Numerical solutions to minimal/maximal operators iterating obstacle problems[*][†]

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March 23, 2017

Abstract

Let $\Omega \subset \mathbb{R}^d$ be a convex polygonal ($d = 2$) or polyhedral ($d = 3$) domain, and let L_i, $i = 1, 2$, be two elliptic operators of the form

$$L_i u(x) := -\operatorname{div}(A_i(x) \nabla u(x)) + c_i(x) u(x) - f_i(x).$$

Motivated by the results in [2], we propose a numerical iterative method to compute the numerical solution of the minimal problem

$$\begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

The convergence of the method is proved, and numerical examples illustrating our results are included.

1 Introduction

Let $\Omega \subset \mathbb{R}^d$ be a convex polygonal ($d = 2$) or polyhedral ($d = 3$) domain, and let L_i, $i = 1, 2$, be two elliptic operators of the form

$$L_i u(x) := -\operatorname{div}\ (A_i(x) \nabla u(x)) + c_i(x) u(x) - f_i(x),$$

where $A_i := [a_{jk}]_{d\times d}$ with $a_{jk} \in C^1(\overline{\Omega})$, $0 \le c_i \in L^\infty(\Omega)$ and $f_i \in L^p(\Omega)$ with $p > d$. Assume also that there exists some $\Lambda > 0$ such that $\langle A_i(x)\xi, \xi\rangle \ge \Lambda |\xi|^2$ for all $\xi \in \mathbb{R}^d$.

Our interest here is to find a numerical solution for the problem

$$(P) := \begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

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[*]2010 *Mathematics Subject Clasification.* 65N30, 47F05, 35R35

[†]*Key words and phrases.* Maximal operators, numerical approximations, obstacle problems

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Analogous results can be obtained for

$$\begin{cases} \max\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

but we concentrate on (P).

Maximal and minimal operators appear naturally in the literature as prototypes of fully nonlinear second order PDEs. For example, when one considers the family of uniformly elliptic second order operators of the form $-tr(AD^2u)$ and looks for maximal operators, one finds the so-called Pucci maximal operators, $P^+_{\lambda,\Lambda}(D^2u) = \max_{A\in\mathcal{A}} -tr(AD^2u)$ and $P^-_{\lambda,\Lambda}(D^2u) = \min_{A\in\mathcal{A}} -tr(AD^2u)$, where $\mathcal{A}$ is the set of uniformly elliptic matrices with ellipticity constant between λ and Λ. This maximal operator plays a crucial role in the regularity theory for uniformly elliptic second order operators, see [7].

In [2], the authors show that one can obtain a solution to (P) by taking the limit of a sequence constructed iterating an obstacle problem alternating the involved operators L_1 and L_2 with the previous term in the sequence as obstacle. More precisely, let u_1 be the unique solution of

$$\begin{cases} L_1 u_1 = 0 & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

and let $u_2 := \mathcal{O}(L_2, u_1)$ be the unique solution of the obstacle problem with L_2 as operator and u_1 as obstacle, that is,

$$(P_{L_2,u_1}) := \begin{cases} u_2 \geq u_1 & \text{in } \Omega, \\ L_2 u_2 \geq 0 & \text{in } \Omega, \\ L_2 u_2 = 0 & \text{in } \{u_2 > u_1\}, \\ u_2 = 0 & \text{on } \partial\Omega; \end{cases}$$

or equivalently,

$$\begin{cases} \min\{L_2 u_2, u_2 - u_1\} = 0 & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega. \end{cases}$$

Inductively, let us define u_n, $n \geq 2$, to be the solution of the obstacle problem

$$u_n := \begin{cases} \mathcal{O}(L_1, u_{n-1}) & \text{if } n \text{ is odd}, \\ \mathcal{O}(L_2, u_{n-1}) & \text{if } n \text{ is even}. \end{cases}$$

It was proved in [2] that u_n is an increasing sequence that converges uniformly to a solution u of the problem associated to the minimal operator that appears in (P).

In this work, inspired by the ideas in [2], we propose a numerical iterative method to compute an approximated solution of (P). Furthermore, we prove that the numerical solution converges to the solution of (P). More precisely, given some partition $\mathcal{T}_h$ of Ω, let us denote by S^h the standard piecewise linear finite element space, and let $u_1^h \in S^h$ be the approximation of the exact solution u_1, that is,

$$\begin{cases} L_1 u_1^h = 0 & \text{in } \Omega, \\ u_1^h = 0 & \text{on } \partial\Omega, \end{cases}$$

where the solution is understood in a suitable weak sense (see Section 2.2 below). Analogously, we set

$$u_n^h := \begin{cases} \mathcal{O}^h\left(L_1, u_{n-1}^h\right) & \text{if } n \text{ is odd}, \\ \mathcal{O}^h\left(L_2, u_{n-1}^h\right) & \text{if } n \text{ is even}, \end{cases} \tag{1.2}$$

where by $\mathcal{O}^h\left(L,\phi^h\right)$ we denote the discretization of $\mathcal{O}\left(L,\phi\right)$. We remark that $u_n^h \in S^h$ and the condition $u_n^h \geq u_{n-1}^h$ is imposed only at the nodes of the triangulation. For the precise definitions and more details see Section 2.2.

We shall show that if u is the solution of problem (P) and u_n^h is given by (1.2), then there exists $h_n > 0$ with $h_n \to 0$ such that

$$\lim_{n\to\infty} \left\|u_n^{h_n} - u\right\|_{L^\infty} = 0.$$

To finish this introduction we remark that there is a large number of references dealing with numerical approximations of obstacle problems, we quote the recent papers [3, 11, 17, 20] and references therein. Observe that any numerical scheme that approximates solutions to obstacle problems (including finite elements) can be iterated to obtain a numerical method for (P). Therefore, the idea presented here is quite flexible. On the other hand, note that, in general, maximal or minimal operators are fully nonlinear ones (due to the presence of the max or min) and hence they are not in divergence form. This makes that classical second order finite element methods are not directly applicable to approximate (P) (instead one has to use finite differences to approximate this problem directly).

The rest of this article is organized as follows. In Section 2 we give the precise formulations for the discrete and continuous problems. In Section 3 we collect some necessary L^∞-error estimates, and we establish a key lemma concerning the stability of the discrete obstacle problem. In Section 4 we prove our main results, and in the last section we present two numerical examples illustrating the behavior of our iterative process.

2 Preliminaries

2.1 Weak formulation of the problems

Let $\mathcal{B}_i : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ and $F_i : H_0^1(\Omega) \to \mathbb{R}$, for $i = 1, 2$, be given by

$$\mathcal{B}_i(u,v) := \int_\Omega \langle A_i(x)\nabla u, \nabla v\rangle + c_i(x)\, uv \quad \text{and} \quad F_i(v) := \int_\Omega f_i(x)\, v.$$

As usual, a function $u \in H_0^1(\Omega)$ is called a weak solution of (1.1) if

$$\mathcal{B}_1(u,v) = F_1(v), \qquad \text{for every } v \in H_0^1(\Omega). \tag{2.1}$$

The assumptions on the coefficients of the matrix A_1 and on c_1 guarantee the continuity and coercivity of the bilinear form $\mathcal{B}_1$ in $H_0^1(\Omega)\times H_0^1(\Omega)$ and therefore this elliptic problem admits a unique weak solution u. Moreover, by standard regularity arguments, since the source $f \in L^p(\Omega)$, we have that $u \in W^{2,p}(\Omega)$.

On the other hand, given $\phi \in H_0^1(\Omega)$, we call a function $u := \mathcal{O}(L_i, \phi) \in K_\phi := \{w \in H_0^1(\Omega) : w \geq \phi\}$ a weak solution of the obstacle problem $(P_{L_i,\phi})$ if

$$\mathcal{B}_i(u, u-v) \leq F_i(u-v), \qquad \text{for every } v \in K_\phi. \tag{2.2}$$

It is well known that the obstacle problem admits a unique solution u, see e.g. [14, Chapter II]. Furthermore, if the source $f \in L^p(\Omega)$ and the obstacle $\phi \in W^{2,p}(\Omega)$, then $u \in W^{2,p}(\Omega)$ (see e.g. [14, 15, 6]).

2.2 Finite element discretization and formulation of the discrete problems

Let $\mathcal{T}_h$, $h > 0$, be a conforming triangulation of the domain $\Omega \subset \mathbb{R}^d$, that is, a partition of Ω into d-simplices T, such that if two elements intersect, they do so at a full edge/face of both elements. Also, assume that each triangulation $\mathcal{T}_h$ satisfies

$$\max\{\text{diam}(T) : T \in \mathcal{T}_h\} \leq h \, \text{diam}(\Omega),$$

where $\text{diam}(A)$ is the diameter of the set A; and that the family of triangulations $\{\mathcal{T}_h\}_{h>0}$ is shape regular, that is,

$$\sup_{h>0} \sup_{T \in \mathcal{T}_h} \frac{\text{diam}(T)}{\rho_T} < \infty,$$

where ρ_T is the radius of the largest ball contained in T.

The standard piecewise linear finite element space $S^h \subset H^1(\Omega)$ is defined by

$$S^h := \{v \in C(\overline{\Omega}) : v_{|T} \text{ is linear } \forall\, T \in \mathcal{T}_h\}.$$

For the discretization of the continuous problems we consider the space

$$S_0^h := \{v \in S^h : v = 0 \text{ on } \partial\Omega\}.$$

Observe that $S_0^h \subset H_0^1(\Omega)$.

The discrete counterpart of (2.1) reads:

$$\text{Find } u^h \in S_0^h \text{ such that } \quad \mathcal{B}_1\left(u^h, v^h\right) = F_1\left(v^h\right), \qquad \text{for every } v^h \in S_0^h. \tag{2.3}$$

Clearly, this discrete problem has a unique solution for each mesh; the system matrix is not affected by the right-hand side and is invertible because the assumptions on the coefficients guarantee the coercivity of the bilinear form $\mathcal{B}_1(\cdot,\cdot)$ in $S_0^h \times S_0^h$.

Now, let $\mathcal{I}_h : \mathcal{C}(\overline{\Omega}) \to S^h$ be the Lagrange interpolation operator. In the case of the obstacle problem $(P_{L_i,\phi})$ (i.e., (2.2)), the discrete formulation is the following:

$$\begin{aligned} &\text{Find } u^h \in K_\phi^h \text{ such that } \quad \mathcal{B}_i\left(u^h, u^h - v^h\right) \leq F_i\left(u^h - v^h\right), \\ &\qquad\qquad\qquad\qquad\qquad\qquad \text{for every } v^h \in K_\phi^h, \end{aligned} \tag{2.4}$$

where $K_\phi^h := \{w^h \in S_0^h : w^h \geq \phi^h\}$ and $\phi^h := \mathcal{I}_h \phi$. It is also well known that the problem (2.4) admits a unique solution u^h (see e.g. [1]), which we denote by $\mathcal{O}^h\left(L_i, \phi^h\right)$.

3 Stability and error analysis for the discrete problems

In this section we establish some pointwise a priori error estimates for both the elliptic and the obstacle problem, and, under an additional condition on $\mathcal{T}_h$, we prove a key stability result for the discrete obstacle problem with respect to the obstacle.

In the sequel, we shall denote by C (or C_i) positive constants which are independent of h (but which may depend on the data of the given problems).

3.1 L^∞-error estimates for the elliptic problem

We start with the following lemma concerning the elliptic problem. The proof can be found for instance in [12, Remark 3.25] or [18, Remark 6.2.3].

Lemma 3.1. *Let $u_1 \in W^{2,2}(\Omega)$ be the solution of* (2.1) *and $u_1^h \in \mathcal{S}_0^h$ be the solution of* (2.3)*. Then, there exists $C_1 > 0$ such that*

$$\left\| u_1 - u_1^h \right\|_{L^\infty(\Omega)} \le C_1 h^{2-d/2} \left\| u_1 \right\|_{W^{2,2}(\Omega)} . \tag{3.1}$$

3.2 Stability and L^∞-error estimates for the obstacle problem

The goal of this subsection is to prove a stability result and give an analogue pointwise a priori error estimates as the one given in (3.1) for the discretized obstacle problem. To obtain these results, we have to restrict our analysis to triangulations of a special kind.

Given a fixed triangulation $\mathcal{T}_h$ of the domain Ω, denote by $x_1, \ldots, x_{n+m}$ its vertices, where

$$x_l \in \partial\Omega \quad \Leftrightarrow \quad n+1 \le l \le n+m.$$

Let $\varphi_1, \ldots, \varphi_{n+m}$ be a nodal basis of the space S^h with the property that

$$\varphi_j(x_l) = \delta_{l,j}, \qquad 1 \le l, j \le n+m.$$

With respect to the nodal basis, a function $v^h \in S^h$ can be written as

$$v^h = \sum_{j=1}^{n+m} v_j \varphi_j, \quad \text{with } v_j = v^h(x_j) \qquad \text{for all } j \in \{1, \ldots, n+m\}.$$

Therefore, if v^h and w^h are functions in S^h,

$$\mathcal{B}_i\left(w^h, v^h\right) = \sum_{l=1}^{n+m} \sum_{j=1}^{n+m} w_l v_j \mathcal{B}_i\left(\varphi_l, \varphi_j\right).$$

Definition 3.2. A triangulation $\mathcal{T}_h$ of the domain Ω is said to satisfy the condition (M) if for all $j \ne l$ with $j, l \in \{1, \ldots, n\}$, it holds that

$$\mathcal{B}_i\left(\varphi_l, \varphi_j\right) = \int_\Omega \langle A_i(x) \nabla\varphi_l, \nabla\varphi_j\rangle + c_i(x) \varphi_l\varphi_j \le 0. \tag{3.2}$$

Remark 3.3. It is worth mentioning that condition (M) is strongly related to the discrete maximun principle. It is well know that this is a sufficient condition for the validity of the discrete maximum principle for a fully discrete linear simplicial finite element discretization of a reaction-diffusion problem, see [4, 19]. The validity of the condition (M) is connected with the dihedral angles of the used simplices and hence it translates into geometric issues. Let us be more precise. Suppose $A_i(x) = a_i(x) I$, where I denotes the identity matrix. For a given d-simplex T with facets F_i and F_j, denote their proper volumes by $|F_i|$, $|F_j|$, and $|T|$. The interior dihedral angle α_{ij} between F_i and F_j is defined as $\alpha_{ij} = \pi - \gamma_{ij}$, where $\gamma_{ij} \in [0, \pi]$ is the angle between outward normals η_i and η_j to F_i and F_j, respectively. To stress the dependence on the facets, we will write $cos(F_i, F_j)$ for $cos(\alpha_{ij})$. Finally, we write σ_j for the (positive) height of T above F_j , which satisfies $\sigma_j = \frac{d|T|}{|F_j|}$, relating the volume of T to that of its facets. With this notation, given a d-simplex $T \in \mathcal{T}_h$ and for $l, j \in \{1, \ldots, n\}$, $l \neq j$, we can express the key integrals as follows:

$$\int_T \varphi_l \varphi_j = \frac{|T|}{(d+1)(d+2)} \quad \text{and} \quad \int_T \langle \nabla\varphi_l, \nabla\varphi_j \rangle = \frac{-cos(F_l, F_j)}{\sigma_l \sigma_j} |T|.$$

Using the above notation and writing $a_i^T := \int_T a_i$, we have that a triangulation $\mathcal{T}_h$ satisfies condition (M) if for all $T \in \mathcal{T}_h$,

$$-a_i^T \frac{cos(F_l, F_j)}{\sigma_l \sigma_j} + \|c_i\|_{L^\infty(\Omega)} \frac{|T|}{(d+1)(d+2)} \leq 0. \tag{3.3}$$

In general, condition (3.3) is satisfied provided all dihedral angles are acute and the mesh is sufficiently fine. In the case of the Poisson problem or pure diffusion problem ($c \equiv 0$), the crucial condition (3.3) reduces to

$$cos(F_l, F_j) \geq 0. \tag{3.4}$$

This corresponds to the well-known requirement of nonobtuseness of all dihedral angles in the triangulation $\mathcal{T}_h$. In [19], a condition sharper than (3.3) is given in terms of the stiffness matrices.

In order to prove the stability of the discrete obstacle problem with respect to the obstacle, we need to introduce the concept of discrete supersolutions for problem (2.4). We note that the following definition extends the notion of supersolutions utilized in [14] to the discrete setting.

Definition 3.4. A function $g^h \in S^h$ is a discrete supersolution of problem (2.4) if it holds:

(i) $\mathcal{B}_i\left(g^h, v^h\right) \leq F_i\left(v^h\right)$, for every $v^h \in S^h$ with $v^h \leq 0$,

(ii) $g^h \geq \phi^h$ in Ω,

(iii) $g^h \geq 0$ on $\partial\Omega$.

The next two lemmas are adaptations of [9, Theorems 3.4 and 3.5], where similar results are proved in the case of the Laplacian operator. Let us point out that the continuous counterpart of Lemma 3.5 below can be found in [14, Theorem 6.4, Chapter II].

Lemma 3.5. *Assume that $\mathcal{T}_h$ satisfies the condition (M). Let u^h the solution of* (2.4) *with obstacle $\phi \in S_0^h$. Then, for every discrete supersolution g^h of* (2.4) *it holds that $u_h \leq g^h$ in Ω.*

Proof. Let $v_h \in S_0^h$ be defined by

$$v_h(x_l) := \min(u^h(x_l), g^h(x_l)), \qquad \text{for every } \ l \in \{1, \ldots, n+m\},$$

where $\{x_l\}$ denotes the set of all vertices of the triangulation $\mathcal{T}_h$. It is clear from the construction that $\phi^h \leq v^h \leq u^h$, and therefore $v^h \in K_{\phi^h}$.

Now, since u^h is solution of problem (2.4), it satisfies

$$\mathcal{B}_i\left(u^h, u^h - v^h\right) \leq F_i\left(u^h - v^h\right), \tag{3.5}$$

and on the other hand, from the first property in Definition 3.4 we have that

$$\mathcal{B}_i\left(g^h, u^h - v^h\right) \geq F_i\left(u^h - v^h\right). \tag{3.6}$$

Then, substracting (3.6) to (3.5) we obtain

$$\mathcal{B}_i\left(u^h - g^h, u^h - v^h\right) \leq 0.$$

Let $y_l := u^h(x_l) - g^h(x_l)$ for $l = 1, \ldots, n+m$. Then,

$$\begin{aligned}
0 &\geq \mathcal{B}_i\left(u^h - g^h, u^h - v^h\right) \\
&= \sum_{l=1}^{n+m} y_l \max(0, y_l) \mathcal{B}_i\left(\varphi_l, \varphi_l\right) + \sum_{l\neq j}^{n+m} y_l \max(0, y_j) \mathcal{B}_i\left(\varphi_l, \varphi_j\right) \\
&= \sum_{l=1}^{n} \max(0, y_l)^2 \mathcal{B}_i\left(\varphi_l, \varphi_l\right) + \sum_{l\neq j}^{n} y_l \max(0, y_j) \mathcal{B}_i\left(\varphi_l, \varphi_j\right).
\end{aligned} \tag{3.7}$$

Now, from the condition (M) we know that for all $l \neq j$ with $j, l \in \{1, \ldots, n\}$ it holds that

$$y_l \max(0, y_j) \mathcal{B}_i\left(\varphi_l, \varphi_j\right) \geq \max(0, y_l) \max(0, y_j) \mathcal{B}_i\left(\varphi_l, \varphi_j\right).$$

Thus, (3.7) implies

$$0 \geq \sum_{l=1}^{n} \sum_{j=1}^{n} \max(0, y_l) \max(0, y_j) \mathcal{B}_i\left(\varphi_l, \varphi_l\right) = \mathcal{B}_i\left(u^h - v^h, u^h - v^h\right) \geq 0$$

and consequently

$$u^h(x_l) - v^h(x_l) = \max(0, u^h(x_l) - g^h(x_l)) = 0 \qquad \forall\, l \in \{1, \ldots, n+m\}.$$

Using again the piecewise linearity of the involved functions, we deduce $u^h \leq g^h$ in Ω and this ends the proof. ■

Now, we prove a key stability result for the discrete obstacle problem with respect to the obstacle. This lemma will be useful in the next section.

Lemma 3.6. *Assume $\mathcal{T}_h$ is a triangulation satisfying condition (M). Let ψ, ϕ be two obstacles in S_0^h, and let $u_\psi^h := \mathcal{O}^h\left(L_i, \psi\right)$ and $u_\phi^h := \mathcal{O}^h\left(L_i, \phi\right)$. Then,*

$$\left\|u_\psi^h - u_\phi^h\right\|_{L^\infty(\Omega)} \leq \|\psi - \phi\|_{L^\infty(\Omega)}.$$

Proof. Let $g^h := u_\phi^h + \|\psi - \phi\|_{L^\infty(\Omega)}$. Then, it clearly holds that

$$g^h \in S^h, \quad g^h \geq 0 \text{ on } \partial\Omega, \quad \text{and} \quad g^h \geq u_\phi^h + \psi - \phi \geq \psi.$$

From the definition of the bilinear form $\mathcal{B}_i$ and the variational inequality (2.4), for all $v^h \in S_0^h$ with $v^h \leq 0$ in Ω, we have that

$$\mathcal{B}_i\left(g^h, v^h\right) \leq \mathcal{B}_i\left(u_\phi^h, v^h\right) = \mathcal{B}_i\left(u_\phi^h, u_\phi^h - \left(u_\phi^h - v^h\right)\right) \leq F_i\left(v^h\right). \tag{3.8}$$

Thus, g^h is a discrete supersolution for the discrete obstacle problem with obstacle ψ. Hence, by Lemma 3.5, we obtain $u^h_\psi \le g^h = u^h_\phi + \|\psi - \phi\|_{L^\infty(\Omega)}$ in Ω and therefore,

$$u^h_\psi - u^h_\phi \le \|\psi - \phi\|_{L^\infty(\Omega)} \quad \text{in } \Omega.$$

Since interchanging the roles of ψ and ϕ we may derive that

$$u^h_\phi - u^h_\psi \le \|\psi - \phi\|_{L^\infty(\Omega)} \quad \text{in } \Omega,$$

the lemma follows. ■

Let us observe that the estimate in the above lemma holds also in the continuous setting for similar obstacle problems, see [14, Theorem 8.5, Chapter 4].

We conclude this section with the following pointwise apriori error estimate for the obstacle problem, for a proof see e.g. [1, 16].

Lemma 3.7. *Let $\mathcal{T}_h$ be a triangulation satisfying condition (M) and an obstacle $\phi \in W^{2,p}(\Omega)$, $p \ge 2$. Let $u \in W^{2,p}(\Omega)$ be the solution of (2.2), and let $u^h \in S^h_0$ be the solution of (2.4). Then there exists a constant $C > 0$ such that*

$$\left\|u - u^h\right\|_{L^\infty(\Omega)} \le C h^{2-d/p} \left|\log h\right| \left(\|u\|_{W^{2,p}(\Omega)} + \|\phi\|_{W^{2,p}(\Omega)}\right).$$

4 Convergence of the discrete iteration

We are now in position to prove our main results. Recall that u_1 and u^h_1 are the solutions of (2.1) and (2.3) respectively, and that for $n \ge 2$,

$$\begin{aligned} u_n &:= \begin{cases} \mathcal{O}\left(L_1, u_{n-1}\right) & \text{if } n \text{ is odd,} \\ \mathcal{O}\left(L_2, u_{n-1}\right) & \text{if } n \text{ is even,} \end{cases} \\ u^h_n &:= \begin{cases} \mathcal{O}^h\left(L_1, u^h_{n-1}\right) & \text{if } n \text{ is odd,} \\ \mathcal{O}^h\left(L_2, u^h_{n-1}\right) & \text{if } n \text{ is even.} \end{cases} \end{aligned} \tag{4.1}$$

Theorem 4.1. *Let $\{\mathcal{T}_h\}$ be a family of triangulations satisfying condition (M). Let $u_n \in W^{2,p}(\Omega)$, $p \ge 2$, and $u^h_n \in S^h_0$ be as in (4.1). Then*

$$\lim_{h \to 0} \left\|u^h_n - u_n\right\|_{L^\infty(\Omega)} = 0.$$

Proof. For $n \ge 2$, let $\widetilde{u}^h_n \in S^h_0$ be defined by

$$\widetilde{u}^h_n := \begin{cases} \mathcal{O}^h\left(L_1, \mathcal{I}_h u_{n-1}\right) & \text{if } n \text{ is odd,} \\ \mathcal{O}^h\left(L_2, \mathcal{I}_h u_{n-1}\right) & \text{if } n \text{ is even.} \end{cases} \tag{4.2}$$

That is, $\widetilde{u}^h_n$ is the solution of the discrete obstacle problem with obstacle $\mathcal{I}_h u_{n-1}$. By Lemma 3.7 we have that

$$\left\|\widetilde{u}^h_n - u_n\right\|_{L^\infty(\Omega)} \le C_n h^{2-d/p} \left|\log h\right|, \tag{4.3}$$

where C_n may depend on $\|u_n\|_{W^{2,p}(\Omega)}$ and $\|u_{n-1}\|_{W^{2,p}(\Omega)}$.

Taking into account Lemma 3.6 and (4.3) we deduce that

$$\begin{aligned}\left\|u_n^h - u_n\right\|_{L^\infty(\Omega)} &\le \left\|u_n^h - \widetilde{u}_n^h\right\|_{L^\infty(\Omega)} + \left\|\widetilde{u}_n^h - u_n\right\|_{L^\infty(\Omega)} \\ &\le \left\|u_{n-1}^h - \mathcal{I}_h u_{n-1}\right\|_{L^\infty(\Omega)} + C_n h^{2-d/p} \left|\log h\right| \\ &\le \left\|u_{n-1}^h - u_{n-1}\right\|_{L^\infty(\Omega)} + \left\|u_{n-1} - \mathcal{I}_h u_{n-1}\right\|_{L^\infty(\Omega)} \\ &+ C_n h^{2-d/p} \left|\log h\right| .\end{aligned}$$

Now, since for any $v \in W^{2,p}(\Omega)$, $p \ge 1$, there exists a constant $C > 0$ such that the Lagrange intepolation satisfies the following estimates (see [5, Remark 4.4.27]):

$$\left\|v - \mathcal{I}_h v\right\|_{L^\infty(\Omega)} \le C h^{2-d/p} \left\|v\right\|_{W^{2,p}(\Omega)} ,$$

we obtain

$$\left\|u_n^h - u_n\right\|_{L^\infty} \le \left\|u_{n-1}^h - u_{n-1}\right\|_{L^\infty} + \widetilde{C}_{n-1} h^{2-d/p} + C_n h^{2-d/p} \left|\log h\right| .$$

Repeating this $n-1$ times and applying Lemma 3.1 we arrive to

$$\begin{aligned}\left\|u_n^h - u_n\right\|_{L^\infty(\Omega)} &\le \left\|u_1^h - u_1\right\|_{L^\infty(\Omega)} + 2(n-1)\overline{C}_n h^{2-d/p}(1 + \left|\log h\right|) \\ &\le h^2 \left(C_1 h^{-d/2} + 2(n-1)\overline{C}_n h^{-d/p}(1 + \left|\log h\right|)\right) ,\end{aligned} \tag{4.4}$$

where $\overline{C}_n := \max\{C_2, \ldots, C_n, \widetilde{C}_1, \ldots \widetilde{C}_{n-1}\}$. Finally, letting $h \to 0$ the theorem follows. ■

As a direct consequence of the above theorem and the convergence result in [2], we have the following corollary. Let us point out that in [2] the solutions are considered in the viscosity sense. However, since our weak solutions lie in $W^{2,p}(\Omega)$, $p \ge d$, an immediate application of the strong maximum principle for strong solutions (e.g. [13, Theorem 9.6]) shows that they are also viscosity solutions (for general theory of viscosity solutions we refer the reader to [8, 10]).

Corollary 4.2. *Let u be the solution of (P) and let u_n^h be as in* (4.1)*. Then, there exists $h_n > 0$ with $h_n \to 0$ such that*

$$\lim_{n\to\infty} \left\|u_n^{h_n} - u\right\|_{L^\infty(\Omega)} = 0.$$

Proof. We observe that

$$\left\|u_n^{h_n} - u\right\|_{L^\infty(\Omega)} \le \left\|u_n^h - u_n\right\|_{L^\infty(\Omega)} + \left\|u_n - u\right\|_{L^\infty(\Omega)} .$$

Let us define

$$\widehat{C}_n := \max\left\{C_1, 2(n-1)\overline{C}_n\right\}, \qquad \tau := 2 - d/2.$$

Taking h_n such that $\widehat{C}_n h_n^\tau \left(2 + \left|\log h_n\right|\right)) \le 1/2n$, using (4.4) and applying the convergence result Theorem 1.1 in [2], we see that

$$\left\|u_n^{h_n} - u\right\|_{L^\infty(\Omega)} \le \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n},$$

and the corollary is proved. ■

5 Numerical experiments

In this section we consider two different numerical examples that document the behavior of the iterative process. We point out that we shall consider two simple problems in which we know the exact solution of (P), in order to be able to compare such solution with the numerical approximation.

Example 1 Let $\Omega := [0,1] \times [0,1]$. We consider the following operators,

$$L_1 u := -\Delta u + f_1(x,y), \qquad L_2 u := -\Delta u + f_2(x,y),$$

where

$$\begin{aligned}
f_1(x,y) &:= \begin{cases} 13 & \text{if } x \in \left(0, \frac{7}{10}\right), \\ -\frac{512}{9}(h(x)g(y) + h(y)g(x)) & \text{if } x \in \left(\frac{7}{10}, 1\right), \end{cases} \\
f_2(x,y) &:= \begin{cases} -\frac{512}{9}(h(x)g(y) + h(y)g(x)) & \text{if } x \in \left(0, \frac{7}{10}\right), \\ 13 & \text{if } x \in \left(\frac{7}{10}, 1\right). \end{cases} \\
g(z) &:= -\frac{256}{9}(z - \frac{1}{4})^2(z - \frac{3}{4})^2 + 1, \\
h(z) &:= (z - \frac{1}{4})^2 + 4(z - \frac{1}{4})(z - \frac{3}{4}) + (z - \frac{3}{4})^2.
\end{aligned}$$

One can see that solving the problem

$$(P_1) := \begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

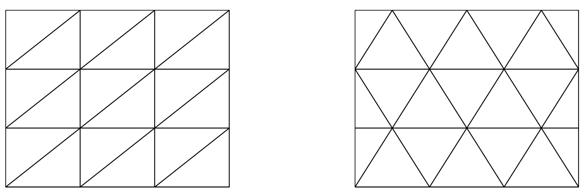
is equivalent to finding the unique solution of the problem

$$\begin{cases} \Delta u = -\frac{512}{9}(h(x)g(y) + h(y)g(x)) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, the solution is given by

$$u(x,y) = g(x)\,g(y).$$

For this first example, we consider a uniform fixed mesh with $N+1$ nodes at each boundary, dividing the unit square into N^2 subsquares and then each subsquare is divided into two triangles. Therefore, we have a triangulation $\mathcal{T}_h$ with $h = \frac{\sqrt{2}}{N}$. See Figure 1(a).



(a) (b)

Figure 1: **(a)** Triangulation $\mathcal{T}_h$ considered in Example 1. **(b)** Triangulation $\mathcal{T}_h$ considered in Example 2.

Let us observe that in this example, since L_1 and L_2 are Poisson problems, the triangulation $\mathcal{T}_h$ satisfies (3.4) and the condition (M) holds.

Next we examine the performance of the iterative process for different values of h and n.

In first place, we ran the algorithm in order to get the numerical solution u_n^h and we compared it with the known exact solution u. In Figure 2 we show $N = 40$ and $n = 50$: at the top left the exact solution u, at the right the approximate solution u_1^h, and at the bottom left u_{50}^h. One can observe that, in spite of starting with a poor initial data, the algorithm is able to give a good approximation of the exact solution. In fact, at the bottom right we plot the $\left\|u - u_n^h\right\|_{L^\infty}$ error versus the number of iterations, and we can see how this error decreases when n increases.

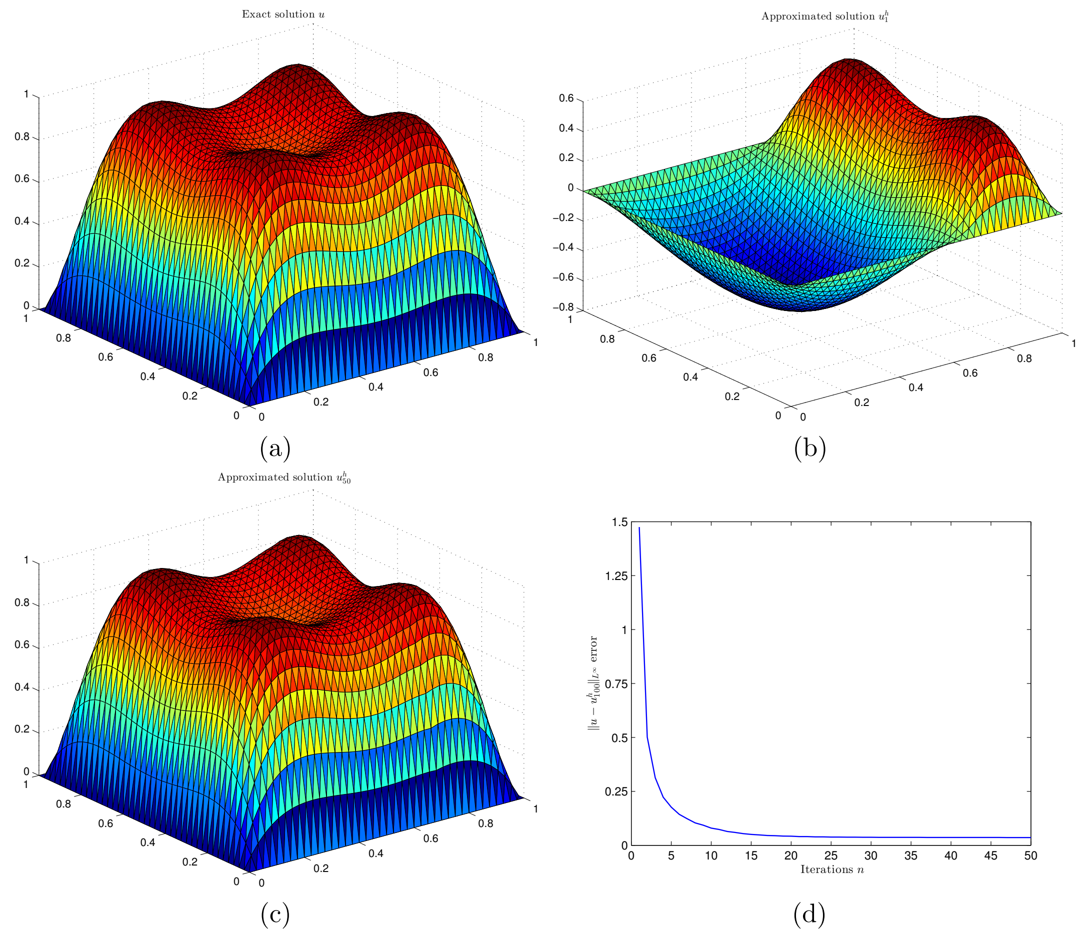


Figure 2: **(a)**–**(d)** Iterative process considering $h = \frac{\sqrt{2}}{40}$ fixed (41 nodes at each boundary side) and varying n from 1 to 50. **(a)** Exact solution $u(x, y) = g(x)g(y)$. **(b)** Approximated solution u_1^h. **(c)** Approximated solution u_{50}^h. In spite of starting with a poor initial data, the algorithm is able to give a good approximation of the exact solution. **(d)** Error $\left\|u - u_n^h\right\|_{L^\infty}$ versus the number of iterations n.

Finally, in Figure 3, we plot $\left\|u - u_{50}^h\right\|_{L^\infty}$ for several choices of h. One can also observe, as expected from the theoretical results, that this error gets smaller as h decreases.

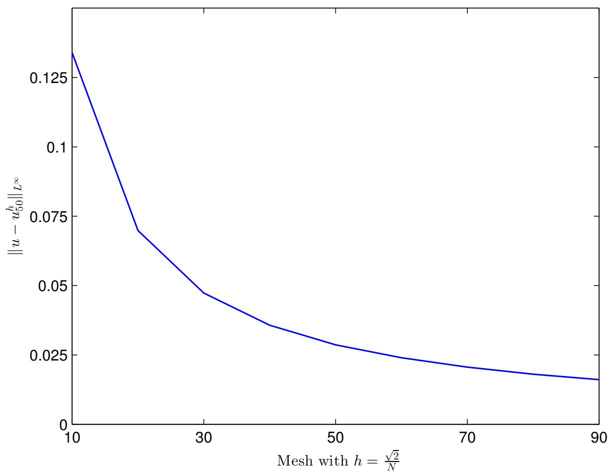


Figure 3: Plot of the errors $\left\|u - u_n^h\right\|_{L^\infty}$ corresponding to Example 1. We considered $n = 50$ fixed and varied the mesh diameter $h = \frac{\sqrt{2}}{N}$ from $N = 10$ to $N = 90$.

Example 2. Let $\Omega := [0,1] \times [0,1]$. We consider the following operators,

$$L_1 u := -\Delta u + c_1(x,y)\, u + f_1(x,y), \qquad L_2 u := -\Delta u + c_2(x,y)\, u + f_2(x,y),$$

where

$$f_1(x,y) := \begin{cases} -3\pi^2 \sin(\pi x)\sin(\pi y) & \text{if } x \in \left(0, \frac{3}{10}\right), \\ 0 & \text{if } x \in \left(\frac{3}{10}, 1\right), \end{cases}$$
$$f_2(x,y) := \begin{cases} 0 & \text{if } x \in \left(0, \frac{3}{10}\right), \\ -3\pi^2 \sin(\pi x)\sin(\pi y) & \text{if } x \in \left(\frac{3}{10}, 1\right), \end{cases}$$

and

$$c_1(x,y) := \begin{cases} \pi^2 & \text{if } x \in \left(0, \frac{3}{10}\right), \\ 2\pi^2(1-x) & \text{if } x \in \left(\frac{3}{10}, 1\right), \end{cases}$$
$$c_2(x,y) := \begin{cases} 2\pi^2 x & \text{if } x \in \left(0, \frac{3}{10}\right), \\ \pi^2 & \text{if } x \in \left(\frac{3}{10}, 1\right). \end{cases}$$

One can see that solving the problem

$$(P_2) := \begin{cases} \min\{L_1 u, L_2 u\} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

is equivalent to finding the unique solution of the problem

$$\begin{cases} \Delta u = 2\pi^2 \sin(\pi x)\sin(\pi y) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, the solution is given by

$$u(x,y) = \sin(\pi x)\sin(\pi y).$$

For this second example we built a triangulation in which for every T such that either $T \cap \partial\Omega = \emptyset$ or $T \cap \partial\Omega = x_j$, the interior angles are acute. The height and base of each T is $\frac{1}{N}$, and so $h = \sqrt{\frac{5}{4}}\frac{1}{N}$. See Figure 1(b).

We point out that a simple computation shows that when $h \leq \frac{1}{4}$, (3.3) holds and therefore the triangulation $\mathcal{T}_h$ satisfies the condition (M).

Here we examined the performance of the iterative process for different values of h and n, doing a similar analysis to the one made for Example 1. The results are shown in Figures 4 and 5.

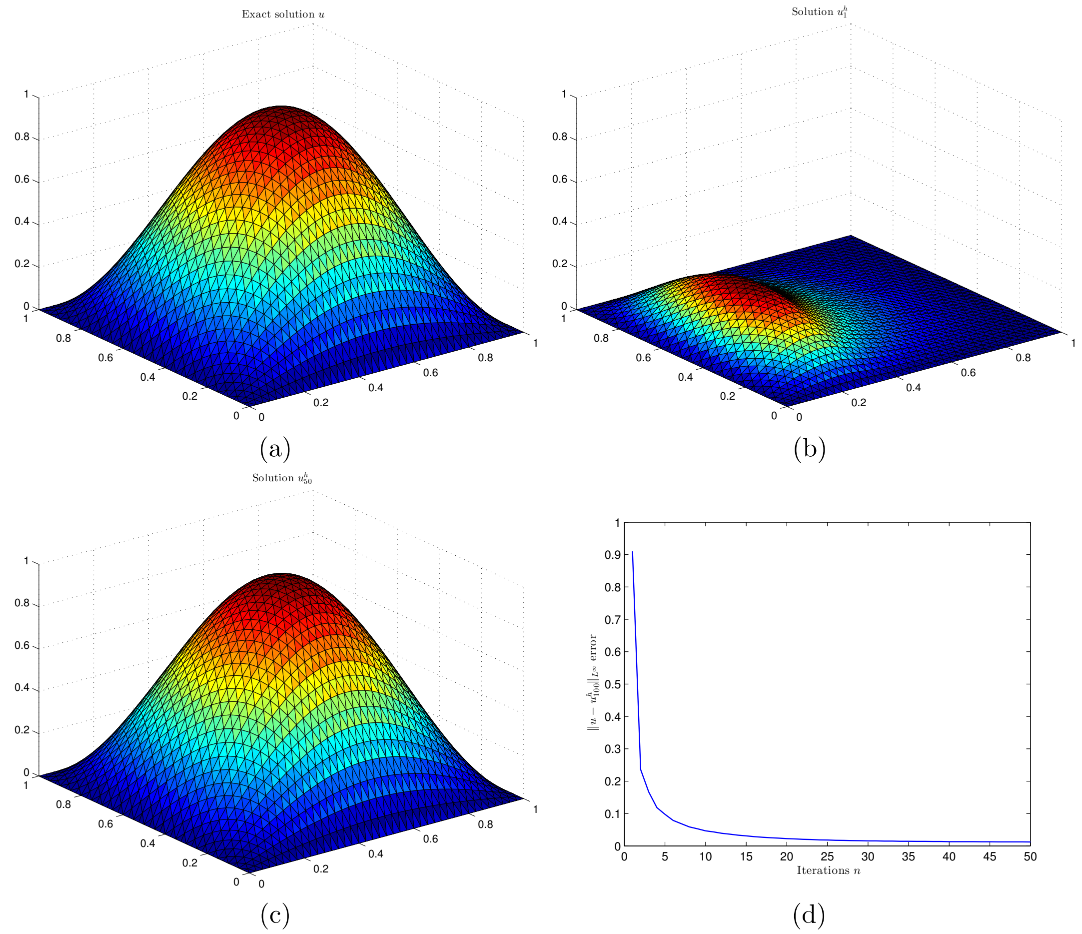


Figure 4: **(a)**–**(d)** Iterative process considering $h = \sqrt{\frac{5}{4}}\frac{1}{40}$ fixed and varying n from 1 to 50. **(a)** Exact solution $u(x, y) = sin(\pi x)sin(\pi y)$. **(b)** Approximated solution u_1^h. **(c)** Approximated solution u_{50}^h. In spite of starting with a poor initial data, the algorithm is able to give a good approximation of the exact solution. **(d)** Error $\left\|u - u_n^h\right\|_{L^\infty}$ versus the number of iterations n.

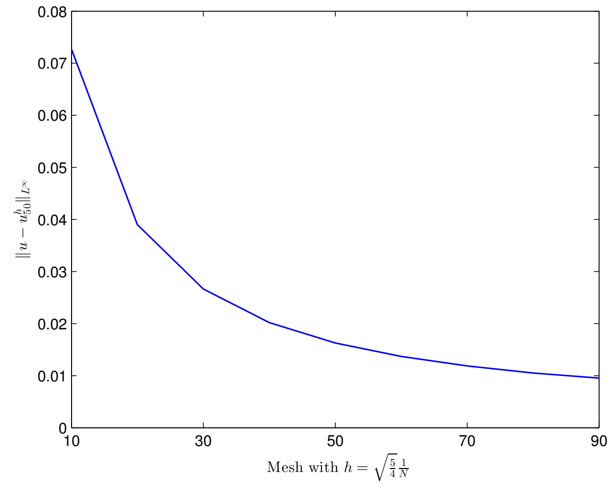


Figure 5: Plot of the errors $\left\|u - u^h_n\right\|_{L^\infty}$ corresponding to Example 2. We considered $n = 50$ fixed and varying the mesh diameter $h = \sqrt{\frac{5}{4}}\frac{1}{N}$ from $N = 10$ to $N = 90$.

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