AN OPTIMAL MATCHING PROBLEM WITH CONSTRAINTS

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Abstract. We deal with an optimal matching problem with constraints, that is, we want to transport two measures with the same total mass in $\mathbb{R}^N$ to a given place (the target set), where they will match and in which we have constraints on the amount of matter we can take to points in the target set. This transport has to be done optimally, minimizing the total transport cost, that in our case is given by the sum of the Euclidean distances that each measure is transported. Here we show that such a problem has a solution. First, we solve the problem using mass transport arguments and next we perform a method to approximate the solution of the problem taking limit as $p \to \infty$ in a p–Laplacian type variational problem.

In the particular case in which the target set is contained in a hypersurface, we deal with an optimal transport problem through a membrane, that is, we want to transport two measures which are located in different locations separated by a membrane (the hypersurface) which only let through a predetermined amount of matter.

1. Introduction

The classical optimal matching problem (see [7], [8]) consists in transporting two commodities (say nuts and screws, we assume that we have the same total number of nuts and screws) to a prescribed location, the target set (say factories where we ensemble the nuts and the screws) in such a way that they match there (each factory receive the same number of nuts and of screws) and the total cost of the operation is minimized.

Optimal matching problems for uniformly convex costs where analyzed in [4], [5], [7], [8] and have implications in economic theory (hedonic markets and equilibria), see [8], [9], [10], [11], [7] and references therein. We studied the case in which one considers the Euclidean distance as cost in [19], and in [20] we consider the case of a general Finsler distace as cost. For numerical approximations of this kind of problems we refer to [2].

Here we are interested in the more realistic case in which we have some constraints on the amount of matter we can transport to points in the target set. For example, suppose that the target set consist in some factories where we ensemble the nuts and the screws and the restriction represents the limit of production of each factory. Another example can be the optimal transport problem through a membrane in which case the restriction represents the permeability of the membrane.

Clearly, the optimal transport problem under consideration is related to the classical Monge-Kantorovich's mass transport problem. Using tools from this theory, we first prove the existence of a solution of the optimal transport problem through a membrane. We remark that, in fact, the existence of solution holds changing the Euclidean norm by any norm in $\mathbb{R}^N$. Next, as one of our main contributions in this paper, we perform a method to solve the problem by taking limit as $p \to \infty$ in a p–Laplacian type variational problem. In this approach we are lead to consider a system of PDE's of p–Laplacian type, nontrivially coupled through a measure supported inside the target set. Passing to the limit in this system (and also in the coupling measure) allows us to give more information about our original optimal mass transport problem. In particular, we can obtain the measure that describes the mass that goes to any point in the target set and the Kantorovich potentials for the involved transport. This procedure to solve mass transport problems (taking limit as $p \to \infty$ in a $p-$Laplacian type equation) was introduced by Evans and Gangbo in [14] and reveals to be quite fruitful, see [1], [16], [18], [19]. We have to remark that the limit as $p \to \infty$ in the problem requires some care since we must handle two variables nontrivially coupled and therefore the estimates for one component are related to the ones for the other. The analysis of this limit is interesting by its own.

1.1. **The optimal matching problem with constraints.** To write the optimal matching problem with constraints in mathematical terms, we fix two non-negative compactly supported functions $f^+,\ f^- \in L^\infty(\mathbb{R}^N)$, with supports $X_+,\ X_-$, respectively, satisfying the mass balance condition

$$M_0 := \int_{X_+} f^+ = \int_{X_-} f^- > 0.$$

We take a bounded $C^{1,1}$ domain $\Omega \subset \mathbb{R}^N$ such that it contains all the relevant sets, the supports of f_+ and f_-, and the target set Γ. We assume that

$$X_+ \cap\ X_- = \emptyset, \quad \big(X_+ \cup X_-\big) \cap \Gamma = \emptyset$$

and that Γ is compact.

The matching problem we are interested in is to send the measures $f^+\,dx$ and $f^-\,dx$ to the target set Γ is such a way that they match and the total cost of the transport operation (measured in terms of the distance that we have to transport $f^+\,dx$ and $f^-\,dx$ to the target set) is minimized. We refer to [18] for such kind of transport problems without constraints.

Ou aim here is to add a restriction on the amount of matter that can be send to a point in Γ. Let Θ be a nonnegative Radon measure in Ω with support included in Γ, representing the maximal amount of matter that we can transport to Γ punctually. That is, we assume that the transport is made in such a way that the amount of mass that any set $E \subset \Gamma$ receives does not exceeds $\int_E d\Theta$. Of course, we assume that

$$\int_\Gamma d\Theta > M_0,$$

since in the case $\int_\Gamma d\Theta < M_0$ the transport problem is impossible and when $\int_\Gamma d\Theta = M_0$, the problem trivializes since the only possibility is to send $f^+\,dx$ and $f^-\,dx$ to Θ.

Now we need to introduce some notations. Let $B \subset \mathbb{R}^N$. We denote by $\mathcal{M}(B)$ the set of all Radon measures on B and by $\mathcal{M}^+(B)$ the non-negative elements of $\mathcal{M}(B)$. Whenever T is a map from a measure space (X, μ) to an arbitrary space Y, we denote by $T\#\mu$ the pushforward measure of μ by T. Explicitly, $(T\#\mu)[K] = \mu[T^{-1}(B)]$. When we write $T\#f = g$, where f and g are nonnegative functions, this means that the measure having density f is pushed-forward to the measure having density g. Given $\mu, \nu \in \mathcal{M}^+(B)$ satisfying the mass balance condition

$$\mu(B) = \nu(B) \tag{1.1}$$

we denote by $\mathcal{A}_B(\mu, \nu)$ the set of transport maps pushing μ to ν, that is, the set of Borel maps $T : B \to B$ such that $T\#\mu = \nu$. In the case $\mu = f\mathcal{L}^N \llcorner B$ and $\nu = g\mathcal{L}^N \llcorner B$, we write $\mathcal{A}_B(f, g)$.

For Borel functions $T_\pm : \Omega \to \Omega$ such that $T_+\#f^+ = T_-\#f^-$, we consider the functional

$$\mathcal{F}(T_+, T_-) := \int_\Omega |x - T_+(x)| f^+(x) dx + \int_\Omega |y - T_-(y)| f^-(y) dy.$$

The optimal matching problem with constraints in which we are interestd can be stated as the minimization problem

$$\min_{(T_+, T_-) \in \mathcal{A}_{\Gamma,\Theta}(f^+, f^-)} \mathcal{F}(T_+, T_-), \tag{1.2}$$

where

$$\mathcal{A}_{\Gamma,\Theta}(f^+, f^-) := \Big\{ (T_+, T_-) : \, T_\pm : \Omega \to \Omega \text{ are Borel functions}, \\ 0 \le T_+\#f^+ = T_-\#f^- \le \Theta \Big\}.$$

By

$$0 \le T_+\#f^+ = T_-\#f^- \le \Theta$$

we mean that, for every Borel set $E \subset \Gamma$, we have

$$0 \le \int_{T_+^{-1}(E)} f^+(x)\,dx = \int_{T_-^{-1}(E)} f^-(y)\,dy \le \int_E d\Theta.$$

Oberve that $T_\pm(X_\pm) \subset \Gamma$.

If $(T_+^*, T_-^*) \in \mathcal{A}_{\Gamma,\Theta}(f^+, f^-)$ is a minimizer of the optimal problem (1.2), we shall call the measure $\mu^* := T_+^* \# f^+ = T_-^* \# f^-$ a Θ-*optimal matching measure*. Note that this measure μ^* encodes the amount of material that has to be transported to any subset of Γ and that the constraint reads as $0 \leq \mu^* \leq \Theta$.

Let us denote by

$$\mathcal{M}(\Gamma, \Theta, M_0) := \{\mu \in \mathcal{M}^+(\Omega) \ : \ \mu(\Omega) = M_0, \ 0 \leq \mu \leq \Theta\}$$

the set of all possible Θ-optimal matching measures. Observe that $\operatorname{supp}(\mu) \subset \Gamma$ since $0 \leq \mu \leq \Theta$ and Θ is supported in Γ.

Given $\mu \in \mathcal{M}(\Gamma, \Theta, M_0)$, we can consider the following minimization problem

$$\inf_{\mu \in \mathcal{M}(\Gamma,\Theta,M_0)} \ \inf_{(T_+,T_-) \in \mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+, T_-),$$

where

$$\mathcal{A}(f^+, f^-, \mu) := \{(T_+, T_-) \ : \ T_+ \in \mathcal{A}_\Omega(f^+, \mu), \, T_- \in \mathcal{A}_\Omega(f^-, \mu)\}.$$

We have that this is one possible way to rewrite our original problem, in fact,

$$\inf_{\mu \in \mathcal{M}(\Gamma,\Theta,M_0)} \ \inf_{(T_+,T_-) \in \mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+, T_-) = \inf_{(T_+,T_-) \in \mathcal{A}_{\Gamma,\Theta}(f^+,f^-)} \mathcal{F}(T_+, T_-). \tag{1.3}$$

Indeed, observe that given $(T_+, T_-) \in \mathcal{A}_{\Gamma,\Theta}(f^+, f^-)$, if we define

$$\mu(E) := \int_{T_+^{-1}(E)} f^+,$$

we have that $\mu \in \mathcal{M}(\Gamma, \Theta, M_0)$ and $(T_+, T_-) \in \mathcal{A}(f^+, f^-, \mu)$.

We will call

$$W_{f^\pm}^{\Gamma,\Theta} := \inf_{(T_+,T_-) \in \mathcal{A}_{\Gamma,\Theta}(f^+,f^-)} \mathcal{F}(T_+, T_-).$$

We provide two different proofs of the existence of minimizer to problem (1.2). The first one is more direct but does not provide a constructive way of getting the optimal matching measure on Γ, which is one of the relevant unknowns in this problem; consequently, the construction of optimal transport maps (that are proved to exist) remains a difficult task. The main tool in this first proof is the use of ingredients from the classical Monge-Kantorovich theory. The second proof is by approximation of the associated Kantorovich potentials by a system of $p-$Laplacian type problems when p goes to infinity. This approach provides an approximation of the potentials but also allows us to obtain the Θ-optimal matching measure in the limit.

Let us now introduce some notations, concepts and results from the Monge-Kantorovich Mass Transport Theory (see [1], [13], [22] and [23]) that will be used in the rest of the paper.

1.2. **Monge-Kantorovich's Mass Transport Theory.**

The Monge problem. *Given $\mu, \nu \in \mathcal{M}^+(B)$ satisfying the mass balance condition* (1.1)*. The Monge problem, associated with the measures μ and ν, is to find a map $T^* \in \mathcal{A}_B(\mu,\nu)$ which minimizes the cost functional*

$$\mathcal{F}_B(T) := \int_B |x - T(x)| \, d\mu(x)$$

in the set $\mathcal{A}_B(\mu,\nu)$. A map $T^* \in \mathcal{A}_B(\mu,\nu)$ satisfying $\mathcal{F}_B(T^*) = \min\{\mathcal{F}_B(T) \, : \, T \in \mathcal{A}_B(\mu,\nu)\}$, is called an optimal transport map of μ to ν.

In general, the Monge problem is ill-posed. To overcome the difficulties of the Monge problem, in 1942, L. V. Kantorovich in [17] proposed to study a relaxed version of the Monge problem and, what is more relevant here, introduced a dual variational principle. Let us define $\pi_t(x,y) := (1-t)x + ty$. Given a Radon measure γ in $B \times B$, its marginals are defined by $proj_x(\gamma) := \pi_0 \# \gamma$, $proj_y(\gamma) := \pi_1 \# \gamma$.

The Monge-Kantorovich problem. *Fix $\mu, \nu \in \mathcal{M}^+(B)$ satisfying the mass balance condition* (1.1)*. The Monge-Kantorovich problem is the minimization problem*

$$\int_{B\times B} |x-y| \, d\gamma^*(x,y) = \min \left\{ \int_{B\times B} |x-y| \, d\gamma(x,y) \, : \, \gamma \in \Pi_B(\mu,\nu) \right\},$$

where $\Pi_B(\mu,\nu) := \{$Radon measures γ in $B \times B : \pi_0\#\gamma = \mu, \pi_1\#\gamma = \nu\}$. The elements $\gamma \in \Pi_B(\mu,\nu)$ are called transport plans between μ and ν, and a minimizer γ^* an optimal transport plan. These minimizers always exist.

The Monge-Kantorovich problem has a dual formulation that can be stated in this case as follows (see for instance [22, Theorem 1.14]).

Kantorovich-Rubinstein Theorem. *Let $\mu, \nu \in \mathcal{M}^+(B)$ be two measures satisfying the mass balance condition* (1.1)*. Then,*

(1.4)

$$\min \left\{ \int_{B\times B} |x-y| \, d\gamma(x,y) \, : \, \gamma \in \Pi_B(\mu,\nu) \right\} = \sup \left\{ \int_\Omega u \, d(\mu - \nu) \, : \, u \in K_1(B) \right\},$$

where $K_1(B) := \{u : B \to \mathbb{R} \, : \, |u(x) - u(y)| \leq |x-y| \ \ \forall \, x,y \in B\}$ is the set of 1-Lipschitz functions in B.

The maximizers u^* of the right hand side of (1.4) are called *Kantorovich potentials*.

We can see the optimal problem (1.3) as a kind of Monge's problem (recall the results gathered in the previous section). The corresponding Monge-Kantorovich's problem is the following

$$\inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)}\ \inf_{(\gamma_+,\gamma_-)\in\Pi(f^+,f^-,\mu)}\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|d\gamma_+ + \int_{\overline{\Omega}\times\overline{\Omega}}|x-y|d\gamma_-,$$

where

$$\Pi(f^+,f^-,\mu) := \left\{(\gamma_+,\gamma_-)\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})^2 : \gamma_+\in\Pi_{\overline{\Omega}}(f^+,\mu), \gamma_-\in\Pi_{\overline{\Omega}}(f^-,\mu)\right\}.$$

For this problem, similarly to (1.3), we have that

$$\begin{aligned}\inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)}\ \inf_{(\gamma_+,\gamma_-)\in\Pi(f^+,f^-,\mu)}&\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|d\gamma_+ + \int_{\overline{\Omega}\times\overline{\Omega}}|x-y|d\gamma_-\\ &= \inf_{(\gamma_+,\gamma_-)\in\Pi_{\Gamma,\Theta}(f^+,f^-)}\int_{\overline{\Omega}\times\overline{\Omega}}|x-y|d\gamma_+ + \int_{\overline{\Omega}\times\overline{\Omega}}|x-y|d\gamma_-,\end{aligned}$$

where

$$\begin{aligned}\Pi_{\Gamma,\Theta}(f^+,f^-) = \Big\{(\gamma_+,\gamma_-)\in\mathcal{M}^+(\overline{\Omega}\times\overline{\Omega})^2 : \pi_0\#\gamma_\pm = f^\pm,\ \pi_1\#\gamma_+ = \pi_1\#\gamma_-,&\\ \operatorname{supp}(\pi_1\#\gamma_\pm)\subset\Gamma,\quad 0\le\pi_1\#\gamma_\pm\le\Theta\Big\}&.\end{aligned}$$

For two measures $\mu,\nu\in\mathcal{M}^+(B)$ satisfying the mass balance condition (1.1), the 1–Wasserstein distance (also called Kantorovich-Rubinstein distance) between μ and ν is defined as

$$W_1^B(\mu,\nu) := \inf\left\{\int_{B\times B}|x-y|\,d\gamma(x,y)\ :\ \gamma\in\Pi_B(\mu,\nu)\right\}.$$

In the case μ has no atom, by [1, Theorem 2.1], we have that

$$W_1^B(\mu,\nu) = \inf\left\{\int_B|x-T(x)|\,d\mu(x)\ :\ T\in\mathcal{A}_B(\mu,\nu)\right\}. \tag{1.5}$$

Remark 1.1. Observe that, by (1.3) and (1.5), we have:

$$W_{f^\pm}^{\Gamma,\Theta} = \inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)}[W_1^\Omega(f^+,\mu)+W_1^\Omega(f^-,\mu)].$$

Let us briefly summarize the contents of this paper: Section 2 is devoted to the first proof of Theorem 2.1 (using ideas from optimal mass transport theory); in Section 3 we study the limit as $p\to\infty$ in a $p-$Laplacian system and give a different proof of Theorem 2.1; in Section 4 we present, as an example where our existence result applies, an optimal transport problem with a permeable membrane.

2. The existence result

We have the following existence theorem.

Theorem 2.1. *The optimal problem* (1.2) *has a solution, that is, there exist Borel functions* $(T_+^*, T_-^*) \in \mathcal{A}_{\Gamma,\Theta}(f^+, f^-)$ *such that*

$$\mathcal{F}(T_+^*, T_-^*) = \inf_{(T_+,T_-)\in\mathcal{A}_{\Gamma,\Theta}(f^+,f^-)} \mathcal{F}(T_+, T_-).$$

Proof. We have

$$W_{f^\pm}^{\Gamma,\Theta} = \inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} [W_1^\Omega(f^+,\mu) + W_1^\Omega(f^-,\mu)].$$

Take a minimizing sequence $\mu_n \in \mathcal{M}(\Gamma,\Theta,M_0)$, then by the weak compactness of the convex set $\mathcal{M}(\Gamma,\Theta,M_0)$ there exists a subsequence, denoted equal, that converges weakly in the sense of measures to a measure $\mu_\infty \in \mathcal{M}(\Gamma,\Theta,M_0)$. Hence, by the weakly lower semi-continuity of the function $\nu \mapsto W_1^{\overline{\Omega}}(\mu,\nu)$, we have

$$W_1^{\overline{\Omega}}(f^+,\mu_\infty) + W_1^{\overline{\Omega}}(f^-,\mu_\infty) \le \lim_n \left(W_1^{\overline{\Omega}}(f^+,\mu_n) + W_1^{\overline{\Omega}}(f^-,\mu_n) \right) = W_{f^\pm}^{\Gamma,\Theta}.$$

Therefore,

$$W_1^{\overline{\Omega}}(f^+,\mu_\infty) + W_1^{\overline{\Omega}}(f^-,\mu_\infty) = W_{f^\pm}^{\Gamma,\Theta}.$$

Now, by [1, Theorem 6.2], which states the existence of an optimal transport map T_+^* transferring f^+ to μ_∞, and an optimal transport map T_-^* transferring f^- to μ_∞, we obtain that

$$\mathcal{F}(T_+^*, T_-^*) = W_{f^\pm}^{\Gamma,\Theta}.$$

This finishes the proof. □

Corollary 2.2. *There exists* $\mu \in \mathcal{M}(\Gamma,\Theta,M_0)$ *such that*

$$W_{f^\pm}^{\Gamma,\Theta} = W_1^{\overline{\Omega}}(f^+,\mu) + W_1^{\overline{\Omega}}(f^-,\mu).$$

Remark 2.3. Having in mind the results in [6], let us point out that Theorem 2.1 is also true in the case that we change in the cost function the Euclidean norm by any norm in $\mathbb{R}^N$.

3. The limit as $p \to \infty$ in a $p-$Laplacian system

In this section we show that we can follow the ideas of Evans-Gangbo ([14]) to get at the same time a Θ-optimal matching measure and Kantorovich potentials for the transports involved. Let us begin with the following result. We will use the following notation for shortness:

$$E_\Gamma = E \cap \Gamma;$$

and $\int_{E_\Gamma} g d\Theta$ even if not necessary since the support of Θ is included in Γ. We will also write, for example, $\{f \le g\}$ to denote $\{x \in \Omega : f(x) \le g(x)\}$.

Theorem 3.1.

$$\begin{aligned} W_{f^\pm}^{\Gamma,\Theta} &= \sup_{(v,w)\, \in\, K_1(\overline{\Omega}) \times K_1(\overline{\Omega})} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_{\{w-v<0\}_\Gamma} (w-v) d\Theta \right\} \\ &= \sup_{\substack{(v,w)\, \in\, K_1(\overline{\Omega}) \times K_1(\overline{\Omega}), \\ \int_{\{w-v<0\}_\Gamma} d\Theta \le M_0 \le \int_{\{w-v\le 0\}_\Gamma} d\Theta}} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_{\{w-v<0\}_\Gamma} (w-v) d\Theta \right\}. \end{aligned}$$

Proof. For a fixed $\mu \in \mathcal{M}(\Gamma, \Theta, M_0)$, it is well known (see for instance [1, 22]) that

$$\min_{T \in \mathcal{A}_{\overline{\Omega}}(f^+,\mu)} \int_{\overline{\Omega}} |x - T(x)| f^+(x) dx = \max_{u \in K_1(\overline{\Omega})} \int_{\overline{\Omega}} u(f^+ - \mu),$$

and

$$\min_{T \in \mathcal{A}_{\overline{\Omega}}(f^-,\mu)} \int_{\overline{\Omega}} |x - T(x)| f^-(x) dx = \max_{u \in K_1(\overline{\Omega})} \int_{\overline{\Omega}} u(f^- - \mu).$$

Therefore,

$$\begin{aligned} &\min_{(T_+,T_-) \in \mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+,T_-) \\ &\quad = \sup_{(v,w)\, \in\, K_1(\overline{\Omega}) \times K_1(\overline{\Omega})} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_\Gamma (w-v) \mu \right\} \\ &\quad \ge \sup_{(v,w)\, \in\, K_1(\overline{\Omega}) \times K_1(\overline{\Omega})} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_{\{w-v<0\}_\Gamma} (w-v) d\Theta \right\} \\ &\quad \ge \sup_{\substack{(v,w)\, \in\, K_1(\overline{\Omega}) \times K_1(\overline{\Omega}), \\ \int_{\{w-v<0\}_\Gamma} d\Theta \le M_0 \le \int_{\{w-v\le 0\}_\Gamma} d\Theta}} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_{\{w-v<0\}_\Gamma} (w-v) d\Theta \right\}. \end{aligned}$$

Consequently,

(3.1)

$$\begin{aligned} W_{f^\pm}^{\Gamma,\Theta} &= \inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \min_{(T_+,T_-)\in\mathcal{A}(f^+,f^-,\mu)} \mathcal{F}(T_+,T_-) \\ &\geq \sup_{(v,w)\,\in\, K_1(\overline{\Omega})\times K_1(\overline{\Omega})} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_{\{w-v<0\}_\Gamma} (w-v)d\Theta \right\} \\ &\geq \sup_{\substack{(v,w)\,\in\, K_1(\overline{\Omega})\times K_1(\overline{\Omega}),\\ \int_{\{w-v<0\}_\Gamma} d\Theta \leq M_0 \leq \int_{\{w-v\leq 0\}_\Gamma} d\Theta}} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_{\{w-v<0\}_\Gamma} (w-v)d\Theta \right\}. \end{aligned}$$

On the other hand, by Fan's Minimax Theorem ([15]),

$$\begin{aligned} W_{f^\pm}^{\Gamma,d\Theta} &= \inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \sup_{(v,w)\,\in\, K_1(\overline{\Omega})\times K_1(\overline{\Omega})} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_\Gamma (w-v)\mu \right\} \\ &= \sup_{(v,w)\,\in\, K_1(\overline{\Omega})\times K_1(\overline{\Omega})} \inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_\Gamma (w-v)\mu \right\}. \end{aligned}$$

Now, we observe that there exists a number s_0 such that

$$\int_{\{w-v<s_0\}_\Gamma} d\Theta \leq M_0 \leq \int_{\{w-v\leq s_0\}_\Gamma} d\Theta. \tag{3.2}$$

Indeed, consider

$$g(s) := \int_{\{w-v<s\}_\Gamma} d\Theta.$$

This function is non–decreasing, $\{s : g(s) < M_0\} \neq \emptyset$ and $\{s : g(s) > M_0\} \neq \emptyset$. Therefore, there exists s_0 such that

$$g(s) \leq M_0 \qquad \forall s < s_0$$

and

$$g(s) \geq M_0 \qquad \forall s > s_0.$$

Now, since

$$\{w-v<s\}_\Gamma = \bigcup_{n\in\mathbb{N}} \left\{ w-v < s - \frac{1}{n} \right\}_\Gamma,$$

we get

$$\int_{\{w-v<s_0\}_\Gamma} d\Theta = \lim_n g\left(s_0 - \frac{1}{n}\right) \leq M_0.$$

On the other hand, using that

$$\{w-v\leq s\}_\Gamma = \bigcap_{n\in\mathbb{N}} \left\{ w-v < s + \frac{1}{n} \right\}_\Gamma,$$

we obtain

$$M_0 \leq \lim_n g\left(s_0 + \frac{1}{n}\right) = \int_{\{w-v\leq s_0\}_\Gamma} d\Theta.$$

Therefore (3.2) holds.

Using (3.2) we have that

$$\begin{aligned}
&\inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \left\{\int_\Omega vf^+ - \int_\Omega wf^- + \int_\Gamma (w-v)\mu\right\} \\
&\quad = \inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \left\{\int_\Omega vf^+ - \int_\Omega (w-s_0)f^- + \int_\Gamma (w-s_0-v)\mu\right\} \\
&\quad \leq \sup_{\substack{(\tilde v,\tilde w)\in K_1(\overline{\Omega})\times K_1(\overline{\Omega}),\\ \int_{\{\tilde w-\tilde v<0\}_\Gamma} d\Theta \leq M_0 \leq \int_{\{\tilde w-\tilde v\leq 0\}_\Gamma} d\Theta}} \inf_{\mu\in\mathcal{M}(\Gamma,d\Theta,M_0)} \left\{\int_\Omega \tilde v f^+ - \int_\Omega \tilde w f^- + \int_\Gamma (\tilde w-\tilde v)\mu\right\}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
W_{f^\pm}^{\Gamma,\Theta} &= \sup_{\substack{(v,w)\in K_1(\overline{\Omega})\times K_1(\overline{\Omega}),\\ \int_{\{w-v<0\}_\Gamma} d\Theta \leq M_0 \leq \int_{\{w-v\leq 0\}_\Gamma} d\Theta}} \inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \left\{\int_\Omega vf^+ - \int_\Omega wf^- + \int_\Gamma (w-v)\mu\right\} \\
&= \sup_{\substack{(v,w)\in K_1(\overline{\Omega})\times K_1(\overline{\Omega}),\\ \int_{\{w-v<0\}_\Gamma} d\Theta \leq M_0 \leq \int_{\{w-v\leq 0\}_\Gamma} d\Theta}} \left\{\int_\Omega vf^+ - \int_\Omega wf^- + \inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \int_\Gamma (w-v)\mu\right\}.
\end{aligned}$$

For a fixed choice of v and w as in the above supremum,

$$\inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \int_\Gamma (w-v)\mu \leq \int_\Gamma (w-v)\mu_0$$

for $\mu_0 \in \mathcal{M}(\Gamma,\Theta,M_0)$ such that $\mu_0 = \Theta$ in $\{w-v<0\}_\Gamma$ and $\mu_0 = 0$ in $\{w-v>0\}_\Gamma$. Now, since

$$\int_\Gamma (w-v)\mu_0 = \int_{\{w-v<0\}_\Gamma} (w-v)d\Theta,$$

we have

$$\inf_{\mu\in\mathcal{M}(\Gamma,\Theta,M_0)} \int_\Gamma (w-v)\mu \leq \int_{\{w-v<0\}_\Gamma} (w-v)d\Theta.$$

Therefore,

$$W_{f^\pm}^{\Gamma,\Theta} \leq \sup_{\substack{(v,w)\in K_1(\overline{\Omega})\times K_1(\overline{\Omega}),\\ \int_{\{w-v<0\}_\Gamma} d\Theta \leq M_0 \leq \int_{\{w-v\leq 0\}_\Gamma} d\Theta}} \left\{\int_\Omega vf^+ - \int_\Omega wf^- + \int_{\{w-v<0\}_\Gamma} (w-v)d\Theta\right\}.$$

Joining this fact with (3.1) we get the conclusion. $\square$

By Theorem 3.1, in order to approximate our problem using the Evans-Gangbo method, we need to consider the functional

$$\Psi_p(v,w) := \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v f^+ + \int_\Omega w f^- + \int_\Gamma (w-v)^- d\Theta$$

and the variational problem associated with this functional:

$$\min_{(v,w)\in W^{1,p}(\Omega)\times\in W^{1,p}(\Omega)} \Psi_p(v,w). \tag{3.3}$$

To study the variational problem (3.3) we need the following inequality of Poincaré type.

Lemma 3.2. *Assume $N<p<\infty$. Then there exists a constant $C>0$ such that*

$$\|v\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)} \le C\left(\|\nabla v\|_{L^p(\Omega)} + \|\nabla w\|_{L^p(\Omega)} + \left|\int_\Omega v + \int_\Omega w\right|\right)$$

for every $v\in W^{1,p}(\Omega)$ and $w\in W^{1,p}(\Omega)$ such that $\{w-v=0\}\neq\emptyset$.

Proof. Suppose the result is not true. Then, there exists $v_n\in W^{1,p}(\Omega)$, $w_n\in W^{1,p}(\Omega)$ and $x_n\in\Omega$ such that $w_n(x_n)-v_n(x_n)=0$ and

$$\|v_n\|_{L^p(\Omega)} + \|w_n\|_{L^p(\Omega)} > n\left(\|\nabla v_n\|_{L^p(\Omega)} + \|\nabla w_n\|_{L^p(\Omega)} + \left|\int_\Omega v_n + \int_\Omega w_n\right|\right), \tag{3.4}$$

for every $n\in\mathbb{N}$. By homogeneity we can assume that

$$\|v_n\|_{L^p(\Omega)} + \|w_n\|_{L^p(\Omega)} = 1$$

for all $n\in\mathbb{N}$. Thus, by (3.4), we get

$$\left(\|\nabla v_n\|_{L^p(\Omega)} + \|\nabla w_n\|_{L^p(\Omega)} + \left|\int_\Omega v_n + \int_\Omega w_n\right|\right) \le \frac{1}{n} \quad \forall\, n\in\mathbb{N}. \tag{3.5}$$

Then, we have $\{v_n\}$ is a bounded sequence in $W^{1,p}(\Omega)$ and $\{w_n\}$ is a bounded sequence in $W^{1,p}(\Omega)$. Therefore, we can assume, after taking a subsequence, that

$$v_n \rightharpoonup w \quad \text{weakly in } \; W^{1,p}(\Omega)$$

and

$$w_n \rightharpoonup w \quad \text{weakly in } \; W^{1,p}(\Omega).$$

By Rellich-Kondrakov's Theorem, we can assume that (for a subsequence, denoted equal)

$$v_n \rightrightarrows v \quad \text{uniformly in } \overline{\Omega} \qquad \text{and} \qquad w_n \rightrightarrows w \quad \text{uniformly in } \overline{\Omega}.$$

We also get $x\in\overline{\Omega}$, a limit of a subsequence of x_n, such that

$$w(x) = v(x). \tag{3.6}$$

Now,

$$N(v,w) := \left| \int_\Omega v + \int_\Omega w \right|$$

is continuous in $L^p(\Omega) \times L^p(\Omega)$, hence

$$N(v,w) = \lim_{n\to\infty} N(v_n, w_n) = 0.$$

Moreover, by the weak convergence of $\nabla v_n \rightharpoonup \nabla v$ and $\nabla w_n \rightharpoonup \nabla w$ in L^p, by (3.5), we deduce that v and w are constant. Now, since (3.6) holds, we conclude that $v = w = c$. On the other hand, since $N(v,w) = 0$, we have

$$0 = \int_\Omega v + \int_\Omega w = c\left(|\Omega| + |\Omega|\right),$$

and then $0 = c = v = w$, but this contradicts the fact that that $\|v\|_{L^p(\Omega)} + \|w\|_{L^p(\Omega)} = 1$ must be also true. $\square$

Theorem 3.3. *Assume $p > N$. There exists a minimizer (v_p, w_p) of* (3.3).

Proof. To prove this result we apply the Direct Method of Calculus of Variations. Let $\tilde\Omega \subset \Omega$ be a compact and connected set such that $\Gamma \subset \tilde\Omega$. We first observe that

$$\inf_{(v,w)\in W^{1,p}(\Omega)\times\in W^{1,p}(\Omega)} \Psi_p(v,w) = \inf_{(v,w)\in\mathcal{B}_p} \Psi_p(v,w), \tag{3.7}$$

where

$$\mathcal{B}_p := \Big\{ (v,w) \in W^{1,p}(\Omega) \times W^{1,p}(\Omega) : \{w - v = 0\}_{\tilde\Omega} \neq \emptyset \Big\}.$$

Indeed, if $(v,w) \in W^{1,p}(\Omega)\times \in W^{1,p}(\Omega)$ does not satisfy $\{w - v = 0\}_{\tilde\Omega} \neq \emptyset$ then, since $\tilde\Omega$ is connected, we have $\{w - v > 0\}_{\tilde\Omega} = \tilde\Omega$ or $\{w - v < 0\}_{\tilde\Omega} = \tilde\Omega$. Now in the first case, since the functions v and w are continuous and Γ is compact, there is a constant $\alpha > 0$ such that $\{w - (v+\alpha) \le 0\}_\Gamma = \Gamma$. Then, since $\int_\Gamma (w-v)^- d\Theta = 0$ and $\int_\Gamma (w-(v+\alpha))^- d\Theta = 0$,

$$\Psi_p(v+\alpha, w) = \Psi_p(v,w) - \alpha M_0 < \Psi_p(v,w).$$

In the second case, there is a constan $\alpha > 0$ such that $\{w + \alpha - v \le 0\}_\Gamma = \Gamma$. Now,

$$\int_\Gamma (w+\alpha - v)^- d\Theta - \int_\Gamma (w-v)^- d\Theta = -\alpha \int_\Gamma d\Theta,$$

and then

$$\Psi_p(v, w+\alpha) = \Psi_p(v,w) + \alpha M_0 - \alpha \int_\Gamma d\Theta < \Psi_p(v,w).$$

Therefore (3.7) holds.

On the other hand, since

$$\Psi_p(v,w) = \Psi_p(v-c, w-c) \quad \text{for any constant } c,$$

by taking

$$c = \frac{1}{2|\Omega|}\left(\int_\Omega v + \int_\Omega w\right),$$

we can minimize $\Psi_p(v,w)$ between functions (v,w) with the constraint

$$\int_\Omega v + \int_\Omega w = 0.$$

Let (v_n, w_n) be a minimizing sequence in $\mathcal{B}_p$ with $\int_\Omega v_n + \int_\Omega w_n = 0$. Then

$$\lim_{n\to\infty} \Psi_p(v_n, w_n) = \inf_{(v,w)\in\mathcal{B}_p} \Psi_p(v,w),$$

and we have that there exists a constant $C_1 > 0$ such that $\Psi_p(v_n, w_n) \le C_1$. Then, by Lemma 3.2, we get that $\{v_n\}$ is bounded in $W^{1,p}(\Omega)$ and $\{w_n\}$ is bounded in $W^{1,p}(\Omega)$. Therefore, by Rellich-Kondrakov's Theorem, taking a subsequence if necessary, we have that

$$v_n \rightharpoonup v_p \quad \text{weakly in } W^{1,p}(\Omega) \quad \text{and} \quad w_n \rightharpoonup w_p \quad \text{weakly in } W^{1,p}(\Omega),$$
$$v_n \rightrightarrows v_p \quad \text{uniformly in } \overline{\Omega} \quad \text{and} \quad w_n \rightrightarrows w_p \quad \text{uniformly in } \overline{\Omega},$$

and

$$\{w_p - v_p = 0\}_{\tilde{\Omega}} \neq \emptyset.$$

Then, $(v_p, w_p) \in \mathcal{B}_p$ and

$$\Psi_p(v_p, w_p) \le \liminf_{n\to\infty} \Psi_p(v_n, w_n),$$

from where it follows that

$$\Psi_p(v_p, w_p) = \min_{(v,w)\in\mathcal{B}_p} \Psi_p(v,w).$$

□

Theorem 3.4. *Let (v_p, w_p) be minimizer functions of (3.3) with*

$$\int_\Omega v_p + \int_\Omega w_p = 0.$$

Then, up to a subsequence,

$$\lim_{p\to\infty} (v_p, w_p) = (v_\infty, w_\infty) \quad \textit{uniformly,}$$

where (v_∞, w_∞) is a solution of the variational problem

$$\sup_{(v,w)\,\in\, K_1(\Omega)\times K_1(\Omega)} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- - \int_\Gamma (w-v)^- d\Theta \right\}. \tag{3.8}$$

Proof. Let us take $(v_p, w_p) \in \mathcal{B}_p$ a minimizer of (3.3). For $(v, w) \in K_1(\Omega) \times K_1(\Omega)$, we have that

$$\begin{aligned}
& -\int_\Omega v_p f^+ + \int_\Omega w_p f^- + \int_\Gamma (w_p - v_p)^- d\Theta \\
& \quad \le \frac{1}{p}\int_\Omega |\nabla v_p|^p + \frac{1}{p}\int_\Omega |\nabla w_p|^p - \int_\Omega v_p f^+ + \int_\Omega w_p f^- + \int_\Gamma (w_p - v_p)^- d\Theta \\
& \quad \le \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v f^+ + \int_\Omega w f^- + \int_\Gamma (w - v)^- d\Theta \\
& \quad \le 2\frac{|\Omega|}{p} - \int_\Omega v f^+ + \int_\Omega w f^- + \int_\Gamma (w - v)^- d\Theta \,.
\end{aligned} \tag{3.9}$$

Let $x_p \in \Omega$ be such that $v_p(x_p) = w_p(x_p)$. Note that we can also assume that there exists $z_\infty \in \Omega$ such that $v_p(z_\infty) = 0$ for all $p > N$.

Let us see now that

$$\|v_p\|_{L^\infty(\Omega)} \le C_1 \|\nabla v_p\|_{L^p(\Omega)}, \tag{3.10}$$

and

$$\|w_p\|_{L^\infty(\Omega)} \le C_1 \left(\|\nabla w_p\|_{L^p(\Omega)} + \|\nabla v_p\|_{L^p(\Omega)} \right), \tag{3.11}$$

with C_1 not depending on p. Indeed, first note that since Ω is bounded, in Morrey's inequality (see, e.g., [12] or [3]) we can take the constant independent of $p > N$. Then, given $x \in \Omega$, we have

$$|v_p(x)| = |v_p(x) - v_p(z_\infty)| \le C_1 \|\nabla v_p\|_{L^p(\Omega)},$$

being C_i independent of p. On the other hand,

$$|w_p(x)| = |w_p(x) - w_p(x_p)| + |v_p(x_p)| \le C_1 \|\nabla w_p\|_{L^p(\Omega)} + |v_p(x_p)|.$$

From (3.9), using Hölder's inequality and having in mind (3.10) and (3.11), we get

$$\begin{aligned}
\frac{1}{p}\int_\Omega |\nabla v_p|^p + \frac{1}{p}\int_\Omega |\nabla w_p|^p &\le C_2(\|v_p\|_{L^p(\Omega)} + \|w_p\|_{L^p(\Omega)} + 1) \\
&\le C_3(\|\nabla v_p\|_{L^p(\Omega)} + \|\nabla w_p\|_{L^p(\Omega)} + 1),
\end{aligned}$$

with C_i independent of p. Hence,

$$\|\nabla v_p\|^{p-1}_{L^p(\Omega)},\ \|\nabla w_p\|^{p-1}_{L^p(\Omega)} \le p\, C_4 \qquad \forall p > N, \tag{3.12}$$

with C_4 independent of p.

Therefore, $\|v_p\|_{W^{1,p}(\Omega)}$ and $\|w_p\|_{W^{1,p}(\Omega)}$ are bounded uniformly in p, and, by Morrey's inequality (e.g. [3] or [12])

$$\begin{cases} |v_p(x) - v_p(y)| \le C_5|x-y|^{1-\frac{N}{p}}, \\ \\ |w_p(x) - w_p(y)| \le C_5|x-y|^{1-\frac{N}{p}}, \end{cases}$$

for some constant C_5 not depending on p. Then, by Arzela-Ascoli's compactness criterion we can extract a sequence $p_i \to \infty$ such that

$$v_{p_i} \rightrightarrows v_\infty \quad \text{uniformly in } \overline{\Omega},$$

$$w_{p_i} \rightrightarrows w_\infty \quad \text{uniformly in } \overline{\Omega},$$

and, so,

$$\{w_\infty - v_\infty = 0\}_\Gamma \neq \emptyset.$$

Moreover, by (3.12), we have

$$\|\nabla v_\infty\|_{L^\infty(\Omega)}, \, \|\nabla w_\infty\|_{L^\infty(\Omega)} \le 1.$$

Finally, passing to the limit in (3.9), we get

$$\begin{aligned} &\int_\Omega v_\infty f^+ - \int_\Omega w_\infty f^- - \int_\Gamma (w_\infty - v_\infty)^- d\Theta \\ &\qquad = \sup_{(v,w)\in\mathcal{A}} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- - \int_\Gamma (w-v)^- d\Theta \right\}. \end{aligned}$$

This ends the proof. □

Theorem 3.5. *Let (v_∞, w_∞) as in Theorem 3.4. If μ^* is a Θ-optimal matching measure for problem* (1.2)*, then:*

1. The measure μ^* satisifies that

$$supp(\mu^*) \subset \{w_\infty - v_\infty \le 0\}_\Gamma, \tag{3.13}$$

and

$$\mu^* \llcorner \{w_\infty - v_\infty < 0\}_\Gamma = \Theta \llcorner \{w_\infty - v_\infty < 0\}_\Gamma. \tag{3.14}$$

2. v_∞ is Kantorovich potential for the optimal mass transport problem of $f^+\mathcal{L}^N \llcorner \Omega$ to μ^*, and w_∞ is Kantorovich potential for the optimal mass transport problem of μ^* to $f^-\mathcal{L}^N \llcorner \Omega$.

3. The following relation holds:

$$\int_{\{w_\infty - v_\infty < 0\}_\Gamma} d\Theta \le M_0 \le \int_{\{w_\infty - v_\infty \le 0\}_\Gamma} d\Theta. \tag{3.15}$$

Proof. By Corollary 2.2, (3.8) and having in mind that $\mu^* \leq \Theta$, we obtain that

$$
\begin{aligned}
W^{\Gamma,\Theta}_{f^\pm} &= W^\Omega_1(f^+,\mu^*) + W^\Omega_1(f^-,\mu^*) \\
&= \sup_{(v,w)\,\in\, K_1(\Omega)\times K_1(\Omega)} \left\{ \int_\Omega v f^+ - \int_\Omega w f^- + \int_\Gamma (w-v)\mu^* \right\} \\
&\geq \int_\Omega v_\infty f^+ - \int_\Omega w_\infty f^- + \int_\Gamma (w_\infty - v_\infty)\mu^* \\
&\geq \int_\Omega v_\infty f^+ - \int_\Omega w_\infty f^- + \int_{\{w_\infty - v_\infty < 0\}_\Gamma} (w_\infty - v_\infty) d\Theta = W^{\Gamma,\Theta}_{f^\pm}.
\end{aligned}
\tag{3.16}
$$

Consequently, all the inequalities in (3.16) are equalities. From where it follows that (3.13) and (3.14) holds, and also that v_∞ is Kantorovich potential for the optimal mass transport problem of f^+ to μ^*, and w_∞ is Kantorovich potential for the optimal mass transport problem of μ^* to f^-. Finally, (3.15) is an easy consequence of *1.* □

Let us now see that, taking limits in the variational problem (3.3), we also obtain a Θ-optimal matching measure.

Theorem 3.6.

1. Let (v_p, w_p) be a minimizer of (3.3). Set $\mathcal{V}_p := |\nabla v_p|^{p-2}\nabla v_p$ and $\mathcal{W}_p := |\nabla w_p|^{p-2}\nabla w_p$. Define the distributions $\mathcal{V}^\eta_p$, $\mathcal{W}^\eta_p$ in $\mathbb{R}^N$ as

$$
\langle \mathcal{V}^\eta_p, \varphi\rangle := -\int_\Omega \mathcal{V}_p \cdot \nabla\varphi + \int_\Omega f^+\varphi \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N),
$$

$$
\langle \mathcal{W}^\eta_p, \varphi\rangle := \int_\Omega \mathcal{W}_p \cdot \nabla\varphi + \int_\Omega f^-\varphi \quad \forall \varphi \in \mathcal{D}(\mathbb{R}^N).
$$

Then, $\mathcal{V}^\eta_p = \mathcal{W}^\eta_p$; and this distribution is given by a positive Radon measure supported on $\{w_p - v_p \leq 0\}_\Gamma$.

2. There exist Radon measures $\mathcal{V}$, $\mathcal{W}$ in Ω and $\mathcal{X}$ in Γ, and a sequence $p_i \to +\infty$, such that

$$
\begin{aligned}
\mathcal{V}_{p_i} &\to \mathcal{V} \quad \text{weakly* in the sense of measures in } \Omega, \\
\mathcal{W}_{p_i} &\to \mathcal{W} \quad \text{weakly* in the sense of measures in } \Omega, \\
\mathcal{V}^\eta_{p_i} &\to \mathcal{X} \quad \text{weakly* in the sense of measures in } \Gamma.
\end{aligned}
$$

3. $\mathcal{X}$ is a Θ-optimal matching measure.

Proof. Let (v_p, w_p) be a minimizer of (3.3). Given $\varphi \in \mathcal{D}(\mathbb{R}^N)$, we have that the function

$$
I(t) := \Psi_p(v_p + t\varphi, w_p + t\varphi)
$$

has a minimum at $t = 0$. Thus, $I'(0) = 0$, from where it follows that

$$\int_\Omega |\nabla v_p|^{p-2}\nabla v_p \nabla\varphi + \int_\Omega |\nabla w_p|^{p-2}\nabla w_p \nabla\varphi = \int_\Omega f^+\varphi - \int_\Omega f^-\varphi.$$

Therefore, we have

$$\mathcal{V}_p^\eta = \mathcal{W}_p^\eta \quad \text{as distributions in } \mathbb{R}^N.$$

Given $\varphi \in \mathcal{D}(\mathbb{R}^N)$ such that $\operatorname{supp}(\varphi) \cap \{w_p - v_p \le 0\}_\Gamma = \emptyset$, we have

$$\Psi_p(v_p, w_p) \le \Psi_p(v_p + t\varphi, w_p).$$

Then

$$\begin{aligned} \int_\Omega f^+(v_p + t\varphi)\,dx - \int_\Omega f^+ v_p\,dx \le \frac{1}{p}\int_\Omega |\nabla v_p + t\nabla\varphi|^p\,dx - \frac{1}{p}\int_\Omega |\nabla v_p|^p\,dx \\ + \int_\Gamma \left((w_p - v_p - t\varphi)^- - (w_p - v_p)^-\right) d\Theta, \end{aligned} \tag{3.17}$$

Now, if we divide by $t > 0$ we get

$$\begin{aligned} \int_\Omega f^+\varphi dx \le \frac{1}{p}\int_\Omega \frac{|\nabla v_p + t\nabla\varphi|^p - |\nabla v_p|^p}{t}\,dx \\ + \frac{1}{t}\int_\Gamma \left((w_p - v_p - t\varphi)^- - (w_p - v_p)^-\right) d\Theta. \end{aligned} \tag{3.18}$$

Hence, since

$$\begin{aligned} \frac{1}{t}\int_\Gamma \left((w_p - v_p - t\varphi)^- - (w_p - v_p)^-\right) d\Theta &= \frac{1}{t}\int_{\{0 < w_p - v_p < t\varphi\}_\Gamma} (t\varphi - (w_p - v_p))\,d\Theta \\ &\le \int_{\{0 < w_p - v_p < t\varphi\}_\Gamma} \varphi d\Theta \ \to 0 \quad \text{ as } t \to 0, \end{aligned}$$

taking limits in (3.18) we get

$$\int_\Omega f^+\varphi\,dx \le \int_\Omega \mathcal{V}_p \cdot \nabla\varphi\,dx. \tag{3.19}$$

Now, if we divide by $t < 0$ in (3.17)

$$\begin{aligned} \int_\Omega f^+\varphi\,dx \ge \frac{1}{p}\int_\Omega \frac{|\nabla v_p + t\nabla\varphi|^p - |\nabla v_p|^p}{t}\,dx \\ + \frac{1}{t}\int_\Gamma \left((w_p - v_p - t\varphi)^- - (w_p - v_p)^-\right) d\Theta. \end{aligned} \tag{3.20}$$

Since

$$\frac{1}{t}\int_\Gamma \left((w_p - v_p - t\varphi)^- - (w_p - v_p)^-\right) d\Theta = \frac{1}{t}\int_{\{0<w_p-v_p<t\varphi\}_\Gamma} (t\varphi - (w_p - v_p))\, d\Theta$$
$$\geq \int_{\{0<w_p-v_p<t\varphi\}_\Gamma} \varphi\, d\Theta \ \geq 0,$$

taking limits in (3.20) we get

$$\int_\Omega f^+\varphi\, dx \geq \int_\Omega \mathcal{V}_p \cdot \nabla\varphi \ dx. \tag{3.21}$$

Therefore, putting together (3.19) and (3.21), we get

$$\langle \mathcal{V}_p^\eta, \varphi\rangle = 0,$$

which implies

$$\operatorname{supp}(\mathcal{V}_p^\eta) \subset \{w_p - v_p \leq 0\}_\Gamma. \tag{3.22}$$

Given $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $\varphi \geq 0$, and $t > 0$, we have

$$\Psi_p(v_p, w_p) \leq \Psi_p(v_p - t\varphi, w_p).$$

Now, we have that $(w_p - v_p + t\varphi)^- - (w_p - v_p)^- \leq 0$, then

$$0 \leq t\int_\Omega f^+\varphi\, dx + \frac{1}{p}\int_\Omega |\nabla v_p - t\nabla\varphi|^p\, dx - \frac{1}{p}\int_\Omega |\nabla v_p|^p\, dx.$$

Dividing by t and taking limit as $t \to 0$, we get

$$-\int_\Omega \mathcal{V}_p \cdot \nabla\varphi\, dx + \int_\Omega f^+\varphi\, dx \geq 0,$$

from where it follows that $\mathcal{V}_p^\eta$ is a positive Radon measure.

Now, we have that $(w_p - v_p + t\varphi)^- - (w_p - v_p)^- \leq 0$, then

$$\int_\Omega f^+\varphi\, dx - \frac{1}{p}\int_\Omega \frac{|\nabla v_p + t\nabla\varphi|^p - |\nabla v_p|^p}{t}\, dx$$
$$\leq \frac{1}{t}\int_\Gamma \left((w_p - v_p - t\varphi)^- - (w_p - v_p)^-\right) d\Theta$$
$$\leq \int_{\{w_p-v_p\leq 0\}_\Gamma} \varphi d\Theta + \int_{\{0<w_p-v_p<t\varphi\}_\Gamma} \varphi d\Theta\,,$$

for, as before, $\varphi \in \mathcal{D}(\mathbb{R}^N)$, $\varphi \geq 0$, and $t > 0$.

Taking limit as $t \to 0$, we get that

$$\mathcal{V}_p^\eta \leq \Theta \llcorner \{w_p - v_p \leq 0\}_\Gamma.$$

Hence,

$$0 \leq \mathcal{V}_p^\eta \leq \Theta \llcorner \{w_p - v_p \leq 0\}_\Gamma. \tag{3.23}$$

Observe that we have

$$\int_\Omega \mathcal{V}_p \cdot \nabla\varphi = \int_\Omega f^+\varphi - \int_\Gamma \varphi d\mathcal{V}_p^\eta \qquad \forall\, \varphi \in W^{1,p}(\Omega). \tag{3.24}$$

and

$$-\int_\Omega \mathcal{W}_p \cdot \nabla\varphi = \int_\Omega f^-\varphi - \int_\Gamma \varphi d\mathcal{W}_p^\eta \qquad \forall\, \varphi \in W^{1,p}(\Omega). \tag{3.25}$$

Taking $\varphi = \chi_\Omega$ in (3.24), we get

$$\int_\Gamma d\mathcal{V}_p^\eta = \int_\Gamma d\mathcal{W}_p^\eta = M_0. \tag{3.26}$$

If we take $\varphi = v_p$ in (3.24), we have

$$\int_\Omega |\nabla v_p|^p = \int_\Omega f^+ v_p - \int_\Gamma v_p d\mathcal{V}_p \cdot \eta \leq \|f^+\|_{L^{p'}(\Omega)} \|v_p\|_{L^p(\Omega)} + \|v_p\|_{L^\infty(\Omega)} M_0 \leq C \tag{3.27}$$

with C independent of $p > N$. As consequence of (3.27), we have $\{\mathcal{V}_p : p > N\}$ is bounded in $L^1(\Omega)$. Therefore, we can assume there exists a subsequence $p_i \to +\infty$ such that

$$\mathcal{V}_{p_i} \rightharpoonup \mathcal{V} \quad \text{weakly}^* \text{ as measures in } \Omega. \tag{3.28}$$

Similarly, we get that

$$\mathcal{W}_{p_i} \rightharpoonup \mathcal{W} \quad \text{weakly}^* \text{ as measures in } \Omega. \tag{3.29}$$

Moreover, by (3.26), we have that

$$\mathcal{V}_{p_i}^\eta \rightharpoonup \mathcal{X} \quad \text{weakly}^* \text{ as measures in } \Gamma. \tag{3.30}$$

Hence, by (3.26), we obtain

$$\int_\Gamma d\mathcal{X} = M_0.$$

For the above sequence $\{p_i\}$, Theorem 3.4 states

$$v_{p_i} \rightrightarrows v_\infty \quad \text{uniformly in } \overline{\Omega}, \quad \text{with } \|\nabla v_\infty\|_{L^\infty(\Omega)} \leq 1$$

and

$$w_{p_i} \rightrightarrows w_\infty \quad \text{uniformly in } \overline{\Omega}, \quad \text{with } \|\nabla w_\infty\|_{L^\infty(\Omega)} \leq 1.$$

Hence, by (3.22), we get

$$\operatorname{supp}(\mathcal{X}) \subset \{w_\infty - v_\infty \leq 0\}_\Gamma.$$

Moreover, by (3.23),

$$0 \le \mathcal{X} \le \Theta \llcorner \{w_\infty - v_\infty \le 0\}_\Gamma. \tag{3.31}$$

Since $|\xi|^p - |\eta|^p \le p|\xi|^{p-2}\xi \cdot (\xi - \eta)$ for any ξ, $\eta \in \mathbb{R}^N$, we have

$$\begin{aligned} \frac{1}{p}\int_\Omega |\nabla v_p|^p - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)v_p + \int_\Omega \mathcal{V}_p \cdot (\nabla\varphi - \nabla v_p) \\ - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)(\varphi - v_p) \le \frac{1}{p}\int_\Omega |\nabla\varphi|^p - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)\varphi \end{aligned} \tag{3.32}$$

for every $\varphi \in W^{1,p}(\Omega)$. Now, by (3.24), we have

$$\int_\Omega \mathcal{V}_p \cdot (\nabla\varphi - \nabla v_p) - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)(\varphi - v_p)) = 0,$$

and using this in (3.32) we arrive to

$$\frac{1}{p}\int_\Omega |\nabla v_p|^p - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)v_p \le \frac{1}{p}\int_\Omega |\nabla\varphi|^p - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)\varphi \quad \forall\, \varphi \in W^{1,p}(\Omega).$$

Therefore, for any $v \in W^{1,\infty}(\Omega)$ with $\|\nabla v\|_{L^\infty(\Omega)} \le 1$,

$$\begin{aligned} -\int_\Omega (f^+ - d\mathcal{V}_p^\eta)v_p \quad &\le \frac{1}{p}\int_\Omega |\nabla v_p|^p - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)v_p \\ &\le \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)v \quad \le \frac{1}{p}|\Omega| - \int_\Omega (f^+ - d\mathcal{V}_p^\eta)v. \end{aligned}$$

Taking $p = p_i$ and taking limit as $i \to \infty$ in the last inequality, we get

$$\int_\Omega (f^+ - d\mathcal{X})v \le \int_\Omega (f^+ - d\mathcal{X})v_\infty,$$

from where it follows that

$$\int_\Omega (f^+ - d\mathcal{X})v_\infty = \sup_{\substack{v \in W^{1,\infty}(\Omega) \\ |\nabla v|_\infty \le 1}} \int_\Omega v(f^+ - d\mathcal{X}).$$

Similarly we get

$$\int_\Omega (d\mathcal{X} - f^-)w_\infty = \sup_{\substack{w \in W^{1,\infty}(\Omega) \\ |\nabla w|_\infty \le 1}} \int_\Omega w(d\mathcal{X} - f^-).$$

Adding up this two last expressions we get

$$\int_\Omega f^+ v_\infty - \int_\Omega f^- w_\infty - \int_\Gamma (w_\infty - v_\infty)^- d\mathcal{X}$$

$$= \sup_{(v,w)\in(v,w)\in K_1(\Omega)\times K_1(\Omega)} \int_\Omega f^+ v - \int_\Omega f^- w - \int_\Gamma (w-v)^- d\mathcal{X}$$

$$\geq \sup_{(v,w)\in(v,w)\in K_1(\Omega)\times K_1(\Omega)} \int_\Omega f^+ v - \int_\Omega f^- w - \int_\Gamma (w-v)^- d\Theta$$

$$= \int_\Omega f^+ v_\infty - \int_\Omega f^- w_\infty - \int_\Gamma (w_\infty - v_\infty)^- d\Theta.$$

From where it follows, by (3.31), that

$$\mathcal{X} \llcorner \{w_\infty - v_\infty < 0\}_\Gamma = \Theta \llcorner \{w_\infty - v_\infty < 0\}_\Gamma$$

and that $\mathcal{X}$ is a Θ-optimal matching measure. □

Remark 3.7. As consequence of the proof of the above theorem we have that if (v_p, w_p) be a minimizer of (3.3), then $(v_p, w_p, \mathcal{V}_p^\eta)$ is a weak solution of the following problem:

$$\left\{ \begin{array}{ll} -\Delta_p v = f^+ - \Xi & \text{with homogeneous Neumann B.C.} \\ -\Delta_p w = \Xi - f^- & \text{with homogeneous Neumann B.C.} \\ \Xi \in \mathcal{M}^+(\mathbb{R}^N), \quad \Xi \leq \Theta \llcorner \{w - v \leq 0\}_\Gamma, & \Xi(\Gamma) = M_0. \end{array} \right. \tag{3.33}$$

Let us see now that if (v, w, Ξ) is a solution of (3.33) and satifies

$$\int (w-v)^- d\Xi = \int (w-v)^- d\Theta, \tag{3.34}$$

then

$$(v, w) \text{ is a minimizer of (3.3).}$$

In fact, working as in the proof of the previous theorem, for any $\tilde{v}, \tilde{w} \in W^{1,p}(\Omega)$, we have

$$\frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega v f^+ + \int_\Gamma v d\Xi \leq \frac{1}{p}\int_\Omega |\nabla \tilde{v}|^p - \int_\Omega \tilde{v} f^+ + \int_\Gamma \tilde{v} d\Xi$$

and

$$\frac{1}{p}\int_\Omega |\nabla w|^p + \int_\Omega w f^- - \int_\Gamma w d\Xi \leq \frac{1}{p}\int_\Omega |\nabla \tilde{v}|^p + \int_\Omega \tilde{w} f^- - \int_\Gamma \tilde{w} - d\Xi.$$

Adding these inequalities we get

$$\begin{aligned}\frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v f^+ + \int_\Omega w f^- - \int_\Gamma (w-v) d\Xi \\ \le \frac{1}{p}\int_\Omega |\nabla \tilde{v}|^p + \frac{1}{p}\int_\Omega |\nabla \tilde{w}|^p - \int_\Omega \tilde{v} f^+ + \int_\Omega \tilde{w} f^- - \int_\Gamma (\tilde{w}-\tilde{v}) d\Xi.\end{aligned}$$

Now, since $0 \le \Xi \le \Theta \llcorner \{w - v \le 0\}_\Gamma \le \Theta$ and $-r \le r^-$, from the last inequality, we obtain that

$$\begin{aligned}\frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v f^+ + \int_\Omega w f^- + \int_\Gamma (w-v)^- d\Xi \\ \le \frac{1}{p}\int_\Omega |\nabla \tilde{v}|^p + \frac{1}{p}\int_\Omega |\nabla \tilde{w}|^p - \int_\Omega \tilde{v} f^+ + \int_\Omega \tilde{w} f^- + \int_\Gamma (\tilde{w}-\tilde{v})^- d\Theta.\end{aligned}$$

Finally, having in mind (3.34),

$$\begin{aligned}\frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v f^+ + \int_\Omega w f^- + \int_\Gamma (w-v)^- d\Theta \\ \le \frac{1}{p}\int_\Omega |\nabla \tilde{v}|^p + \frac{1}{p}\int_\Omega |\nabla \tilde{w}|^p - \int_\Omega \tilde{v} f^+ + \int_\Omega \tilde{w} f^- + \int_\Gamma (\tilde{w}-\tilde{v})^- d\Theta,\end{aligned}$$

from where it follows that (v, w) is a minimizer of (3.3).

Remark 3.8. Given $\varphi \in C^1(\overline{\Omega})$ and $\psi \in C^1(\overline{\Omega})$ taking limits in (3.24) and (3.25) for $p = p_i$, on account of (3.28), (3.29) and (3.30), we get

$$\int_\Omega \nabla\varphi \; d\mathcal{V} = \int_\Omega f^+\varphi - \int_\Gamma \varphi d\mathcal{X},$$

and

$$\int_\Omega \nabla\psi \; d\mathcal{W} = -\int_\Omega f^-\psi + \int_\Gamma \psi d\mathcal{X}.$$

Therefore, we have

$$\begin{cases} -\mathrm{div}(\mathcal{V}) = f^+ - \mathcal{X} & \text{in } \Omega \\ \nabla\mathcal{V}\cdot\eta = 0 & \text{on } \partial\Omega \end{cases}$$

and

$$\begin{cases} -\mathrm{div}(\mathcal{W}) = -f^- + \mathcal{X} & \text{in } \Omega \\ \nabla \mathcal{W} \cdot \eta = 0 & \text{on } \partial\Omega \end{cases}$$

Observe also that we can also get

$$\lim_{i\to\infty} \int_\Omega \mathcal{V}_{p_i} \cdot \nabla v_\infty = \int_\Omega f^+ v_\infty - \int_\Gamma v_\infty d\mathcal{X},$$

$$\lim_{i\to\infty} \int_\Omega \mathcal{W}_{p_i} \cdot \nabla w_\infty = -\int_\Omega f^- w_\infty + \int_\Gamma w_\infty d\mathcal{X}.$$

That is, formally,

$$\int_\Omega \nabla v_\infty \, d\mathcal{V} = \int_\Omega f^+ v_\infty - \int_\Gamma v_\infty d\mathcal{X},$$

and

$$\int_\Omega \nabla w_\infty \, d\mathcal{W} = -\int_\Omega f^- w_\infty + \int_\Gamma w_\infty d\mathcal{X}.$$

Let us finish this section going back to the variational problem that defines the approximations and discuss uniqueness of minimizers (v_p, w_p) of (3.3). Remark that we can pass to the limit as $p \to \infty$ as we did in Theorem 3.4, without any assumption concerning uniqueness of the minimizers.

In fact, it is immediate to see that if (v, w) is a minimizer then $(\tilde{v}, \tilde{w}) = (v_p + k, w_p + k)$ is also a minimizer for every constant k, therefore uniqueness does not hold in general. However, one may ask if uniqueness of minimizers hold under the additional constraint

$$\int_\Omega v + \int_\Omega w = 0. \tag{3.35}$$

In the next result we show that this is not necessarily the case and moreover we characterize when this uniqueness hold.

Theorem 3.9. *Let* (v, w) *be a minimizer of* (3.3) *satisfying* (3.35)*. Then*

$$\int_{\{w-v\le 0\}_\Gamma} d\Theta \ge M_0.$$

Moreover, (3.3) *has another minimizer satisfying* (3.35) *if and only if there exists* $c \neq 0$ *such that*

$$\begin{cases} \displaystyle\int_{\{w-v\le 0\}_\Gamma} d\Theta = \int_{\{w-v<c\}_\Gamma} d\Theta = M_0 & \text{if } c>0, \\ \displaystyle\int_{\{w-v\le c\}_\Gamma} d\Theta = \int_{\{w-v<0\}_\Gamma} d\Theta = M_0 & \text{if } c<0. \end{cases}$$

Proof. Let $\mathcal{V}_p^\eta$ be defined as in Theorem 3.6 for the pair (v,w). Remember that by (3.23) and (3.26) we have

$$0 \le \mathcal{V}_p^\eta \le \Theta \llcorner \{w-v\le 0\}_\Gamma$$

and

$$\int_\Gamma d\mathcal{V}_p^\eta = M_0.$$

Consequently, also

$$\int_{\{w-v\le 0\}_\Gamma} d\Theta \ge M_0.$$

Suppose that $(\tilde v, \tilde w)$ is another minimizer of (3.3) satisfying (3.35). Then, since Ψ_p is convex, and $\|\cdot\|_{L^p(\Omega)}$ is strictly convex, there exists constants c_1, c_2 such that

$$(\tilde v, \tilde w) = (v + c_1, w + c_2).$$

Now, since (v,w) and $(\tilde v, \tilde w)$ satisfy (3.35), we get

$$c_1 = -c_2;$$

that is, every possible minimizer satisfying (3.35) is of the form

$$(\tilde v, \tilde w) = (v + c_1, w - c_1).$$

Now,

$$\begin{aligned} \Psi_p(v,w) &= \Psi_p(v+c_1, w-c_1) \\ &= \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v f^+ + \int_\Omega w f^- \\ &\qquad -2c_1 M_0 + \int_\Gamma (w - c_1 - (v+c_1))^- d\Theta \\ &= \Psi_p(v,w) - 2c_1 M_0 + \int_\Gamma (w - v - 2c_1)^- d\Theta - \int_\Gamma (w-v)^- d\Theta. \end{aligned}$$

Hence

$$\int_\Gamma (w - v - 2c_1)^- d\Theta - \int_\Gamma (w-v)^- d\Theta = 2c_1 M_0.$$

Note that $(v+c_1, w-c_1)+(c_1,c_1)=(v+2c_1,w)$ is also a minimizer of (3.3) and, by convexity, using that (v,w) and $(v+2c_1,w)$ are minimizers, we get that every pair of the form $(v+c,w)$, with c between 0 and $2c_1$, is also a minimizer of (3.3). From our previous arguments we obtain that

$$\int_\Gamma (w-v-c)^- d\Theta - \int_\Gamma (w-v)^- d\Theta = cM_0,$$

for every c between 0 and $2c_1$.

Now, if $c_1>0$ then we have

$$\begin{aligned} M_0 &= \int_\Gamma \frac{(w-v-2c_1)^- - (w-v)^-}{2c_1} d\Theta \\ &= \int_{\{w-v\le 0\}} d\Theta \ + \int_{\{0<w-v<2c_1\}} \frac{(w-v-2c_1)^-}{2c_1} d\Theta \\ &\ge \int_{\{w-v\le 0\}} d\Theta \ge \int_{\{w-v\le 0\}_\Gamma} d\mathcal{V}_p^\eta = M_0, \end{aligned}$$

from where it follows that

$$\int_{\{w-v\le 0\}_\Gamma} d\Theta = M_0, \quad \text{and} \quad \int_{\{w-v<2c_1\}_\Gamma} d\Theta = M_0. \tag{3.36}$$

Remark that, since

$$M_0 \le \int_{\{w-v\le 0\}_\Gamma} d\Theta \le \int_{\{w-v<2c_1\}_\Gamma} d\Theta,$$

we have that (3.36) is equivalent to

$$\int_{\{w-v<2c_1\}_\Gamma} d\Theta \le M_0.$$

On the other hand, if $c_1<0$ then, for $2c_1\le c<0$, we have

$$\begin{aligned} M_0 &= \int_\Gamma \frac{(w-v-c)^- - (w-v)^-}{c} d\Theta \\ &= \int_{\{w-v\le c\}_\Gamma} d\Theta + \int_{\{c<w-v<0\}} \frac{-(w-v)^-}{c} d\Theta \\ &\ge \int_{\{w-v\le c\}_\Gamma} d\Theta. \end{aligned}$$

Taking $c \to 0$ we obtain

$$\int_{\{w-v<0\}_\Gamma} d\Theta \le M_0.$$

Let us see that

$$\int_{\{w-v<0\}_\Gamma} d\Theta = M_0 \quad \text{and} \quad \int_{\{w-v\le 2c_1\}_\Gamma} d\Theta = M_0. \tag{3.37}$$

Indeed,

$$\begin{aligned} M_0 &= \int_\Gamma \frac{(w-v-2c_1)^- - (w-v)^-}{2c_1} d\Theta \\ &= \int_{\{w-v\le 2c_1\}_\Gamma} d\Theta + \int_{\{2c_1<w-v<0\}} \frac{-(w-v)^-}{2c_1} d\Theta \\ &\le \int_{\{w-v\le 2c_1\}_\Gamma} d\Theta + \int_{\{2c_1<w-v<0\}} d\Theta \\ &= \int_{\{w-v<0\}_\Gamma} d\Theta \le M_0. \end{aligned}$$

Then, since $\frac{-(w-v)^-}{2c_1} < 1$ on $\{2c_1 < w - v < 0\}$, the integrals

$$\int_{\{2c_1<w-v<0\}} d\Theta \qquad \text{and} \qquad \int_{\{2c_1<w-v<0\}} \frac{-(w-v)^-}{2c_1} d\Theta$$

vanish, and consequently

$$\int_{\{w-v<0\}_\Gamma} d\Theta = M_0 \qquad \text{and} \qquad \int_{\{w-v\le 2c_1\}_\Gamma} d\Theta = M_0.$$

Let us see that the reciprocal also holds. Let (v, w) be a minimizer of (3.3) and let $c_1 > 0$ satisfying (3.36). Then we have

$$\begin{aligned} \Psi_p(v + c_1, w - c_1) = \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v f^+ + \int_\Omega w f^- \\ -2c_1 M_0 + \int_\Gamma (w - v - 2c_1)^- d\Theta \end{aligned}$$

$$\begin{aligned} &= \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega v f^+ + \int_\Omega w f^- \\ &\quad -2c_1 M_0 + \int_{\{w-v\le 0\}} (w - v - 2c_1)^- d\Theta + \int_{\{0<w-v<2c_1\}} (w - v - 2c_1)^- d\Theta. \end{aligned}$$

Now, from (3.36) we deduce that

$$-2c_1M_0 + \int_{\{w-v\le 0\}} (w-v-2c_1)^- d\Theta = \int_{\{w-v\le 0\}} (w-v)^- d\Theta.$$

Also from (3.36), we have

$$\int_{\{0<w-v<2c_1\}_\Gamma} d\Theta = 0.$$

Therefore,

$$\int_{\{0<w-v<2c_1\}} (w-v-2c_1)^- d\Theta = 0,$$

and hence,

$$\Psi_p(v+c_1, w-c_1) = \Psi_p(v,w).$$

Now, let (v,w) be a minimizer of (3.3) and $c_1 < 0$ satisfying (3.37). Then we have

$$\begin{aligned}\Psi_p(v+c_1, w-c_1) &= \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega vf^+ + \int_\Omega wf^- \\ &\qquad -2c_1M_0 + \int_\Gamma (w-v-2c_1)^- d\Theta \\ &= \frac{1}{p}\int_\Omega |\nabla v|^p + \frac{1}{p}\int_\Omega |\nabla w|^p - \int_\Omega vf^+ + \int_\Omega wf^- \\ &\qquad -2c_1M_0 + \int_{\{w-v\le 2c_1\}} (w-v-2c_1)^- d\Theta.\end{aligned}$$

Now, from (3.37) we obtain that

$$-2c_1M_0 + \int_{\{w-v\le 2c_1\}} (w-v-2c_1)^- d\Theta = \int_{\{w-v\le 2c_1\}} -(w-v) d\Theta,$$

and also that

$$\int_{\{2c_1<w-v<0\}_\Gamma} d\Theta = 0.$$

Therefore,

$$\int_{\{w-v\le 2c_1\}} -(w-v) d\Theta = \int_{\{w-v<0\}} -(w-v) d\Theta = \int_{\{w-v\le 0\}} (w-v)^- d\Theta.$$

Hence, also in this case we conclude that

$$\Psi_p(v+c_1, w-c_1) = \Psi_p(v,w),$$

and the proof is finished. $\square$

Example 3.10. Let Ω be an open bounded subset of $\mathbb{R}^N$. We take $f_+, f_- \in L^\infty(\Omega)$ with disjoint supports, X_+ and X_-, respectively, such that

$$\int_{X_+} f^+ = \int_{X_-} f^- = M_0.$$

Let $x_0 \in \Omega \setminus (X_+ \cup X_-)$. Consider $\Gamma = \{x_0\}$ and $\Theta = k\delta_{x_0}$, with $k \geq M_0$. For these data, let (v, w) be a minimizer of (3.3) satisfying (3.35). By Theorem 3.9, we have

$$\int_{\{w-v\leq 0\}_\Gamma} d\Theta \geq M_0.$$

Then, since Θ is concentrated on $\{x_0\}$, we have

$$x_0 \in \{w - v \leq 0\}_\Gamma.$$

Therefore, if $k = M_0$, we have

$$\int_{\{w-v\leq 0\}_\Gamma} d\Theta = \int_{\{w-v<c\}_\Gamma} d\Theta = M_0 \qquad \forall c > 0.$$

Hence, by Theorem 3.9, for any $c > 0$, $(v+c, w-c)$ is a minimizer of (3.3) satisfying (3.35) and consequently, in this case, there is no uniqueness of minimizers.

On te other hand, if $k > M_0$, then we have uniqueness. In fact, in this case

$$\int_{\{w-v<c\}_\Gamma} d\Theta = k > M_0 \qquad \forall c > 0.$$

Moreover, if $x_0 \in \{w - v < 0\}_\Gamma$, then

$$\int_{\{w-v<0\}_\Gamma} d\Theta = k > M_0,$$

and if $x_0 \notin \{w - v < 0\}_\Gamma$, then

$$\int_{\{w-v\leq c\}_\Gamma} d\Theta = 0 \neq M_0 \qquad \forall c < 0.$$

Therefore, by Theorem 3.9, we have uniqueness a minimizer of (3.3) satisfying (3.35) in this case.

4. An example. An optimal transport problem with a permeable membrane

4.1. **Optimal mass transport problem through a partially permeable membrane.**

In this section we are interested in the optimal mass transport problem in which the mass transported must cross a partially permeable membrane, that is, we want to transport one measure into another that are located in different places separated by a membrane which only let through a point in the membrane a predetermined amount of matter. We can see that this problem fits into our general optimal matching with constraints. In fact, the membrane can be seen as the target set and the permeability of the membrane is understood as the amount of mass that each point on the membrane can receive. Of course, note that we need that the total amount of mass that the membrane can aloud must be bigger than the mass that we have to transport (otherwise the transport problem is impossible). Using our previous results we have that there exists a solution to this problem, that is, there is a way of transporting the two involved measures to the membrane (a hypersurface that separates the two measures) in such a way that they match, the total amount of mass that can be located at a point on the membrane is bounded by a given measure (the permeability of the membrane) and the total cost (measured in terms of the distance that the measures have to be transported) is minimized.

Note that if we assume that the membrane separates the domain into two subdomains Ω_+ (where f^+ is supported) and Ω_- (where f^- is supported) our $p-$Laplacian approximation reads as look for (v_p, w_p, Ξ), a weak solution of the following problem:

$$\begin{cases} -\Delta_p v = f^+ - \Xi \qquad \text{in } \Omega_+ \text{ with } \ \text{Neumann B.C.} \\ -\Delta_p w = \Xi - f^- \qquad \text{in } \Omega_- \text{ with } \ \text{Neumann B.C.} \\ \Xi \in \mathcal{M}^+(\mathbb{R}^N), \quad \Xi \leq \Theta \llcorner \{w - v \leq 0\}_\Gamma, \quad \Xi(\Gamma) = M_0. \end{cases}$$

Here we have two $p-$Laplacian equations in two different domains (Ω_+ and Ω_-) that have some part of their boundaries in common (the membrane Γ) and the coupling in the system is given by the existence of the measure Ξ supported on the common boundary Γ. Remark that we can formally write

$$|\nabla v_p|^{p-2} \frac{\partial v_p}{\partial \eta} = -|\nabla v_p|^{p-2} \frac{\partial v_p}{\partial \eta} = -\Xi \qquad \text{on } \Gamma,$$

where η is the normal vector field to Γ pointing to the exterior of Ω_+, and homogeneous Neumann boundary condition on the rest of the boundary of Ω_+ and that of Ω_-.

Next, we will see that in a particular case (a membrane with only two holes) we can give a geometrical characterization of an optimal transport.

4.2. **A membrane with two holes.**

Here we face the following situation, we have a quantity of resources that has to be delivered to the consumers, but between both there is a frontier (or a river) with only two frontier passes (or only two bridges) that allow a maximum amount of mass to pass through each one of them (each bridge has a limited amount of weight that can support). Our goal in this situation is to distribute the resources determining which part of them has to be delivered to which consumer (and also we need to specify through which pass (or bridge) has to be send). This distribution is to be optimized in the sense that we want to minimize the total cost of the operation (measured in terms of the distance that me have to move the mass). Note that in this setting it may perfectly happen that sending al the resources through one frontier point is more convenient (since the distances to that point are shorter than the distances to the other point) but this strategy is impossible since we have more resources than the ones that this frontier point allows. Therefore, in this case, we have to select which mass has to be send to the second frontier point (and doing so in such a way that we minimize the total cost).

Now, we put this problem into mathematical terms. Assume that we have two uniform measures given by

$$\mu = \chi_A \qquad \text{and} \qquad \nu = \chi_B$$

for two disjoint sets A and B with the same measure $|A| = |B| = M_0$. The set A encodes the resources and B the consumers. Separating these two sets we have a membrane Γ and furthermore assume that we only have two bridges on Γ, that is, we have two points $x_1, x_2 \in \Gamma$ and the restriction measure is

$$\Theta = k_1 \delta_{x_1} + k_2 \delta_{x_2}, \quad k_1 > 0, k_2 > 0.$$

(to simplify the exposition, we do not care too much about geometric restrictions, but we suppose that the membrane is such that for going from A to each x_i it must be crossed only once, the same about B). Note that we need to assume

$$M_0 < k_1 + k_2 = \int_\Gamma d\Theta.$$

Now, we have to determine which points in A and B has to be send to x_1 and x_2, or equivalently, which points in A has to be send to x_1 and x_2, and which points in B must receive mass from x_1 and x_2, for an optimal transport.

Now, for an optimal transport map pair $(T_+, T_-) \in \mathcal{A}_{\Gamma,\Theta}(f^+, f^-)$, set

$$A_1 := \{a \in T_+^{-1}(\{x_1\}) : a \text{ is a point of density 1 of } A\}$$

and

$$A_2 := \{a \in T_-^{-1}(\{x_2\}) : a \text{ is a point of density 1 of } A\}.$$

If $|A_1| > 0$ and $|A_2| > 0$, for $a_1 \in A_1$ and $a_2 \in A_2$ we must have

$$|a_1 - x_1| + |a_2 - x_2| \le |a_2 - x_1| + |a_1 - x_2|.$$

In fact, if we have

$$|a_1 - x_1| + |a_2 - x_2| > |a_2 - x_1| + |a_1 - x_2|,$$

for some $a_1 \in A_1$ and $a_2 \in A_2$, we argue as follows: by continuity of the distance we have that the same relation

$$|\tilde{a}_1 - x_1| + |\tilde{a}_2 - x_2| > |\tilde{a}_2 - x_1| + |\tilde{a}_1 - x_2|,$$

holds for every $\tilde{a}_1 \in B_{\delta_1}(a_1)$ and for every $\tilde{a}_2 \in B_{\delta_2}(a_2)$. We can choose δ_1, δ_2 (decreasing one of them if necessary) in such a way that $|B_{\delta_1}(a_1) \cap A| = |B_{\delta_2}(a_2) \cap A| > 0$. Now we redefine T_+ as

$$\tilde{T}_+(a) = \begin{cases} x_2, & \text{if } a \in B_{\delta_1}(a_1) \cap A \\ x_1, & \text{if } a \in B_{\delta_2}(a_2) \cap A \\ T_+(a), & \text{if } a \in A \setminus (B_{\delta_1}(a_1) \cup B_{\delta_2}(a_2)). \end{cases}$$

Note that since we have $|B_{\delta_1}(a_1) \cap A| = |B_{\delta_2}(a_2) \cap A| > 0$ then the total mass that we are sending to x_1 (and to x_2) with T_+ is the same that we are sending with $\tilde{T}_+$.

Now, when we compute the cost we have that

$$\mathcal{F}(T_+, T_-) > \mathcal{F}(\tilde{T}_+, T_-)$$

due to the inequality

$$|\tilde{a}_1 - x_1| + |\tilde{a}_2 - x_2| > |\tilde{a}_2 - x_1| + |\tilde{a}_1 - x_2|,$$

that holds for every $\tilde{a}_1 \in B_{\delta_1}(a_1)$ and for every $\tilde{a}_2 \in B_{\delta_2}(a_2)$. Therefore we arrive to a contradiction with we the assumption that (T_+, T_-) is optimal.

Therefore, we need to consider the function $\Lambda_{x_1,x_2} : A \mapsto \mathbb{R}$ given by

$$\Lambda_{x_1,x_2}(a) = |a - x_1| - |a - x_2|.$$

This function encodes which part of A has to be send to x_1 and which part to x_2. Indeed, from

$$|a_1 - x_1| - |a_1 - x_2| \leq |a_2 - x_1| - |a_2 - x_2|$$

we get

$$\Lambda_{x_1,x_2}(a_1) \leq \Lambda_{x_1,x_2}(a_2)$$

for almost any pair of pair of points a_1, a_2 such that a_1 is send to x_1 and a_2 to x_2. We also have the same relation in the case that $|A_1| = 0$ or $|A_2| = 0$.

Let $0 \leq \beta \leq M_0$ be the amount of material that is transported to x_1 (and therefore we transport $M_0 - \beta$ to x_2). From our constraint on the permeability of the membrane we have the following constraints for β,

$$\beta \leq k_1 \qquad \text{and} \qquad M_0 - \beta \leq k_2,$$

that is

$$\max\{0, M_0 - k_2\} \leq \beta \leq \min\{k_1, M_0\}.$$

Let us define

$$\lambda_A(\theta) := \Big|\{x \in A : \Lambda_{x_1,x_2}(x) < \theta\}\Big|,$$

which is non-decreasing as a function of θ.

We have that there exists a value $\theta(\beta)$ such that

$$\lambda_A(\theta(\beta)) = \beta$$

and

$$\lambda_A(\theta) \geq \beta \quad \forall \theta > \theta(\beta).$$

Then the set

$$\{x \in A : \Lambda_{x_1,x_2}(x) < \theta(\beta)\}$$

has to be sent to x_1, and

$$\{x \in A : \Lambda_{x_1,x_2}(x) > \theta(\beta)\}$$

has to be sent to x_2.

In an analogous way we have, for B, that the set

$$\{x \in B : \Lambda_{x_1,x_2}(x) < \gamma(\beta)\}$$

has to be sent to x_1, and the set

$$\{x \in A : \Lambda_{x_1,x_2}(x) > \gamma(\beta)\}$$

has to be sent to x_2, if we choose $\gamma(\beta)$ in such a way that

$$\lambda_B(\gamma(\beta)) \leq \beta,$$

and

$$\lambda_B(\gamma) \geq \beta \quad \forall \gamma > \gamma(\beta),$$

where

$$\lambda_B(\gamma) := |\{x \in B : \Lambda_{x_1,x_2}(x) < \gamma\}|.$$

The total cost of this transport is given by

$$\begin{aligned} W_{x_1,x_2}(\beta) = & \int_{\{x\in A:\Lambda_{x_1,x_2}(x)<\theta(\beta)\}} \Lambda_{x_1,x_2}(x)dx + \theta(\beta)\Big(\beta - \lambda_A(\theta(\beta))\Big) \\ & + \int_{\{x\in B:\Lambda_{x_1,x_2}(x)<\gamma(\beta)\}} \Lambda_{x_1,x_2}(x)dx + \gamma(\beta)\Big(\beta - \lambda_B(\gamma(\beta))\Big) \\ & + \int_A |x - x_2|dx + \int_B |x - x_2|dx \end{aligned} \tag{4.1}$$

in dimension $N = 1$; and, for $N \geq 2$, by

$$W_{x_1,x_2}(\beta) = \int_{\{x\in A:\Lambda_{x_1,x_2}(x)<\theta(\beta)\}} \Lambda_{x_1,x_2}(x)dx + \int_{\{x\in B:\Lambda_{x_1,x_2}(x)<\gamma(\beta)\}} \Lambda_{x_1,x_2}(x)dx + \int_A |x-x_2|dx + \int_B |x-x_2|dx. \tag{4.2}$$

Indeed:

$$\begin{aligned} W_{x_1,x_2}(\beta) = &\int_{\{x\in A:\Lambda_{x_1,x_2}(x)<\theta(\beta)\}} |x-x_1|dx \\ &+ \int_{\{x\in A:\Lambda_{x_1,x_2}(x)=\theta(\beta),\ x \text{ is sent to } x_1\}} |x-x_1|dx + \int_{\{x\in A:\Lambda_{x_1,x_2}(x)>\theta(\beta)\}} |x-x_2|dx \\ &+ \int_{\{x\in A:\Lambda_{x_1,x_2}(x)=\theta(\beta),\ x \text{ is sent to } x_2\}} |x-x_2|dx + \int_{\{x\in B:\Lambda_{x_1,x_2}(x)<\gamma(\beta)\}} |x-x_1|dx \\ &+ \int_{\{x\in B:\Lambda_{x_1,x_2}(x)=\gamma(\beta),\ x \text{ is sent to } x_1\}} |x-x_1|dx + \int_{\{x\in B:\Lambda_{x_1,x_2}(x)>\gamma(\beta)\}} |x-x_2|dx \\ &+ \int_{\{x\in B:\Lambda_{x_1,x_2}(x)=\gamma(\beta),\ x \text{ is sent to } x_2\}} |x-x_2|dx. \end{aligned}$$

Now, we replace

$$\int_{\{x\in A:\Lambda_{x_1,x_2}(x)>\theta(\beta)\}} |x-x_2|dx$$

by the equivalent expression

$$\begin{aligned} &\int_A |x-x_2|dx - \int_{\{x\in A:\Lambda_{x_1,x_2}(x)<\theta(\beta)\}} |x-x_2|dx \\ &- \int_{\{x\in A:\Lambda_{x_1,x_2}(x)=\theta(\beta)\ x \text{ is sent to } x_1\}} |x-x_2| - \int_{\{x\in A:\Lambda_{x_1,x_2}(x)=\theta(\beta)\ x \text{ is sent to } x_2\}} |x-x_2|, \end{aligned}$$

and the similar one for

$$\int_{\{x\in B:\Lambda_{x_1,x_2}(x)>\gamma(\beta)\}} |x-x_2|dx$$

to get,

$$\begin{aligned} W_{x_1,x_2}(\beta) = &\int_{\{x\in A:\Lambda_{x_1,x_2}(x)<\theta(\beta)\}} \Lambda_{x_1,x_2}(x)dx + \int_B |x-x_2|dx \\ &+ \int_{\{x\in A:\Lambda_{x_1,x_2}(x)=\theta(\beta),\ x \text{ is sent to } x_1\}} \Lambda_{x_1,x_2}(x)dx + \int_A |x-x_2|dx \\ &+ \int_{\{x\in B:\Lambda_{x_1,x_2}(x)<\gamma(\beta)\}} \Lambda_{x_1,x_2}(x)dx + \int_{\{x\in B:\Lambda_{x_1,x_2}(x)=\gamma(\beta),\ x \text{ is sent to } x_1\}} \Lambda_{x_1,x_2}(x)dx. \end{aligned}$$

But now, in the above expression we use that

$$\begin{aligned}\int_{\{x\in A:\Lambda_{x_1,x_2}(x)=\theta(\beta),\ x \text{ is sent to } x_1\}} &\Lambda_{x_1,x_2}(x)dx \\ &= \theta(\beta)\big|\{x\in A:\Lambda_{x_1,x_2}(x)=\theta(\beta),\ x \text{ is sent to } x_1\}\big| \\ &= \theta(\beta)(\beta-\lambda_A(\theta(\beta))),\end{aligned}$$

which is null if $N\geq 2$, and

$$\int_{\{x\in B:\Lambda_{x_1,x_2}(x)=\gamma(\beta),\ x \text{ is sent to } x_1\}} \Lambda_{x_1,x_2}(x)dx = \gamma(\beta)(\beta-\lambda_B(\gamma(\beta))),$$

which is also null for $N\geq 2$; from where we obtain the expression (4.1), or (4.2), for $W_{x_1,x_2}(\beta)$.

Now our task is to search for β solving

$$\min_{\max\{0,M_0-k_2\}\leq\beta\leq\min\{k_1,M_0\}} W_{x_1,x_2}(\beta). \tag{4.3}$$

This problem has the following simple geometric interpretation for $N\geq 2$: consider that the points x_1 and x_2 are located at $(-h,0)$ and $(h,0)$, for some $h>0$, and $0\in\mathbb{R}^{N-1}$. There is no loss of generality in localizing the two points in this way since the transport problem under consideration is invariant under translations and rotations. Then from our previous arguments the problem reduces to make an hyperbolic foliation of the sets A and B to obtain,

$$A_\theta=\{x\in A:\Lambda_{x_1,x_2}(x)<\theta\},\quad B_\gamma=\{x\in B:\Lambda_{x_1,x_2}(x)<\gamma\}.$$

Now, for sets of equal measure $\beta=|A_{\theta(\beta)}|=|B_{\gamma(\beta)}|$ between $\max\{0,M_0-k_2\}$ and $\min\{k_1,M_0\}$, we have to compute the sum of the integrals of Λ_{x_1,x_2}, that is,

$$w_{x_1,x_2}(\beta)=\int_{A_{\theta(\beta)}}\Lambda_{x_1,x_2}(x)+\int_{B_{\gamma(\beta)}}\Lambda_{x_1,x_2}(x),$$

and we have to select β such that $w_{x_1,x_2}(\beta)$ is minimized. Note that in doing this we also obtain the subsets of A and B that have to be connected to each point x_i.

4.2.1. *Monotonicity:* $\theta(\beta)$ and $\gamma(\beta)$ are increasing. Of course, $-h<\theta(\beta)<h$ and $-h<\gamma(\beta)<h$.

Let us call

$$\beta_i:=\max\{0,M_0-k_2\},\ \beta_s:=\min\{k_1,M_0\}.$$

4.2.2. *Continuity of* $w_{x_1,x_2}(\beta)$*:* Let $\beta \in \big]\max\{0, M_0-k_2\}, \min\{k_1, M_0\}\big[$. Now, for $t>0$ such that $\beta+t \in \big]\max\{0, M_0-k_2\}, \min\{k_1, M_0\}\big[$ we have

$$\begin{aligned}\theta(\beta)\Big|\big\{x\in A:\theta(\beta)<\Lambda_{x_1,x_2}(x)<\theta(\beta+t)\big\}\Big| \\ \le \int_{\{x\in A:\theta(\beta)<\Lambda_{x_1,x_2}(x)<\theta(\beta+t)\}}\Lambda_{x_1,x_2}(x)\,dx \\ \le \theta(\beta+t)\Big|\big\{x\in A:\theta(\beta)<\Lambda_{x_1,x_2}(x)<\theta(\beta+t)\big\}\Big|\end{aligned}$$

and

$$\Big|\big\{x\in A:\theta(\beta)<\Lambda_{x_1,x_2}(x)<\theta(\beta+t)\big\}\Big| = \beta+t-\beta=t.$$

Hence, using a similar property for B and $\gamma(\beta)$ we get

$$\big(\theta(\beta)+\gamma(\beta)\big)t \le w_{x_1,x_2}(\beta+t)-w_{x_1,x_2}(\beta) \le \big(\theta(\beta+t)+\gamma(\beta+t)\big)t. \tag{4.4}$$

Also, for $t>0$ such that $\beta-t \in \big]\max\{0, M_0-k_2\}, \min\{k_1, M_0\}\big[$,

$$\big(\theta(\beta-t)+\gamma(\beta-t)\big)t \le w_{x_1,x_2}(\beta)-w_{x_1,x_2}(\beta-t) \le \big(\theta(\beta)+\gamma(\beta)\big)t.$$

Observe that for β_i and for $t>0$ such that $\beta_i+t \in \big]\max\{0, M_0-k_2\}, \min\{k_1, M_0\}\big[$ we also have,

$$\big(\theta(\beta_i)+\gamma(\beta_i)\big)t \le w_{x_1,x_2}(\beta_i+t)-w_{x_1,x_2}(\beta_i) \le \big(\theta(\beta_i+t)+\gamma(\beta_i+t)\big)t; \tag{4.5}$$

ando for β_s and $t>0$ such that $\beta_s-t \in \big]\max\{0, M_0-k_2\}, \min\{k_1, M_0\}\big[$,

$$\big(\theta(\beta_s-t)+\gamma(\beta_s-t)\big)t \le w_{x_1,x_2}(\beta_s)-w_{x_1,x_2}(\beta_s-t) \le \big(\theta(\beta_s)+\gamma(\beta_s)\big)t. \tag{4.6}$$

Then, from the boundedness of $\theta(\beta)$ and $\gamma(\beta)$, we get the continuity of $w_{x_1,x_2}(\beta)$ in $\big[\max\{0, M_0-k_2\}, \min\{k_1, M_0\}\big]$.

4.2.3. *Minimizers:*

1. Observe that

$$\text{if } \theta(\beta_i)+\gamma(\beta_i)\ge 0 \text{ then, from (4.5), } \beta_i \text{ is a minimizer of (4.3).}$$

This is the case, for example, when A and B are located in $[0,+\infty[\times\mathbb{R}^{N-1}$.

2. And

$$\text{if } \theta(\beta_s)+\gamma(\beta_s)\le 0 \text{ then, from (4.6), } \beta_s \text{ is a minimizer of (4.3).}$$

This is the case, for example, when A and B are located in $]-\infty,0]\times\mathbb{R}^{N-1}$.

3. On the other hand, if $\theta(\beta_i)+\gamma(\beta_i)<0$ and $\theta(\beta_s)+\gamma(\beta_s)>0$, we have that, if $\theta(\beta)+\gamma(\beta)$ is continuous then there exists $\beta_0\in\big]\max\{0,M_0-k_2\},\,\min\{k_1,M_0\}\big[$ such that

$$\theta(\beta_0)+\gamma(\beta_0)=0,$$

and this is a critical point of $w_{x_1,x_2}(\beta)$ as can be seen in Section 4.2.4.

Nevertheless, without using continuity of $\theta(\beta)+\gamma(\beta)$, if there exists $\beta_0\in\big]\max\{0,M_0-k_2\},\,\min\{k_1,M_0\}\big[$ such that

$$\theta(\beta_0)+\gamma(\beta_0)=0 \text{ then } \beta_0 \text{ is a a minimizer of (4.3).}$$

This follows directly from (4.5) and (4.6). However, the existence of such a β_0 can not be guaranteed, but what can be assured is the existence of $\beta_0\in\big]\max\{0,M_0-k_2\},\,\min\{k_1,M_0\}\big[$ such that

$$\theta(b)+\gamma(b)\le 0\le\theta(c)+\gamma(c) \text{ for all } b<\beta_0<c,$$

and this implies, by (4.5) and (4.6), that

$$\beta_0 \text{ is a a minimizer of (4.3).}$$

4.2.4. *Continuous differentiability of $w_{x_1,x_2}(\beta)$ for $\theta(\beta)+\gamma(\beta)$ continuous.* Note that, if A and B are domains then $\theta(\beta)$ and $\gamma(\beta)$ are continuous.

For $\beta\in\big]\max\{0,M_0-k_2\},\,\min\{k_1,M_0\}\big[$, we have

$$\left(\frac{d}{d\beta}\right)^{+} w_{x_1,x_2}(\beta)=\theta(\beta)+\gamma(\beta). \tag{4.7}$$

In fact, from (4.4), we obtain, for $t>0$ small enough,

$$\theta(\beta)+\gamma(\beta)\le\frac{1}{t}\Big(w_{x_1,x_2}(\beta+t)-w_{x_1,x_2}(\beta)\Big)\le\theta(\beta+t)+\gamma(\beta+t).$$

Then, from the continuity of $\theta(\beta)+\gamma(\beta)$, we get (4.7). In an analogous way we get

$$\left(\frac{d}{d\beta}\right)^{-} w_{x_1,x_2}(\beta)=\theta(\beta)+\gamma(\beta).$$

Therefore,

$${w_{x_1,x_2}}'(\beta)=\theta(\beta)+\gamma(\beta),$$

which moreover is continuous.

The next simple example shows how formula (4.8) can be used in one dimension.

Example 4.1. Let $f^+ = \chi_{[0,1]}$ and $f^- = \chi_{[a,a+1]}$ with $a > 1$, and $\Theta = k_1\delta_{x_1} + k_2\delta_{x_2}$, $k_1 > 0, k_2 > 0$, $k_1 + k_2 > 1$, with $x_1 < 0$ and $1 < x_2 < a$.

We are interested in getting

$$\beta_0 := \operatorname{argmin}\{F(\beta) \ : \ \max\{0, M_0 - k_2\} \leq \beta \leq \min\{k_1, M_0\}\}, \tag{4.8}$$

being

$$\begin{aligned} F(\beta) = &\int_{\{x\in A:\Lambda_{x_1,x_2}(x)<\theta(\beta)\}} \Lambda_{x_1,x_2}(x)dx + \theta(\beta)\Big(\beta - \lambda_A(\theta(\beta))\Big) \\ &+ \int_{\{x\in B:\Lambda_{x_1,x_2}(x)<\gamma(\beta)\}} \Lambda_{x_1,x_2}(x)dx + \gamma(\beta)\Big(\beta - \lambda_B(\gamma(\beta))\Big). \end{aligned}$$

β_0 represent the optimal amount of mass that we must sent to x_1.

We have

$$\Lambda_{x_1,x_2}(x) = 2x - (x_1 + x_2) \quad \text{for } x \in A \quad \text{and} \quad \Lambda_{x_1,x_2}(x) = x_2 - x_1 \quad \text{for } x \in B.$$

Then,

$$\lambda_A(s) = \begin{cases} 0 & \text{if } \ s < -(x_1 + x_2) \\ \dfrac{x_1 + x_2 + s}{2} & \text{if } \ -(x_1 + x_2) \leq s \leq 2 - (x_1 + x_2) \\ 1 & \text{if } \ s > 2 - (x_1 + x_2) \end{cases}$$

and

$$\lambda_B(r) = \begin{cases} 0 & \text{if } \ r \leq x_2 - x_1 \\ 1 & \text{if } \ r > x_2 - x_1. \end{cases}$$

Now,

$$\lambda_A(\theta(\beta)) = \frac{x_1 + x_2 + \theta(\beta)}{2} \leq \beta \quad \text{and} \quad \lambda_A(\theta) \geq \beta \quad \forall\theta > \theta(\beta),$$

implies that $\theta(\beta) = 2\beta - (x_1 + x_2)$. Moreover, $\gamma(\beta) = x_2 - x_1$. Then, $F(\beta) = \beta^2 - 2\beta x_1$ and

$$\beta_0 = \operatorname{argmin}\{\beta^2 - 2\beta x_1 \ : \ \max\{0, 1 - k_2\} \leq \beta \leq \min\{k_1, 1\}\} = \max\{0, 1 - k_2\}.$$

Therefore, if $k_2 \geq 1$, then $\beta_0 = 0$, which means that the optimal transport consists in tranport all the masses to x_2. Now if $0 < k_2 < 1$, then $\beta_0 = 1 - k_2$, which means that the optimal transport in this case is the following: transport the masses from A and B to x_2 to saturate, and send the rest to x_1.

Remark 4.2. In this example, when $k_1 > 1$ and $k_2 > 1$ we have uniqueness of our approximations for p finite, that is, there is a unique minimizer of (3.3) verifying

$$\int_\Omega v + \int_\Omega w = 0.$$

In fact, in this case, since we always have

$$\int_{\{w_p - v_p \leq 0\}_\Gamma} d\Theta > 1 = M_0,$$

we can apply Theorem 3.9 to get uniqueness.

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