OPTIMAL MASS TRANSPORTATION FOR COSTS GIVEN BY FINSLER DISTANCES VIA p-LAPLACIAN APPROXIMATIONS

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Abstract. In this paper we find a Kantorovich potential for the mass transport problem of two measures with transport cost given by a Finsler distance. To obtain such a potential we take the limit as p goes to infinity of a family of variational problems of $p-$Laplacian type. This procedure yields not only a Kantorovich potential but also a transport density. We also obtain a characterization of the Kantorovich potentials and a Benamou-Brenier formula for the problem.

1. Introduction and preliminaries

1.1. **Introduction.** This paper deals with an optimal mass transport problem when the cost of moving one unit of mass from one point x to another y is given by a Finsler distance in a bounded domain Ω in $\mathbb{R}^N$.

Our approach to this problem is based on an idea by Evans and Gangbo, [17], that approximates a Kantorovich potential for a transport problem with cost given by the Euclidean distance using the limit as p goes to infinity of a family of $p-$Laplacian type problems. This limit procedure turns out to be quite flexible and allowed us to deal with different transport problems in which the cost is given by the Euclidean distance or variants of it. For example, optimal matching problems (here one deals with systems of $p-$Laplacian type), optimal import/export problems (here one considers Dirichlet or Neumann boundary conditions), and optimal transport with the help of a courier (this is related to the double obstacle problem for the $p-$Laplacian). We refer to [10], [21], [22], [23], [24], [25]. Here we extend the previous results considering a more delicate structure, that is given in terms of a Finsler metric that may change from one point to another in the domain (this is what is called a Finsler structure in the literature). Our ideas can also be extended to manifolds, but, to simplify the presentation, we prefer to state and prove our results just in a bounded domain Ω in $\mathbb{R}^N$.

Date: June 12, 2014.

Key words and phrases. Mass transport, Monge-Kantorovich problems, Finsler metric $p-$Laplacian equation.

2010 *Mathematics Subject Classification.* 49J20,53C60, 49J45,90C08.

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On the other hand, at the end of the paper we present how the obtained results read on a Riemannian manifold.

Now, let us introduce some terminology and general results from optimal mass transportation theory. The *Monge transportation problem* consists in moving one distribution of mass into another one minimizing a given transportation cost. In mathematical terms, the problem can be stated as follows: let Ω a open bounded subset of $\mathbb{R}^N$, given $f^+, f^- \in L^1(\Omega)$, two nonnegative compactly supported functions with the same total mass, find a measurable map $T : \Omega \to \Omega$ such that $T\#f^+ = f^-$, i.e.,

$$\int_{T^{-1}(A)} f^+(x)dx = \int_A f^-(x)dx \qquad \forall\, A \subset \Omega \ \text{ measurable,}$$

and in such a way that T minimizes the total transport cost, that is,

$$\int_\Omega c(x, T(x))f^+(x)dx = \min_{S:\, S\#f^+=f^-} \int_\Omega c(x, S(x))f^+(x)dx,$$

where $c : \Omega \times \Omega \to \mathbb{R}$ is a given cost function. The map T is called an *optimal transport map*. The difficulties in solving such problem motivated Kantorovich to introduce a relaxed formulation, called the *Monge-Kantorovich problem*, that consists in looking for plans instead of transport maps, that is, we look for nonnegative Radon measures μ in $\Omega \times \Omega$ such that $\mathrm{proj}_x(\mu) = f^+(x)dx$ and $\mathrm{proj}_y(\mu) = f^-(y)dy$. Denoting by $\Pi(f^+, f^-)$ the set of plans, the Monge-Kantorovich problem consists in minimizing the total cost functional

$$\mathcal{K}_c(\mu) := \int_{\Omega\times\Omega} c(x, y)\, d\mu(x, y)$$

in $\Pi(f^+, f^-)$. If μ is a minimizer of the above problem we say that it is an *optimal plan*. When c is lower-semicontinuous, it is well known that

$$\inf_{T\#f^+=f^-} \int_\Omega c(x, T(x))f^+(x)dx = \min_{\mu\in\Pi(f^+,f^-)} \mathcal{K}_c(\mu).$$

For notation and general results on Mass Transport Theory we refer to [1, 4, 16, 17, 31] and [32], below we summarize our main concern in this paper.

Here we will deal with a cost c given by a *Finsler distance* (see Subsection 1.2 for a precise definition) that can be non-symmetric. However, since the cost satisfies the triangular inequality, the following duality result holds (see [31]):

$$\min \left\{ \mathcal{K}_c(\mu) : \mu \in \Pi(f^+, f^-) \right\} = \sup \left\{ \int_\Omega v(f^- - f^+) : v \in K_c(\Omega) \right\}, \tag{1.1}$$

where $K_c(\Omega) := \{u : \Omega \mapsto \mathbb{R} : u(y) - u(x) \leq c(x,y)\}$. Moreover, there exists $u \in K_c(\Omega)$ such that

$$\int_\Omega u(f^- - f^+) = \sup\left\{\int_\Omega v(f^- - f^+) : v \in K_c(\Omega)\right\}.$$

Such maximizers are called *Kantorovich potentials.*

When c is symmetric, it holds that

$$\min\{\mathcal{K}_c(\mu) \,:\, \mu \in \Pi(f^+, f^-)\} = \sup\left\{\int_\Omega v(f^+ - f^-) \,:\, v \in K_c(\Omega)\right\}, \tag{1.2}$$

since $v \in K_c(\Omega)$ iff $-v \in K_c(\Omega)$. For $c(x,y) = |x-y|$, the Euclidean distance, Evans and Gangbo found in [17] a maximizer in (1.2) taking limits as $p \to \infty$ of the solutions of certain p–Laplacian problems. As we have already mentioned, our main goal here is the same, we aim to find the Kantorovich potentials taking limits of some kind of p-Laplacian problems as $p \to \infty$.

Now, we state precisely what is our cost function. In order to do this we introduce briefly the definition of Finsler structures (see Section 2 for details and properties). Finsler functions are *grosso modo* extensions of norms. Basic references in Finsler geometry are [6, 29].

From now on, Ω will be a bounded domain in $\mathbb{R}^N$, and $f^+, f^- \in L^2(\Omega)$ are non-negative, compactly supported functions with the same total mass. We also assume that $\operatorname{supp}(f^+) \cup \operatorname{supp}(f^-) \subset\subset \Omega$.

1.2. **The cost function.** We will denote by $\langle \xi; \eta\rangle$ the Euclidean inner product between ξ and η in $\mathbb{R}^N$ and by $|\xi| = \sqrt{\langle \xi, \xi\rangle}$ the Euclidean norm in $\mathbb{R}^N$.

A Finsler function Φ in $\mathbb{R}^N$ is a function that is non-negative, continuous, convex, positively homogeneous of degree 1,

$$\Phi(t\xi) = t\Phi(\xi) \quad \text{for any } t \geq 0,\ \xi \in \mathbb{R}^N,$$

and vanishes only at 0. The *dual function (or polar function)* of a Finsler function Φ is defined as

$$\Phi^*(\xi^*) := \sup\{\langle \xi^*; \xi\rangle : \Phi(\xi) \leq 1\} \ \text{ for } \xi^* \in \mathbb{R}^N.$$

It is immediate to verify that Φ^* is also a Finsler function.

A *Finsler structure* F on Ω is a measurable function $F : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ such that for any $x \in \Omega$, $F(x, \cdot)$ a Finsler function in $\mathbb{R}^N$. For a Finsler structure F on Ω, we define the dual structure $F^* : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ by

$$F^*(x, \xi) := \sup\{\langle \eta; \xi\rangle : F(x, \eta) \leq 1\}.$$

Important examples of Finsler structures on Ω are those of the form $\Phi(B(x)\xi)$, being Φ a Finsler function and $B(x)$ a symmetric $N \times N$ matrix, positive definite. Such type of Finsler structures are known as *deformations of Minkowski norms.*

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Let us now introduce the cost function. Given a Finsler structure F on Ω, we define the following cost function c:

$$c_F(x,y) := \inf_{\sigma\in\Gamma^{\Omega}_{x,y}} \int_0^1 F(\sigma(t),\sigma'(t))\,dt, \tag{1.3}$$

where, for $x,y\in\Omega$, the set $\Gamma^{\Omega}_{x,y}$ is given by,

$$\Gamma^{\Omega}_{x,y} := \{\sigma\in C^1([0,1],\Omega),\ \sigma(0)=x,\ \sigma(1)=y\}.$$

We have that c_F is a *Finsler distance*. We make emphasis on the fact that c_F is not necessary symmetric (i.e., $c_F(x,y)\neq c_F(y,x)$ may happen) because F is merely positively homogeneous.

Remark 1.1. In the particular case of $F(x,\xi)=\Phi(\xi)$ and Ω convex, we have that

$$c_F(x,y)=\Phi(y-x).$$

In fact, given $\sigma\in\Gamma^{\Omega}_{x,y}$, since Φ is convex, applying Jensen's inequality, we get

$$\Phi(y-x)=\Phi\left(\int_0^1\sigma'(t)\,dt\right)\le\int_0^1\Phi(\sigma'(t))\,dt.$$

Therefore, taking infimum, we get $\Phi(y-x)\le c_F(x,y)$. On the other hand, if $\sigma(t)=x+t(y-x)$, we have

$$c_F(x,y)\le\int_0^1\Phi(\sigma'(t))\,dt=\Phi(y-x).$$

Let us remark that when c_F is not symmetric, then (1.2) is not true in general. For example, if $\Phi(\xi):=a\xi^-+b\xi^+$, with $0<a<b$, then for $f_+=\chi_{(0,1)}$ and $f_-=\chi_{(1,2)}$, we have that an optimal transport map is $T(x)=x+1$, so

$$\begin{aligned}\min\{\mathcal{K}_{c_F}(\mu)\,:\,\mu\in\Pi(f^+,f^-)\} &= \int c(x,T(x))f_+(x)dx\\ &= \int\Phi(T(x)-x))f_+(x)dx=b=\int u(x)(f_-(x)-f_+(x))dx,\end{aligned}$$

where $u(x)=bx$ is the Kantorovich potential. On the other hand, an optimal transport map for the transport of f_- to f_+ is $S(x)=x-1$, and consequently

$$\begin{aligned}\sup\left\{\int_\Omega v(f^+-f^-)\,:\,v\in K_{c_F}(\Omega)\right\} &= \int c_F(x,S(x))f_-(x)dx\\ &= \int\Phi(S(x)-x))f_-(x)dx=a=\int v(x)(f_+(x)-f_-(x))dx,\end{aligned}$$

where $u(x)=-ax$ is a Kantorovich potential.

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1.3. **Main results.** We will denote by $\mathcal{M}(\overline{\Omega},\mathbb{R}^N)$ the set of all $\mathbb{R}^N$-valued Radon measures in $\overline{\Omega}$, which, by the Riesz representation Theorem, can be identify with the dual of the space $C(\overline{\Omega},\mathbb{R}^N)$ endowed with the supremum norm.

Our main result reads as follows:

Theorem 1.2. *Let F a continuous Finsler structure such that*

$$\alpha|\xi| \leq F^*(x,\xi) \leq \beta|\xi| \quad \text{for any } \xi \in \mathbb{R}^N \quad \text{and } x \in \Omega$$

(here α, β are positive constants), and $F^*(x,\cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$. For $p > N$, let u_p be a solution to the variational problem

$$\min_{u \in S_p} \int_\Omega \frac{[F^*(x,Du)]^p}{p} - \int_\Omega uf,$$

where $f = f^- - f^+$ and $S_p = \{u \in W^{1,p}(\Omega) : \int_\Omega u = 0\}$.

Then, there exists a uniform limit as $p \to \infty$ of u_p (extracting a sequence $p_j \to \infty$ if necessary), u_∞, that is a Kantorovich potential for the mass transport problem of f_+ to f_- with cost given by the Finsler distance given in (1.3)*. Moreover, there exists $\mathcal{X}_\infty \in \mathcal{M}(\overline{\Omega},\mathbb{R}^N)$ such that*

$$\int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} Dv\, d\mathcal{X}_\infty \quad \forall\, v \in C^1(\overline{\Omega}),$$

and

$$|\mathcal{X}_\infty|_F = \int_\Omega u_\infty(f^- - f^+) = \min\{\mathcal{K}_{c_F}(\mu) \,:\, \mu \in \Pi(f^+,f^-)\}$$

where $|\mathcal{X}_\infty|_F$ is defined by

$$|\mathcal{X}_\infty|_F := \sup\left\{\int_{\overline{\Omega}} \Phi\, d\mathcal{X}_\infty \;:\; \Phi \in C(\overline{\Omega},\mathbb{R}^N), \quad F^*(x,\Phi(x)) \leq 1 \ \forall x \in \Omega\right\}.$$

If in addition we assume that

$$F^*(x,D_\mu u(x)) \leq 1 \quad \mu - a.e. \text{ in } \overline{\Omega},$$

then

$$\begin{cases} \displaystyle\int_\Omega (f^- - f^+)v = \int_{\overline{\Omega}} \frac{\partial F^*}{\partial \xi}(\cdot, D_\mu u_\infty) \cdot Dv\, d\mu \quad \forall\, v \in C^1(\overline{\Omega}), \\ \\ F^*(x,D_\mu u(x)) = 1 \quad \mu - a.e. \text{ in } \overline{\Omega}, \end{cases}$$

where $D_\mu u_\infty$ is the tangential gradient of u_∞ respect to the transport density $\mu = F(x,\mathcal{X}_\infty)$.

For the particular case of quadratic cost $c(x,y) = |x-y|^2$, Benamou and Brenier in [9] introduced the *Eulerian* point of view of the mass transport problem and obtained what is usually known as *Benamou-Brenier formula.* This point of view has been generalized in different directions (see for instance, [1], [14], [3]). Following Brenier, see [14], we consider the paths

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$f:[0,1]\to\mathcal{M}(\overline{\Omega},\mathbb{R})^+$ and the vector fields $E:[0,1]\to\mathcal{M}(\overline{\Omega},\mathbb{R}^N)$ satisfying

$$\left\{\begin{array}{l}\displaystyle\frac{d}{dt}\int_{\overline{\Omega}}\phi df(t)+\int_{\overline{\Omega}}\nabla\phi dE(t)=0 \quad \text{in } \mathcal{D}'(0,1), \quad \forall\phi\in C^1(\overline{\Omega}),\\[2mm] f(0)=f^+, \quad \text{and} \quad f(1)=f^-.\end{array}\right. \tag{1.4}$$

Given a solution (f,E) of (1.4), we define its energy as

$$J_F(f,E):=\int_0^1|E(t)|_F\,dt.$$

We have the following relation between the Monge-Kantorovich problem and the equation (1.4), that provides a Benamou-Brenier formula for this kind of transport problems.

Theorem 1.3. *Assume that F is continuous and that $F^*(x,\cdot)\in C^1(\mathbb{R}^N\setminus\{0\})$ and consider $\mathcal{X}_\infty$ the flux given in Theorem 1.2. Then, given $f(t):=f^++t(f^--f^+)$ and $E(t):=\mathcal{X}_\infty$ for $t\in[0,1]$, (f,E) is a solution of problem* (1.4)*. Moreover,*

$$\begin{aligned}&\min\{J_F(f,E)\ :\ (f,E) \textit{ is a solution of } (1.4)\}\\ &\qquad=\min\{\mathcal{K}_{c_F}(\mu)\,:\,\mu\in\Pi(f^+,f^-)\}.\end{aligned}$$

The paper is organized as follows: in the next section we give some preliminaries on Finsler structures; in Section 3 we introduce the $p-$Laplacian problems that we use to approximate a Kantorovich potential of our mass transport problem, and we prove that we can take limits as $p\to\infty$ along subsequences of the solutions obtaining in the limit a Lipschitz function; in Section 4 we show that this limit is in fact a Kantorovich potential for our problem and moreover, we find a PDE that is verified by the limit, this PDE involves a transport density. In Section 5 we see that the results obtained in Section 4 characterize the Kantorovich potentials for the transport problem we study. Section 6 is devoted to get a Benamou-Brenier formula for the problem. Finally, in Section 7 we briefly comment on the extension of our results to a general Riemannian manifold.

2. Preliminaries on Finsler structures

In this section we collect some properties of Finsler functions in $\mathbb{R}^N$ that will be used in the sequel. Recall from the introduction that a Finsler function Φ is a non-negative continuous convex function, positively homogeneous of degree 1,

$$\Phi(t\xi)=t\Phi(\xi) \quad \text{for any } t\geq 0,\ \xi\in\mathbb{R}^N,$$

that vanishes only at 0. Observe that Φ satisfies

$$\alpha|\xi|\leq\Phi(\xi)\leq\beta|\xi| \quad \text{for any } \xi\in\mathbb{R}^N,$$

for some positive constants α, β.

Note that Finsler functions are extensions of norms. In fact, any norm in $\mathbb{R}^N$ is a Finsler function, and any symmetric Finsler function is a norm. Moreover, for any Finsler function, convexity is equivalent to the triangular inequality.

Let

$$B_\Phi := \{\xi \in \mathbb{R}^N : \Phi(\xi) \leq 1\}.$$

This set B_Φ is a closed bounded convex set with $0 \in \mathrm{int}(B)$. It is symmetric around the origin if Φ is a norm. Conversely, for any closed bounded convex set K with $0 \in \mathrm{int}(K)$, $\phi_K(\xi) := \inf\{\alpha > 0 : \xi \in \alpha K\}$ is a Finsler function with $B_{\phi_K} = K$; when K is centrally symmetric, we have a norm. In the literature the Finsler functions are also denominated as Minkowski norms.

The *dual function (or polar function)* of a Finsler function Φ is defined as

$$\Phi^*(\xi^*) := \sup\{\langle \xi^*; \xi \rangle : \xi \in B_\Phi\} \quad \text{for } \xi^* \in \mathbb{R}^N.$$

It is immediate to verify that Φ^* is also a Finsler function; and a norm when Φ is a norm. We also have

$$\Phi^*(\xi^*) = \sup_{\xi \neq 0} \frac{\langle \xi^*; \xi \rangle}{\Phi(\xi)}.$$

Therefore, the following inequality of Cauchy-Schwarz type holds,

$$\langle \xi^*; \xi \rangle \leq \Phi(\xi)\Phi^*(\xi^*). \tag{2.1}$$

If Φ is a norm, we have

$$|\langle \xi^*; \xi \rangle| \leq \Phi(\xi)\Phi^*(\xi^*). \tag{2.2}$$

Now, for general Finsler functions the inequality (2.2) is not true. An example of a Finsler function that is not a norm in $\mathbb{R}$ is given by $\Phi(\xi) := a\xi^- + b\xi^+$, with $0 < a < b$.

It is not difficult to see that

$$\Phi^{**}(\xi) = \Phi(\xi), \qquad \forall\, \xi \in \mathbb{R}^N.$$

Hence,

$$\Phi(\xi) = \sup_{\xi^* \neq 0} \frac{\langle \xi; \xi^* \rangle}{\Phi^*(\xi^*)}. \tag{2.3}$$

If we assume that the Finsler function Φ is differentiable at ξ, then by Euler's Theorem,

$$\Phi(\xi) = \langle D\Phi(\xi); \xi \rangle. \tag{2.4}$$

Moreover, if we assume Φ is differentiable in $K \subset \mathbb{R}^N$, since Φ is convex and satisfies the triangle inequality, we have

$$\langle D\Phi(\xi); \eta \rangle \leq \Phi(\eta) \quad \forall \xi, \eta \in K, \tag{2.5}$$

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and consequently

$$|\langle D\Phi(\xi);\eta\rangle| \leq \sup\{\Phi(\eta),\Phi(-\eta)\} \leq \beta|\eta| \quad \forall \xi,\eta \in K. \tag{2.6}$$

If we assume Φ is differentiable in $\mathbb{R}^N \setminus \{0\}$, by Lagrange multipliers, from $\Phi^*(\xi^*) = \sup_{\Phi(\xi)=1}\langle \xi;\xi^*\rangle$, we get that

$$\begin{gathered}\text{if} \quad \Phi(\xi)=1 \quad \text{and} \quad \Phi^*(\xi^*) = \langle \xi;\xi^*\rangle \\ \text{then there exists } \lambda\in\mathbb{R} \text{ such that } \xi^* = \lambda D\Phi(\xi).\end{gathered}$$

Therefore, by (2.4), we get that

$$\text{if} \quad \Phi(\xi)=1 \quad \text{and} \quad \Phi^*(\xi^*) = \langle \xi;\xi^*\rangle, \text{ then } \xi^* = \Phi^*(\xi^*)D\Phi(\xi). \tag{2.7}$$

From (2.4) and (2.5), we also have

$$\Phi^*(D\Phi(\xi)) = 1 \quad \forall\, \xi \neq 0. \tag{2.8}$$

A *Finsler structure* F on Ω is a measurable function $F : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ such that for any $x \in \Omega$, $F(x,\cdot)$ a Finsler function in $\mathbb{R}^N$. For a Finsler structure F on Ω, we define the dual structure $F^* : \Omega \times \mathbb{R}^N \to \mathbb{R}_+$ by

$$F^*(x,\xi) := \sup\{\langle \eta;\xi\rangle : F(x,\eta) \leq 1\}.$$

Finally, besides Finsler structures, let us remark that we will identify the elements $\eta \in L^1(\Omega,\mathbb{R}^N)$ as elements of $\mathcal{M}(\overline{\Omega},\mathbb{R}^N)$ by means of

$$\langle \eta,\Phi\rangle := \int_{\overline{\Omega}} \langle \Phi(x),\overline{\eta}(x)\rangle\, dx,$$

where

$$\overline{\eta}(x) := \begin{cases} \eta(x) & \text{if } x\in\Omega \\ 0 & \text{if } x \in \overline{\Omega}\setminus\Omega. \end{cases}$$

3. A $p-$Laplacian problem

From now on, we will assume that F is Finsler structure on Ω, continuous in $\Omega \times \mathbb{R}^N$, satisfying

$$\alpha|\xi| \leq F^*(x,\xi) \leq \beta|\xi| \quad \text{for any } \xi\in\mathbb{R}^N \text{ and } x\in\Omega, \tag{3.1}$$

being α and β positive constants. Condition (3.1) is satisfied, for example, if we impose that $F^2(x,\cdot)$ is twice differentiable (for $\xi \neq 0$) and the matrix

$$\frac{\partial^2}{\partial\xi_i\partial\xi_j}\left(\frac{1}{2}F^2(x,\xi)\right)$$

is uniformly elliptic (see [6]). Let us remark that due to the fact that f^+ and f^- are compactly supported inside Ω, condition (3.1) can be relaxed. For example, for the Poincaré disk, that is, the unit disc with the Finsler structure

$$F(x,\xi) = \frac{2|\xi|}{1-|x|^2},$$

for which the distance c_F is given by

$$c_F(x,y) = \operatorname{argcosh}\left(1 + \frac{2|x-y|^2}{(1-|x|^2)(1-|y|^2)}\right),$$

our results can be applied.

For $p > N$, we consider the variational problem

$$\min_{u \in S_p} \int_\Omega \frac{[F^*(x,Du)]^p}{p} - \int_\Omega uf. \tag{3.2}$$

where $f \in L^2(\Omega)$, $\int_\Omega f = 0$, and $S_p = \{u \in W^{1,p}(\Omega) : \int_\Omega u = 0\}$.

Lemma 3.1. *For $p > N$, there exists a continuous solution u_p to the variational problem* (3.2)*.*

Proof. Note that under the conditions on F^*, we have

$$\alpha|Du| \leq F^*(\cdot, Du) \leq \beta|Du|. \tag{3.3}$$

Hence, for every $u \in W^{1,p}(\Omega)$,

$$\alpha \int_\Omega \frac{|Du|^p}{p} \leq \int_\Omega \frac{[F^*(x,Du)]^p}{p} \leq \beta \int_\Omega \frac{|Du|^p}{p}$$

and therefore the functional

$$\Theta_{p,f}(u) = \int_\Omega \frac{[F^*(x,Du)]^p}{p} - \int_\Omega uf, \tag{3.4}$$

is well defined in the set S_p which is convex, weakly closed and non empty. On the other hand, $\Theta_{p,f}$ is coercive, bounded below and lower semicontinuous in S_p. Then, there is a minimizing sequence $u_n \in S_p \subset W^{1,p}(\Omega)$ such that $u_n \rightharpoonup u \in S_p$ and

$$\inf_S \Theta_{p,f} = \liminf_{n \to +\infty} \Theta_{p,f}(u_n) \geq \Theta_{p,f}(u).$$

Hence the minimum of $\Theta_{p,f}$ in S_p is attained. $\square$

Remark 3.2. When $F^*(x,\cdot)$ is strictly convex, we get uniqueness of u_p. Observe that we have $\int_\Omega u_p = 0$. As usually happens for homogeneous Neumann problems there are infinitely many solutions to (3.2) but any two of them differ by an additive constant.

Assuming that $F^*(x,\cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$, then, via standard arguments like the ones used in [7], we have that u_p is a weak solution of the following problem of $p-$Laplacian type

$$\begin{cases} -\operatorname{div}\left([F^*(x,Du(x))]^{p-1} \dfrac{\partial F^*}{\partial \xi}(x,Du(x))\right) = f & \text{in } \Omega, \\ [F^*(x,Du(x))]^{p-1} \langle \dfrac{\partial F^*}{\partial \xi}(x,Du(x)); \eta \rangle = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.5}$$

Here η is the exterior normal vector on $\partial\Omega$, and $\frac{\partial F^*}{\partial \xi}$ is the gradient of $F^*(x,\xi)$ with respect the second variable ξ.

In the particular case $F(x,\xi)=\Phi(A(x)\xi)$, with Φ a Finsler function and, $A(x)$ a symmetric $N\times N$ matrix, positive definite, that depends smoothly on x, equation (3.5) becomes

$$\begin{cases} -\mathrm{div}\left(\left[\Phi^*(A^{-1}Du)\right]^{p-1} A^{-1}D\Phi^*(A^{-1}Du)\right)=f & \text{in } \Omega,\\ \left[\Phi^*(A^{-1}Du)\right]^{p-1}\langle A^{-1}D\Phi^*(A^{-1}Du);\eta\rangle=0 & \text{on } \partial\Omega. \end{cases} \tag{3.6}$$

Note that in the particular case of the Euclidan norm $\Phi(\xi)=|\xi|$, equation (3.6) reads as

$$\begin{cases} -\mathrm{div}\left(\left|A^{-1}Du\right|^{p-2} A^{-2}Du\right)=f & \text{in } \Omega,\\ \left|A^{-1}Du\right|^{p-2}\langle A^{-2}Du;\eta\rangle=0 & \text{on } \partial\Omega. \end{cases}$$

Finally, if $A=I$, equation (3.6) is given by

$$\begin{cases} -\Delta_{p,\Phi^*}u=f & \text{in } \Omega,\\ [\Phi^*(Du)]^{p-1}\langle D\Phi^*(Du);\eta\rangle=0 & \text{on } \partial\Omega, \end{cases}$$

being

$$\Delta_{p,\Phi^*}u:=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left([\Phi^*(Du)]^{p-1}\frac{\partial\Phi^*}{\partial\xi_i}(Du)\right).$$

In particular, for Φ^* an ℓ^q-norm, that is,

$$\Phi^*(\xi)=\|\xi\|_q:=\left(\sum_{k=1}^N|\xi_k|^q\right)^{\frac{1}{q}},$$

the operator Δ_{p,Φ^*} becomes

$$\Delta_{p,\Phi^*}u=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left(\left[\sum_{k=1}^N\left|\frac{\partial u}{\partial x_k}\right|^q\right]^{\frac{p-q}{q}}\left|\frac{\partial u}{\partial x_i}\right|^{q-2}\frac{\partial u}{\partial x_i}\right),$$

and consequently, if $q=2$, we get the classical $p-$Laplacian operator

$$\Delta_p u:=\mathrm{div}\left(|Du|^{p-2}Du\right).$$

Now, let us see that we can extract a sequence of solutions to (3.2), $\{u_{p_j}\}_j$, with $p_j\to\infty$, that converges uniformly as $j\to\infty$.

Lemma 3.3. *Let u_p be a solution to* (3.2) *indexed by p with $p>N$. Then, there exists a subsequence $p_j\to\infty$ such that*

$$u_{p_j}\rightrightarrows u_\infty$$

uniformly in $\overline{\Omega}$. Moreover, the limit u_∞ is Lipschitz continuous.

Proof. Along this proof we will denote by C a constant independent of p that may change from one line to another.

Our first aim is to prove that the L^p-norm of the gradient of u_p is bounded independently of p.

Let v a fixed Lipschitz function with $F^*(x, Dv(x)) \leq 1$ for a.e. $x \in \Omega$ and $\int_\Omega v = 0$, then we have that $v \in S_p$. Hence, since u_p is a minimizer of the functional $\Theta_{p,f}$ in S_p, we have

$$\begin{aligned} \int_\Omega \frac{[F^*(x, Du_p(x))]^p}{p} - \int_\Omega f u_p &\leq \int_\Omega \frac{[F^*(x, Dv(x))]^p}{p} - \int_\Omega f v \\ &\leq \int_\Omega \frac{1}{p} - \int_\Omega f v. \end{aligned}$$

Consequently,

$$\int_\Omega \frac{[F^*(x, Du_p(x))]^p}{p} \leq \frac{1}{p} |\Omega| - \int_\Omega f v + \int_\Omega f u_p.$$

Now, thanks to the fact that $\int_\Omega u_p = 0$ and that the constant in the inequality $\|u_p\|_{L^p(\Omega)} \leq C \|Du_p\|_{L^p(\Omega)}$ can be chosen independent of p (see [23]) we get

$$\int_\Omega f u_p \leq C \|u_p\|_{L^p(\Omega)} \leq C \|Du_p\|_{L^p(\Omega)},$$

and then we obtain

$$\int_\Omega \frac{[F^*(x, Du_p(x))]^p}{p} \leq C + C \|Du_p\|_{L^p(\Omega)}.$$

Then, by (3.3), we get

$$\int_\Omega [F^*(x, Du_p(x))]^p \leq pC + pC \left(\int_\Omega [F^*(x, Du_p(x))]^p \right)^{\frac{1}{p}}.$$

From this inequality we can obtain that there exists C, independent of p, such that

$$\left(\int_\Omega [F^*(x, Du_p(x))]^p \right)^{\frac{1}{p}} \leq (Cp)^{\frac{1}{p-1}}. \tag{3.7}$$

Then, from (3.3) we obtain that there exists C, independent of p, such that

$$\left(\int_\Omega |Du_p|^p \right)^{\frac{1}{p}} \leq C.$$

Now, using this uniform bound, we prove uniform convergence of a sequence u_{p_j}. In fact, we take m such that $N < m \leq p$ and obtain the following bound

$$\begin{aligned} \|Du_p\|_{L^m(\Omega)} &= \left(\int_\Omega |Du_p|^m \cdot 1\right)^{\frac{1}{m}} \\ &\le \left[\left(\int_\Omega |Du_p|^p\right)^{\frac{m}{p}} \left(\int_\Omega 1\right)^{\frac{p-m}{p}}\right]^{\frac{1}{m}} \\ &\le C_1 \, |\Omega|^{\frac{p-m}{pm}} \le C_2, \end{aligned}$$

the constant C_2 being independent of p. We have proved that $\{u_p\}_{p>N}$ is bounded in $W^{1,m}(\Omega)$, and we know that $\int_\Omega u_p = 0$, so we can obtain a subsequence $u_{p_j} \rightharpoonup u_\infty \in W^{1,m}(\Omega)$ with $p_j \to +\infty$. Since $W^{1,p}(\Omega) \hookrightarrow C^{0,\alpha}(\bar{\Omega})$ and $u_{p_j} \rightharpoonup u_\infty \in W^{1,p}(\Omega)$, we obtain $u_{p_j} \to u_\infty$ in $C^{0,\alpha}(\Omega)$, and in particular $u_{p_j} \rightrightarrows u_\infty$ uniformly in $\overline{\Omega}$. As $u_{p_j} \in C(\overline{\Omega})$, then $u_\infty \in C(\overline{\Omega})$.

Finally, let us show that the limit function u_∞ is Lipschitz. In fact, we proved that,

$$\left(\int_\Omega |Du_\infty|^m\right)^{\frac{1}{m}} \le \liminf_{p_j \to +\infty} \left(\int_\Omega \left|Du_{p_j}\right|^m\right)^{\frac{1}{m}} \le C_1 \, |\Omega|^{\frac{1}{m}} \le C_2.$$

Now, we take $m \to \infty$ to obtain $\|Du_\infty\|_{L^\infty(\Omega)} \le C_2$. So, we have proved $u_\infty \in W^{1,\infty}(\Omega)$, that is, u_∞ is a Lipschitz function. □

Remark 3.4. *All the results of this section remains true if we assume that* $f = f_p$ *and*

$$f_p \rightharpoonup f \quad \text{weakly in } L^2(\Omega).$$

4. Mass transport interpretation of the limit

4.1. **Kantorovich potentials.** The goal of this section is to show that the limit u_∞ of u_p that we proved to exist in the previous section, for $f = f^- - f^+$, is a Kantorovich potential for the mass transport problem of f^+ to f^- with the cost given by the Finsler distance

$$c_F(x,y) := \inf_{\sigma \in \Gamma^\Omega_{x,y}} \int_0^1 F(\sigma(t), \sigma'(t))\, dt.$$

The key idea is contained in the following result.

Lemma 4.1. $u \in W^{1,\infty}(\Omega)$ *if and only if* $Lip(u, c_F) < \infty$*, where*

$$Lip(u, c_F) := \sup\left\{ \frac{u(y) - u(x)}{c_F(x,y)} : x, y \in \Omega, x \neq y \right\};$$

and

$$esssup_{x \in \Omega} F^*(x, Du(x)) = Lip(u, c_F).$$

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Proof. The first assertion is an easy consequence of (3.1).

Now let $\sigma \in \Gamma_{x,y}^{\Omega}$, then, by (2.1),

$$\begin{aligned} u(y) - u(x) &= \int_0^1 \langle Du(\sigma(t)); \sigma'(t) \rangle dt \\ &\leq \int_0^1 F^*(\sigma(t), Du(\sigma(t))) F(\sigma(t), \sigma'(t)) dt \\ &\leq \operatorname{esssup}_{x\in\Omega} F^*(x, Du(x)) \int_0^1 F(\sigma(t), \sigma'(t)) dt. \end{aligned}$$

Taking the infimun in $\sigma \in \Gamma_{x,y}^{\Omega}$ we get

$$u(y) - u(x) \leq \operatorname{esssup}_{x\in\Omega} F^*(x, Du(x)) c_F(x,y)$$

from where it follows that

$$Lip(u,c) \leq \operatorname{esssup}_{x\in\Omega} F^*(x, Du(x)).$$

Let us now consider $u \in W^{1,\infty}(\Omega)$, then, for a.e. $x \in \Omega$,

$$\begin{aligned} \frac{\langle Du(x); \xi \rangle}{F(x,\xi)} &= \lim_{h\to 0^+} \frac{u(x+h\xi) - u(x)}{F(x,h\xi)} \\ &\leq Lip(u,c_F) \liminf_{h\to 0^+} \frac{c(x,x+h\xi)}{F(x,h\xi)} \\ &\leq Lip(u,c_F) \liminf_{h\to 0^+} \frac{1}{F(x,h\xi)} \int_0^1 F\left(x + th\xi, h\xi\right) dt \\ &= Lip(u,c_F). \end{aligned}$$

Consequently, by (2.3), we get the reverse inequality:

$$\operatorname{esssup}_{x\in\Omega} F^*(x, Du(x)) \leq Lip(u,c_F).$$

This ends the proof. □

Observe that if $F^*(x,\cdot)$ is a norm then, as usual,

$$Lip(u,c_F) = \sup\left\{ \frac{|u(y) - u(x)|}{c_F(x,y)} : x,y \in \Omega,\ x \neq y \right\}.$$

Therefore, we have the following corollary:

Corollary 4.2. *Assume $F^*(x,\cdot)$ is a norm. Then, for $u \in W^{1,\infty}(\Omega)$, we have*

$$F^*(x, Du(x)) \leq 1 \quad a.e.\ in\ \ \Omega \iff |u(x) - u(y)| \leq c_F(x,y).$$

As consequence of Lemma 4.1, we have that the set of functions

$$K_{c_F}(\Omega) = \left\{ u \in W^{1,\infty}(\Omega)\ :\ u(y) - u(x) \leq c_F(x,y) \right\}$$

coincides with the set

$$K_F^*(\Omega) := \left\{ u \in W^{1,\infty}(\Omega)\ :\ \operatorname{esssup}_{x\in\Omega} F^*(x, Du(x)) \leq 1 \right\}.$$

Hence, we have that (1.1) can be written as

$$\min\{\mathcal{K}_{c_F}(\mu) : \mu \in \Pi(f^+, f^-)\} = \sup\left\{ \int_\Omega v(f^- - f^+) : v \in K_F^*(\Omega) \right\}. \tag{4.1}$$

Theorem 4.3. *Any limit u_∞, of a sequence u_{p_j}, is a Kantorovich potential for the optimal transport problem of f^+ to f^- with the cost given by*

$$c_F(x,y) = \inf_{\sigma \in \Gamma^\Omega_{x,y}} \int_0^1 F((\sigma(t)), \sigma'(t))\, dt,$$

that is, the supremum in (4.1) is attained at u_∞.

Moreover, if $F^*(x,\cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$, then there exists $\mathcal{X}_\infty \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ such that

$$\int_\Omega (f^- - f^+) v = \int_{\overline{\Omega}} Dv\, d\mathcal{X}_\infty \quad \forall\, v \in C^1(\overline{\Omega}). \tag{4.2}$$

In particular,

$$-\mathrm{div}(\mathcal{X}_\infty) = f^- - f^+ \quad \textit{in the sense of distributions,} \tag{4.3}$$

Proof. From the proof of Lemma 3.3, for every $v \in K_F^*(\Omega)$, we have

$$\begin{aligned} -\int_\Omega u_p(f^- - f^+) &\leq \int_\Omega \frac{[F^*(x, Du_p(x))]^p}{p} - \int_\Omega u_p(f^- - f^+) \\ &\leq \int_\Omega \frac{[F^*(x, v(x))]^p}{p} - \int_\Omega v(f^- - f^+) \\ &\leq \frac{|\Omega|}{p} - \int_\Omega v(f^- - f^+). \end{aligned}$$

Taking limits as $p_j \to \infty$ we obtain

$$\int_\Omega u_\infty(f^- - f^+) \geq \sup\left\{ \int_\Omega v(f^- - f^+) \,:\, v \in K_F^*(\Omega) \right\}.$$

Then, it only remains to prove that $u_\infty \in K_F^*(\Omega)$. Now, using again (3.7) from the previous computations, we have that

$$\left(\int_\Omega [F^*(x, Du_p(x))]^p \right)^{\frac{1}{p}} \leq (Cp)^{\frac{1}{p-1}}.$$

Then, as above, if take $N < m \leq p$, we get

$$\|F^*(x, Du_p(x))\|_{L^m(\Omega)} \leq (C_1 p)^{\frac{1}{p-1}},$$

the constant C_1 being independent of p. Hence, having in mind that

$$u_{p_j} \rightrightarrows u_\infty \quad \text{uniformly in } \Omega,$$

we can assume that $Du_{p_j} \rightharpoonup Du_\infty$ in $(L^m(\Omega))^N$. Then, by Mazur's Theorem [15, Corollary 3.8], there exists $\lambda_i^j \geq 0$, with $\sum_{i=1}^{k_j} \lambda_i^j = 1$ such that

$$\sum_{i=1}^{k_j} \lambda_i^j Du_{p_i} \to Du_\infty \text{ stronly in } (L^m(\Omega))^N \text{ and a.e. in } \Omega.$$

Then, by the continuity of F^*, we have

$$F^*\left(\cdot, \sum_{i=1}^{k_j} \lambda_i^j Du_{p_i}\right) \to F^*\left(\cdot, Du_\infty\right) \text{ stronly in } L^m(\Omega) \text{ and a.e. in } \Omega.$$

Therefore,

$$\begin{aligned} \|F^*(\cdot, Du_\infty)\|_{L^m(\Omega)} &\leq \liminf_{j\to\infty} \left\| F^*\left(\cdot, \sum_{i=1}^{k_j} \lambda_i^j Du_{p_i}\right)\right\|_{L^m(\Omega)} \\ &\leq \liminf_{j\to\infty} \sum_{i=1}^{k_j} \lambda_i^j \left\|F^*\left(\cdot, Du_{p_i}\right)\right\|_{L^m(\Omega)} \leq \liminf_{j\to\infty} \sum_{i=1}^{k_j} \lambda_i^j (C_1 p_i)^{\frac{1}{p_i - 1}} = 1. \end{aligned}$$

Taking limit as $m \to \infty$, we get that

$$\|F^*(\cdot, Du_\infty)\|_{L^\infty(\Omega)} \leq 1,$$

and we conclude that $u_\infty \in K_F^*(\Omega)$.

Finally, if $F^*(x, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$, since u_p is a weak solution of problem (3.5), if we define

$$\mathcal{X}_p := [F^*(x, Du_p(x))]^{p-1} \frac{\partial F^*}{\partial \xi}(x, Du_p(x)),$$

then

$$\int_\Omega \langle \mathcal{X}_p; Dv\rangle = \int_\Omega (f^- - f^+) v \quad \forall\, v \in W^{1,p}(\Omega). \tag{4.4}$$

Let us see that

$$\{\mathcal{X}_p \ : \ p \geq N\}$$

is bounded in $L^1(\Omega, \mathbb{R}^N)$. In fact, first, taking u_p as test function in (4.4) and having in mind (2.4),we have

$$\int_\Omega [F^*(x, Du_p(x))]^p \, dx \leq C_1, \qquad \forall p > N.$$

Then, by Hölder's inequality, we get

$$\int_\Omega [F^*(x, Du_p(x))]^{p-1} \, dx \leq C_2, \qquad \forall p > N. \tag{4.5}$$

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On the other hand, given $\varphi \in L^\infty(\Omega, \mathbb{R}^N)$, from (2.6), (3.1) and (4.5), we have

$$\begin{aligned}\left|\int_\Omega \langle \mathcal{X}_p; \varphi\rangle\right| &\leq \int_\Omega [F^*(x, Du_p(x))]^{p-1} \left|\left\langle \frac{\partial F^*}{\partial \xi}(x, Du_p(x)); \varphi(x)\right\rangle\right| dx \\ &\leq \beta \|\varphi\|_{L^\infty(\Omega)} \int_\Omega [F^*(x, Du_p(x))]^{p-1}\, dx \\ &\leq C_2 M \|\varphi\|_{L^\infty(\Omega)},\end{aligned}$$

from where it follows that $\{\mathcal{X}_p \ : \ p \geq N\}$ is bounded in $L^1(\Omega, \mathbb{R}^N)$. Therefore, there exists $\mathcal{X}_\infty \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ such that

$$\mathcal{X}_{p_i} \rightharpoonup \mathcal{X}_\infty \quad \text{weakly}^* \text{ as measures in } \overline{\Omega}.$$

Thus, for any $v \in C^1(\overline{\Omega})$, having in mind (4.4), we get

$$\int_\Omega (f^- - f^+)v = -\int_\Omega \operatorname{div}(\mathcal{X}_{p_i})v = \int_\Omega \langle \mathcal{X}_{p_i}; Dv\rangle \to \int_{\overline{\Omega}} Dv d\mathcal{X}_\infty.$$

Hence we have proved (4.2). $\square$

For the next theorem we need to introduce, given a measure, two new measures using the Finsler structure. First, given a measure $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$, we define its total variation respect the Finsler structure F as follows: for an open set $A \subset \overline{\Omega}$, we define

$$|\mathcal{X}|_F(A) :=$$

$$\sup\left\{\int_{\overline{\Omega}} \Phi d\mathcal{X} : \Phi \in C(\overline{\Omega}, \mathbb{R}^N),\ \operatorname{supp}(\Phi) \subset A, \quad \Phi(x) \in B_{F^*(x,\cdot)}\ \forall x \in \Omega\right\}.$$

Lemma 4.4. *The extension of $|\mathcal{X}|_F$ to every Borel set $B \subset \overline{\Omega}$ given by*

$$|\mathcal{X}|_F(B) := \inf\{|\mathcal{X}|_F(A) \ : \ A \text{ open},\ B \subset A\}$$

is a Radon measure in $\overline{\Omega}$.

Proof. By the De Giorgi-Letta Theorem [2, Theorem 1.53], it is enough to show that $|\mathcal{X}|_F$ is subadditive, superadditive and inner regular. In fact, given open sets $A, B \subset \overline{\Omega}$ and $\Phi \in C(\overline{\Omega}, \mathbb{R}^N)$, $\operatorname{supp}(\Phi) \subset A \cup B$, $\Phi(x) \in B_{F^*(x,\cdot)}\ \forall x \in \Omega$, let $\{\eta_i : i = 1, 2, 3\}$ a partition of unity such that $\operatorname{supp}(\eta_1) \subset A$, $\operatorname{supp}(\eta_2) \subset B$ and $\operatorname{supp}(\eta_3) \subset \overline{\Omega} \setminus \operatorname{supp}(\Phi)$. Then,

$$\int_{\overline{\Omega}} \Phi d\mathcal{X} = \int_{\overline{\Omega}} \eta_1 \Phi d\mathcal{X} + \int_{\overline{\Omega}} \eta_2 \Phi d\mathcal{X} + \int_{\overline{\Omega}} \eta_3 \Phi d\mathcal{X} \leq |\mathcal{X}|_F(A) + |\mathcal{X}|_F(B).$$

Hence, taking supremum in Φ, we obtain that

$$|\mathcal{X}|_F(A \cup B) \leq |\mathcal{X}|_F(A) + |\mathcal{X}|_F(B).$$

The other two properties are easy to prove. $\square$

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Since F is non-negative, positively 1–homogenous and convex in the second variable, given $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$, we can also define (see for instance [5], [2]) the measure $F(x, \mathcal{X})$ as

$$\int_B F(x,\mathcal{X}) := \int_B F(x, \mathcal{X}^a(x))dx + \int_B F\left(x, \frac{d\mathcal{X}^s}{d|\mathcal{X}^s|}(x)\right) d|\mathcal{X}^s|$$
$$= \int_B F\left(x, \frac{d\mathcal{X}}{d|\mathcal{X}|}(x)\right) d|\mathcal{X}|,$$

for all Borel set $B \subset \overline{\Omega}$, being $\mathcal{X} = \mathcal{X}^a + \mathcal{X}^s$ the Lebesgue decomposition of $\mathcal{X}$, and $\frac{d\mathcal{X}}{d|\mathcal{X}|}$ the Radon-Nikodym derivative of $\mathcal{X}$ respect to $|\mathcal{X}|$. Since $|\mathcal{X}|$ is absolutely continuous respect to the measure $|\mathcal{X}|_F$, by [2, Proposition 2.37], we have

$$\int_B F(x,\mathcal{X}) = \int_B F\left(x, \frac{d\mathcal{X}}{d|\mathcal{X}|_F}(x)\right) d|\mathcal{X}|_F \quad \text{for all Borel set } B \subset \overline{\Omega}. \tag{4.6}$$

Having in mind (4.6) and following the proof of the continuity Reshetnyak Theorem given in [30], we get the following result.

Lemma 4.5. *Let $\mathcal{X}_n, \mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ such that*

$$\mathcal{X}_n \rightharpoonup \mathcal{X} \quad \text{in } \mathcal{M}(\overline{\Omega}, \mathbb{R}^N) \quad \text{and} \quad |\mathcal{X}_n|_F(\overline{\Omega}) \to |\mathcal{X}|_F(\overline{\Omega}).$$

Then

$$\lim_{n\to\infty} \int_{\overline{\Omega}} F(x, \mathcal{X}_n) = \int_{\overline{\Omega}} F(x, \mathcal{X}).$$

We will also use the following approximation result. For this result, and from now on, we will assume that F is a continuous Finsler structure in a larger domain, $F : \Omega' \times \mathbb{R}^N \to [0, \infty[$ with $\Omega \subset\subset \Omega'$, which is not a loss of generality since we are assuming that $\mathrm{supp}(f^+) \cup \mathrm{supp}(f^-) \subset\subset \Omega$.

Lemma 4.6. *For any $u \in W^{1,\infty}(\Omega)$, such that $Du(x) \in B_{F^*(x,\cdot)}$ a.e. $x \in \Omega$, there exists $u_\epsilon \in C^1(\overline{\Omega})$, such that $u_\epsilon \to u$ uniformly in any compact subset K of Ω, and*

$$\limsup_{\epsilon\to 0} \sup_{\overline{\Omega}} F^*(x, Du_\epsilon(x)) \leq 1.$$

Proof. Since $Du(x) \in B_{F^*(x,\cdot)}$ a.e. $x \in \Omega$, we can take the McShane-Whitney extension

$$\overline{u}(x) := \inf_{y\in\Omega} \{u(y) + c_F(y, x)\}, \quad x \in \Omega',$$

and then we have that $\overline{u}(x) - \overline{u}(y) \leq c_F(y, x)$. Let $u_\epsilon = \overline{u} * \rho_\epsilon \in C^1(\overline{\Omega})$ (we can extend $\overline{u}$ as zero outside Ω'). Then $u_\epsilon \to u$ uniformly in any compact subset K of Ω. On the other hand, by continuity, there exists $x_\epsilon \in \overline{\Omega}$, such that

$$\sup_{\overline{\Omega}} F^*(x, Du_\epsilon(x)) = F^*(x_\epsilon, Du_\epsilon(x_\epsilon)).$$

By Lemma 4.1, $\operatorname{esssup}_{x\in\Omega'} F^*(x, D\overline{u}(x)) \leq 1$. Then, by Jensen's inequality, we have, for ϵ small,

$$\begin{aligned}
&F^*(x_\epsilon, Du_\epsilon(x_\epsilon)) \leq \int_{\mathbb{R}^N} F^*(x_\epsilon, D\overline{u}(y))\rho_\epsilon(x_\epsilon - y)dy \\
&= \int_{\mathbb{R}^N} F^*(x_\epsilon, D\overline{u}(y))\rho_\epsilon(x_\epsilon - y)dy - \int_{\mathbb{R}^N} F^*(y, D\overline{u}(y))\rho(x_\epsilon - y)dy \\
&\qquad + \int_{\mathbb{R}^N} F^*(y, D\overline{u}(y))\rho_\epsilon(x_\epsilon - y)dy \\
&\leq \int_{\mathbb{R}^N} \Big(F^*(x_\epsilon, D\overline{u}(y)) - F^*(y, D\overline{u}(y))\Big)\rho_\epsilon(x_\epsilon - y)dy + 1 \ .
\end{aligned}$$

Now, there exists a subsequence such that $x_{\epsilon_n} \to x_0$, and, for this subsequence,

$$\int_{\mathbb{R}^N} \Big(F^*(x_{\epsilon_n}, D\overline{u}(y)) - F^*(y, D\overline{u}(y))\Big)\rho_{\epsilon_n}(x_{\epsilon_n} - y)dy \to 0$$

as $n \to +\infty$. $\square$

Theorem 4.7. *Let u_∞ and $\mathcal{X}_\infty$ be as in Theorem 4.3. Then,*

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) = \int_{\overline{\Omega}} F(x, \mathcal{X}_\infty) = \int_\Omega u_\infty(f^- - f^+). \tag{4.7}$$

Proof. Let v_ϵ be the approximation given in Lemma 4.6 for $u = u_\infty$, then

$$\begin{aligned}
&\int_\Omega (f^- - f^+)u_\infty \, dx = \lim_{\epsilon\to 0} \int_\Omega (f^- - f^+)v_\epsilon \, dx = \lim_{\epsilon\to 0} \int_{\overline{\Omega}} Dv_\epsilon d\mathcal{X}_\infty \\
&\qquad \leq \limsup_{\epsilon\to 0} \sup_{\overline{\Omega}} F^*(x, Dv_\epsilon(x))|\mathcal{X}_\infty|_F(\overline{\Omega}) \leq |\mathcal{X}_\infty|_F(\overline{\Omega}).
\end{aligned} \tag{4.8}$$

Let now $\Phi \in C(\overline{\Omega}, \mathbb{R}^N)$ with $\Phi(x) \in B_{F^*(x,\cdot)}$ for all $x \in \Omega$. By (2.1), we have

$$\int_\Omega \Phi\mathcal{X}_{p_i} dx \leq \int_\Omega F^*(x, \Phi(x))F(x, \mathcal{X}_{p_i}(x))dx \leq \int_\Omega F(x, \mathcal{X}_{p_i}(x))dx. \tag{4.9}$$

Therefore

$$\int_{\overline{\Omega}} \Phi d\mathcal{X}_\infty = \lim_i \int_\Omega \Phi\mathcal{X}_{p_i} \leq \limsup_i \int_\Omega F(x, \mathcal{X}_{p_i}(x))dx,$$

and, taking supremum in Φ,

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) \leq \limsup_i \int_\Omega F(x, \mathcal{X}_{p_i}(x))dx. \tag{4.10}$$

Now, applying Hölder's inequality, (2.8), (2.4) and (4.4),we get

$$\begin{aligned}
&\limsup_{i\to\infty} \int_\Omega F(x,\mathcal{X}_{p_i}(x))dx \\
&= \limsup_{i\to\infty} \int_\Omega [F^*(x,Du_{p_i}(x))]^{p_i-1}\, F\left(x, \frac{\partial F^*}{\partial \xi}(x,Du_{p_i}(x))\right) dx \\
&\le \limsup_{i\to\infty} \left(\int_\Omega [F^*(x,Du_{p_i}(x))]^{p_i}\, dx\right)^{\frac{p_i-1}{p_i}} \\
&= \limsup_{i\to\infty} \left(\int_\Omega [F^*(x,Du_{p_i}(x))]^{p_i-1} \left\langle \frac{\partial F^*}{\partial \xi}(x,Du_{p_i}(x)); Du_{p_i}(x)\right\rangle dx\right)^{\frac{p_i-1}{p_i}} \\
&= \limsup_{i\to\infty} \left(\int_\Omega \langle \mathcal{X}_{p_i}; Du_{p_i}\rangle\right)^{\frac{p_i-1}{p_i}} = \lim_{i\to\infty} \int_\Omega \langle \mathcal{X}_{p_i}; Du_{p_i}\rangle \\
&= \lim_{i\to\infty} \int_\Omega (f^- - f^+)u_{p_i} = \int_\Omega (f^- - f^+)u_\infty,
\end{aligned}$$

that is,

$$\limsup_{i\to\infty} \int_\Omega F(x,\mathcal{X}_{p_i}(x))dx \le \int_\Omega (f^- - f^+)u_\infty. \tag{4.11}$$

Then, by (4.8), (4.10) and (4.11),

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) = \int_\Omega (f^- - f^+)u_\infty dx. \tag{4.12}$$

Let us see now that

$$|\mathcal{X}_{p_i}|_F(\overline{\Omega}) \to |\mathcal{X}_\infty|_F(\overline{\Omega}). \tag{4.13}$$

By (4.9), taking supremum in Φ,

$$|\mathcal{X}_{p_i}|_F(\overline{\Omega}) \le \int_\Omega F(x,\mathcal{X}_{p_i}).$$

Then, by (4.11) and (4.12), we get

$$\limsup_{i\to\infty} |\mathcal{X}_{p_i}|_F(\overline{\Omega}) \le \limsup_{i\to\infty} \int_\Omega F(x,\mathcal{X}_{p_i}) = \int_\Omega (f^- - f^+)u_\infty = |\mathcal{X}_\infty|_F(\overline{\Omega}).$$

On the other hand, given $\Phi \in C(\overline{\Omega},\mathbb{R}^N)$ with $\Phi(x) \in B_{F^*(x,\cdot)}$ for all $x \in \Omega$,

$$\int_\Omega \Phi\mathcal{X}_{p_i} \le |\mathcal{X}_{p_i}|_F(\overline{\Omega}),$$

then

$$\int_\Omega \Phi\mathcal{X}_\infty \le \liminf_i |\mathcal{X}_{p_i}|_F(\overline{\Omega}),$$

and from here, we get that

$$|\mathcal{X}_\infty|_F(\overline{\Omega}) \le \liminf_i |\mathcal{X}_{p_i}|_F(\overline{\Omega}),$$

and the proof of (4.13) is finished.

Finally, since $\mathcal{X}_{p_i} \rightharpoonup \mathcal{X}_\infty$ in $\mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ and we have (4.13), by Lemma 4.5, we get

$$\int_{\overline{\Omega}} F(x, \mathcal{X}_\infty) = \lim_{n\to\infty} \int_{\overline{\Omega}} F(x, \mathcal{X}_{p_i}) = \int_\Omega (f^- - f^+) u_\infty. \quad \square$$

Let us see now that $F(x, \mathcal{X}_\infty)$ is the transport density of the transport problem we are dealing with. To do that we need to recall the concept of tangential derivative respect a Radon measure (see for instance, [12], [13] or [11]). Given $\mu \in \mathcal{M}(\overline{\Omega})^+$, we define

$$\mathcal{N} := \left\{ \begin{array}{c} \xi \in L^\infty_\mu(\overline{\Omega}, \mathbb{R}^N) \ : \ \exists u_n \in C^\infty(\overline{\Omega}), \ u_n \to 0 \text{ uniformly}, \\ Du_n \rightharpoonup \xi \text{ in } \sigma(L^\infty_\mu, L^1_\mu) \end{array} \right\}.$$

The orthogonal of $\mathcal{N}$ in $L^1_\mu(\overline{\Omega}, \mathbb{R}^N)$ is characterized in [13] as

$$\mathcal{N}^\perp = \left\{ \sigma \in L^1_\mu(\overline{\Omega}) \ : \ \sigma(x) \in T_\mu(x) \ \ \mu - a.e \right\},$$

where T_μ is a closed valued μ-measurable multifunction, that is called the *tangent space* to the measure μ. For a function $u \in C^1(\overline{\Omega})$, its *tangential gradient* $D_\mu u(x)$ is defined as the projection $P_\mu(x) Du(x)$ on $T_\mu(x)$. In [13] it is proved that the linear operator $u \in C^1(\overline{\Omega}) \mapsto D_\mu u \in L^\infty_\mu(\overline{\Omega}, \mathbb{R}^N)$ can be extended in a unique way as a linear continuous operator

$$D_\mu : Lip(\overline{\Omega}) \to L^\infty_\mu(\overline{\Omega}, \mathbb{R}^N),$$

where $Lip(\overline{\Omega})$ is equipped with the uniform convergence and $L^\infty_\mu(\overline{\Omega}, \mathbb{R}^N)$ with the weak topology. Consequently, there exists $v_\epsilon \in \mathcal{C}^1(\overline{\Omega})$ such that

$$\left\{ \begin{array}{l} v_\epsilon \to u_\infty \quad \text{uniformly} \\ \\ D_\mu v_\epsilon \rightharpoonup D_\mu u_\infty \quad \sigma(L^\infty_\mu, L^1_\mu). \end{array} \right. \tag{4.14}$$

Following [28], given $u \in W^{1,\infty}(\Omega)$, we define the *$\mu$-tangential gradient* of u respect to F in the following form: for $x \in \Omega$ such that there exists $D_\mu u(x)$, we define

$$\partial_{F,\mu} u(x) := \left\{ \frac{D_\mu u(x) \cdot \hat{v}}{F(x, \hat{v})^2} \hat{v} \ : \ \hat{v} \in \underset{\substack{v \in T_\mu(x) \\ |v| = 1}}{\operatorname{argmax}} \frac{D_\mu u(x) \cdot v}{F(x, v)} \right\}.$$

In case $F(x, \cdot)$ is strictly convex, then there is a unique maximum

$$\hat{v} \in \operatorname{argmax} \left\{ \frac{D_\mu u(x) \cdot v}{F(x, v)} \ : \ v \in T_\mu(x), \ |v| = 1 \right\},$$

and consequently $\partial_{F,\mu} u(x)$ has a unique element that we denote by $\nabla_{F,\mu} u(x)$ that is called the *μ-tangential gradient* of u at x respect to F, that is,

$$\nabla_{F,\mu} u(x) = \frac{D_\mu u(x) \cdot \hat{v}}{F(x, \hat{v})^2} \hat{v}.$$

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Observe that

$$\partial_{F,\mu} u(x) = \left\{ (D_\mu u(x) \cdot \hat{v})\hat{v} \, : \, \hat{v} \in \underset{\substack{v \in T_\mu(x) \\ F(x,v) = 1}}{\operatorname{argmax}} \; D_\mu u(x) \cdot v \right\}.$$

Theorem 4.8. *Let u_∞ and $\mathcal{X}_\infty$ be as in Theorem 4.3. If we set $\mu := F(x, \mathcal{X}_\infty)$, then*

$$\begin{cases} \displaystyle\int_\Omega (f^- - f^+) v = \int_{\overline{\Omega}} \frac{d\mathcal{X}_\infty}{d\mu} \cdot Dv \, d\mu \quad \forall \, v \in C^1(\overline{\Omega}), \\[2ex] \displaystyle\frac{d\mathcal{X}_\infty}{d\mu}(x) \in \partial_{F,\mu} u_\infty(x) \quad \text{and} \quad F\left(x, \frac{d\mathcal{X}_\infty}{d\mu}(x)\right) = 1 \quad \mu - a.e. \ in \ \overline{\Omega}. \end{cases}$$

Moreover, if $F(x,.)$ is strictly convex, then

$$\begin{cases} \displaystyle\int_\Omega (f^- - f^+) v = \int_{\overline{\Omega}} \nabla_{F,\mu} u_\infty \cdot Dv \, d\mu \quad \forall \, v \in C^1(\overline{\Omega}), \\[2ex] F(x, \nabla_{F,\mu} u_\infty(x)) = 1 \quad \mu - a.e. \ in \ \overline{\Omega} \end{cases}$$

Proof. Since $\mathcal{X}_\infty$ is absolutely continuos respect to μ, we have the Radon-Nikodym derivative $\frac{d\mathcal{X}_\infty}{d\mu} \in L^1_\mu(\overline{\Omega}, \mathbb{R}^N)$. On the other hand, by (4.3),

$$-\text{div}\left(\mu \frac{d\mathcal{X}_\infty}{d\mu}\right) = f^- - f^+ \quad \text{in the sense of distributions.}$$

Then, from [13, Proposition 3.5], it follows that

$$\frac{d\mathcal{X}_\infty}{d\mu}(x) \in T_\mu(x) \quad \mu - \text{a.e.} \tag{4.15}$$

We claim now that

$$D_\mu u_\infty(x) \cdot v(x) \leq F\left(x, v(x)\right) \quad \mu - \text{a.e.} \tag{4.16}$$

for any $v(x) \in T_\mu(x) \quad \mu - \text{a.e.}$.

Let u_ϵ be the functions given in Lemma 4.6. Then, by (2.1), if $v(x) \in T_\mu(x) \quad \mu - \text{a.e.}$, we have

$$D_\mu u_\epsilon(x) \cdot v(x) = Du_\epsilon(x) \cdot v(x) \leq F^*(x, Du_\epsilon(x)) F\left(x, v(x)\right),$$

for μ-almost all x. By contradiction, if (4.16) does not hold, then the set $A := \{\in \overline{\Omega} \ : \ D_\mu u_\infty(x) \cdot v(x) > F\left(x, v(x)\right)\}$ has positive μ-measure. Now, integrating in the above inequality and taking limits as $\epsilon \to 0$, we get

$$\int_A D_\mu u_\infty(x) \cdot v(x) \, d\mu(x) \leq \int_A F\left(x, v(x)\right) \, d\mu(x),$$

which is a contradiction, and therefore (4.16) holds.

From (4.16) and (4.15), we can write

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$$D_\mu u_\infty(x) \cdot \frac{d\mathcal{X}_\infty}{d\mu}(x) \leq F\left(x, \frac{d\mathcal{X}_\infty}{d\mu}(x)\right) \quad \mu - \text{a.e.} \tag{4.17}$$

Now, since

$$F\left(x, \frac{d\mathcal{X}_\infty}{d\mu}(x)\right) = 1 \quad \mu - \text{a.e.},$$

inequality (4.17) reads as

$$D_\mu u_\infty \cdot \frac{d\mathcal{X}_\infty}{d\mu} \leq 1 \quad \mu - \text{a.e.} \tag{4.18}$$

Now, taking v_ϵ as in (4.14) and having in mind (4.15), we get

$$\int_{\overline{\Omega}} D_\mu v_\epsilon \frac{d\mathcal{X}_\infty}{d\mu} d\mu = \int_{\overline{\Omega}} D v_\epsilon d\mathcal{X}_\infty = \int_\Omega (f^- - f^+) v_\epsilon.$$

Therefore, taking limits as $\epsilon \to 0$, we obtain that

$$\int_{\overline{\Omega}} D_\mu u_\infty \frac{d\mathcal{X}_\infty}{d\mu} d\mu = \int_\Omega (f^- - f^+) u_\infty = \int_{\overline{\Omega}} d\mu,$$

where the last equality is a consequence of (4.7). Then, by (4.18),

$$D_\mu u_\infty \cdot \frac{d\mathcal{X}_\infty}{d\mu} = 1 \quad \mu - \text{a.e.} \tag{4.19}$$

On account of (4.16) and (4.19), we have

$$\frac{d\mathcal{X}_\infty}{d\mu}(x) \in \operatorname{argmax}\left\{ D_\mu u_\infty(x) \cdot v \ : \ v \in T_\mu(x),\ F(x,v) = 1 \right\},$$

and consequently

$$\frac{d\mathcal{X}_\infty}{d\mu}(x) \in \partial_{F,\mu} u_\infty(x).$$

Assuming that $F(x,\cdot)$ is strictly convex, then we have

$$\frac{d\mathcal{X}_\infty}{d\mu}(x) = \nabla_{F,\mu} u_\infty(x),$$

and the proof concludes. □

Corollary 4.9. *Let u_∞ and $\mathcal{X}_\infty$ be as in Theorem 4.3. If in addition we assume that*

$$F^*(x, D_\mu u_\infty(x)) \leq 1 \quad \mu - a.e. \ in \ \overline{\Omega}, \tag{4.20}$$

then

$$\begin{cases} \displaystyle\int_\Omega (f^- - f^+) v = \int_{\overline{\Omega}} \frac{\partial F^*}{\partial \xi}(., D_\mu u_\infty) \cdot Dv \, d\mu \quad \forall\, v \in C^1(\overline{\Omega}), \\ \\ F^*(x, D_\mu u_\infty(x)) = 1 \quad \mu - a.e. \ in \ \overline{\Omega}, \end{cases} \tag{4.21}$$

Proof. Since

$$1 = D_\mu u_\infty \cdot \frac{d\mathcal{X}_\infty}{d\mu} \le F^*\left(x, D_\mu u_\infty(x)\right) \;\; \mu - \text{a.e.},$$

by (4.20), we have that in fact,

$$F^*(x, D_\mu u(x)) = 1 \quad \mu - \text{a.e.} \tag{4.22}$$

On the other hand,

$$D_\mu u_\infty \cdot \frac{d\mathcal{X}_\infty}{d\mu} = 1 = F\left(x, \frac{d\mathcal{X}_\infty}{d\mu}(x)\right) \quad \mu - \text{a.e.} \tag{4.23}$$

Now, having in mind (4.23) and (4.22), applying (2.7), we deduce that

$$\begin{aligned}\frac{d\mathcal{X}_\infty}{d\mu}(x) &= F\left(x, \frac{d\mathcal{X}_\infty}{d\mu}(x)\right)\frac{\partial F^*}{\partial \xi}(x, D_\mu u_\infty(x)) \\ &= \frac{\partial F^*}{\partial \xi}(x, D_\mu u_\infty(x)) \quad \mu - \text{a.e..}\end{aligned}$$

Then, by the above theorem we get (4.21). □

Remark 4.10. If $F(x,\xi) = |A(x)\xi|$ with $A(x)$ a symmetric matrix, positive definite, then

$$D_\mu u_\infty(x) \in B_{F^*(x,\cdot)}, \;\; \mu - \text{a.e. in } \overline{\Omega}.$$

In fact, we have $F^*(x,\xi) = |A(x)^{-1}\xi|$, then, since $A(x)^{-1}$ preserves the orthogonality, by the Pythagoras Theorem, we have

$$|A(x)^{-1}Du_\infty(x)|^2 = |A(x)^{-1}D_\mu u_\infty(x)|^2 + |A(x)^{-1}(Du_\infty(x) - D_\mu u_\infty(x))|^2.$$

Therefore,

$$F^*(x, D_\mu u_\infty(x)) \le F^*(x, Du_\infty(x)) \le 1.$$

Let us remark that in this case, it is known that, in fact, μ is absolutely continuous w.r.t. the Lebesgue measure (see [28] and [18]), and then $D_\mu u_\infty = Du_\infty$.

In the case $F(\cdot, \mathcal{X}_\infty(\cdot)) \in L^1(\Omega)$, we can write the following result.

Corollary 4.11. *Let u_∞ and $\mathcal{X}_\infty$ be as in Theorem 4.3. If $F(\cdot, \mathcal{X}_\infty(\cdot)) \in L^1(\Omega)$, then*

$$\text{for almost every } x, \;\; F(x, \mathcal{X}_\infty(x)) > 0 \text{ implies } F^*(x, Du_\infty(x)) = 1, \tag{4.24}$$

and

$$\begin{aligned}\int_\Omega F(x, \mathcal{X}_\infty(x))\left\langle \frac{\partial F^*}{\partial \xi}(x, Du_\infty(x)); Dv(x)\right\rangle dx \\ = \int_\Omega (f^-(x) - f^+(x))v(x)\, dx\end{aligned}$$

for all $v \in C^1(\overline{\Omega})$; in particular,

$$-\text{div}\left(F(\cdot, \mathcal{X}_\infty(\cdot))\frac{\partial F^*}{\partial \xi}(\cdot, Du_\infty)\right) = (f^- - f^+) \tag{4.25}$$

holds in the sense of distributions. And

$$\int_\Omega F(x, \mathcal{X}_\infty(x))\, dx = \int_\Omega u_\infty(x)(f^-(x) - f^+(x))\, dx. \tag{4.26}$$

Remark 4.12. Let us give an interpretation of equation (4.25) in terms of the Finsler manifold (Ω, F). For that we need to recall the concept of gradient vector in a Finsler manifold (see, for example, [27]). Let us suppose that $\frac{1}{2}F^2(x,\cdot)$ is differentiable for $\xi \neq 0$. Let $J : \Omega \times \mathbb{R}^N \to \mathbb{R}^N$ be the transfer map of the Finsler structure F defined in $\alpha \in \mathbb{R}^N$ as the unique maximizer of the function $\xi \mapsto \langle \alpha, \xi\rangle - \frac{1}{2}F^2(x,\xi)$. The vector $J(x,\alpha)$ can be given by

$$J(x,\alpha) = F^*(x,\alpha)\frac{\partial F^*}{\partial \xi}(x,\alpha).$$

The gradient vector in the Finsler manifold (Ω, F) of a smooth function $u : \Omega \to \mathbb{R}$ is defined as

$$\nabla u(x) := J(x, Du(x)) = F^*(x, Du(x))\frac{\partial F^*}{\partial \xi}(x, Du(x)).$$

Let us remark that the gradient vector ∇u coincides with $\nabla_{F,\mu} u$ when μ is absolutely continuous with respect to the Lebesgue measure.

Then, by (4.24), let us call $a(x) = F(x, \mathcal{X}_\infty)$, we have

$$a(x)\nabla u_\infty(x) = a(x)\frac{\partial F^*}{\partial \xi}(x, Du_\infty(x)).$$

Therefore, we can write equation (4.25) as

$$-\mathrm{div}\,(a\nabla u_\infty) = f^- - f^+ \quad \text{in the sense of distributions,}$$

with $\mathrm{esssup}_{x\in\Omega} F(x, \nabla u_\infty(x)) \leq 1$. Moreover,

$$\text{for almost every } x,\ a(x) > 0 \text{ implies } F(x, \nabla u_\infty(x)) = 1.$$

Indeed, by (2.8),

$$\begin{aligned} F(x, \nabla u_\infty(x)) &= F\left(x, F^*(x, Du_\infty(x))\frac{\partial F^*}{\partial \xi}(x, Du_\infty(x))\right) \\ &= F^*(x, Du_\infty(x))F\left(x, \frac{\partial F^*}{\partial \xi}(x, Du_\infty(x))\right) = F^*(x, Du_\infty(x)), \end{aligned}$$

and, by (4.24), we have that $F(x, \nabla u_\infty(x)) = F^*(x, Du_\infty(x)) = 1$ for almost every x such that $a(x) > 0$.

We have been dealing with a mass transport problem in the Finsler metric space (Ω, F, dx), with a quite general Finsler structure F, for the distance induced by such structure. This general structure includes the case $F(x,\xi) = \Phi(A(x)\xi)$, with Φ a Finsler function and $A(x)$ a symmetric $N \times N$ matrix, positive definite, that depends smoothly on x, in particular the Riemannian structures $F(x,\xi) = |A(x)\xi|$ with $|\cdot|$ the Euclidean norm. Let us see how these results can be interpreted in the context of optimal transportation on Riemannian manifolds.

4.2. **Example.** In the particular case in which $F(x,\xi) = |A(x)\xi|$ with $|\cdot|$ the Euclidean norm and with $A(x)$ a symmetric $N \times N$ matrix, positive definite, that depends smoothly on $x \in \overline{\Omega}$, we have

$$\begin{aligned} c_F(x,y) &= \inf_{\sigma \in \Gamma^{\Omega}_{x,y}} \int_0^1 \sqrt{\langle A(\sigma(t))\sigma'(t); A(\sigma(t))\sigma'(t)\rangle dt} \\ &= \inf_{\sigma \in \Gamma^{\Omega}_{x,y}} \int_0^1 \sqrt{\langle A^2(\sigma(t))\sigma'(t); \sigma'(t)\rangle dt}. \end{aligned}$$

Therefore, writing $A^2(z) = (g_{i,j}(z))_{i,j} =: g(z)$, the cost function c is given by

$$c_F(x,y) = d_{g,\Omega}(x,y) := \inf_{\sigma \in \Gamma^{\Omega}_{x,y}} \int_0^1 \sqrt{\sum_{i,j} g_{i,j}(\sigma(t))\sigma_i'(t)\sigma_j'(t)dt}.$$

That is, in this case the cost function c is the distance induced by the metric tensor g.

When $A(z) = b(z)I_N$ (here I_N denotes the $N \times N$ identity matrix), we have that the cost is given by

$$c_F(x,y) = \inf_{\sigma \in \Gamma^{\Omega}_{x,y}} \int_{\sigma} b(z)ds.$$

This case has been studied in [25].

The results obtained can be interpreted in the context of optimal transportation on Riemannian manifolds with cost function the distance induced by the metric tensor. Let us illustrate this with the following example.

N-dimensional parameterized manifolds in $\mathbb{R}^M$. Let $\mathcal{S}$ be a N-dimensional parameterized manifold in $\mathbb{R}^M$ $(M \geq N)$, that is $\mathcal{S} = \psi(\Omega)$, where Ω is an open bounded set of $\mathbb{R}^N$ and $\psi : \Omega \to \mathbb{R}^M$ is a smooth map such that for each $x \in \Omega$, the $M \times N$ Jacobian matrix $J_\psi(x)$ has rank N. We denote by g the metric tensor $g := J_\psi^t \cdot J_\psi$ and by $|g|$ the determinant of g. Consider in $\mathcal{S}$ the Riemaniann distance induced by the the Euclidean distance in $\mathbb{R}^M$, i.e,

$$d_{I_M,\mathcal{S}}(\xi,\eta) = \inf_{\sigma \in \Gamma^{\mathcal{S}}_{\xi,\eta}} \int_0^1 |\sigma'(t)|\, dt,$$

where I_M is the $M \times M$ identity matrix.

One can think for example on $\mathcal{S}$ the sphere of radius R in $\mathbb{R}^3$, parameterized by $\psi :]0,2\pi[\times]0,\pi[\to \mathbb{R}^3$ given by

$$\psi(\theta,\phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi),$$

which is a non Euclidean Riemann manifold with metric g defined by

$$g(\theta,\phi)=\begin{pmatrix} R^2\sin^2\phi & 0 \\ 0 & R^2 \end{pmatrix}.$$

Suppose we have two functions $\tilde f^{\pm}\in L^1(\mathcal{S},dvol)$ both with equal mass

$$\begin{aligned}\int_{\mathcal{S}}\tilde f^+(z)\,dvol(z)&=\int_\Omega\sqrt{|g|(x)}\tilde f^+(\psi(x))dx\\ =\int_{\mathcal{S}}\tilde f^-(z)\,dvol(z)&=\int_\Omega\sqrt{|g|(x)}\tilde f^-(\psi(x))dx,\end{aligned}$$

and we want to transport $\tilde f^+$ to $\tilde f^-$ on $\mathcal{S}$ with cost function the distance $d_{I_M,\mathcal{S}}$. If we take

$$f^{\pm}(x)=\sqrt{|g|(x)}\tilde f^{\pm}(\psi(x)),$$

we have

$$\int_\Omega f^+(x)dx=\int_\Omega f^-(x)dx.$$

A simple calculation shows that

$$d_{I_M,\mathcal{S}}(\xi,\eta)=d_{g,\Omega}(\psi^{-1}(\xi),\psi^{-1}(\eta))\quad\text{for all}\quad \xi,\eta\in\mathbb{R}^M. \tag{4.27}$$

Moreover, if $\tilde T\#\tilde f^+=\tilde f^-$ and $T:=\psi^{-1}\circ\tilde T\circ\psi$, then $T\#f^+=f^-$ and

$$\begin{aligned}&\int_{\mathcal{S}}d_{I_M,\mathcal{S}}(\xi,\tilde T(\xi))\tilde f^+(\xi)\,dvol(\xi)\\ &\qquad=\int_\Omega\sqrt{|g|(x)}d_{g,\Omega}(x,\psi^{-1}(\tilde T(\psi(x))\tilde f^+(\psi(x))dx\\ &\qquad=\int_\Omega d_{g,\Omega}(x,T(x))f^+(x)dx.\end{aligned}$$

Similarly, if $T\#f^+=f^-$ and $\tilde T:=\psi\circ T\circ\psi^{-1}$, then $\tilde T\#\tilde f^+=\tilde f^-$ and

$$\int_\Omega d_{g,\Omega}(x,T(x))f^+(x)dx=\int_{\mathcal{S}}d_{I_M,\mathcal{S}}(\xi,\tilde T(\xi))\tilde f^+(\xi)\,dvol(\xi).$$

Therefore, for the Monge problems, we have

$$\begin{aligned}&\min_{\tilde T\#\tilde f^+=\tilde f^-}\left\{\int_{\mathcal{S}}d_{I_M,\mathcal{S}}(\xi,\tilde T(\xi))\tilde f^+(\xi)\,dvol(\xi)\right\}\\ &\qquad=\min_{T\#f^+=f^-}\left\{\int_\Omega d_{g,\Omega}(x,T(x))f^+(x)dx\right\}.\end{aligned}$$

Consider now the Kantorovich potential u_∞ obtained in Theorem 4.3 for $F^*(x,\xi) = |A^{-1}(x)\xi|$, A an square root of g and the masses $f^\pm$. Then,

$$\begin{aligned}
&\sup\left\{\int_\Omega v(x)(f^-(x)-f^+(x))dx : v \in K_{d_{g,\Omega}}(\Omega)\right\}\\
&\quad = \int_\Omega u_\infty(x)(f^-(x)-f^+(x))dx\\
&\quad = \int_\Omega u_\infty(\psi^{-1}(\psi(x)))(\sqrt{|g|(x)}\tilde{f}^-(\psi(x)) - \sqrt{|g|(x)}\tilde{f}^+(\psi(x)))dx\\
&\quad = \int_{\mathcal{S}} u_\infty(\psi^{-1}(z))(\tilde{f}^-(z)-\tilde{f}^+(z))\,dvol(z)\,.
\end{aligned}$$

On the other hand, by (4.27), it is easy to see that

$$v \in K_{d_{g,\Omega}}(\Omega) \iff v(\psi^{-1}(z)) \in K_{d_{I_M,\mathcal{S}}}(\mathcal{S}).$$

Thus,

$$\begin{aligned}
&\sup\left\{\int_\Omega v(x)(f^-(x)-f^+(x))dx : v \in K_{d_{g,\Omega}}(\Omega)\right\}\\
&\qquad = \sup\left\{\int_{\mathcal{S}} w(z)(\tilde{f}^-(z)-\tilde{f}^+(z))\,dvol(z) : w \in K_{d_{I_M,\mathcal{S}}}(M)\right\}.
\end{aligned}$$

Consequently, for $\tilde{u}_\infty(z) := u_\infty(\psi^{-1}(z))$,

$$\begin{aligned}
&\int_{\mathcal{S}} \tilde{u}_\infty(z)(\tilde{f}^-(z)-\tilde{f}^+(z))\,dvol(z)\\
&\qquad = \sup\left\{\int_{\mathcal{S}} w(z)(\tilde{f}^-(z)-\tilde{f}^+(z))\,dvol(z) : w \in K_{d_{I_M,\mathcal{S}}}(\mathcal{S})\right\},
\end{aligned}$$

and $\tilde{u}_\infty$ is a Kantorovich potential for the transport of $\tilde{f}^+$ to $\tilde{f}^-$ on the manifold $\mathcal{S}$ with respect to the Riemaniann distance $d_{I_M,\mathcal{S}}$.

When $N = M$ we are considering a change of variables. In this case, $\sqrt{|g(x)|} = |J_\psi(x)|$. Now a square root A of g can be J_ψ between others.

Corollary 4.11 reads now as follows. Let us call the transport density $F(x, \mathcal{X}_\infty(x))$ as $a(x)$. Then

$$-\mathrm{div}(ag^{-1}Du_\infty) = f^- - f^+ \quad \text{in } \Omega, \tag{4.28}$$

and

$$\text{for a.e. } x,\ a(x) > 0 \text{ implies } \langle g^{-1}(x)Du_\infty(x); Du_\infty(x)\rangle = 1. \tag{4.29}$$

If we define

$$\tilde{a} := \frac{a}{\sqrt{|g|}} \circ \psi^{-1},$$

from (4.26), we have

$$\begin{aligned}
&\int_{\mathcal{S}} \tilde{u}_\infty(z)(\tilde{f}^-(z) - \tilde{f}^+(z))dS \\
&\qquad = \int_\Omega \sqrt{|g|(x)}\tilde{u}_\infty(\psi(x))(\tilde{f}^-(\psi(x)) - \tilde{f}^+(\psi(x)))dx \\
&\qquad = \int_\Omega u_\infty(x)(f^-(x) - f^+(x)\,dx = \int_\Omega a(x)\,dx = \int_{\mathcal{S}} \tilde{a}(z)\,dvol(z).
\end{aligned}$$

Recall that $w \in W^{1,\infty}(\mathcal{S})$ if $w \circ \psi \in W^{1,\infty}(\Omega)$. For $w \in W^{1,\infty}(\mathcal{S})$, the gradient of w at $z \in \mathcal{S}$ is denoted by $\nabla w(z) \in T_z\mathcal{S}$ and is defined, for $v \in T_z\mathcal{S}$, as

$$\langle \nabla w(z), v\rangle = \frac{d}{dt}(w\circ\alpha)_{|t=0},$$

where $\alpha :]-\epsilon,\epsilon[\to \mathcal{S}$ is a smooth path such that $\alpha(0) = z$ and $\alpha'(0) = v$. Then, we have

$$\langle \nabla w(\psi(x)), J_\psi(x)u\rangle = \langle D(w\circ\psi)(x), u\rangle \quad \text{for all } x\in\Omega,\ u\in\mathbb{R}^N. \tag{4.30}$$

In fact, if we defined $\alpha(t) := \psi(x+tu) = (\psi\circ r)(t)$, applying the change rule, we have

$$\langle \nabla w(\psi(x)), J_\psi(x)u\rangle = \frac{d}{dt}(w\circ\alpha)_{|t=0} = \frac{d}{dt}((w\circ\psi)\circ r)_{|t=0} = \langle D(w\circ\psi)(x), u\rangle.$$

Given $\varphi \in W^{1,\infty}(\mathcal{S})$, multiplying in (4.28) by $\varphi\circ\psi$ and integrating by parts, we get

$$\begin{aligned}
&\int_\Omega a(x)\langle g^{-1}(x)Du_\infty(x); D(\varphi\circ\psi)(x)\rangle dx = \int_\Omega \varphi(\psi(x))(f^-(x) - f^+(x))dx \\
&\qquad = \int_{\mathcal{S}} \varphi(z)(\tilde{f}^-(z) - \tilde{f}^+(z))\,dvol(z).
\end{aligned}$$

On the other hand, applying two times (4.30), we get

$$\begin{aligned}
&\int_\Omega a(x)\langle g^{-1}(x)Du_\infty(x); D(\varphi\circ\psi)(x)\rangle dx \\
&\qquad = \int_\Omega a(x)\langle J_\psi(x)(J_\psi(x)^tJ_\psi(x))^{-1}Du_\infty(x); \nabla\varphi(\psi(x))\rangle dx \\
&\qquad = \int_\Omega a(x)\langle J_{\psi^{-1}}(\psi(x))^tDu_\infty(x); \nabla\varphi(\psi(x))\rangle dx \\
&\qquad = \int_\Omega \sqrt{|g|(x)}\tilde{a}(\psi(x))\langle \nabla\tilde{u}_\infty(\psi(x)); \nabla\varphi(\psi(x))\rangle dx \\
&\qquad = \int_S \tilde{a}(z)\langle\nabla\tilde{u}_\infty(z); \nabla\varphi(z)\rangle\,dvol(z).
\end{aligned}$$

Consequently,

$$-\mathrm{div}(\tilde{a}\nabla\tilde{u}_\infty) = \tilde{f}^- - \tilde{f}^+ \quad \text{in the weak sense.}$$

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Moreover, by (4.29), if $\tilde{a}(z) > 0$ then $\langle \nabla \tilde{u}_\infty(z); \nabla \tilde{u}_\infty(z) \rangle = 1$. Observe that this is the formulation given in [18].

4.3. **Optimal mass transport maps.** Let us point out that Feldman and McCann in [18], by using Kantorovich potentials, find an optimal transport map $\tilde{T}_0 : \mathcal{S} \to \mathcal{S}$ which solves the Monge's problem

$$\min_{\tilde{T}\#\tilde{f}^+ = \tilde{f}^-} \left\{ \int_{\mathcal{S}} d_{I_M,\mathcal{S}}(\xi, \tilde{T}(\xi)) \tilde{f}^+(\xi) \, dvol(\xi) \right\}.$$

Here we have presented a way to obtain Kantorovich potentials taking limit of p–Laplacian type problems by using the idea of Evans and Gangbo in [17].

On existence of optimal transport maps see also [8] and [19] for Tonelli Lagrangians with superlinear growth. Existence of an optimal transport map in Finsler manifolds is obtained in [26] in the case that the Finsler structure is independent of x and for quadratic cost functions. The Lagrangian $F(x, \xi)$ treated here has not superlinear growth.

5. Characterization of the Kantorovich potentials

In this section we shall see that the results obtained in Section 4 characterize the Kantorovich potentials for the transport problem we are dealing here. Similar results have been obtained by A. Pratelli in [28], with different methods, in the context of Riemannian manifold, and for symmetric Finsler structures.

Remark 5.1. Thanks to Remark 3.4, the results of Theorems 4.3 and 4.7 remain true if we assume that $f^\pm = f_p^\pm$ and

$$f_p^\pm \rightharpoonup f^\pm, \quad \text{weakly in } L^2(\Omega).$$

Lemma 5.2. *Let v_p be the solution of*

$$\tilde{\Theta}_{p,g}(v_p) = \min_{v \in S_p} \tilde{\Theta}_{p,g}(v),$$

where

$$\tilde{\Theta}_{p,g}(v) = \int_\Omega \frac{[F^*(x, Dv)]^p}{p} - \frac{1}{2} \int_\Omega |v - g|^2$$

and $g \in L^2(\Omega)$ is a given function with $\int_\Omega g = 0$. Then, there exists a subsequence $p_j \to \infty$ such that

$$v_{p_j} \to v_\infty = I\!P_{K_F^*(\Omega)}(g), \quad \text{uniformly in } \Omega,$$

where $I\!P_{K_F^*(\Omega)}$ is the projection in $L^2(\Omega)$ on the convex set $K_F^*(\Omega)$.

Proof. It is easy to see that v_p is bounded in $L^2(\Omega)$, so that there exists a subsequence $p_j \to \infty$, such that $v_{p_j} \rightharpoonup v_\infty$ in weakly in $L^2(\Omega)$. Note that

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v_p is a minimizer of the functional Θ_{p,f_p}, defined by (3.4)), for $f_p = g - v_p$. Then, applying Theorem 4.3 (see Remark 5.1), we have

$$v_\infty \in K_F^*(\Omega), \tag{5.31}$$

and also that, there exists $\mathcal{X}_\infty \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ such that

$$\int_\Omega (g - v_\infty) v = \int_{\overline{\Omega}} Dv \, d\mathcal{X}_\infty \quad \forall\, v \in C^1(\overline{\Omega}). \tag{5.32}$$

On the other hand, by Theorem 4.7 (see Remark 5.1),

$$\int_{\overline{\Omega}} F(x, \mathcal{X}_\infty) = \int_\Omega (g - v_\infty) v_\infty.$$

From (5.32), for $v \in K_F^*(\Omega)$, we obtain that (after a regularization approach using Lemma 4.6):

$$\int_\Omega (g - v_\infty) v \le \int_{\overline{\Omega}} F(x, \mathcal{X}_\infty) = \int_\Omega (g - v_\infty) v_\infty. \tag{5.33}$$

Now, (5.31) and (5.33) gives

$$v_\infty = I\!P_{K_F^*(\Omega)} g,$$

as we wanted to show. $\square$

Theorem 5.3. *The following assertions are equivalent:*

1. u is a Kantorovich potential for the mass transport problem of f^+ to f^- with cost given by the Finsler distance given in (1.3).

2. $v \in K_{F^*}$ and there exists $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$, satisfying

$$(C1) \quad \left\{ \begin{array}{l} \displaystyle\int_\Omega (f^- - f^+) v = \int_{\overline{\Omega}} Dv \, d\mathcal{X} \quad \forall\, v \in C^1(\overline{\Omega}), \\ \displaystyle\int_\Omega (f^- - f^+)\, u = \int_{\overline{\Omega}} F(x, \mathcal{X}). \end{array} \right.$$

3. $u \in K_{F^*}$ and there exist $\nu \in \mathcal{M}(\overline{\Omega})^+$ and $\Lambda \in L^1_\nu(\overline{\Omega}, \mathbb{R}^N)$ such that

$$(C2) \quad \left\{ \begin{array}{l} \displaystyle\int_\Omega (f^- - f^+) v = \int_{\overline{\Omega}} \Lambda \cdot Dv \, d\nu \quad \forall\, v \in C^1(\overline{\Omega}), \\ \Lambda(x) \in \partial_{F,\nu} u(x) \quad \text{and} \quad F\left(x, \Lambda(x)\right) = 1 \quad \nu - a.e. \text{ in } \overline{\Omega}. \end{array} \right.$$

Proof. First of all observe that

$$u \text{ is a Kantorovich potential} \iff u = I\!P_{K_F^*}(f + u). \tag{5.34}$$

$\boxed{2 \Rightarrow 1}$ From $(C1)$, using Lemma 4.6, it is not difficult to see that

$$\int_\Omega (f^- - f^+) v \le \int_{\overline{\Omega}} F(x, \mathcal{X}) = \int (f^- - f^+) u \quad \forall v \in K_{F^*},$$

then v is a Kantorovich potential.

$\boxed{1 \Rightarrow 2}$ Take v_p a weak solution of the following problem of $p-$Laplacian type

$$\begin{cases} v_p - \operatorname{div}\left([F^*(x, Dv_p(x))]^{p-1} \dfrac{\partial F^*}{\partial \xi}(x, Dv_p(x)) \right) = f + u & \text{in } \Omega, \\ [F^*(x, Dv_p(x))]^{p-1} \langle \dfrac{\partial F^*}{\partial \xi}(x, Dv_p(x)); \eta \rangle = 0 & \text{on } \partial\Omega. \end{cases}$$

Then, by Lemma 5.2 and (5.34), we have that

$$\lim_{p\to\infty} v_p(x) = I\!\!P_{K_{F^*}}(u+f) = u \quad \text{uniformly in } \Omega.$$

Finally, taking into account Remark 5.1, we can also get $\mathcal{X} \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^N)$ satisfying $(C1)$.

$\boxed{3 \Rightarrow 2}$ If we set $\mathcal{X} := \Lambda\nu$, it is enough to show that

$$\int_{\overline{\Omega}} F(x, \mathcal{X}) = \int_{\overline{\Omega}} F(x, \Lambda) d\nu = \int (f^- - f^+)\, u.$$

By (4.14), there exist smooth functions v_ϵ such that

$$\begin{cases} v_\epsilon \to u & \text{uniformly} \\ D_\mu v_\epsilon \rightharpoonup D_\mu u & \sigma(L^\infty_\mu, L^1_\mu). \end{cases}$$

Then, taking $v = v_\epsilon$ in $(C2)$,

$$\int_\Omega (f^- - f^+) v_\epsilon = \int_{\overline{\Omega}} \Lambda \cdot Dv_\epsilon \, d\nu = \int_{\overline{\Omega}} \Lambda \cdot D_\nu v_\epsilon \, d\nu,$$

and taking limits, we get

$$\int_\Omega (f^- - f^+) u = \int_{\overline{\Omega}} \Lambda \cdot D_\nu u \, d\nu. \tag{5.35}$$

Now, working as in the proof of (4.16) we get

$$D_\nu u(x) \cdot v(x) \le F(x, v(x)) \quad \nu - \text{a.e.}$$

for any $v(x) \in T_\nu(x) \quad \nu -$ a.e. This implies that

$$F(x, \Lambda(x)) = \Lambda(x) \cdot D_\nu u(x) \quad \nu - \text{a.e. in } \overline{\Omega}.$$

Going back to (5.35) and using again $(C2)$, we get

$$\int_\Omega (f^- - f^+) u = \int_{\overline{\Omega}} F(x, \Lambda) \, d\nu.$$

$\boxed{2 \Rightarrow 3}$ Take $\nu = F(x, \mathcal{X})$ and $\Lambda = \frac{d\mathcal{X}}{d\nu}$. We only need to show that

$$\Lambda(x) \in \partial_{F,\nu} u(x) \quad \text{and} \quad F(x, \Lambda(x)) = 1 \quad \nu - \text{a.e. in } \overline{\Omega}.$$

Now, this can be prove as in Theorem 4.8 changing u_∞ by u. $\square$

6. The Benamou-Brenier approach

Proof of Theorem 1.3. By (4.3), we have that for $f(t) := f^+ + t(f^- - f^+)$, and $E(t) := \mathcal{X}_\infty$ for $t \in [0,1]$, (f, E) is a solution of problem (1.4). Then, from (4.7), it follows that

$$\min\{J_F(f,E) \,:\, (f,E) \text{ is a solution of } (1.4)\}$$
$$\leq |\mathcal{X}_\infty|_F = \min\{\mathcal{K}_{c_F}(\mu) \,:\, \mu \in \Pi(f^+, f^-)\}.$$

To prove the reverse inequality, take v_ϵ the approximation given in Lemma 4.6 for $u = u_\infty$. Then, given (f, E) a solution of (1.4), we have

$$\begin{aligned} &\min\{\mathcal{K}_c(\mu) \,:\, \mu \in \Pi(f^+, f^-)\} = \int_\Omega u_\infty (f^- - f^+) \\ &= -\int_\Omega \int_0^1 u_\infty \frac{\partial f}{\partial t} = -\lim_{\epsilon \to 0} \int_\Omega \int_0^1 v_\epsilon \frac{\partial f}{\partial t} = \lim_{\epsilon \to 0} \int_0^1 \int_{\overline{\Omega}} \nabla v_\epsilon dE(t) \\ &\leq \int_0^1 |E(t)|_F \leq J_F(f,E), \end{aligned}$$

and consequently

$$\min\{\mathcal{K}_{c_F}(\mu) \,:\, \mu \in \Pi(f^+, f^-)\}$$
$$\leq \min\{J_F(f,E) \,:\, (f,E) \text{ is a solution of } (1.4)\}.$$

This ends the proof. □

We say that the Finsler strucrure F is *geodesically complete* if for any $x, y \in \Omega$ there exists $\sigma_{x,y} \in \Gamma^\Omega_{x,y}$ such that

$$c_F(x,y) = \inf_{\sigma \in \Gamma^\Omega_{x,y}} \int_0^1 F((\sigma(t)), \sigma'(t))\, dt = \int_0^1 F((\sigma_{x,y}(t)), \sigma'_{x,y}(t))\, dt.$$

Theorem 6.1. *Assume $F^*(x, \cdot) \in C^1(\mathbb{R}^N \setminus \{0\})$ and also that F is geodesically complete. For any transport plan $\gamma \in \Pi(f^+, f^-)$ we define the measures*

$$f(t) := \pi_t \# \gamma, \quad E(t) := \pi_t \# \left(\sigma'_{x,y}(t)\gamma\right),$$

with $\pi_t(x,y) := \sigma_{x,y}(t)$. Then (f, E) is a solution of (1.4). Moreover, if γ is an optimal transport plan, then

$$J_F(f,E) = \min\{\mathcal{K}_{c_F}(\mu) \,:\, \mu \in \Pi(f^+, f^-)\}. \tag{6.36}$$

Proof. Let $\gamma \in \Pi(f^+, f^-)$ a transport plan. Given $\phi \in C^1(\overline{\Omega})$,

$$\begin{aligned} &\frac{d}{dt} \int_{\overline{\Omega}} \phi df(t) = \frac{d}{dt} \int_{\overline{\Omega} \times \overline{\Omega}} \phi(\sigma_{x,y}(t)) d\gamma(x,y) \\ &= \int_{\overline{\Omega} \times \overline{\Omega}} \nabla \phi(\sigma_{x,y}(t)) \sigma'_{x,y}(t) d\gamma(x,y) = \int_{\overline{\Omega}} \nabla \phi dE(t), \end{aligned}$$

hence (f,E) is a solution of (1.4). Suppose now that γ is an optimal transport plan. Then,

$$
\begin{aligned}
&|E(t)|_F \\
&= \sup\left\{ \int_{\overline{\Omega}} \Phi dE(t) \,:\, \Phi \in C(\Omega,\mathbb{R}^N),\ \text{with } \Phi(x) \in B_{F^*(x,\cdot)},\ \forall\, x \in \Omega \right\} \\
&= \sup\Big\{ \int_{\overline{\Omega}\times\overline{\Omega}} \langle \Phi(\sigma_{x,y}(t)), \sigma'_{x,y}(t)\rangle d\gamma(x,y) \ :\ \Phi \in C(\Omega,\mathbb{R}^N), \\
&\qquad\qquad\qquad\qquad \text{with } \Phi(x) \in B_{F^*(x,\cdot)},\ \ \forall\, x \in \Omega \Big\}.
\end{aligned}
$$

Now, by (2.1), we have

$$
\begin{aligned}
&\langle \Phi(\sigma_{x,y}(t)), \sigma'_{x,y}(t)\rangle \\
&\le F(\sigma_{x,y}(t), \sigma'_{x,y}(t)) F^*(\sigma_{x,y}(t), \Phi(\sigma_{x,y}(t)) \\
&\le F(\sigma_{x,y}(t), \sigma'_{x,y}(t)).
\end{aligned}
$$

Thus,

$$
|E(t)|_F \le \int_{\overline{\Omega}\times\overline{\Omega}} F(\sigma_{x,y}(t), \sigma'_{x,y}(t)) d\gamma(x,y),
$$

and then

$$
\begin{aligned}
J_F(f,E) &= \int_0^1 |E(t)|_F \, dt \le \int_0^1 \int_{\overline{\Omega}\times\overline{\Omega}} F(\sigma_{x,y}(t), \sigma'_{x,y}(t)) d\gamma(x,y) \\
&= \int_{\overline{\Omega}\times\overline{\Omega}} c_F(x,y) d\gamma(x,y) = \min\{\mathcal{K}_{c_F}(\mu) \,:\, \mu \in \Pi(f^+, f^-)\}.
\end{aligned}
$$

Therefore, by Theorem 1.3, we get (6.36). □

7. Extensions to Riemannian manifolds

In this section we briefly comment on the extension of our results to the case in which the optimal transport problem takes place on a Riemannian manifold. For such extension we use ingredients of the general theory of Sobolev spaces on Riemannian manifolds and we refer to [20] for details.

We deal with a Riemannian manifold M of dimension N with a metric tensor g_{ij} and a compatible measure μ (that is, a measure such that the measure of a geodesic ball of radius r is comparable with r^N). The manifold M is assumed to be compact but we let that it may have boundary or not. We also have that $Vol_\mu(M) = \int_M d\mu$ is finite.

On this manifold we have a Finsler structure, that is, a function $F(x,\xi)$ that for each $x \in M$ is a Finsler function on $\xi \in T_xM$. Using the Riemannian inner product in the tangent plane we can define the dual Finsler structure $F^*(x,\xi)$ (that gives also a Finsler function on T_xM for every $x \in M$).

Associated to this Finsler structure we can define the cost c exactly as we did before. Given $x, y \in M$, let

$$\Gamma_{x,y}^{M} := \{\sigma \in C^1([0,1], M),\ \sigma(0) = x,\ \sigma(1) = y\},$$

and define

$$c_F(x,y) := \inf_{\sigma \in \Gamma_{x,y}^M} \int_0^1 F((\sigma(t)), \sigma'(t))\, dt. \tag{7.1}$$

Now, our mass transport problem reads as follows: given f_+ and f_- with the same total mass, find T an optimal transport map, that is, a minimizer of

$$\min_{T\# f^+ = f^-} \int_M c_F(x, T(x)) f^+(x)\, d\mu.$$

In this setting we can consider the following variational problem: for $p > N$, minimize

$$\int_M \frac{[F^*(x, Du)]^p}{p} d\mu - \int_M uf\, d\mu.$$

in the set $S_p = \{u \in W^{1,p}(M) : \int_M u\, d\mu = 0\}$. Here, as before, $f = f^- - f^+$.

For minimizers of this functional (that can be proved to exists as in Lemma 3.1) one can show with the same computations of Lemma 3.3 that there exists a subsequence $p_j \to \infty$ such that

$$u_{p_j} \rightrightarrows u_\infty$$

uniformly in M. Moreover, the limit u_∞ is Lipschitz continuous.

In addition, it can be proved as in Section 4 that u_∞ is a Kantorovich potential for the mass transport problem of f^+ to f^- with cost given by the Finsler distance given in (7.1), that is, u_∞ maximizes

$$\int_M v(f^- - f^+)\, d\mu,$$

in the set $K_{c_F}(M) := \{u : M \mapsto \mathbb{R} : u(y) - u(x) \le c_F(x,y)\}$.

Acknowledgments: We would like to thanks to Sergio Segura de León for some suggestions. J.M.M. and J.T. have been partially supported by the Spanish MEC and FEDER, project MTM2012-31103. J.D.R. has been partially supported by MEC MTM2010-18128 and MTM2011-27998 (Spain).

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