We present sharp lower bounds for eigenvalues of the one-dimensional $p$-Laplace operator. The method of proof is rather elementary, based on a suitable generalization of the Lyapunov inequality.

1. Introduction

In [9], Krein obtained sharp lower bounds for eigenvalues of weighted second-order Sturm-Liouville differential operators with zero Dirichlet boundary conditions. In this paper, we give a new proof of this result and we extend it to the one-dimensional $p$-Laplacian

$$\begin{align*}
-\left(|u'(x)|^{p-2}u'(x)\right)' &= \lambda r(x)|u(x)|^{p-2}u(x), \quad x \in (a,b), \\
u(a) &= 0, \quad u(b) = 0,
\end{align*}$$

(1.1)

where $\lambda$ is a real parameter, $p > 1$, and $r$ is a bounded positive function. The method of proof is based on a suitable generalization of the Lyapunov inequality to the nonlinear case, and on some elementary inequalities. Our main result is the following theorem.

**Theorem 1.1.** Let $\lambda_n$ be the $n$th eigenvalue of problem (1.1). Then,

$$\frac{2^n n^p}{(b-a)^{p-1} \int_a^b r(x)dx} \leq \lambda_n.$$  

(1.2)

We also prove that the lower bound is sharp.

Eigenvalue problems for quasilinear operators of $p$-Laplace type like (1.1) had received considerable attention in the last years (see, e.g., [1, 2, 3, 5, 8, 13]). The asymptotic behavior of eigenvalues was obtained in [6, 7].
2 Lower bounds for eigenvalues

Lyapunov inequalities have proved to be useful tools in the study of qualitative nature of solutions of ordinary linear differential equations. We recall the classical Lyapunov’s inequality.

**Theorem 1.2 (Lyapunov).** Let \( r : [a,b] \rightarrow \mathbb{R} \) be a positive continuous function. Let \( u \) be a solution of

\[
-u''(x) = r(x)u(x), \quad x \in (a,b), \quad u(a) = 0, \quad u(b) = 0.
\]

Then, the following inequality holds:

\[
\int_a^b r(x)dx \geq \frac{4}{b-a}.
\]

For the proof, we refer the interested reader to [10, 11, 12]. We wish to stress the fact that those proofs are based on the linearity of (1.3), by direct integration of the differential equation. Also, in [12], the special role played by the Green function \( g(s,t) \) of a linear differential operator \( L(u) \) was noted, by reformulating the Lyapunov inequality for

\[
L(u)(x) - r(x)u(x) = 0
\]

as

\[
\int_a^b r(x)dx \geq \frac{1}{\text{Max} \{g(s,s) : s \in (b-a)\}}.
\]

The paper is organized as follows. Section 2 is devoted to the Lyapunov inequality for the one-dimensional \( p \)-Laplace equation. In Section 3, we focus on the eigenvalue problem and we prove Theorem 1.1.

2. The Lyapunov inequality

We consider the following quasilinear two-point boundary value problem:

\[
-(|u'|^{p-2}u')' = r|u|^{p-2}u, \quad u(a) = 0 = u(b),
\]

where \( r \) is a bounded positive function and \( p > 1 \). By a solution of problem (2.1), we understand a real-valued function \( u \in W_0^{1,p}(a,b) \), such that

\[
\int_a^b |u'|^{p-2}u'v' = \int_a^b r|u|^{p-2}uv \quad \text{for each} \ v \in W_0^{1,p}(a,b).
\]

The regularity results of [4] imply that the solutions \( u \) are at least of class \( C^{1,\alpha}_{\text{loc}} \), and satisfy the differential equation almost everywhere in \((a,b)\).

Our first result provides an estimation of the location of the maxima of a solution in \((a,b)\). We need the following lemma,
Lemma 2.1. Let \( r : [a, b] \to \mathbb{R} \) be a bounded positive function, let \( u \) be a solution of problem (2.1), and let \( c \) be a point in \((a, b)\) where \(|u(x)|\) is maximized. Then, the following inequalities holds:

\[
\int_a^c r(x)dx \geq \left( \frac{1}{c-a} \right)^{p/q}, \\
\int_c^b r(x)dx \geq \left( \frac{1}{b-c} \right)^{p/q},
\]

where \( q \) is the conjugate exponent of \( p \), that is, \( 1/p + 1/q = 1 \).

Proof. Clearly, by using Hölder’s inequality,

\[
u(c) = \int_a^c u'(x)dx \leq (c-a)^{1/q} \left( \int_a^c |u'(x)|^p dx \right)^{1/p}.
\]

We note that \( u'(c) = 0 \). So, integrating by parts in (2.1) after multiplying by \( u \) gives

\[
\int_a^c |u'(x)|^p dx = \int_a^c r(x) |u(x)|^p dx.
\]

Thus,

\[
u(c) \leq (c-a)^{1/q} \left( \int_a^c r(x) |u(x)|^p dx \right)^{1/p} \leq (c-a)^{1/q} |u(c)| \left( \int_a^c r(x)dx \right)^{1/p}.
\]

Then, the first inequality follows after cancelling \( u(c) \) in both sides while the second is proved in a similar fashion. \(\square\)

Remark 2.2. The sum of both inequalities shows that \( c \) cannot be too close to \( a \) or \( b \). We have \( \int_a^b r(x)dx < \infty \), but

\[
\lim_{c \to a^+} \left[ \left( \frac{1}{c-a} \right)^{p/q} + \left( \frac{1}{b-c} \right)^{p/q} \right] = \lim_{c \to b^-} \left[ \left( \frac{1}{c-a} \right)^{p/q} + \left( \frac{1}{b-c} \right)^{p/q} \right] = \infty.
\]

Our next result gives the Lyapunov inequality.

Theorem 2.3. Let \( r : [a, b] \to \mathbb{R} \) be a bounded positive function, let \( u \) be a solution of problem (2.1), and let \( q \) be the conjugate exponent of \( p \in (1, +\infty) \). The following inequality holds:

\[
\frac{2^p}{(b-a)^{p/q}} \leq \int_a^b r(x)dx.
\]

Proof. For every \( c \in (a, b) \), we have

\[
2 |u(c)| = \left| \int_a^c u'(x)dx \right| + \left| \int_c^b u'(x)dx \right| \leq \int_a^b |u'(x)| dx.
\]
4 Lower bounds for eigenvalues

By using Hölder’s inequality,

\[
2 |u(c)| \leq (b - a)^{1/q} \left( \int_a^b |u'(x)|^p \, dx \right)^{1/p} \\
= (b - a)^{1/q} \left( \int_a^b r(x) |u(x)|^p \, dx \right)^{1/p}.
\]

(2.10)

We now choose \( c \) in \((a, b)\) such that \(|u(x)|\) is maximized. Then,

\[
2 |u(c)| \leq (b - a)^{1/q} |u(c)| \left( \int_a^b r(x)dx \right)^{1/p}.
\]

(2.11)

After cancelling, we obtain

\[
\frac{2p}{(b - a)^{p/q}} \leq \int_a^b r(x)dx,
\]

(2.12)

and the theorem is proved. □

Remark 2.4. We note that, for \( p = 2 = q \), inequality (2.8) coincides with inequality (1.4).

3. Eigenvalues bounds

In this section, we focus on the following eigenvalue problem:

\[
-\left(|u'|^{p-2}u'\right)' = \lambda r|u|^{p-2}u,
\]

\[
u(a) = 0 = u(b),
\]

(3.1)

where \( r \in L^\infty(a, b) \) is a positive function, \( \lambda \) is a real parameter, and \( p > 1 \).

Remark 3.1. The eigenvalues could be characterized variationally:

\[
\lambda_k(\Omega) = \inf_{F \in C_k^\Omega} \sup_{u \in F} \int_\Omega |u'|^p,
\]

(3.2)

where

\[
C_k^\Omega = \{ C \subset M^\Omega : C \text{ compact, } C = -C, y(C) \geq k \},
\]

\[
M^\Omega = \left\{ u \in W_0^{1,p} (\Omega) : \int_\Omega |u'|^p = 1 \right\},
\]

(3.3)

and \( y : \Sigma \to \mathbb{N} \cup \{ \infty \} \) is the Krasnoselskii genus,

\[
y(A) = \min \{ k \in \mathbb{N} \text{ there exist } f \in C(A, \mathbb{R}^k \setminus \{0\}), f(x) = -f(-x) \}. \]

(3.4)

The spectrum of problem (1.1) consists of a countable sequence of nonnegative eigenvalues \( \lambda_1 < \lambda_2 < \cdots < \lambda_k < \cdots \), and coincides with the eigenvalues obtained by Ljusternik-Schnirelmann theory.
Now, we prove the lower bound for the eigenvalues of problem (3.1) for every $p \in (1, +\infty)$. We recall Theorem 1.1.

**Theorem 3.2.** Let $\lambda_n$ be the $n$th eigenvalue of problem (3.1). Then,

$$
\frac{2^p n^p}{(b-a)^{p-1} \int_a^b r(x) \, dx} \leq \lambda_n.
$$

(3.5)

**Proof.** Let $\lambda_n$ be the $n$th eigenvalue of problem (3.1) and let $u_n$ be an associate eigenfunction. As in the linear case, $u_n$ has $n$ nodal domains in $[a, b]$ (see [2, 13]).

Applying inequality (2.8) in each nodal domain, we obtain

$$
\sum_{k=1}^{n} \frac{2^p}{(x_k - x_{k-1})^{p/q}} \leq \lambda_n \sum_{k=1}^{n} \left( \int_{x_{k-1}}^{x_k} r(x) \, dx \right) \leq \lambda_n \int_a^b r(x) \, dx,
$$

(3.6)

where $a = x_0 < x_1 < \cdots < x_n = b$ are the zeros of $u_n$ in $[a, b]$.

Now, the sum on the left-hand side is minimized when all the summands are the same, which gives the lower bound

$$
2^p n \left( \frac{n}{b-a} \right)^{p/q} \leq \lambda_n \int_a^b r(x) \, dx.
$$

(3.7)

The theorem is proved. □

Finally, we prove that the lower bound is sharp.

**Theorem 3.3.** Let $\epsilon \in \mathbb{R}$ be a positive number. There exist a family of weight functions $r_{n, \epsilon}$ such that

$$
\lim_{\epsilon \to 0^+} \left( \lambda_{n, \epsilon} - \frac{2^p n^p}{(b-a)^{p-1} \int_a^b r_{n, \epsilon}} \right) = 0,
$$

(3.8)

where $\lambda_{n, \epsilon}$ is the $n$th eigenvalue of

$$
-(|u'|^{p-2}u')' = \lambda r_{n, \epsilon} |u|^{p-2}u,
$$

$$
u(a) = 0 = u(b).
$$

(3.9)

**Proof.** We begin with the first eigenvalue $\lambda_1$. We fix $\int_a^b r(x) \, dx = M$, and let $c$ be the midpoint of the interval $(a,b)$.

Let $r_1$ be the delta function $M\delta_c(x)$. We obtain

$$
\lambda_1 = \min_{u \in W_{0}^{1,p}} \frac{\int_a^b |u'|^p}{\int_a^b \delta_c u} = \min_{u \in W_{0}^{1,p}} \frac{2\int_a^c |u'|^p}{M u(c)} = \frac{2\mu_1}{M},
$$

(3.10)

where $\mu_1$ is the first Steklov eigenvalue in $[a, c]$,

$$
-(|u'(x)|^{p-2}u'(x))' = 0,
$$

$$
|u'(c)|^{p-2}u'(c) = \mu |u(c)|^{p-2}u(c), \quad u(a) = 0.
$$

(3.11)
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A direct computation gives

\[ \mu_1 = \frac{2^{p-1}}{(b-a)^{p-1}}. \tag{3.12} \]

Now, we define the functions \( r_{1, \varepsilon} \):

\[
r_{1, \varepsilon} = \begin{cases} 
0 & \text{for } x \in \left[ a, \frac{a+b}{2} - \varepsilon \right], \\
M \varepsilon & \text{for } x \in \left[ \frac{a+b}{2} - \varepsilon, \frac{a+b}{2} + \varepsilon \right], \\
0 & \text{for } x \in \left[ \frac{a+b}{2} + \varepsilon, b \right],
\end{cases} \tag{3.13}
\]

and the result follows by testing, in the variational formulation (3.2), the first Steklov eigenfunction

\[
u(x) = \begin{cases} 
x - a & \text{if } x \in \left[ a, \frac{a+b}{2} \right], \\
b - x & \text{if } x \in \left[ \frac{a+b}{2}, b \right].
\end{cases} \tag{3.14}
\]

Thus, the inequality is sharp for \( n = 1 \).

We now consider the case \( n \geq 2 \). We divide the interval \((a, b)\) into \( n \) subintervals \( I_i \) of equal length, and let \( c_i \) be the midpoint of the \( i \)th subinterval.

By using a symmetry argument, the \( n \)th eigenvalue corresponding to the weight

\[
r_n(x) = \frac{M}{n} \sum_{i=1}^{n} \delta_{c_i}(x), \tag{3.15}
\]

restricted to \( I_i \), is the first eigenvalue in this interval, that is,

\[ \lambda_n = \frac{2n \mu_1}{M} = \frac{2^p n^p}{M(b-a)^{p-1}}. \tag{3.16} \]

The proof is now completed. \( \square \)

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Juan Pablo Pinasco: Departamento de Matemática, Universidad de Buenos Aires, Pabellon 1, Ciudad Universitaria, 1428 Buenos Aires, Argentina

Current address: Instituto de Ciencias, Universidad Nacional de General Sarmiento, J.M. Gutierrez 1150, Los Polvorines, 1613 Buenos Aires, Argentina

E-mail address: jpinasco@dm.uba.ar

We changed the format of the first author in [3] according to the MathSciNet database.

We changed the year in [10] and “Lyapunov” to “Liapounoff” according to the MathSciNet database.
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