u(y) + 1,$$

so it also is nondecreasing in this interval. So, taking into account again (5.2), u is nondecreasing in all $\Omega = (0, 2)$. Consequently, we have

$$u(x) = \frac{1}{2}\left(x - \frac{3}{2}\right) + \frac{1}{2} u\left(x + \frac{3}{2}\right) \qquad \text{if } x \in (0, 1/2) \tag{5.3}$$

and

$$u(x) = \frac{1}{2} u\left(x - \frac{3}{2}\right) + 1 \qquad \text{if } x \in (1/2, 2). \tag{5.4}$$

Now, if $x \in (3/2, 2)$, since $x - \frac{3}{2} \in (0, 1/2)$, by (5.3) and (5.4), we have

$$u(x) = \frac{1}{2}\left(\frac{1}{2}(x-3) + \frac{1}{2}u(x)\right) + 1,$$

from where it follows that

$$u(x) = \frac{1}{3} + \frac{1}{3}x, \quad x \in (3/2, 2).$$

Similarly, if $x \in (0, 1/2)$, since $x + \frac{3}{2} \in (3/2, 2)$, by (5.3) and (5.4), we have

$$u(x) = \frac{1}{2}\left(x - \frac{3}{2}\right) + \frac{1}{2}\left(\frac{1}{2}u(x) + 1\right),$$

from where it follows that

$$u(x) = \frac{2}{3}x - \frac{1}{3}, \quad x \in (0, 1/2).$$

Let us see now how u is in $[\frac{1}{2}, \frac{3}{2})$. If $x \in [\frac{1}{2}, \frac{7}{6})$, since $\inf_{\overline{B}_{\frac{3}{2}}(x)} u(y)$ is taken in $x - \frac{3}{2}$, we have

$$u(x) = \frac{1}{2}\left(x - \frac{3}{2}\right) + \frac{1}{2}2 = \frac{1}{2}x + \frac{1}{4}.$$

And, if $x \in [\frac{7}{6}, \frac{3}{2})$ since $\inf_{\overline{B}_{\frac{3}{2}}(x)} u(y) = -\frac{1}{3}$, we have

$$u(x) = \frac{5}{6}, \quad x \in [7/6, 3/2).$$

Finally $u(1/2) = 1/2$. And we have arrived at

$$u(x) = \begin{cases} \frac{2}{3}x - \frac{1}{3}, & x \in (0, 1/2), \\ \frac{1}{2}x + \frac{1}{4}, & x \in [1/2, 7/6), \\ \frac{5}{6}, & x \in [7/6, 3/2), \\ \frac{1}{3} + \frac{1}{3}x, & x \in [3/2, 2). \end{cases}$$

Observe that $u(x) < 0$ for $0 < x < \frac{1}{2}$; u is increasing in Ω but not in the whole Ω_ε.

Acknowledgments: J.M.M and J.T. has been partially supported by the MICINN project MTM2008-03176. J.D.R. has been partially supported by MICINN project MTM2008-05824 and CONICET, Argentina.

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